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# Continuous-Time Optimal Portfolio Selection Using Mean-CaR Models

Zhong-Fei Li [1*], Kai W. Ng [2] and Xiao-Tie Deng [3]

[1] *Department of Risk Management and Insurance, Lingnan (University) College, Sun Yat-Sen University, Guangzhou 510275, People's Republic of China*
[2] *Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong*
[3] *Department of Computer Science, City University of Hong Kong, Kowloon, Hong Kong*

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**Abstract:** This paper studies continuous-time optimal portfolio selection under the setting of Black-Scholes financial markets and constant re-balanced portfolio (CRP) investment strategies. Three mean-CaR models are formulated, which minimize the risk measured by capital-at-risk (CaR) under the constraint that the expected terminal wealth is not lower than a pre-assigned level. These models are converted into bi-level optimization problems by virtue of a decomposition of the feasible solution set and, as a result, explicit optimal strategies and efficient frontiers are obtained in closed-form. A comparison of the three mean-CaR models and a numerical example illustrating the results are presented. Some economic implications of the results are also examined.

**Keywords:** *Continuous-time portfolio selection; Capital-at-Risk (CaR); Black-Scholes financial markets; constant-rebalanced portfolios (CRP); mean-CaR models.*

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* Corresponding author: lnslzf@zsu.edu.cn

## 1 Introduction

The pioneering work of Markowitz [13] introduced the mean-variance framework for portfolio selection and risk management. The mean-variance approach has become the foundation of modern finance theory and inspired a substantial number of extensions and applications in literature. From a theoretical point of view, there are two challenges. The first is the extension of the classical single-period mean-variance analysis to a multi-period or continuous-time mean-variance analysis. There is a considerable volume of literature on dynamic asset allocation. The main focus, however, is on maximizing some time-additive utility of terminal wealth and/or consumption (see, e.g., Merton [15, 16], Samuelson [18] and Smith [20]). At the same time, enormous difficulties in solving dynamic mean-variance problems were reported (see, e.g., Chen, Jen and Zionts [4]). Consequently, Markowitz's mean-variance formulation has not been fully exploited in dynamic cases for quite a long time since the dynamic mean-variance problems were set in a very general approach, by Schweizer [19] among others. Up to recently, the dynamic mean-variance problems have been solved analytically by Li and Ng [12] and Zhou and Li [21], respectively, in a discrete-time and a continuous-time frameworks.

The second challenge lies on appropriate measures of risk. The classical risk measure is the variance, as used in the mean-variance approach. However, the variance as a measure of risk has the drawback that it penalizes equally both upside and downside movement in the portfolio value. Realizing this, Markowitz [14] proposed semivariance as an alternative that measures risk as deviations below the mean only. Unfortunately he did not resolve the difficulties of the mean-semivariance framework caused by the non-differentiability in the setup. Consequently, other alternatives have been suggested, such as downside risk (see, e.g., Fishburn [5] and Harlow [9]), coherence risk (see, e.g., Artzner *et al.* [1]), the limited expected loss (see, e.g., Basak and Shapiro [2]), and so on. Among them, value-at-risk (VaR) (see, e.g., Jorior [11]) is the most prominent risk measure and has become an industry benchmark, which has been accepted by the regulators and banks in more than 100 countries around the world for controlling market risk.

Recently, Emmer, Klüppelberg and Korn [6, 7] developed the classic mean-variance method along the two clues mentioned above. In continuous-time financial markets with a Black-Scholes setting, they proposed a VaR-based related risk concepts known as capital-at-risk (CaR), which includes mainly three kinds of measures. Under constant rebalanced portfolio (CRP) investment strategies, they formulated two mean-CaR portfolio optimization models using the first two kinds of CaR as a replacement of the variance in a continuous-time mean-variance portfolio selection model, and derived analytically the optimal solutions for their models and the mean-variance model. Their solutions, however, involve a parameter that is a solution of a nonlinear algebra equation. In this sense, their solutions are not close-form. A possible reason for this is that they maximize the expected terminal wealth under the constraint that the CaR or the variance of the terminal wealth is not higher than a prescribed level.

In this paper, we reformulate the continuous-time mean-CaR portfolio selection models so as to minimize the risk measured by CaR under the constraint that the expected terminal wealth is not lower than a pre-assigned level. We aim at explicit expressions for optimal solutions and efficient frontiers in closed-form. We solve the mean-CaR model associated with the third kind of CaR and compare the three mean-CaR models. In addition to closed-form solutions, our approach has the advantage of easily comparing the optimal strategies to different mean-CaR models and the convenience of solving different mean-CaR models as they have the same set of feasible portfolios and hence can use the same decomposition of the feasible set. It is believed that the approach can be applied to some other continuous-time portfolio selection problems.

## 2 The financial market and CaR

Consider a standard Black-Scholes type financial market in which $n+1$ assets (or securities) are traded continuously in the horizon $[0,T]$ and indexed by $i = 0, 1, \dots, n$. One of the assets, say $i = 0$, is the riskless bond whose price process $P_0(t)$ evolves according to the following (deterministic) ordinary differential equation

$$dP_0(t) = P_0(t)rdt \quad \text{for} \quad t \in [0,T],$$

where $r$ is the rate of interest and is assumed to be constant. The other $n$ assets are risky stocks whose price processes $P_1(t), \dots, P_n(t)$ follow the following stochastic differential equations

$$dP_i(t) = P_i(t)\left(b_i dt + \sum_{j=1}^{n} \sigma_{ij} dB_j(t)\right) \quad \text{for} \quad t \in [0,T], \quad i = 1, \dots, n,$$

where $b = (b_1, \dots, b_n)'$ is the vector of stock-appreciation rate, $\sigma = (\sigma_{ij})_{n\times n}$ is the matrix of stock-volatilities and $B(t) = (B_1(t), \dots, B_n(t))'$ is a standard $n$-dimensional Brownian motion. Here $b$ and $\sigma$ are assumed to be constant in time. As usual, we further assume that $\sigma$ is invertible and that $b_i \geq r$ for each $i$.

