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Stable Communication Topologies of a Formation of Satellites

M. Dellnitz, O. Junge, A. Krishnamurthy and R. Preis *

Institute for Mathematics, University of Paderborn, Germany

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Abstract: Several currently planned space missions consist of a set of satellites flying in formation. While increasing the functionality, this concept introduces several new challenges with respect to the design of the mission. The topology of the sensing or communication network among the satellites can be a bottleneck in the operation because the transmission of information and the coordination of the formation relies on it. Here we study the robustness of the formation dynamics with respect to changes in the communication topology (like the failure of some communication links). Moreover, we propose a special variant of the notion of stability radius in order to measure the robustness of a certain topology.

Keywords: *Formation; stability; stability radius.*

Mathematics Subject Classification (2000): 70M20, 70K20.

1 Introduction

Space missions with several spacecraft flying in formation have received a lot of attention recently. Increased functionality and robustness of the mission are two key characteristics of this approach. Several currently planned space missions consist of a set of satellites flying in formation, like, e.g., the NASA mission *Terrestrial Planet Finder (TPF)* and the ESA mission *Darwin*. In both missions, a network of formation flying spacecraft builds up an infrared interferometer in order to detect and study planets in outer space.

One key challenge in the design of these missions is the question on how to efficiently attain and accurately maintain the desired formation. In [1, 5] it has been shown that formation-stabilizing control laws can be derived for the individual spacecraft that rely on local information only. The key idea is that, together with the stability properties of the dynamics of the individual spacecraft, the spectrum of the Laplacian associated

* Corresponding author: {dellnitz,arvind,robsy,junge}@upb.de

to the graph that describes which spacecraft communicate with which other spacecraft plays a crucial role in the design of this control law.

In this paper, by means of a network of six spacecraft with simple linear dynamics, we study the robustness of certain communication graphs with respect to the removal of edges (i.e. the failure of some communication links). Based on stabilizability statements from [1, 5] we analyse how many communication links can fail before the dynamics of the overall system becomes unstable. For more complicated situations we propose an adapted version of the concept of stability radius of a linear system in order to measure this robustness.

2 Model

In the current mission design for Darwin and TPF it is planned to inject the spacecraft into a Libration orbit around the Lagrange point L_2. Close to this orbit, a time-dependent linear model may be used in order to describe the motion of the spacecraft [4]. For the purposes of this paper we restrict ourselves to a time-independent model as used in [5], i.e. we model the dynamics of each of the N vehicles by

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 1, \dots, N, \tag{1}$$

where $x_i \in R^6$ is the state and $u_i \in R^p$ for some p is the control of vehicle i and A and B are real matrices of appropriate size. As shown in [5], a linear local feedback can be designed which drives the system asymptotically into a prescribed *formation*, i.e. the vehicles attain prescribed distances relative to each other as well as the same velocity. We follow [5] in the following description.

The feedback law is local in the sense that each vehicle i can generate its own control u_i from the determination of its state relative to the states of some subset $S_i \subset \{1, \dots, N\}$ of all vehicles (obtained, e.g., by communicating with the vehicles in S_i). The i-th vehicle computes

$$z_i = (x_i - h_i) - \frac{1}{|S_i|} \sum_{j \in S_i} (x_j - h_j), \tag{2}$$

where $h_i \in R^6$ is some reference state for the i-th vehicle, and sets $u_i = Fz_i$ for some feedback matrix F.

Viewing the system as an undirected graph $G = (V, E)$, where the set of nodes $V = \{1, \dots, N\}$ represent the vehicles and the set of edges E represent communication links (i.e. $E = \{(i,j)\colon\ j \in S_i\}$), one can compactly write the system in the form

$$\dot{x} = \hat{A}x + \hat{B}\hat{F}\hat{L}(x - h). \tag{3}$$

Here $x = (x_1, \dots, x_N) \in R^{6N}$, $\hat{A} = I_N \otimes A$, $\hat{B} = I_N \otimes B$, $\hat{F} = I_N \otimes F$, $h = (h_1, \dots, h_N)$ and $\hat{L} = L \otimes I_6$, with L being the Laplacian of the graph G, i.e.

$$L_{ij} = \begin{cases} 1: & i = j, \\ -\frac{1}{|S_i|}: & j \in S_i, \\ 0: & j \notin S_i. \end{cases} \tag{4}$$

