



Elements of Stability Theory of A.M. Liapunov for Dynamic Equations on Time Scales

(Devoted to the 150th birthday of A.M. Liapunov)

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Abstract: Stability of dynamic equations on time scales is investigated in this paper. The main results are new conditions for stability, uniform stability, and uniform asymptotic stability of quasilinear and nonlinear systems.

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1 Introduction

The sixth of June, 2007 is the 150th birthday anniversary of the outstanding Russian mathematician and mechanical scientist, Academician Liapunov. A brief outline of the life and activities of Alexander Mikhaylovich Liapunov is contained in [30], while a more detailed outline is given in [22]. The main directions of Liapunov's scientific activities are as follows:

- stability of equilibrium and motion of mechanical systems with a finite number of degrees of freedom;
- equilibrium figures of uniformly rotating liquids;
- stability of equilibrium figures of rotating liquids;
- equations of mathematical physics;

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- probability theory;
- lecture courses on theoretical mechanics.

For a detailed analysis of Liapunov’s papers in the above mentioned directions see the survey [33].

Liapunov started publication of his works on the problems of motion stability of systems with a finite number of degrees of freedom in 1888. In 1892 he formulated a strict definition of stability which crowned his intensive work during 1889–1892. The notion of “Liapunov stability” adopted nowadays denotes stability of solutions with respect to perturbation of the initial data over infinite time intervals. The exact definition of stability was of principal importance for further determination of stability criteria of the equilibrium and/or motion of mechanical or other kinds of systems.

In 1892 the Kharkov Mathematical Society published Liapunov’s paper “General Problem of Motion Stability” [11]. This work was defended by Liapunov as his doctoral thesis at Moscow University in 1892. In this paper Liapunov considered differential equations of perturbed motion in a quite general form and developed two general methods of analysis of their solutions. The first method is based on the integration of the above mentioned equations by special series. The second technique is based on the application of an auxiliary function whose properties together with properties of its total time derivative along solutions of the system under consideration allow the conclusion on dynamical behavior of solutions for the system.

Alongside these two methods of qualitative analysis of motion equations, Liapunov introduced the notion of a function’s characteristic number and applied it to stability analysis of solutions for linear systems of differential equations with variable coefficients. Liapunov completely solved the problem of stability by the first approximation and studied stability of solutions to perturbed motion equations in some critical cases.

The list of references (see [9–23]) presents the papers by Liapunov published to date which deal with stability of systems with a finite number of degrees of freedom, general theory of ordinary differential equations, and classical mechanics. Note that many of Liapunov’s papers still remain unpublished.

The aim of our paper is to present some results of stability analysis of solutions for a new class of perturbed motion equations referred to as *dynamic equations on time scales*. Equations on time scales provide a possibility for a simultaneous description of dynamics of continuous-time and discrete-time systems. Such two-mode systems occur in some problems on impulsive control in the description of some technological processes with discrete effects of a catalyst. Some necessary introduction to the mathematical analysis on time scales is presented here in accordance with [2, 3], with vast bibliography therein.

2 Elements of Calculus on Time Scales

2.1 Description of Time Scales

An arbitrary nonempty closed subset of the set of real numbers \mathbb{R} is referred to as a *time scale* and denoted by \mathbb{T} . Examples of time scales are the reals \mathbb{R} , the integers \mathbb{Z} , the positive integers \mathbb{N} , and the nonnegative integers \mathbb{N}_0 . The most common time scales are $\mathbb{T} = \mathbb{R}$ for continuous calculus, $\mathbb{T} = \mathbb{Z}$ for discrete calculus, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, where $q > 1$, for quantum calculus.

Definition 2.1 • The *forward* and *backward jump operators* σ and ρ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{for all } t \in \mathbb{T}$$

and

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{for all } t \in \mathbb{T},$$

respectively.

- By means of the operators $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ and $\rho : \mathbb{T} \rightarrow \mathbb{T}$, the elements $t \in \mathbb{T}$ are classified as follows: If $\sigma(t) = t$, $\rho(t) = t$, $\sigma(t) > t$, and $\rho(t) < t$, then t is called *right-dense*, *left-dense*, *right-scattered*, and *left-scattered*, respectively. Here it is assumed that $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if \mathbb{T} contains the maximal element t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if \mathbb{T} contains the minimal element t).
- In addition to the set \mathbb{T} , the set \mathbb{T}^κ is defined as follows. If \mathbb{T} contains the left scattered maximum m , then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$, and $\mathbb{T}^\kappa = \mathbb{T}$ in the other cases. Therefore,

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

- The distance from an arbitrary element $t \in \mathbb{T}$ to its follower is called the *graininess* of the time scale \mathbb{T} and is given by the formula

$$\mu(t) = \sigma(t) - t \quad \text{for all } t \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t = \rho(t)$ and $\mu(t) = 0$, and if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\rho(t) = t - 1$, and $\mu(t) = 1$.

In some cases for equations on a time scale \mathbb{T} , the principle of induction on time scales is applied. In the monograph [2], this principle is formulated as follows.

Theorem 2.1 Let $t_0 \in \mathbb{T}$ and $\{S(t) : t \in [t_0, \infty)\}$ be a set of assertions satisfying the conditions:

1. The statement $S(t)$ is true for $t = t_0$.
2. If $t \in [t_0, \infty)$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is true as well.
3. If $t \in [t_0, \infty)$ is right-dense and $S(t)$ is true, then there exists a neighborhood W of t such that $S(s)$ is true for all $s \in W \cap (t, \infty)$.
4. If $t \in (t_0, \infty)$ is left-dense and $S(s)$ is true for all $s \in [t_0, t)$, then $S(t)$ is true.

Then $S(t)$ is true for all $t \in [t_0, \infty)$.

2.2 Differentiation on Time Scales

Further we shall consider a function $f : \mathbb{T} \rightarrow \mathbb{R}$ and determine its Δ -derivative at a point $t \in \mathbb{T}^\kappa$.

Definition 2.2 • The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called Δ -differentiable at a point $t \in \mathbb{T}^\kappa$ if there exists $\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a W -neighborhood of $t \in \mathbb{T}^\kappa$ satisfying

$$|[f(\sigma(t)) - f(s)] - \gamma[\sigma(t) - s]| < \varepsilon|\sigma(t) - s| \quad \text{for all } s \in W.$$

In this case we shall write $f^\Delta(t) = \gamma$.

• If the function f is Δ -differentiable for any $t \in \mathbb{T}^\kappa$, then $f : \mathbb{T} \rightarrow \mathbb{R}$ is called Δ -differentiable on \mathbb{T}^κ .

Some useful properties of the derivative of a function f are found in the results below.

Theorem 2.2 Assume that $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. Then the following assertions are true:

- (1) if f is differentiable at t , then f is continuous at t ;
- (2) if f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)};$$

- (3) if t is right-dense, then f is differentiable at t iff there exists the limit

$$\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}$$

as a finite number, and then

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s};$$

- (4) if f is differentiable at t , then

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

Note that, if $\mathbb{T} = \mathbb{R}$, then $f^\Delta = f'$, which is the Cauchy derivative of f , and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$, which is the forward difference of f .

Further we present the following result.

Theorem 2.3 Assume that the functions $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$. Then the following assertions are valid:

- (1) the sum $f + g : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t and

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t);$$

- (2) for any constant α , the function $\alpha f : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t and

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t);$$

(3) the product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t and

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t));$$

(4) if $f(t)f(\sigma(t)) \neq 0$, then the function $1/f$ is differentiable at t and

$$\left(\frac{1}{f}\right)^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))};$$

(5) if $g(t)g(\sigma(t)) \neq 0$, then the function f/g is differentiable at t and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

2.3 Integration on Time Scales

Further we shall consider functions that are “integrable” on the time scale \mathbb{T} .

Definition 2.3 • A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *regulated* provided its right-sided limit exist (finite) at all right-dense points in \mathbb{T} and its left-sided limits exist (finite) at all left-dense points in \mathbb{T} .

- A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in \mathbb{T} and its left-sided limits exist (finite) at left-dense points in \mathbb{T} .
- The set of all rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$.