Let $\pi_i(t)$ be the fraction of the wealth $W^\pi(t)$ invested in asset $i$ at time $t$. Let $\pi(t) = (\pi_1(t), \dots, \pi_n(t))' \in \mathbb{R}^n$. Then $\pi_0(t) = 1 - \pi(t)'\mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)'$ is the vector whose components are all units. The portfolio process $\pi(t)$ is called a portfolio strategy.

Throughout the paper, we assume that transaction costs and consumption are not considered and that portfolio strategy $\pi(t)$ is self-financing. Thus

$$dW^\pi(t) = W^\pi(t)\left\{((1 - \pi(t)'\mathbf{1})r + \pi(t)'b)dt + \pi(t)'\sigma dB(t)\right\}$$

with $W^\pi(0) = w > 0$ being the initial wealth of an investor.

In what follows, we restrict ourselves to constant-rebalanced portfolio (CRP) strategies. A CRP strategy is an investment strategy which keeps a fixed fraction of the wealth in each of the underlying stocks from time to time. Therefore, a CRP strategy employs the same investment vector $\pi(t) = \pi = (\pi_1, \dots, \pi_n)'$ at each $t$ in the planning horizon $[0,T]$. Such an investment strategy does not imply that there is no trading, since at each time instant $t$ the investment proportions are rebalanced back to the vector $\pi$. See an example in Helmbold *et al.* [10] for the power of CRP investment strategies.

Standard Itô integral and the fact that $E[e^{sB_j(t)}] = e^{ts^2/2}$, where $E$ is the expectation operator, yield the following explicit formulae for the wealth process $W^\pi(t)$ for all $t \in [0,T]$ (see, e.g., [6]).

$$\begin{aligned}
W^\pi(t) &= w\exp((\pi'(b - r\mathbf{1}) + r - \|\pi'\sigma\|^2/2)t + \pi'\sigma B(t)), && (2.1)\\
E[W^\pi(t)] &= w\exp((\pi'(b - r\mathbf{1}) + r)t), && (2.2)\\
Var[W^\pi(t)] &= w^2\exp(2(\pi'(b - r\mathbf{1}) + r)t)(\exp(\|\pi'\sigma\|^2 t) - 1), && (2.3)
\end{aligned}$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $Var$ is the variance operator.

Associated with a real number $\alpha \in (0,1)$, initial wealth $w$, time horizon $T$ and portfolio $\pi$, we denote by $\rho_0(\alpha,\pi,w,T)$ the $\alpha$-quantile of the terminal wealth $W^\pi(T)$, that is, it is implicitly defined by

$$P(W^\pi(T) \le \rho_0(\alpha,\pi,w,T)) = \alpha, \tag{2.4}$$

where $P(\cdot)$ is the probability. Using the notation $\rho_0$, the expected shortfall or more precisely the conditional tail expectation of $W^\pi(T)$ is defined as

$$\rho_1(\alpha,\pi,w,T) = E[W^\pi(T)|W^\pi(T) \le \rho_0(\alpha,\pi,w,T)]. \tag{2.5}$$

Furthermore, the conditional tail semi-standard derivation of $W^\pi(T)$ is defined as

$$\rho_2(\alpha,\pi,w,T) = \sqrt{E[(W^\pi(T))^2|W^\pi(T) \le \rho_0(\alpha,\pi,w,T)]}. \tag{2.6}$$

Using the risk measures $\rho_k(\alpha,\pi,w,T)$, $k=0,1,2$, Emmer, Klüppelberg and Korn [7] defined the Capital-at-Risk with respect to $\rho_k(\alpha,\pi,w,T)$ as its difference to the terminal wealth of the pure bond strategy.

**Definition 2.1 (Capital-at-Risk)** *The* Capital-at-Risk (CaR) *of a CRP investment strategy $\pi$ with respect to $\rho_k$ ($k=0,1,2$) with initial wealth $w$ and time horizon $T$ is the difference between the terminal wealth of the pure bond strategy and the risk measure $\rho_k$, i.e.,*

$$CaR_k(\pi) := w\exp(rT) - \rho_k(\alpha,\pi,w,T). \tag{2.7}$$

Let $z_\alpha$ be the $\alpha$-quantile of the standard normal distribution and $\Phi$ the distribution function of a standard normal random variable.

Since $\pi'\sigma B(T)/(\|\pi'\sigma\|\sqrt{T})$ is a standard normal random variable, by using (2.1) and (2.4)–(2.7), we can express explicitly the risk measures $\rho_k, k=0,1,2$ as (see [7])

$$\rho_0(\alpha,\pi,w,T) = w\exp\left(\left(\pi'(b-r\mathbf{1}) + r - \|\pi'\sigma\|^2/2\right)T + z_\alpha\|\pi'\sigma\|\sqrt{T}\right), \tag{2.8}$$

$$\rho_1(\alpha,\pi,w,T) = w\exp\left((\pi'(b-r\mathbf{1}) + r)T\right)\frac{\Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T})}{\alpha}, \tag{2.9}$$

$$\rho_2(\alpha,\pi,w,T) = w\exp\left((\pi'(b-r\mathbf{1}) + r + \|\pi'\sigma\|^2/2)T\right)\sqrt{\frac{\Phi(z_\alpha - 2\|\pi'\sigma\|\sqrt{T})}{\alpha}}. \tag{2.10}$$

Consequently, closed-form expressions of $CaR_k(\pi)$ for $k=0,1,2$ can be given.

To avoid discussions of some subcases, throughout this paper we make the following assumption.

**Assumption 2.1** *The parameter $\alpha$ satisfies $\alpha < 0.5$ and hence $z_\alpha < 0$.*

Denote by $\varphi$ the density function of a standard normal random variable.