3 Robustness of Communication Topologies

In [5] it is shown that the vehicles are in formation if and only if $\hat{L}(x-h)=0$ and (under certain assumptions) that if the matrix $A+\lambda BF$ is stable for each nonzero eigenvalue λ of L, then $\hat{L}(x(t)-h)\to 0$ as $t\to\infty$, i.e. the vehicles asymptotically attain the desired formation. For a given communication graph, this result thus gives a criterion on how to design the feedback matrix F. In fact, under certain assumptions on the uncontrolled dynamics of a single system, i.e. on the matrix A, one can show that for every connected graph one can find a feedback matrix F which renders the closed loop system stable ([5], Proposition 4.4). What is more, under these assumptions, feedback matrices can in fact be constructed which render the system stable regardless of how the communication graph is chosen — as long as it is connected.

However, the choice of the feedback matrix F will depend on the single system dynamics (i.e. the matrix A) and, in particular in our application context, in a non-autonomous setting it may happen that A is changing in such a way that the overall system dynamics becomes unstable. In this case, the question arises which communication topology is best suited in the sense that it will ensure stability of the formation for the largest "range" of single system dynamics. What is more, taking into account that communication links may fail, the question is which topology is most robust with respect to such failures, i.e. ensures stability of the system even when a certain number of links fail.

In order to make these considerations more precise we focus on the following basic setting from [5]: we assume that each coordinate of the system is modelled by the same second order dynamics, i.e. we have

$$A = I_3 \otimes \begin{pmatrix} 0 & 1 \\ 0 & a_{22} \end{pmatrix} \quad \text{and} \quad B = I_3 \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Using $F = I_3 \otimes (f_1\ f_2)$ as the feedback matrix, our goal will thus be to render the matrix

$$H_\lambda = \begin{pmatrix} 0 & 1 \\ 0 & a_{22} \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 \\ f_1 & f_2 \end{pmatrix}$$

stable for each nonzero eigenvalue λ of the Laplacian L. This matrix will be stable if and only if

$$a_{22} + \lambda f_2 < 0 \quad \text{and} \quad \lambda f_1 < 0,$$

i.e. f_1 has to be chosen negative (since $\lambda \in [0,2]$ for all eigenvalues of L). If the single system dynamics is unstable, i.e. $a_{22} > 0$, then the eigenvalue λ of L which is closest to zero determines how f_2 has to be chosen in order to render the overall system stable.

Since we are assuming that all vehicles have identical dynamics and communication capabilities, it seems natural to restrict the choice of communication graphs to regular ones. Figure 3.1 shows all non-isomorphic connected regular (undirected) graphs with six nodes (see e.g. [6]). In the first column of Table 3.1 we list the corresponding minimal nonzero eigenvalues of the Laplacians (except for the 2-regular graph which becomes disconnected as soon as more than one edge is removed). The other columns show how these eigenvalues change when removing a certain number of edges from the corresponding graph (where we minimized over all possible removals).

From these values, the choice of the full graph appears to be the best one (as one might have expected), since it features the largest minimal nonzero eigenvalue (and thus, for a fixed f_2, allows for the largest value for a_{22}).

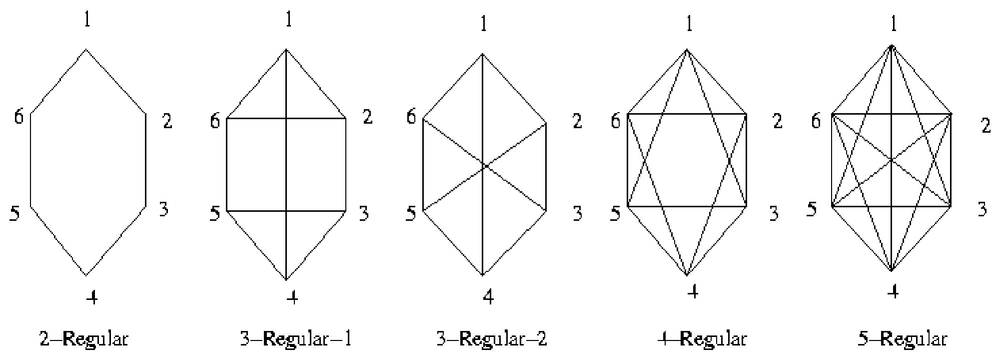

Figure 3.1: All non-isomorphic connected regular undirected graphs with six nodes.

graph \ # of edge failures	0	1	2	3	4
3 regular (1)	0.6670	0.4226	0.2047	0.1960	0.1910
3 regular (2)	1.0000	0.6670	0.5286	0.2929	0.1910
4 regular	1.0000	0.8104	0.6670	0.4610	0.2727
5 regular	1.2000	1.0000	0.8911	0.8104	0.7180

Table 3.1: The minimal nonzero eigenvalues for the communication graphs under consideration in dependence of the number of edge failures.