Theorem 2.4 Assume that $f : \mathbb{T} \rightarrow \mathbb{R}$. Then the following assertions are true:

- (1) If f is continuous on \mathbb{T} , then it is rd-continuous on \mathbb{T} ;
- (2) if f is rd-continuous on \mathbb{T} , then it is regulated on \mathbb{T} ;
- (3) the jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is rd-continuous;
- (4) if f is regulated or rd-continuous on \mathbb{T} , then the function $f \circ \sigma$ possesses the same property;
- (5) if $f : \mathbb{T} \rightarrow \mathbb{R}$ is continuous and $g : \mathbb{T} \rightarrow \mathbb{R}$ is regulated and rd-continuous, then the function $f \circ g$ possesses the same property.

Definition 2.4 • A function $F : \mathbb{T} \rightarrow \mathbb{R}$ such that $F^\Delta = f$ is called an *antiderivative* of the function f .

- If F is an antiderivative of f , then the integral is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

It is well known that any rd-continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$ possesses an antiderivative.

If $f^\Delta(t) \geq 0$ on $[a, b]$ and $s, t \in \mathbb{T}$ with $a \leq s \leq t \leq b$, then

$$f(t) = f(s) + \int_s^t f^\Delta(\tau) \Delta\tau \geq f(s),$$

i.e., the function f is increasing on \mathbb{T} .

Some properties of integration on \mathbb{T} are presented next.

Theorem 2.5 *Let $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{\text{rd}}(\mathbb{T})$. Then*

- (i) $\int_a^b [f(t) + g(t)] \Delta t = \int_a^b f(t) \Delta t + \int_a^b g(t) \Delta t;$
- (ii) $\int_a^b (\alpha f)(t) \Delta t = \alpha \int_a^b f(t) \Delta t;$
- (iii) $\int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t;$
- (iv) $\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t;$
- (v) $\int_a^b f(\sigma(t)) g^\Delta(t) \Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g(t) \Delta t;$
- (vi) $\int_a^a f(t) \Delta t = 0;$
- (vii) $\int_t^{\sigma(t)} f(\tau) \Delta\tau = \mu(t)f(t);$
- (viii) *if $|f| \leq g$ on $[a, b]$, then $\left| \int_a^b f(t) \Delta t \right| \leq \int_a^b g(t) \Delta t;$*
- (ix) *if $f \geq 0$ on $[a, b]$, then $\int_a^b f(t) \Delta t \geq 0.$*

Next we shall present some chain rules. We recall that if $f, g : \mathbb{R} \rightarrow \mathbb{R}$, then

$$(f \circ g)' = (f' \circ g)g'.$$

The following two chain rules hold.

Theorem 2.6 *Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable, $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, and $g : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable on \mathbb{T} . Then there exists c in the real interval $[t, \sigma(t)]$ such that*

$$(f \circ g)^\Delta(t) = f'(g(c))g^\Delta(t).$$

Theorem 2.7 *Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable and $g : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable on \mathbb{T} . Then $(f \circ g) : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable, and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t)) dh \right\} g^\Delta(t)$$

holds.

Definition 2.5 If $\sup \mathbb{T} = \infty$, then the *improper integral* is defined by

$$\int_a^\infty f(t) \Delta t = \lim_{b \rightarrow \infty} F(t) \Big|_a^b \quad \text{for } a \in \mathbb{T}.$$

2.4 The Exponential Function on Time Scales

An rd-continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *regressive* if

$$1 + \mu(t)f(t) \neq 0 \quad \text{for all } t \in \mathbb{T}$$

(we write $f \in \mathcal{R}$) and *positively regressive* if

$$1 + \mu(t)f(t) > 0 \quad \text{for all } t \in \mathbb{T}$$

(we write $f \in \mathcal{R}^+$). For the operation \oplus defined by

$$p \oplus q = p + q + \mu pq \quad \text{on } \mathbb{T},$$

the couple (\mathcal{R}, \oplus) is an Abelian group with inverse element

$$\ominus p = -\frac{p}{1 + \mu p} \quad \text{for } p \in \mathcal{R}.$$

We also define $p \ominus q = p \oplus (\ominus q)$. We note that if $p, q \in \mathcal{R}$, then $\ominus p, \ominus q, p \oplus q, p \ominus q, q \ominus p \in \mathcal{R}$.

For the definition of the exponential function on a time scale \mathbb{T} , we follow [5] and shall consider for some $h > 0$ the strip

$$\mathbb{Z}_h = \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \text{Im}(z) \leq \frac{\pi}{h} \right\}$$

and the set \mathbb{C}_h

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}.$$

For $h = 0$, we let $\mathbb{Z}_h = \mathbb{C} = \mathbb{C}_h$ be the set of complex numbers. Then for $h \geq 0$ we define the *cylinder transformation* $\xi_h : \mathbb{C}_h \rightarrow \mathbb{Z}_h$ by the formula

$$\xi_h = \begin{cases} \frac{1}{h} \text{Log}(1 + zh) & \text{if } h > 0, \\ z & \text{if } h = 0. \end{cases}$$

where Log is the principal logarithm function. The inverse cylinder transformation $\xi_h^{-1} : \mathbb{Z}_h \rightarrow \mathbb{C}_h$ is given by

$$\xi_h^{-1}(z) = \frac{e^{zh} - 1}{h} = (\exp zh - 1)h^{-1}.$$

For a function $p \in \mathcal{R}$, the *exponential function* e_p is defined by the expression

$$e_p(t, s) = \exp \left(\int_s^t \xi_{\mu(t)}(p(\tau)) \Delta \tau \right) \quad \text{for all } (t, s) \in \mathbb{T} \times \mathbb{T}. \tag{2.1}$$

The following properties of the exponential function (2.1) are known (see [2]).

Theorem 2.8 *Let $p, q \in \mathcal{R}$ and $t, r, s \in \mathbb{T}$. Then*

- (i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
- (ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;

- (iii) $\frac{1}{e_p(t, s)} = e_{\ominus p}(t, s);$
- (iv) $e_p(t, s) = \frac{1}{e_p(s, t)} = e_{\ominus p}(s, t);$
- (v) $e_p(t, s)e_p(s, r) = e_p(t, r);$
- (vi) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s);$
- (vii) $\frac{e_p(t, s)}{e_q(t, s)} = e_{p \ominus q}(t, s);$
- (viii) if $\mathbb{T} = \mathbb{R}$, then $e_p(t, s) = e^{\int_s^t p(\tau) d\tau};$
- (ix) if $\mathbb{T} = \mathbb{R}$ and $p(t) \equiv \alpha$, then $e_p(t, s) = e^{\alpha(t-s)};$
- (x) if $\mathbb{T} = \mathbb{Z}$, then $e_p(t, s) = \prod_{\tau=s}^{t-1} (1 + p(\tau));$
- (xi) if $\mathbb{T} = h\mathbb{Z}$ with $h > 0$ and $p(t) \equiv \alpha$, then $e_p(t, s) = (1 + h\alpha)^{\frac{t-s}{h}}.$

2.5 Variation of Constants on Time Scales

In terms of the exponential function (2.1), there are two variation of constants formulas that read as follows.

Theorem 2.9 *Let $f \in C_{\text{rd}}$, $p \in \mathcal{R}$, $t_0 \in \mathbb{T}$, and $x_0 \in \mathbb{R}$. Then the unique solution of the initial value problem*

$$x^\Delta(t) = -p(t)x(\sigma(t)) + f(t), \quad x(t_0) = x_0$$

is

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t, \tau)f(\tau)\Delta\tau,$$

and the unique solution of the initial value problem

$$x^\Delta(t) = p(t)x(t) + f(t), \quad x(t_0) = x_0$$

is

$$x(t) = e_p(t, t_0)x_0 + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau.$$

3 Method of Integral Inequalities on Time Scales

The method of integral inequalities for stability analysis of solutions of continuous systems is well developed and its main results are presented in a series of publications, of which we note [26, 31]. The development of this method for stability analysis of solutions on a time scale \mathbb{T} is associated with obtaining appropriate inequalities on time scales.