**Lemma 2.1** *Let $x > 0$. Then*

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\varphi(x) < \Phi(-x) < \frac{\varphi(x)}{x}.$$

***Proof*** See Gänssler and Stute [8]. □

## 3 Mean-CaR portfolio selection

Emmer, Klüppelberg and Korn [6] solved the portfolio optimization problem that maximizes the expected terminal wealth under a given level of $CaR_0$, i.e.,

$$\max_{\pi\in\mathbb{R}^n} \; E[W^\pi(T)] \quad \text{subject to} \quad CaR_0(\pi) \le C. \tag{3.1}$$

Emmer, Klüppelberg and Korn [7] solved the portfolio optimization problem that maximizes the expected terminal wealth under a given level of $CaR_1$, i.e.,

$$\max_{\pi\in\mathbb{R}^n} \; E[W^\pi(T)] \quad \text{subject to} \quad CaR_1(\pi) \le C. \tag{3.2}$$

The two models are analogues of the Markowitz's mean-variance model that maximizes the expected terminal wealth under a given level of the variance of the terminal wealth. In this paper, we solve the portfolio optimization problem associated with $CaR_2$. However, our model is to minimize $CaR_2$ of the terminal wealth under a given level of the expected terminal wealth. This is an analogue of the Markowitz's mean-variance model that minimizes the variance of the terminal wealth under a given level of the expected terminal wealth. As an application of our method, we also solve the portfolio optimization models that minimizes respectively $CaR_0$ and $CaR_1$ under a given level of the expected terminal wealth. We refer the three portfolio optimization models as mean-CaR models.

### 3.1 Mean-$CaR_2$ portfolio selection

Consider the following mean-CaR model associated with $CaR_2$:

$$\min_{\pi\in\mathbb{R}^n} \; CaR_2(\pi) \quad \text{subject to} \quad E[W^\pi(T)] \ge C, \tag{3.3}$$

where $C > 0$ is a predetermined level of the expected terminal wealth $E[W^\pi(T)]$. Since the pure bond policy (i.e., the one that invests all of the wealth in the bond for the entire investment period) yields a deterministic terminal wealth of $w\exp(rT)$, throughout this paper we assume that the expected wealth level $C$ satisfies the following lower bound condition:

$$C \ge w\exp(rT). \tag{3.4}$$

Obviously, this is a reasonable assumption, for the solution of problem (3.3) under $C < w\exp(rT)$ is foolish for rational investors.

In the following we derive analytically the best CRP investment strategy; i.e., the optimal solution to problem (3.3). As a by-product, we also obtain a closed-form expression for the efficient frontier of the mean-CaR model.

Let $\theta := \|\sigma^{-1}(b - r\mathbf{1})\|$ and denote $a^+ = \max\{a, 0\}$ for a real number $a$.

**Theorem 3.1** *Assume that $b \ne r\mathbf{1}$. Assume furthermore that $C$ satisfies*

$$C \ge w\exp(rT + (\theta\sqrt{T} + z_\alpha)^+\theta\sqrt{T}). \tag{3.5}$$

*Then the unique optimal policy of the mean-CaR model (3.3) is*

$$\pi^* = (\varepsilon^*/\theta)\,(\sigma\sigma')^{-1}(b - r\mathbf{1}), \tag{3.6}$$

*where*

$$\varepsilon^* = (\ln(C/w) - rT)\,/(\theta T). \tag{3.7}$$

*The corresponding expected terminal wealth is* $E[W^{\pi^*}(T)] = C$ *and Capital-at-Risk is*

$$CaR_2(\pi^*) = w\exp(rT) - C\sqrt{\exp\left(\varepsilon^{*2}T\right)\Phi(z_\alpha - 2\varepsilon^*\sqrt{T})/\alpha}. \tag{3.8}$$

***Proof*** With the help of expressions (2.2) and (2.10) and the definition of $CaR_2$, problem (3.3) cab be equivalently written as

$$\begin{cases} \max & w\exp\left((\pi'(b-r\mathbf{1}) + r + \|\pi'\sigma\|^2/2)T\right)\sqrt{\Phi(z_\alpha - 2\|\pi'\sigma\|\sqrt{T})/\alpha} \\ \text{s.t.} & w\exp\left((\pi'(b-r\mathbf{1})+r)T\right) \geq C. \end{cases} \tag{3.9}$$

The feasible set of the problem is

$$\Pi = \left\{\pi : (b-r\mathbf{1})'\pi T \geq \ln(C/w) - rT\right\}.$$

Given $\varepsilon > 0$, the intersection of $\Pi$ and the ellipsoid $\|\pi'\sigma\| = \varepsilon$ is

$$\Pi(\varepsilon) = \left\{\pi : \|\pi'\sigma\| = \varepsilon, \quad (b-r\mathbf{1})'\pi T \geq \ln(C/w) - rT\right\}.$$

The hyperplane $(b-r\mathbf{1})'\pi T = \ln\frac{C}{w} - rT$ is tangent to the ellipsoid $\|\pi'\sigma\| = \varepsilon$ if and only if $\varepsilon\theta T = \ln(C/w) - rT$, that is $\varepsilon = \varepsilon^* := (\ln(C/w) - rT)/(\theta T) > 0$. Consequently $\Pi(\varepsilon) = \emptyset$ if $\varepsilon < \varepsilon^*$ and hence $\Pi = \bigcup_{\varepsilon\geq\varepsilon^*}\Pi(\varepsilon)$. Thus problem (3.9) is equivalent to the following bilevel optimization problem

$$\max_{\varepsilon\geq\varepsilon^*}\max_{\pi\in\Pi(\varepsilon)} w\exp\left((\pi'(b-r\mathbf{1}) + r + \varepsilon^2/2)T\right)\sqrt{\Phi(z_\alpha - 2\varepsilon\sqrt{T})/\alpha}. \tag{3.10}$$