The full graph is also optimal if we choose f_2 dependent on the graph and ask for maximal robustness with respect to communication link failures, since the absolute decrease of the minimal nonzero eigenvalue is smallest for this graph. This fact is also visualized in Figure 3.2.

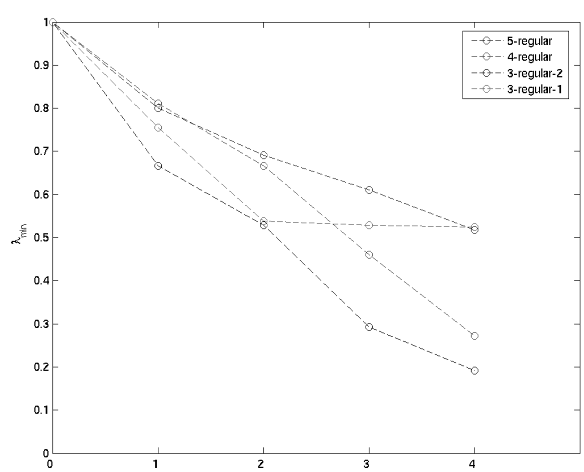

Figure 3.2: The plot of normalized λ_{min}.

4 Conclusion and Outlook

Stability radius. In the preceding section we have been considering a simple model in order to analyse robustness properties of certain communication topologies. In a more general setting it will be necessary to approach this question in a more systematic way. To this end, we propose to use the concept of the *stability radius* from control theory, see [3].

We denote by $\mathcal{U}_n$ the set of unstable real $n \times n$ matrices:

$$\mathcal{U}_n = \{U \in R^{n\times n}\colon\ \sigma(U) \cap \overline{\mathbb{C}}_+ \neq \varnothing\}, \tag{5}$$

where $\overline{\mathbb{C}}_+$ is the closed right half complex plane. For a general matrix $A \in R^{n\times n}$, the stability radius measures the distance $r(A)$ of A from the set $\mathcal{U}_n$ of unstable matrices,

$$r(A) = \inf_{U\in\mathcal{U}_n} \|A - U\|. \tag{6}$$

Proposition 4.1 ([2]) *Let $A \in R^{n\times n}$ be stable and normal with eigenvalues $\lambda_j = -\alpha_j \pm i\omega_j$, $\alpha_1 \geq \cdots \geq \alpha_n > 0$, then $r(A) = \alpha_n$.*

This proposition shows that for normal matrices A, the distance of A from the set of unstable matrices is given by the distance of its spectrum from the imaginary axis. If A is not normal, then the distance of $\sigma(A)$ from the imaginary axis can be a very misleading indicator of the "robustness" of A.

Motivated by an adaptation of this notion to structured systems [2] we we propose the following definition which is adapted to our context. Let $L(G)$ be the Laplacian associated with a given communication graph $G = (V, E)$ and let $\mathbb{L}(G)$ be the set of Laplacians associated with those graphs which result from G by removing some edges, i.e.

$$\mathbb{L}(G) = \{L(G')\colon\ G' = (V, E'), E' \subset E\}.$$

For some Laplacian $L' = L(G') \in \mathbb{L}(G)$ with $G' = (V, E')$ let

$$d(L') = |E| - |E'|$$

be the number of communication link failures. We define the stability radius $r(A, B, F, G)$ of a given system (A, B, F, G) as the minimum number of edges failures such that the system is unstable, i.e.

$$r(A, B, F, G) = \min_{L\in\mathbb{L}(G)} \{d(L)\colon\ \sigma(\hat{A} + \hat{B}\hat{F}\hat{L}) \cap \overline{\mathbb{C}_+} \neq \emptyset\}\,. \tag{7}$$

In future work we will explore the usefulness of this concept for the analysis of the robustness of certain communication topologies within systems with more complicated dynamics.

Conclusion. Using a simple linear model, we explored the robustness of different communication graphs with respect to failures of communication links. We introduced a variant of the notion of the stability radius of a given system as a means of systematically measuring the robustness for more complicated systems. It remains to explore this concept in an application scenario as well as to analyse nonautonomous systems like motivated by missions with formation flying spacecraft on Libration orbits.

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