In this section we introduce the method of integral inequalities to study the behavior of solutions of the system of dynamic equations of the perturbed motion

$$x^\Delta = A(t)x + f(t, x), \quad f(t, 0) = 0, \tag{3.1}$$

where $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ with $n \in \mathbb{N}$, $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $F(t) = f(t, x(t))$ satisfies $F \in C_{\text{rd}}(\mathbb{T})$ whenever x is a differentiable function. These assumptions guarantee that the unique solution $x = x(\cdot; t_0, x_0)$ of (3.1) together with the initial condition $x(t_0) = x_0$, where $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$, may be written in the form (see Theorem 2.9 in Section 2.5)

$$x(t) = x(t; t_0, x_0) = e_A(t, t_0)x_0 + \int_{t_0}^t e_A(t, \sigma(\tau))f(\tau, x(\tau))\Delta\tau. \tag{3.2}$$

In this section, letting $m \in \mathbb{N}$ and subject to the two assumptions

$$\|f(t, x)\| \leq a(t) \|x\|^m \quad \text{for } t \geq t_0, x \in \mathbb{R}^n, \quad \text{where } a \in C_{\text{rd}}(\mathbb{T}) \tag{3.3}$$

and

$$\|e_A(t, s)\| \leq \varphi(t)\psi(s) \quad \text{for } t \geq s \geq t_0, \quad \text{where } \varphi, \psi \in C_{\text{rd}}(\mathbb{T}), \tag{3.4}$$

we derive sufficient criteria for stability, uniform stability, and asymptotical stability of the unperturbed motion of (3.1). In the next subsection below we consider the case $m = 1$ while we study the case $m > 1$ in the subsequent subsection. The case $m = 1$ uses the well-known Gronwall inequality on time scales while for the case $m > 1$, a dynamic version of Stachurska’s inequality [34] is employed. This inequality is a new result for dynamic equations, so it will be proved in Section 3.2 below.

We will use the following standard definition of different types of stability.

Definition 3.1 The unperturbed motion of (3.1) is said to be

(S₁) *stable* if for each $\varepsilon > 0$ and $t_0 \in \mathbb{T}$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|x_0\| < \delta \quad \text{implies} \quad \|x(t; t_0, x_0)\| < \varepsilon \quad \text{for all } t \geq t_0;$$

(S₂) *uniformly stable* if the δ in (S₁) is independent of t_0 ;

(S₃) *asymptotically stable* if it is stable and there exists δ_0 such that

$$\|x_0\| < \delta_0 \quad \text{implies} \quad \lim_{t \rightarrow \infty} x(t; t_0, x_0) = 0.$$

3.1 Stability via Gronwall’s Inequality

We start by recalling Gronwall’s inequality from [2, Theorem 6.4].

Theorem 3.1 (Gronwall’s Inequality) *Let $y, f \in C_{\text{rd}}$ and $p \geq 0$. Then*

$$y(t) \leq f(t) + \int_{t_0}^t y(\tau)p(\tau)\Delta\tau \quad \text{for all } t \geq t_0$$

implies

$$y(t) \leq f(t) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)p(\tau)\Delta\tau \quad \text{for all } t \geq t_0.$$

The version we will use is the following inequality from [2, Corollary 6.7].

Corollary 3.1 *Let $y \in C_{rd}$, $p \geq 0$, and $\alpha \in \mathbb{R}$. Then*

$$y(t) \leq \alpha + \int_{t_0}^t y(\tau)p(\tau)\Delta\tau \quad \text{for all } t \geq t_0$$

implies

$$y(t) \leq \alpha e_p(t, t_0) \quad \text{for all } t \geq t_0.$$

The following main results in this subsection are given for $\mathbb{T} = \mathbb{R}$ in [31, Lemma 2 and Theorem 5].

Lemma 3.1 *Suppose that (3.3) for $m = 1$ and (3.4) hold. Then any solution of (3.1) satisfies the estimate*

$$\|x(t; t_0, x_0)\| \leq \varphi(t)\psi(t_0)e_{\varphi\psi\sigma_a}(t, t_0) \|x_0\| \quad \text{for all } t \geq t_0. \quad (3.5)$$

Proof First note that the assumptions of Corollary 3.1 are satisfied. Let x be a solution of (3.1) so that by (3.2) we have for all $t \geq t_0$ the estimate

$$\|x(t; t_0, x_0)\| \leq \varphi(t)\psi(t_0) \|x_0\| + \int_{t_0}^t \varphi(t)\psi(\sigma(\tau))a(\tau) \|x(\tau; t_0, x_0)\| \Delta\tau.$$

Hence the function $y = \|x(\cdot; t_0, x_0)\| / \varphi$ satisfies

$$y(t) \leq \psi(t_0) \|x_0\| + \int_{t_0}^t \varphi(\tau)\psi(\sigma(\tau))a(\tau)y(\tau)\Delta\tau \quad \text{for all } t \geq t_0.$$

By Corollary 3.1,

$$y(t) \leq \psi(t_0) \|x_0\| e_{\varphi\psi\sigma_a}(t, t_0) \quad \text{for all } t \geq t_0.$$

Using the definition of y , the claim (3.5) follows. \square

Theorem 3.2 *Suppose that (3.3) for $m = 1$ and (3.4) hold.*

(i) *If for all $s \geq t_0$ there exists $K(s) > 0$ such that*

$$\varphi(t)e_{\varphi\psi\sigma_a}(t, s) \leq K(s) \quad \text{for all } t \geq s \geq t_0,$$

then the unperturbed motion of system (3.1) is stable;

(ii) *if there exists $K > 0$ such that*

$$\varphi(t)\psi(s)e_{\varphi\psi\sigma_a}(t, s) \leq K \quad \text{for all } t \geq s \geq t_0,$$

then the unperturbed motion of system (3.1) is uniformly stable;

(iii) *if*

$$\lim_{t \rightarrow \infty} \{\varphi(t)e_{\varphi\psi\sigma_a}(t, s)\} = 0,$$

then the unperturbed motion of system (3.1) is asymptotically stable.

Proof First we prove (1). Let $\varepsilon > 0$ and $t_0 \in \mathbb{T}$. Define

$$\delta(\varepsilon, t_0) = \varepsilon K^{-1}(t_0)\psi^{-1}(t_0)$$

and assume $\|x_0\| < \delta$. Then by Lemma 3.1,

$$\|x(t; t_0, x_0)\| < \varphi(t)\psi(t_0)e_{\varphi\psi\sigma_a}(t, t_0)\delta \leq \psi(t_0)K(t_0)\delta = \varepsilon.$$

Now we prove (2). Let $\varepsilon > 0$. Define

$$\delta(\varepsilon) = \varepsilon K^{-1}$$

and assume $\|x_0\| < \delta$. Then by Lemma 3.1,

$$\|x(t; t_0, x_0)\| < \varphi(t)\psi(t_0)e_{\varphi\psi\sigma_a}(t, t_0)\delta \leq K\delta = \varepsilon.$$

Finally we prove (3). Since $\varphi e_{\varphi\psi\sigma_a}(\cdot, s)$ tends to zero, it is bounded. By (1), we have stability. Let $\delta_0 = 1$ and assume $\|x_0\| < \delta_0$. Then by Lemma 3.1,

$$\|x(t; t_0, x_0)\| < \varphi(t)\psi(t_0)e_{\varphi\psi\sigma_a}(t, t_0) \rightarrow 0$$

as $t \rightarrow \infty$. \square

3.2 Stability via Stachurska’s Inequality

In preparation for Stachurska’s inequality on time scales, we require the following two lemmas.