For each fixed $\varepsilon \geq \varepsilon^*$, we solve the inner-level optimization problem

$$\max_{\pi\in\Pi(\varepsilon)} \quad w\exp\left((\pi'(b-r\mathbf{1}) + r + \varepsilon^2/2)T\right)\sqrt{\Phi(z_\alpha - 2\varepsilon\sqrt{T})/\alpha}$$

or equivalently

$$\max_{\pi\in\Pi(\varepsilon)}(b-r\mathbf{1})'\pi. \tag{3.11}$$

The unique optimal solution is the tangent point

$$\pi^*_\varepsilon = (\varepsilon/\theta)(\sigma\sigma')^{-1}(b-r\mathbf{1})$$

of the hyperplane that parallels $(b-r\mathbf{1})'\pi T = \ln\frac{C}{w} - rT$ to the ellipsoid $\|\pi'\sigma\| = \varepsilon$, with maximum $(b-r\mathbf{1})'\pi^*_\varepsilon = \varepsilon\theta$. Therefore, we obtain the solution of problem (3.10) by solving the problem

$$\max_{\varepsilon\geq\varepsilon^*} \quad w\exp\left((\varepsilon\theta + r + \varepsilon^2/2)T\right)\sqrt{\Phi(z_\alpha - 2\varepsilon\sqrt{T})/\alpha}. \tag{3.12}$$

Consider the functions $h$ on $[0, +\infty)$ defined by

$$h(\varepsilon) = 2\varepsilon\theta T + \varepsilon^2 T + \ln\left(\Phi(z_\alpha - 2\varepsilon\sqrt{T})\right).$$

Noting $1 - \Phi(x) = \Phi(-x)$ and $\varphi(-x) = \varphi(x)$, setting $x = 2\varepsilon\sqrt{T} - z_\alpha$ in the second inequality in Lemma 2.1 yields $\varphi(z_\alpha - 2\varepsilon\sqrt{T}) > \Phi(z_\alpha - 2\varepsilon\sqrt{T})(2\varepsilon\sqrt{T} - z_\alpha)$. Thus

$$
\begin{aligned}
h'(\varepsilon) &= 2\theta T + 2\varepsilon T + \frac{(-2\sqrt{T})\varphi(z_\alpha - 2\varepsilon\sqrt{T})}{\Phi(z_\alpha - 2\varepsilon\sqrt{T})} < 2\sqrt{T}\left[\theta\sqrt{T} + \varepsilon\sqrt{T} - (2\varepsilon\sqrt{T} - z_\alpha)\right] \\
&= 2\sqrt{T}(\theta\sqrt{T} + z_\alpha - \varepsilon\sqrt{T}).
\end{aligned}
$$

If $\theta\sqrt{T} + z_\alpha \le 0$, then obviously $h'(\varepsilon) < 0$ for $\varepsilon \ge 0$. If $\theta\sqrt{T} + z_\alpha > 0$, then condition (3.5) implies that $\varepsilon^* \ge (\theta\sqrt{T} + z_\alpha)/\sqrt{T}$ and hence $h'(\varepsilon) < 0$ for $\varepsilon \ge \varepsilon^*$. Thus, function $h$ is strictly decreasing when $\varepsilon \ge \varepsilon^*$. Consequently, problem (3.12)'s objective function, equal to $\exp\left((h(\varepsilon) + 2rT - \ln\alpha)/2\right)$, is strictly decreasing when $\varepsilon \ge \varepsilon^*$. Therefore, the optimal solution of problem (3.12) is the unique $\varepsilon^*$. This completes the proof. □

As an immediate consequence, the analytic result in Theorem 3.1 provides an explicit relation between the optimal Capital-at-Risk and the expected terminal wealth. Letting $\xi := E[W^{\pi^*}(T)]$, we have

$$
CaR_2(\xi) = we^{rT} - \xi\sqrt{\frac{1}{\alpha}\exp\left(\frac{\left(\ln\frac{\xi}{w} - rT\right)^2}{\theta^2 T}\right)\Phi\left(z_\alpha - \frac{2\left(\ln\frac{\xi}{w} - rT\right)}{\theta\sqrt{T}}\right)} \tag{3.13}
$$

for $\xi \ge w\exp(rT + (\theta\sqrt{T} + z_\alpha)^+\theta\sqrt{T})$. The above relationship is known as the *efficient frontier* of the mean-CaR model associated with $CaR_2$ in mean-CaR space.

### 3.2 Mean-$CaR_1$ portfolio selection

Consider the following mean-CaR model associated with $CaR_1$:

$$
\min_{\pi\in\mathbb{R}^n} CaR_1(\pi) \quad \text{subject to} \quad E[W^\pi(T)] \ge C, \tag{3.14}
$$

where $C$, as in model (3.3), is again the predetermined level of the expected terminal wealth $E[W^\pi(T)]$ and satisfies condition (3.4).

Using a quite similar derivation as that in the proof of Theorem 3.1, we can also obtain a closed-form solution for problem (3.14), which is summarized by the following theorem stated without proof.

**Theorem 3.2** *Assume that $b \ne r\mathbf{1}$. Assume furthermore that $C$ satisfies (3.5). Then the unique optimal policy of the mean-CaR model (3.14) is*

$$
\pi^* = (\varepsilon^*/\theta)\,(\sigma\sigma')^{-1}(b - r\mathbf{1}), \tag{3.15}
$$

*where*

$$
\varepsilon^* = \left(\ln(C/w) - rT\right)/(\theta T). \tag{3.16}
$$

*The corresponding expected terminal wealth is $E[W^{\pi^*}(T)] = C$ and Capital-at-Risk is*

$$
CaR_1(\pi^*) = w\exp(rT) - C\Phi(z_\alpha - \varepsilon^*\sqrt{T})/\alpha. \tag{3.17}
$$

Consequently, the efficient frontier of the mean-CaR model associated with $CaR_1$ in mean-CaR space is given by

$$CaR_1(\xi) = we^{rT} - \frac{\xi}{\alpha}\Phi\left(z_\alpha - \frac{\ln\frac{\xi}{w} - rT}{\theta\sqrt{T}}\right) \tag{3.18}$$

for $\xi := E[W^{\pi^*}(T)] \geq w\exp(rT + (\theta\sqrt{T} + z_\alpha)^+\theta\sqrt{T})$.

It should be pointed out that although Emmer, Klüppelberg and Korn [7] also obtained a solution to (3.2) that has the same representation as (3.15), the parameter $\varepsilon^*$ however was not obtained explicitly as in (3.16). In fact, in their formulation $\varepsilon^*$ is estimated as a value between two expressions representing two real numbers.

### 3.3 Mean-$CaR_0$ portfolio selection

Consider the following mean-CaR model associated with $CaR_0$:

$$\min_{\pi\in\mathbb{R}^n} CaR_0(\pi) \quad \text{subject to} \quad E[W^\pi(T)] \geq C, \tag{3.19}$$

where $C$, as in problem (3.3), is again the predetermined level of the expected terminal wealth $E[W^\pi(T)]$ and satisfies condition (3.4).

The solution to the above optimization problem (3.19) is summarized in the following theorem. We omit the proof since it is very similar to the proof of Theorem 3.1.

**Theorem 3.3** *Assume that $b \neq r\mathbf{1}$. Then the unique optimal policy of mean-CaR model* (3.19) *is*

$$\pi^* = (\varepsilon^*/\theta)\,(\sigma\sigma')^{-1}(b - r\mathbf{1}), \tag{3.20}$$

*where*

$$\varepsilon^* = \max\left\{(\ln(C/w) - rT)\,/(\theta T), \theta + z_\alpha/\sqrt{T}\right\}. \tag{3.21}$$

*The corresponding expected terminal wealth is*

$$E[W^{\pi^*}(T)] = w\exp(\varepsilon^*\theta T + rT) = \max\left\{C, w\exp\left(rT + \theta T\left(\theta + z_\alpha/\sqrt{T}\right)\right)\right\} \tag{3.22}$$

*and the Capital-at-Risk is*

$$CaR_0(\pi^*) = w\exp(rT)\left[1 - \exp\left(\varepsilon^*\theta T - \varepsilon^{*2}T/2 + z_\alpha\varepsilon^*\sqrt{T}\right)\right]. \tag{3.23}$$

Based on this result, the efficient frontier of the mean-CaR model associated with $CaR_0$ in mean-CaR space is given by

$$CaR_0(\xi) = w\exp(rT) - \xi\exp\left(\frac{\ln(\xi/w) - rT}{\theta T}\left(z_\alpha\sqrt{T} - \frac{\ln(\xi/w) - rT}{2\theta}\right)\right) \tag{3.24}$$

for $\xi := E[W^{\pi^*}(T)] \geq w\exp\left(rT + \left(\theta + z_\alpha/\sqrt{T}\right)^+\theta T\right)$.

We noted that the part of the efficient frontier corresponding to those $C$ satisfying

$$w\exp(rT) \leq C \leq w\exp\left(rT + \left(\theta + z_\alpha/\sqrt{T}\right)^+\theta T\right)$$

degenerates to a single point where $\xi = w\exp\left(rT + \left(\theta + z_\alpha/\sqrt{T}\right)^+\theta T\right)$ in mean-CaR space. Hence the whole efficient frontier starts from this point.

## 4 A comparison of the mean-CaR models

Based on the results in the previous section, in this section we compare the optimal behaviors of our mean-$CaR_0$, mean-$CaR_1$, and mean-$CaR_2$ models.

(1) For any given expected terminal wealth level $C \geq C_0 := w\exp\Big(rT+\theta\sqrt{T}\left(\theta\sqrt{T}+z_\alpha\right)^+\Big)$, the three mean-CaR models have the same optimal strategy which does not depend on the confidence level $\alpha$ and the same expected terminal wealth which is equal to the lowest permissible wealth $C$. When the given expected terminal wealth level $C$ is lower than $C_0$, the optimal policy of the mean-$CaR_0$ model does not dependent on the expected terminal wealth level $C$ but depends on the the confidence level $\alpha$.

(2) For each mean-CaR model, the optimal fraction of wealth invested in risky assets $\pi^*$ is increasing with the expected terminal wealth level $C$, indicating that a higher expected terminal wealth level requires more investment in risky assets. (In the low level region $C \leq C_0$, the optimal stock weights of the mean-$CaR_0$ model are invariant with the expected terminal wealth level.)

(3) For the mean-$CaR_2$ and the mean-$CaR_1$ models, the optimal fraction of wealth invested in risky assets $\pi^*$ is decreasing with the investment horizon $T$, exhibiting the reverse time-diversification effect: an investor allocates less to stocks when confronted with a longer investment horizon. For the mean-$CaR_0$ model, however, the optimal fraction of wealth invested in stocks first decreases with $T$ in the region $T \leq T_0 := \left(\frac{-z_\alpha\theta+\sqrt{(z_\alpha\theta)^2+4(r+\theta^2)\ln(C/w)}}{2(r+\theta^2)}\right)^2$, exhibiting the reverse time-diversification effect in the region of short investment horizons $T \leq T_0$, and then increases with $T$ in the region $T \geq T_0$, exhibiting the time-diversification effect in the region of long investment horizons $T \geq T_0$.

(4) For each mean-CaR model, CaR of the optimal strategy is decreasing with confidence level $\alpha$; that is, smaller risk measured by CaR is exposed at the expense of higher confidence level.

(5) For each mean-CaR model, roughly the CaR of the optimal strategy is first increasing and then decreasing with time horizon $T$, implying that more (less) risk measured by CaR is exposed as the horizon extends in the small (large) region of short (long) horizons. This will be illustrated in the example of the next section.