Lemma 3.2 *If $f \leq g$ and $f, g \in \mathcal{R}^+$, then $\ominus f \geq \ominus g$.*

Proof Under the stated assumptions we calculate

$$(\ominus f) - (\ominus g) = -\frac{f}{1 + \mu f} + \frac{g}{1 + \mu g} = \frac{g - f}{(1 + \mu f)(1 + \mu g)} \geq 0. \square$$

Lemma 3.3 *If $f \geq 0$ and $g \in (0, 1]$, then $\ominus(f/g) \geq (\ominus f)/g$.*

Proof Under the stated assumptions we calculate

$$\left(\ominus \frac{f}{g}\right) - \frac{\ominus f}{g} = -\frac{f}{g + \mu f} + \frac{f}{g + \mu fg} = \frac{\mu f^2(1 - g)}{(g + \mu f)(g + \mu fg)} \geq 0. \square$$

Theorem 3.3 (Stachurska’s inequality) *Assume f, g, p are rd-continuous and nonnegative on \mathbb{T} . Let $m \in \mathbb{N} \setminus \{1\}$. If f/p is nondecreasing on \mathbb{T} , then each function x satisfying*

$$x(t) \leq f(t) + p(t) \int_{t_0}^t q(s)x^m(s)\Delta s \quad \text{for all } t \geq t_0 \tag{3.6}$$

satisfies

$$x(t) \leq \frac{f(t)}{\left\{1 + (m - 1) \int_{t_0}^t (\ominus qp f^{m-1})(s)\Delta s\right\}^{1/(m-1)}} \tag{3.7}$$

on $[t_0, t_m)$, where t_m is the first point for which the denominator on the right-hand side of (3.7) is nonpositive.

Proof We prove the claim by induction. First we assume that (3.6) holds for $m = 2$. Define

$$v(t) := \int_{t_0}^t q(s)x^2(s)\Delta s + \frac{f(t)}{p(t)}.$$

Then $x \leq pv$ and

$$v^\Delta = qx^2 + \left(\frac{f}{p}\right)^\Delta \leq qp^2v^2 + \left(\frac{f}{p}\right)^\Delta$$

and therefore by [2, Theorem 6.1]

$$v(t) \leq e_{qp^2v}(t, t_0) \left\{ v(t_0) + \int_{t_0}^t e_{\ominus qp^2v}(\sigma(s), t_0) \left(\frac{f}{p}\right)^\Delta(s) \Delta s \right\} \leq e_{qp^2v}(t, t_0) \frac{f(t)}{p(t)}$$

since $v(t_0) = f(t_0)/g(t_0)$, $(f/p)^\Delta \geq 0$, and $e_{\ominus qp^2v}(\sigma(s), t_0) \leq 1$. Define now

$$V := e_{\ominus qp^2v}(\cdot, t_0)$$

so that $v \leq f/(pV)$ and thus $qp^2v \leq qpf/V$ and hence

$$\ominus qp^2v \geq \ominus \frac{qpf}{V} \geq \frac{\ominus qpf}{V},$$

where we used Lemmas 3.2 and 3.3. Hence

$$V^\Delta = (\ominus qp^2v)V \geq \frac{\ominus qpf}{V}V = \ominus qpf.$$

Thus

$$V(t) \geq V(t_0) + \int_{t_0}^t (\ominus qpf)(s) \Delta s = 1 + \int_{t_0}^t (\ominus qpf)(s) \Delta s$$

and therefore

$$v(t) \leq \frac{f(t)}{p(t)V(t)} \leq \frac{f(t)}{p(t) \left\{ 1 + \int_{t_0}^t (\ominus qpf)(s) \Delta s \right\}}.$$

Plugging this in the inequality $x \leq pv$ yields (3.7) for $m = 2$.

Now we assume that the claim of the theorem holds for some $m \in \mathbb{N} \setminus \{1\}$. Suppose that (3.6) holds with m replaced by $m + 1$. Then

$$x(t) \leq f(t) + p(t) \int_{t_0}^t q(s)x(s)x^m(s) \Delta s$$

and using the induction hypothesis yields

$$x \leq \frac{f}{\{1 + (m-1)u\}^{1/(m-1)}}, \quad \text{where } u(t) := \int_{t_0}^t (\ominus qxp f^{m-1})(s) \Delta s.$$

Now using again Lemmas 3.2 and 3.3, we find

$$u^\Delta = \ominus qxp f^{m-1} \geq \ominus \frac{qpf^m}{\{1 + (m-1)u\}^{1/(m-1)}} \geq \frac{\ominus qpf^m}{\{1 + (m-1)u\}^{1/(m-1)}}.$$

Thus

$$mu^\Delta \{1 + (m - 1)u\}^{1/(m-1)} \geq m(\ominus qpf^m).$$

Let $F(x) = (1 + (m - 1)x)^{m/(m-1)}$ for $x \geq 0$ so that $F'(x) = m(1 + (m - 1)x)^{1/(m-1)}$ is nondecreasing. By Keller’s chain rule, Theorem 2.7, we have

$$\begin{aligned} \left\{ (1 + (m - 1)u)^{m/(m-1)} \right\}^\Delta &= (F \circ u)^\Delta = u^\Delta \int_0^1 F'(u(1 - h) + hu^\sigma) dh \\ &\geq u^\Delta \int_0^1 F'(u) dh = u^\Delta F'(u) \geq m(\ominus qpf^m), \end{aligned}$$

where we used $u^\Delta \leq 0$ and its consequence $u^\sigma \leq u$. Integrating yields

$$\begin{aligned} \{1 + (m - 1)u\}^{m/(m-1)}(t) &= 1 + \int_{t_0}^t \left\{ (1 + (m - 1)u)^{m/(m-1)} \right\}^\Delta(s) \Delta s \\ &\geq 1 + m \int_{t_0}^t (\ominus qpf^m)(s) \Delta s \end{aligned}$$

and therefore

$$\{1 + (m - 1)u(t)\}^{1/(m-1)} \geq \left\{ 1 + m \int_{t_0}^t (\ominus qpf^m)(s) \Delta s \right\}^{1/m}.$$

Plugging this in $x \leq f/(1 + (m - 1)u)^{1/(m-1)}$ gives (3.7) with m replaced by $m + 1$. \square

The following main results in this subsection are given for $\mathbb{T} = \mathbb{R}$ in [31, Lemma 1 and Theorems 1–3].

Lemma 3.4 *Suppose that (3.3) for $m > 1$ and (3.4) hold. Then any solution of (3.1) satisfies the estimate*

$$\|x(t; t_0, x_0)\| \leq \frac{\varphi(t)\psi(t_0) \|x_0\|}{\left\{ 1 - (m - 1) \|x_0\|^{m-1} \psi^{m-1}(t_0) D(t, t_0) \right\}^{1/(m-1)}} \tag{3.8}$$

for all $t \geq t_0$ for which

$$(m - 1) \|x_0\|^{m-1} \psi^{m-1}(t_0) D(t, t_0) < 1,$$

where

$$D(t, t_0) = \int_{t_0}^t \phi^m(\tau)\psi(\sigma(\tau))a(\tau)\Delta\tau.$$

Proof First note that the assumptions of Theorem 3.3 are satisfied. Let x be a solution of (3.1) so that by (3.2) we have for all $t \geq t_0$ the estimate

$$\|x(t; t_0, x_0)\| \leq \varphi(t)\psi(t_0) \|x_0\| + \int_{t_0}^t \varphi(t)\psi(\sigma(\tau))a(\tau) \|x(\tau; t_0, x_0)\|^m \Delta\tau.$$

Hence the function $y = \|x(\cdot; t_0, x_0)\| / \varphi$ satisfies

$$y(t) \leq \psi(t_0) \|x_0\| + \int_{t_0}^t \varphi^m(\tau)\psi(\sigma(\tau))a(\tau)y^m(\tau)\Delta\tau \quad \text{for all } t \geq t_0.$$

By Theorem 3.3, as long as the denominator remains positive,

$$y(t) \leq \frac{\psi(t_0) \|x_0\|}{\left\{1 + (m-1) \int_{t_0}^t (\ominus \phi^m \psi^\sigma a \psi^{m-1}(t_0) \|x_0\|^{m-1})(\tau) \Delta \tau\right\}^{1/(m-1)}}.$$

Since

$$\ominus g = -\frac{g}{1 + \mu g} \geq -g \quad \text{for all } g \geq 0,$$

and using the definition of y , the claim (3.8) follows. \square

Theorem 3.4 *Suppose that (3.3) for $m > 1$ and (3.4) hold.*

(i) *If for all $s \geq t_0$ there exists $K(s) > 0$ such that*

$$\varphi(t) \leq K(s) \quad \text{for all } t \geq s \geq t_0$$

and

$$D(t_0) := \lim_{t \rightarrow \infty} D(t, t_0) < \infty, \tag{3.9}$$

then the unperturbed motion of system (3.1) is stable;

(ii) *if there exist $K_1, K_2 > 0$ such that*

$$\varphi(t)\psi(s) \leq K_1 \quad \text{for all } t \geq s \geq t_0$$

and

$$\psi^{m-1}(s) \left\{ \lim_{t \rightarrow \infty} D(t, s) \right\} \leq K_2 \quad \text{for all } s \geq t_0,$$

then the unperturbed motion of system (3.1) is uniformly stable;

(iii) *if (3.9) and*

$$\lim_{t \rightarrow \infty} \varphi(t) = 0$$

hold, then the unperturbed motion of system (3.1) is asymptotically stable.