(6) As to be expected, in mean-CaR spaces, the three mean-CaR efficient frontiers are all strictly increasing and concave, where the concavity of the mean-$CaR_0$ efficient frontier is true at least in the region

$$\xi \geq w\exp\left(rT+\left(\theta\sqrt{T}+z_\alpha\right)\theta\sqrt{T}+\left(\sqrt{1/4+1/(\theta^2T)}-1/2\right)\theta^2T\right).$$

(7) The mean-$CaR_1$ efficient frontier is higher than the mean-$CaR_2$ efficient frontier, which, in turn, is higher than the mean-$CaR_0$ efficient frontier; that is, for each $\xi = E[W^{\pi^*}(T)] \geq w\exp\left(rT+\left(\theta\sqrt{T}+z_\alpha\right)^+\theta\sqrt{T}\right)$, $CaR_1(\xi) \geq CaR_2(\xi) \geq CaR_0(\xi)$. In other words, for the same expected terminal wealth level, the optimal strategy of the mean-$CaR_1$ model has larger CaR than the one of the mean-$CaR_2$ model, which in turn has larger CaR than the one of the mean-$CaR_0$ model. In fact, it holds that $CaR_1(\pi) \geq CaR_2(\pi) \geq CaR_0(\pi)$ for general strategies $\pi$; see Corollary 2.4 of Emmer, Klüppelberg and Korn [7].

(8) Each of the three mean-CaR efficient frontiers depends only on the stocks via the norm $\|\sigma^{-1}(b - r\mathbf{1})\|$ and has no explicit dependence on the number of different stocks. Therefore, Theorems 3.1, 3.2 and 3.3 can be interpreted as a kind of *mutual fund theorems* since there is no difference between investment in our multi-stock market and a market consisting of the bond and just one stock with appropriate market coefficients $b$ and $\sigma$, as observed by Emmer, Klüppelberg and Korn [6] for their mean-CaR model.

## 5 An example

In this section, a numerical example is presented to demonstrate the results stated in the previous section.

**Example 5.1** Consider a market that consists of the bond and just one stock (i.e., $n = 1$). Assume that the rate of interest of the bond is $r = 0.05$, the stock-appreciation rate is $b = 0.1$, and the stock-volatility is $\sigma = 0.2$, implying $\theta = 0.25$. And assume that the initial wealth of an investor is $w = 1000$.

Figures 5.1 and 5.2 show the dependence of the optimal fraction of wealth invested in the stock on the time horizon $T$, the expected terminal wealth level $C$ and the confidence levels $\alpha$. Figure 5.1 exhibits the reverse time-diversification effect, the increasingness with the expected terminal wealth level, and the invariance with the confidence level of the optimal stock fraction to the mean-$CaR_2$ and the mean-$CaR_1$ models. In Figure 5.2, the optimal stock fraction of the mean-$CaR_0$ model displays the reverse time-diversification effect in a large time horizon region (e.g., $0 < T \leq 16.48$ for $\alpha = 0.20$), the time-diversification effect in a small time horizon region (e.g., $16.48 \leq T \leq 20$ for $\alpha = 0.20$), the increasingness with the expected terminal wealth level, and the increasingness with the confidence level.

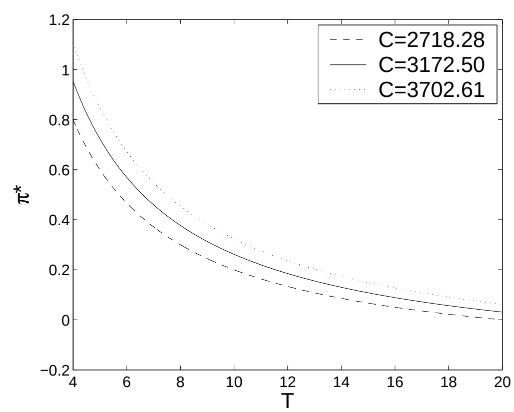
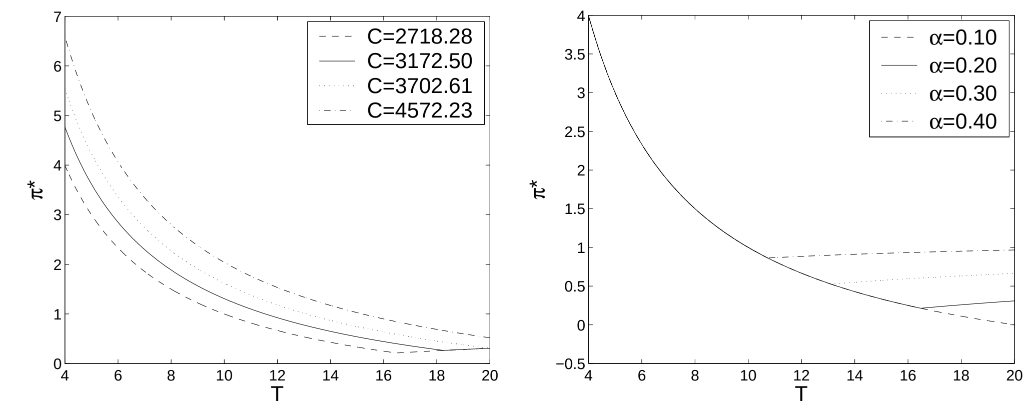



**Figure 5.1:** The optimal stock fraction of the mean-$CaR_2$ and the mean-$CaR_1$ models with any confidence level $\alpha < 0.13$ as a function of the time horizon $T$ ($0 < T \leq 20$) for different expected terminal wealth levels $C$.

The CaR of the optimal strategy as a function of the time horizon $T$ is illustrated graphically in Figure 5.3 for mean-$CaR_2$, Figure 5.4 for mean-$CaR_1$, and Figure 5.5 for mean-$CaR_0$ models, which indicates that more (less) CaR risk is exposed as the horizon extends in a small (large) region of short (long) horizons for each of the mean-CaR models.




(a) with a confidence level $\alpha = 0.20$ for different expected terminal wealth levels $C$

(b) with a expected terminal wealth level $C = w\exp(20r) = 2718.28$ for different confidence levels $\alpha$

**Figure 5.2:** The optimal stock fraction of the mean-$CaR_0$ model as a function of the time horizon $T$ $(0 < T \leq 20)$.

Figures 5.5(a) and 5.5(b) also display some difference of $CaR_0$ of the optimal strategy to the mean-$CaR_0$ model with the same expected terminal wealth levels between different confidence levels. Figure 5.6 plots the CaR of the three mean-CaR models in the same plane to compare them, showing that the optimal $CaR_1$ is larger than the optimal $CaR_2$ which is larger than the optimal $CaR_0$ for the same time horizon.

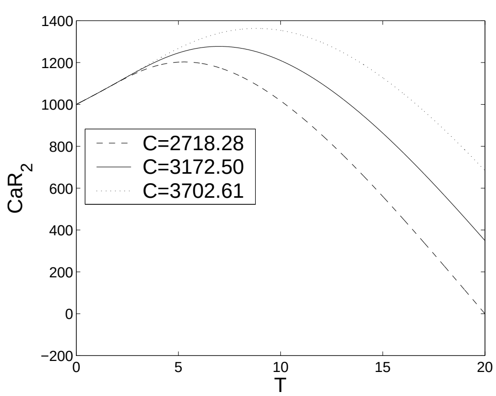



**Figure 5.3:** $CaR_2$ of the optimal strategy to the mean-$CaR_2$ model with a confidence level $\alpha = 0.05$ as a function of the time horizon $T$ $(0 < T \leq 20)$ for different expected terminal wealth levels $C$.

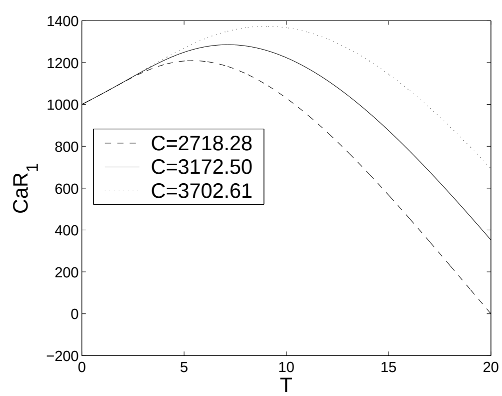
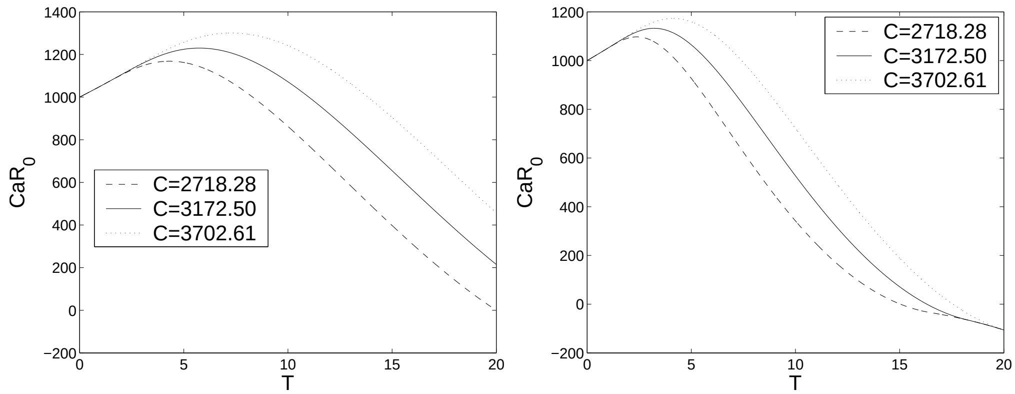


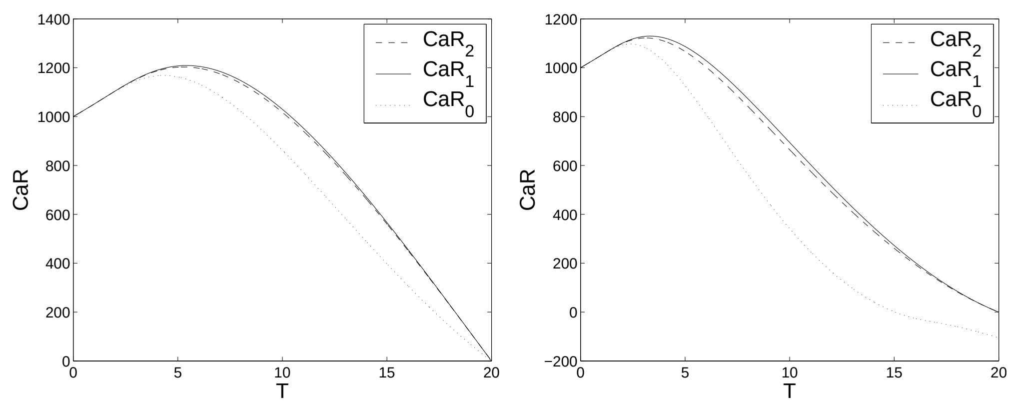

**Figure 5.4:** $CaR_1$ of the optimal strategy to the mean-$CaR_1$ model with a confidence level $\alpha = 0.05$ as a function of the time horizon $T$ $(0 < T \leq 20)$ for different expected terminal wealth levels $C$.

The mean-$CaR_2$, the mean-$CaR_1$ and the mean-$CaR_0$ efficient frontiers are depicted respectively in Figure 5.7, Figure 5.8 and Figure 5.9 with the mean on the horizontal axis and the CaR on the vertical axis for confidence levels $\alpha = 0.01$ (dashed line), $\alpha = 0.05$ (solid line) and $\alpha = 0.1$ (dotted line). Clearly, all the efficient frontiers are increasing




(a) with a confidence level $\alpha = 0.05$ (b) with a confidence level $\alpha = 0.20$

**Figure 5.5:** $CaR_0$ of the optimal strategy to the mean-$CaR_0$ model as a function of the time horizon $T$ $(0 < T \leq 20)$ for different expected terminal wealth levels $C$.