Proof First we prove (1). Let $\varepsilon > 0$ and $t_0 \in \mathbb{T}$. Define

$$\delta(\varepsilon, t_0) = \min \left\{ [2(m-1)\psi^{m-1}(t_0)D(t_0)]^{-1/(m-1)}, \varepsilon\psi^{-1}(t_0)K^{-1}(t_0)2^{-1/(m-1)} \right\}$$

and assume $\|x_0\| < \delta$. Then by Lemma 3.4,

$$\begin{aligned} \|x(t; t_0, x_0)\| &< \frac{\varphi(t)\psi(t_0)\delta}{\{1 - (m-1)\delta^{m-1}\psi^{m-1}(t_0)D(t, t_0)\}^{1/(m-1)}} \\ &\leq \frac{\varphi(t)\psi(t_0)\varepsilon\psi^{-1}(t_0)K^{-1}(t_0)2^{-1/(m-1)}}{\{1 - (m-1)2^{-1}(m-1)^{-1}\psi^{1-m}(t_0)D^{-1}(t_0)\psi^{m-1}(t_0)D(t, t_0)\}^{1/(m-1)}} \\ &\leq \frac{\varepsilon 2^{-1/(m-1)}}{\{1 - 2^{-1}\}^{1/(m-1)}} = \varepsilon. \end{aligned}$$

Now we prove (2). Let $\varepsilon > 0$. Define

$$\delta(\varepsilon) = \min \left\{ [2(m-1)K_2]^{-1/(m-1)}, \varepsilon K_1^{-1} 2^{-1/(m-1)} \right\}$$

and assume $\|x_0\| < \delta$. Then by Lemma 3.4,

$$\begin{aligned} \|x(t; t_0, x_0)\| &< \frac{\varphi(t)\psi(t_0)\delta}{\{1 - (m - 1)\delta^{m-1}\psi^{m-1}(t_0)D(t, t_0)\}^{1/(m-1)}} \\ &\leq \frac{\varphi(t)\psi(t_0)\varepsilon K_1^{-1}2^{-1/(m-1)}}{\{1 - (m - 1)2^{-1}(m - 1)^{-1}K_2^{-1}\psi^{m-1}(t_0)D(t, t_0)\}^{1/(m-1)}} \\ &\leq \frac{\varepsilon 2^{-1/(m-1)}}{\{1 - 2^{-1}\}^{1/(m-1)}} = \varepsilon. \end{aligned}$$

Finally we prove (3). Since φ tends to zero, it is bounded. By (1), we have stability. Let $\delta_0 > 0$ be such that the denominator in (3.8) is positive and assume $\|x_0\| < \delta_0$. Then by Lemma 3.4,

$$\|x(t; t_0, x_0)\| < \frac{\varphi(t)\psi(t_0)\delta_0}{\{1 - (m - 1)\delta_0^{m-1}\psi^{m-1}(t_0)D(t, t_0)\}^{1/(m-1)}} \rightarrow 0$$

as $t \rightarrow \infty$. \square

4 Generalized Direct Liapunov Method on Time Scales

4.1 General Theorems

The direct method of investigation of motion stability of continuous systems with a finite number of degrees of freedom as developed by Liapunov is now extended for many classes of systems of equations. In this section we present the main theorems of the direct Liapunov method for dynamic equations on a time scale \mathbb{T} .

Corresponding to the time scale \mathbb{T} we consider the following sets:

$$\begin{aligned} \mathcal{A} &= \{t \in \mathbb{T} : t \text{ left-dense and right-scattered}\}, \\ \mathcal{B} &= \{t \in \mathbb{T} : t \text{ left-scattered and right-dense}\}, \\ \mathcal{C} &= \{t \in \mathbb{T} : t \text{ left-scattered and right-scattered}\}, \\ \mathcal{D} &= \{t \in \mathbb{T} : t \text{ left-dense and right-dense}\}. \end{aligned}$$

Assume that $\sup \mathbb{T} = a \in \mathcal{A} \cup \mathcal{D}$ and $\inf \mathbb{T} = b \in \mathcal{B} \cup \mathcal{D}$ and designate the Euler derivative of the state vector of system $x : \mathbb{T} \rightarrow \mathbb{R}^n$ in $t \in \mathbb{T}$ by $\dot{x}(t)$, should it exists.

We consider a system of perturbed motion equations

$$x^\Delta(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{4.1}$$

where $x : \mathbb{T} \rightarrow \mathbb{R}^n$, $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, and

$$x^\Delta(t) = \begin{cases} \frac{x(\sigma(t)) - x(t)}{\mu(t)} & \text{if } t \in \mathcal{A} \cup \mathcal{C}, \\ \dot{x}(t) & \text{in other points.} \end{cases}$$

Our assumptions on system (4.1) are as follows:

- H₁ The vector-valued function $F(t) = f(t, x(t))$ satisfies the condition $F \in C_{rd}(\mathbb{T})$ whenever x is a differentiable function with its values in N , where $N \subset \mathbb{R}^n$ is an open connected neighborhood of the state $x = 0$.

H₂ The vector-valued function $f(t, x)$ is component-wise regressive, i.e.,

$$e^T + \mu(t)f(t, x) \neq 0 \text{ for all } t \in [t_0, \infty), \quad \text{where } e^T = (1, \dots, 1)^T \in \mathbb{R}^n.$$

H₃ $f(t, x) = 0$ for all $t \in [t_0, \infty)$ iff $x = 0$.

H₄ The graininess function μ satisfies $0 < \mu(t) \in M$ for all $t \in \mathbb{T}$, where M is a compact set.

For stability analysis of the state $x = 0$ of system (4.1), the matrix-valued function [24]

$$U(t, x) = [v_{ij}(t, x)], \quad i, j = 1, \dots, m \quad (4.2)$$

will be applied as an auxiliary function, where $v_{ii} : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ for $i, j = 1, \dots, m$ and $v_{ij} : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}$ for $i \neq j, i, j = 1, \dots, m$. The elements $v_{ij}(t, x)$ of the matrix-valued function (4.2) are assumed to satisfy the following conditions:

- (1) $v_{ij}(t, x)$ are locally Lipschitzian in x for all $t \in \mathbb{T}$;
- (2) $v_{ij}(t, x) = 0$ for all $t \in \mathbb{T}$ iff $x = 0$;
- (3) $v_{ij}(t, x) = v_{ji}(t, x)$ for all $t \in \mathbb{T}$ and $i, j = 1, \dots, m$.

Along with the function (4.2) we shall use the scalar function

$$v(t, x, \theta) = \theta^T U(t, x) \theta, \quad \theta \in \mathbb{R}_+^m \quad (4.3)$$

and comparison functions of class K . Recall that a real-valued function a belongs to the class K if it is definite continuous and strictly increasing on $[0, r_1]$ with $0 \leq r_1 < +\infty$ and $a(0) = 0$.

Definition 4.1 The matrix-valued function (4.2) is called

- (1) *positive (negative) semidefinite* on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$, if $v(t, x, \theta) \geq 0$ ($v(t, x, \theta) \leq 0$) for all $(t, x, \theta) \in \mathbb{T} \times N \times \mathbb{R}_+^m$, respectively;
- (2) *positive definite* on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$, if there exists a function $a \in K$ such that $v(t, x, \theta) \geq a(\|x\|)$ for all $(t, x, \theta) \in \mathbb{T} \times N \times \mathbb{R}_+^m$;
- (3) *decreascent* on $\mathbb{T} \times N$ if there exists a function $b \in K$ such that $v(t, x, \theta) \leq b(\|x\|)$ for all $(t, x, \theta) \in \mathbb{T} \times N \times \mathbb{R}_+^m$;
- (4) *radially unbounded* on $\mathbb{T} \times N$, if $v(t, x, \theta) \rightarrow +\infty$ for $\|x\| \rightarrow +\infty$, for $(t, x, \theta) \in \mathbb{T} \times N \times \mathbb{R}_+^m$.