(a) with a confidence level $\alpha = 0.05$ and a expected terminal wealth level $C = 2718.28$ (b) with a confidence level $\alpha = 0.20$ and a expected terminal wealth level $C = 2718.28$

**Figure 5.6:** CaR of the optimal strategies to the mean-$CaR_k$ $(k = 0, 1, 2)$ models as a function of the time horizon $T$ $(0 < T \leq 20)$.

and concave; and the higher is the confidence level $\alpha$, the lower is the efficient frontier for each of the three mean-CaR models, implying that CaR of the optimal strategy for each mean-CaR model decreases as the confidence level increases. Furthermore, in order to demonstrates the difference of the three efficient frontiers, the efficient frontiers of mean-$CaR_2$ (dashed line), mean-$CaR_1$ (solid line) and mean-$CaR_0$ (dotted line) models are plotted in the same plane, see Figure 5.10. Obviously, the efficient frontiers of the mean-$CaR_1$, the mean-$CaR_2$ and the mean-$CaR_0$ models fall in turn, again implying that the risk measured by $CaR_1$ is the largest and the one by $CaR_0$ is the smallest, among the three.

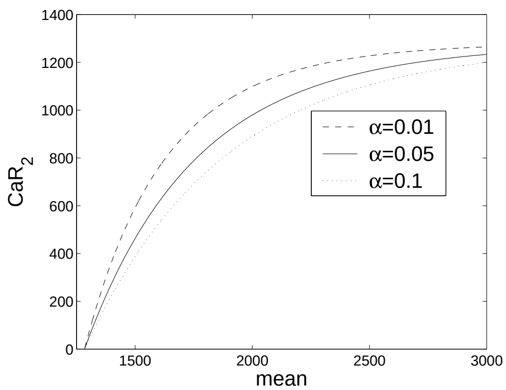



**Figure 5.7:** Mean-$CaR_2$ efficient frontiers for different confidence levels.

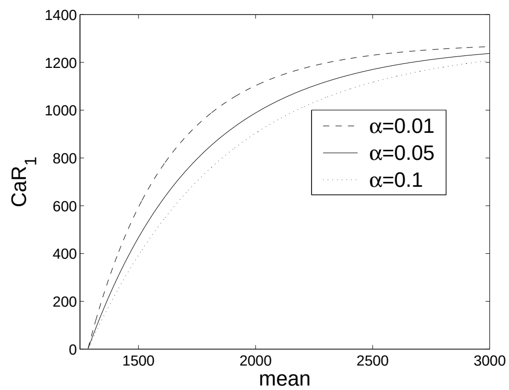



**Figure 5.8:** Mean-$CaR_1$ efficient frontiers for different confidence levels.

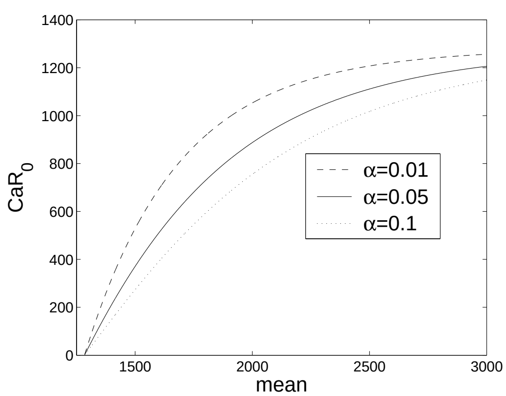



**Figure 5.9:** Mean-$CaR_0$ efficient frontiers for different confidence levels.

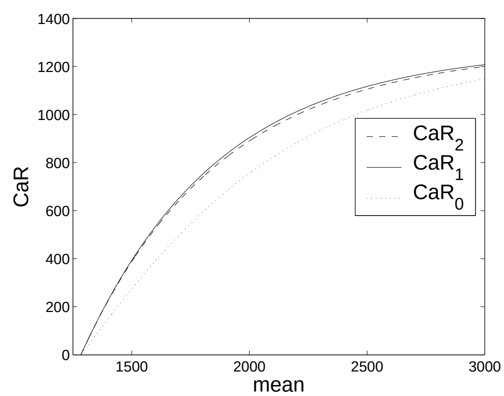



**Figure 5.10:** The mean-$CaR_k$ efficient frontiers with $k = 0, 1, 2$ for $\alpha = 0.10$.

## 6 Conclusions

This paper investigates three continuous-time mean-CaR portfolio selection models under the setting of Black-Scholes financial markets and CRP investment strategies. After converting the portfolio optimization problems we obtain closed-form explicit expressions of optimal strategies and efficient frontiers by virtue of a decomposition of the feasible solution set. This approach facilitates computation and the comparison of results and can be easily used in practice. It unifies the framework of dealing with different mean-CaR portfolio selection models. In an analogous way, it can be shown that the approach can be applied to a mean-variance model, a mean-VaR model, and some expected utility models with a shortfall constraint to obtain closed-form solutions. We also believe that the approach can be applied to some other continuous-time portfolio selection problems.

Note that the derived optimal strategies of the three mean-CaR models are nonnegative under the assumption that each stock-appreciation rate is not smaller than the riskless interest rate. In this case, our results are valid for continuous-time mean-CaR portfolio selection problems where short-selling of risky assets is not allowed. (However, short-selling the riskless asset is still allowed.)

CRP strategies have a variety of optimality properties associated with them for ordinary portfolio problems (see, e.g., Merton [15, 16]) showed that this form of strategies are optimal to portfolio selection problems of maximizing expected utility with constant relative risk-aversion.) and are widely used in asset allocation practice (see, e.g., Perold and Sharpe [17] and Black and Perold [3]). However, since such strategies may not be feedback strategies under general models, the optimal CRP strategy for our models or for the models in Emmer, Klüppelberg and Korn [6, 7] may not be globally optimal in the set of all dynamic strategies. Removing the restriction to strategies with constant proportions would be both mathematically harder and more interesting.

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