Lemma 4.1 The matrix-valued function $U : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ is positive definite on \mathbb{T} iff the function (4.3) can be represented as

$$\theta^T U(t, x) \theta = \theta^T U_+(t, x) \theta + a(\|x\|), \quad t \in \mathbb{T},$$

where U_+ is a positive semidefinite matrix-valued function and $a \in K$.

Lemma 4.2 *The matrix-valued function $U : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ is decrescent on \mathbb{T} iff the function (4.3) can be represented as*

$$\theta^T U(t, x) \theta = \theta^T U_-(t, x) \theta + b(\|x\|), \quad t \in \mathbb{T},$$

where U_- is a negative semidefinite matrix-valued function and $b \in K$.

Further we need the notion of the total Δ -derivative of the function (4.3) along solutions of system (4.1). It reads as

$$v_+^\Delta(t, x, \theta) = \theta^T U_+^\Delta(t, x) \theta, \quad \theta \in \mathbb{R}_+^m, \quad t \in \mathbb{T},$$

where $U_+^\Delta(t, x)$ is calculated element-wise by the formula

$$U_+^\Delta(t) = \begin{cases} \overline{\lim}\{[u_{ij}(t+h) - u_{ij}(t)]h^{-1} : h \rightarrow 0, h+t \in \mathbb{T}\} & \text{if } t = \sigma(t), \\ [u_{ij}(\sigma(t)) - u_{ij}(t)]\mu^{-1}(t) & \text{if } t < \sigma(t), \end{cases}$$

where $u_{ij}(t) = u_{ij}(t, x(t; t_0, x_0))$, $1 \leq i, j \leq m$.

We note that the calculation of the derivative is not easy in general. However, if the function (4.3) is independent of t , then it may be easy to calculate the Δ -derivative.

Example 4.1 Consider the function $v(t, x, \theta) = x^T x$, $x \in \mathbb{R}^n$. Then by Theorem 2.3 (3) we have

$$\begin{aligned} v_+^\Delta(t, x, \theta) &= (x^T x)^\Delta(t) = x^T(t)x^\Delta(t) + (x^T)^\Delta(t)x(\sigma(t)) \\ &= x^T(t)f(t, x(t)) + f^T(t, x(t))[x(t) + \mu(t)f(t, x(t))]. \end{aligned}$$

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and

$$v_+^\Delta(t, x, \theta) = \frac{d}{dt}(x^T x) = x^T f(t, x) + f^T(t, x)x.$$

Example 4.2 Consider the function $U(t, x) = xx^T$, $x \in \mathbb{R}^n$. By Theorem 2.3 (3) we have

$$\begin{aligned} v_+^\Delta(t, x, \theta) &= \theta^T (xx^T)^\Delta(t) \theta \\ &= \theta^T \{x(t)f^T(t, x(t)) + f(t, x)x^T(t) + \mu(t)f(t, x(t))f^T(t, x(t))\} \theta. \end{aligned}$$

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and

$$v_+^\Delta(t, x, \theta) = \theta^T \frac{d}{dt}(xx^T) \theta = \theta^T \{x(t)f^T(t, x) + f(t, x)x^T(t)\} \theta.$$

Next, we shall formulate a general Liapunov-type result on stability of the state $x = 0$ of system (4.1).

Theorem 4.1 *Assume that the vector-valued function $f(t, x)$ in system (4.1) satisfies assumptions H_1 – H_4 on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$. Assume there exist*

- (1) *a matrix-valued function $U : \mathbb{T} \times N \rightarrow \mathbb{R}^{m \times m}$ and a vector $\theta \in \mathbb{R}_+^m$ such that the function $v(t, x, \theta) = \theta^T U(t, x) \theta$ is locally Lipschitzian in x for all $t \in \mathbb{T}$;*

(2) comparison functions $\psi_{i1}, \psi_{i2}, \psi_{i3} \in K$ and symmetric $m \times m$ matrices A_j , $j = 1, 2$, such that for all $(t, x) \in \mathbb{T} \times N$

(a) $\psi_1^T(\|x\|)A_1\psi_1(\|x\|) \leq v(t, x, \theta)$;

(b) $v(t, x, \theta) \leq \psi_2^T(\|x\|)A_2\psi_2(\|x\|)$;

(c) there exists an $m \times m$ matrix $A_3 = A_3(\mu(t))$ such that

$$v_+^\Delta(t, x, \theta) \leq \psi_3^T(\|x\|)A_3\psi_3(\|x\|) \quad \text{for all } (t, x) \in \mathbb{T} \times N;$$

(d) there exists $\mu^* > 0$ such that $\mu^* \in M$ and

$$\frac{1}{2} [A_3^T(\mu(t)) + A_3(\mu(t))] \leq A_3(\mu^*) \quad \text{whenever } 0 < \mu(t) < \mu^*.$$

Then, if the matrices A_1 and A_2 are positive definite and the matrix $A_3^* = A_3(\mu^*)$ is negative semidefinite, then the state $x = 0$ of system (4.1) is stable under conditions 2(a), 2(b), 2(d) and uniformly stable under conditions 2(a)–2(d).

Proof The fact that A_1 and A_2 are positive definite matrices implies that $\lambda_m(A_1) > 0$ and $\lambda_M(A_2) > 0$, where $\lambda_m(A_1)$ and $\lambda_M(A_2)$ are the minimal and maximal eigenvalues of the matrices A_1 and A_2 , respectively. In view of this fact we present the estimates (a) and (b) from condition (2) as

$$\lambda_m(A_1)\bar{\psi}_1(\|x\|) \leq v(t, x, \theta) \leq \lambda_M(A_2)\bar{\psi}_2(\|x\|) \quad \text{for all } (t, x) \in \mathbb{T} \times N,$$

where $\bar{\psi}_1, \bar{\psi}_2 \in K$ so that

$$\bar{\psi}_1(\|x\|) \leq \psi_1^T(\|x\|)\psi_1(\|x\|), \quad \bar{\psi}_2(\|x\|) \geq \psi_2^T(\|x\|)\psi_2(\|x\|) \quad \text{for all } x \in N.$$

Let $\varepsilon > 0$. Let $S(t)$ be the following assertion:

$$\text{There exists } \delta = \delta(\varepsilon) > 0 \text{ such that } \|x_0\| < \delta \text{ implies } \|x(t; t_0, x_0)\| < \varepsilon.$$

Let

$$S^* = \{t \in [t_0, \infty) : S(t) \text{ is false}\}.$$

Let us show that under our assumptions the set S^* is empty. Assume on the contrary $S^* \neq \emptyset$. The fact that S^* is closed and nonempty implies that $\inf S^* = t^* \in S^*$. First notice that $S(t_0)$ is true, since $\|x(t_0; t_0, x_0)\| < \varepsilon$ for $\|x_0\| < \varepsilon$ because $x(t_0; t_0, x_0) = x_0$. Therefore $t^* > t_0$. Then pick $\delta_1 = \delta_1(\varepsilon)$ such that

$$\lambda_M(A_2)\bar{\psi}_2(\delta_1) < \lambda_m(A_1)\bar{\psi}_1(\varepsilon).$$

Define $\delta = \min\{\varepsilon, \delta_1\}$ so that

$$\|x(t^*; t_0, x_0)\| = \varepsilon \quad \text{and} \quad \|x(t; t_0, x_0)\| < \varepsilon \text{ for } t \in [t_0, t^*) \text{ and } \|x_0\| < \delta.$$

By conditions 2(c) and 2(d) we have

$$v_+^\Delta(t, x, \theta) \leq \lambda_M(A_3^*)\bar{\psi}_3(\|x\|) \leq 0 \quad \text{for all } (t, x, \theta) \in \mathbb{T} \times N \times \mathbb{R}_+^m.$$

Hence, for $t = t^*$,

$$\begin{aligned} \lambda_m(A_1)\bar{\psi}_1(\varepsilon) &= \lambda_m(A_1)\bar{\psi}_1(\|x(t^*; t_0, x_0)\|) \leq v(t^*, x(t^*), \theta) \\ &\leq v(t_0, x_0, \theta) < \lambda_M(A_2)\bar{\psi}_2(\delta) \end{aligned}$$

for $\|x_0\| < \delta$. This contradiction yields that $S(t^*)$ is true so that $t^* \notin S^*$. Hence $S^* = \emptyset$ and the proof is complete. \square

Corollary 4.1 (cf. [7]) *Let the vector-valued function f in system (4.1) satisfy hypotheses H_1 – H_4 on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$. Suppose there exist at least one couple of indices $(p, q) \in [1, m]$ for which $(v_{pq}(t, x) \neq 0) \in U(t, x)$ and the function $v(t, x, \theta) = e^T U(t, x)e = v(t, x)$ for all $(t, x) \in \mathbb{T} \times N$ satisfies the conditions*

- (a) $\psi_1(\|x\|) \leq v(t, x)$;
- (b) $v(t, x) \leq \psi_2(\|x\|)$;
- (c) $v^\Delta(t, x)|_{(4.1)} \leq 0$ for all $0 < \mu(t) < \mu^* \in M$,

where ψ_1, ψ_2 are some functions of class K . Then the state $x = 0$ of system (4.1) is stable under conditions (a) and (c) and uniformly stable under conditions (a)–(c).

Theorem 4.2 *Assume that the vector-valued function $f(t, x)$ in system (4.1) satisfies hypotheses H_1 – H_4 on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$. Assume there exist*

- (1) *a matrix-valued function $U : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ and a vector $\theta \in \mathbb{R}_+^m$ such that the function $v(t, x, \theta) = \theta^T U(t, x)\theta$ is locally Lipschitzian in x for all $t \in \mathbb{T}$;*
- (2) *comparison functions $\psi_{i1}, \psi_{i2}, \psi_{i3} \in K$ and symmetric $m \times m$ matrices B_j , $j = 1, 2, 3$ such that*
 - (a) $\psi_1^T(\|x\|)B_1\psi_1(\|x\|) \leq v(t, x, \theta)$;
 - (b) $v(t, x, \theta) \leq \psi_2^T(\|x\|)B_2\psi_2(\|x\|)$ for all $(t, x, \theta) \in \mathbb{T} \times N \times \mathbb{R}_+^m$;
 - (c) *there exists an $m \times m$ matrix $B_3 = B_3(\mu(t))$ such that*

$$v^\Delta_+(t, x, \theta) \leq \psi_3^T(\|x\|)B_3\psi_3(\|x\|) + w(t, \psi_3(\|x\|))$$

for all $(t, x, \theta) \in \mathbb{T} \times N \times \mathbb{R}_+^m$, where $w(t, \cdot)$ satisfies the condition

$$\lim_{\|\psi_3\| \rightarrow 0} \frac{|w(t, \psi_3(\|x\|))|}{\|\psi_3\|} = 0 \quad \text{as} \quad \|\psi_3\| \rightarrow 0$$

uniformly with respect to $t \in \mathbb{T}$;

- (d) *there exists $\mu^* > 0$ such that $\mu^* \in M$ and*

$$\frac{1}{2} [B_3^T(\mu(t)) + B_3(\mu(t))] \leq B_3(\mu^*) \quad \text{for all} \quad 0 < \mu(t) < \mu^*.$$

Then, if the matrices B_1 and B_2 are positive definite and the matrix $B_3^* = B_3(\mu^*)$ is negative definite, then

- (a) *under conditions 2(a) and 2(c) the state $x = 0$ of system (4.1) is asymptotically stable on \mathbb{T} ;*
- (b) *under conditions 2(a)–2(c) the state $x = 0$ of system (4.1) is uniformly asymptotically stable on \mathbb{T} .*

Proof Consider the assertion

$$\{S_1(t): S(t) \text{ for } t \in [t_0, \infty) \text{ and } \lim_{t \rightarrow \infty} \|x(t; t_0, x_0)\| = 0, \text{ if } \|x_0\| < \delta(t_0)\}.$$

Following considerations similar to those in the proof of Theorem 4.1, one can easily verify the assertions. \square

Corollary 4.2 (cf. [7]) *Let the vector-function f in system (4.1) satisfy hypotheses H_1 – H_4 on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$. Suppose there exist at least one couple of indices $(p, q) \in [1, m]$ for which $(v_{pq}(t, x) \neq 0) \in U(t, x)$ and the function $v(t, x, \theta) = e^T U(t, x) e = v(t, x)$ for all $(t, x) \in \mathbb{T} \times N$ satisfies the conditions*

- (a) $\psi_1(\|x\|) \leq v(t, x)$;
- (b) $v(t, x) \leq \psi_2(\|x\|)$;
- (c) for all $0 < \mu(t) < \mu^* \in M$

$$v^\Delta(t, x)|_{(4.1)} \leq -\psi_3(\|x\|) + w(t, \psi_3(\|x\|))$$

and

$$\lim \frac{|w(t, \psi_3(\|x\|))|}{\psi_3(\|x\|)} \quad \text{as } \psi_3 \rightarrow 0$$

uniformly with respect to $t \in \mathbb{T}$, where ψ_1, ψ_2, ψ_3 are comparison functions of class K .

Then, under conditions (a) and (c) the state $x = 0$ of system (4.1) is asymptotically stable and under conditions (a)–(c) the state $x = 0$ of system (4.1) is uniformly asymptotically stable.

Theorem 4.3 *Assume that the vector-valued function $f(t, x)$ in system (4.1) satisfies hypotheses H_1 – H_4 on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$. Suppose*

- (1) *there exist a matrix-valued function $U : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$ and a vector $\theta \in \mathbb{R}_+^m$ such that the function $v(t, x, \theta) = \theta^T U(t, x) \theta$ is locally Lipschitzian in x for all $t \in \mathbb{T}$;*
- (2) *there exist comparison functions $\psi_1, \psi_3 \in K$ and a symmetric $m \times m$ matrix A_1 such that for $(t, x) \in \mathbb{T} \times N$*

$$(a) \quad \psi_1^T(\|x\|) A_1 \psi_1(\|x\|) \leq v(t, x, \theta);$$

$$(b) \quad \text{there exists an } m \times m \text{ matrix } C_3 = C_3(\mu(t)) \text{ such that}$$

$$v_+^\Delta(t, x, \theta) \geq \psi_3^T(\|x\|) C_3 \psi_3(\|x\|) \quad \text{for all } (t, x, \theta) \in \mathbb{T} \times L \times \mathbb{R}_+^m, \quad L \subset N;$$

$$(c) \quad \text{there exists an } m \times m \text{ matrix } C_3(\mu^*) \geq \frac{1}{2}[C_3^T(\mu(t)) + C_3(\mu(t))] \text{ for some } \mu^* \in M \text{ at } t \in \mathbb{T};$$

- (3) *the point $x = 0$ belongs to the boundary L ;*

$$(4) \quad v(t, x, \theta) = 0 \text{ on } \mathbb{T} \times (\partial L \cap B_\Delta), \text{ where } B_\Delta = \{x \in \mathbb{R}^n : \|x\| < \Delta\}.$$

Then, if the matrices A_1 and $C_3(\mu^*)$ are positive definite, then the state $x = 0$ of system (4.1) is unstable.

Proof The proof is based on the assertion

$$\{S_2(t) : \text{there exist } t_1 \in [t_0, \infty) \text{ such that } \|x(t_1; t_0, x_0)\| > \varepsilon \\ \text{for any } 0 < \delta < \varepsilon, \text{ for which } \|x_0\| < \delta\}$$

and follows arguments similar to those of the proof of Theorem 4.1. \square

Corollary 4.3 (cf. [7]) *Let the vector-function f in system (4.1) satisfy hypotheses H_1 – H_4 on $\mathbb{T} \times N$, $N \subset \mathbb{R}^n$. Suppose there exist at least one couple $(p, q) \in [1, m]$ such that for $(v_{pq}(t, x) \neq 0) \in U(t, x)$ and the function $v(t, x, e) = e^T U(t, x)e = v(t, x)$ for all $(t, x) \in \mathbb{T} \times N$ satisfies the conditions*

- (a) $\psi_1(\|x\|) \leq v(t, x)$, $\psi_1 \in K$;
- (b) for all $0 < \mu(t) < \mu^* < M$ the inequality $v_+^\Delta(t, x, \theta)|_{(4.1)} \geq \psi_3(\|x\|)$, $\psi_3 \in K$ holds;
- (c) the point $(x = 0) \in \partial L$;
- (d) $v(t, x) = 0$ on $\mathbb{T} \times (\partial\mathbb{T} \cap B_\Delta)$.

Then the state $x = 0$ of system (4.1) is unstable.

Example 4.3 Consider the perturbed motion equations on \mathbb{T} with the graininess function $0 < \mu(t) < +\infty$

$$\begin{aligned} x^\Delta &= y(x + y), & x(t_0) &= x_0, \\ y^\Delta &= -x(x + y), & y(t_0) &= y_0. \end{aligned} \tag{4.4}$$

For the function $v(x, y) = x^2 + y^2$ we have

$$v_+^\Delta(x(t), y(t))|_{(4.4)} = \mu(t)(x + y)^2(x^2 + y^2) \tag{4.5}$$

which translates for the case $\mathbb{T} = \mathbb{R}$ to

$$\dot{v}(x(t), y(t)) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Condition (4.5) implies that $x = y = 0$ of system (4.4) is stable when $\mathbb{T} = \mathbb{R}$, while $x = y = 0$ of system (4.4) is unstable whenever the graininess function satisfies $0 < \mu(t) < +\infty$.

Example 4.4 Let a system of dynamic equations

$$\begin{aligned} x^\Delta &= -x - y(x^2 + y^2), & x(t_0) &= x_0, \\ y^\Delta &= -y + x(x^2 + y^2), & y(t_0) &= y_0 \end{aligned} \tag{4.6}$$

be given. For the positive definite function $v(x, y) = x^2 + y^2$ we have

$$v_+^\Delta(x(t), y(t))|_{(4.6)} = -2(x^2 + y^2) + \mu(t)[x^2 + y^2 + (x^2 + y^2)^3] \tag{4.7}$$

which translates for the case $\mathbb{T} = \mathbb{R}$ to

$$\dot{v}(x(t), y(t)) = -2(x^2 + y^2) \quad \text{for all } t \in \mathbb{R}.$$

The analysis of (4.7) shows that $x = y = 0$ of the system (4.6) is asymptotically stable when $\mathbb{T} = \mathbb{R}$. If the time scale \mathbb{T} has the graininess $\mu(t) = 1$, i.e., $\mathbb{T} = \mathbb{Z}$, then for the initial values (x_0, y_0) from the domain $x_0^2 + y_0^2 < 1$, the zero solution of system (4.6) is asymptotically stable on \mathbb{Z} . If $\mu(t) = 2$, which corresponds to the time scale $\mathbb{T} = 2\mathbb{N}_0 = \{k_0, k_0 + 2, k_0 + 4, \dots\}$, then

$$v^\Delta(x(t), y(t))|_{(4.6)} = 2(x^2 + y^2)^3,$$

and the state $x = y = 0$ of system (4.6) is unstable.

4.2 Linear Systems

Consider a time scale \mathbb{T} and a linear homogeneous dynamic system

$$x^\Delta(t) = A(t)x(t), \quad t \in \mathbb{T}, \quad (4.8)$$

where the matrix-valued function $A : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$ is rd-continuous and regressive. Together with equation (4.8), we consider the initial value problem

$$x^\Delta(t) = A(t)x(t), \quad x(s) = x_0,$$

where $s \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$.

In some cases the behavior of the solution x of system (4.8) can be investigated by means of the function $v(x) = x^T x$ for which

$$v^\Delta(x(t))|_{(4.8)} = x^T(A^T \oplus A)(t)x,$$

where $(A^T \oplus A)(t) = A^T(t) + A(t) + \mu(t)A^T(t)A(t)$. We define the sets

$$\Lambda_s(\mathbb{T}) = \{A \in \mathcal{R}(\mathbb{T}) : \exists c \in \mathcal{R}^+ \text{ for which } (A^T \oplus A)(t) \leq 2cI < 0 \text{ for all } t \in \mathbb{T}\}$$

and

$$\Lambda_u(\mathbb{T}) = \{A \in \mathcal{R}(\mathbb{T}) : \exists c > 0 \text{ for which } (A^T \oplus A)(t) \geq 2cI \text{ for all } t \in \mathbb{T}\},$$

where I is the $n \times n$ identity matrix and \mathcal{R}^+ is the set of positively regressive functions (see Section 2.4). Let the norm of the matrix M be defined by $\|M\| = \sup_{u \neq 0} \frac{|Mu|}{|u|}$.

The following results are known [1].

Theorem 4.4 *Consider system (4.8). If $A \in \Lambda_s(\mathbb{T})$, then*

- (a) $\|e_A(t, s)\| \leq e_c(t, s)$ for all $s \leq t$;
- (b) $\|e_A(t, s)\| \geq e_c(t, s)$ for all $s \geq t$;
- (c) $\lim_{t \rightarrow \infty} \|e_A(t, s)\| = 0$ for every fixed s and $\lim_{s \rightarrow -\infty} \|e_A(t, s)\| = 0$ for every fixed t .

If $A \in \Lambda_u(\mathbb{T})$, then

- (d) $\|e_A(t, s)\| \geq e_c(t, s)$ for all $t \leq s$;
- (e) $\|e_A(t, s)\| \leq e_c(t, s)$ for all $t \geq s$;
- (f) $\lim_{t \rightarrow -\infty} \|e_A(t, s)\| = \infty$ for every fixed s and $\lim_{s \rightarrow \infty} \|e_A(t, s)\| = 0$ for every fixed t .

The proof of these assertions is based on the analysis of the Δ -derivative of the function $v(x) = x^T x$:

$$v^\Delta(x(t))|_{(4.8)} = (2 \odot c)v(x(t)),$$

where $2 \odot c = c \oplus c = 2c + \mu(t)c^2$.

Now we apply Theorems 4.1, 4.2, 4.3 to system (4.8). Assume that in the matrix-valued function $U(t, x)$ the elements $v_{ij}(t, x)$, $i, j = 1, 2, \dots, n$ are such that $v_{ii}(t, x) = x_i^2$, $i = 1, 2, \dots, n$ and $v_{ij}(t, x) \equiv 0$ for $i \neq j$. In this case, the function (4.3) with $\theta = (1, 1, \dots, 1)^T \in \mathbb{R}_+^n$ is of the form

$$v(t, x, \theta) = \theta^T U(t, x)\theta = x^T x. \quad (4.9)$$

Theorem 4.5 *Let the system (4.1) be of the form (4.8) and the function (4.3) be of the form (4.9). Then, if there exists $\mu^* \in M$ such that the matrix $D_0(t, \mu(t))$ in the expression*

$$v_+^\Delta(t, x(t)) = x^T(t)D_0(t, \mu(t))x(t), \quad \text{where } D_0(t, \mu(t)) = (A^T \oplus A)(t),$$

is negative semidefinite (negative definite) whenever $0 < \mu(t) \leq \mu^$, then the equilibrium state $x = 0$ of system (4.8) is stable (asymptotically stable), respectively.*

Proof The statements of the theorem follow from Theorem 4.1. \square
 Next, we shall consider the case when

$$v(t, x, \theta) = \theta^T U(t, x)\theta = x^T H(t)x, \quad t \in \mathbb{T}^\kappa, \tag{4.10}$$

where $H \in C_{\text{rd}}^1(\mathbb{T}^\kappa, \mathbb{R}^{n \times n})$, and assume that the condition

$$\alpha \|x(t)\|^2 \leq x^T H(t)x \leq \beta \|x(t)\|^2 \quad \text{for all } t \in \mathbb{T}^\kappa, \tag{4.11}$$

is satisfied, where $\alpha, \beta > 0$ are constants.

Theorem 4.6 (cf. [4]) *Let the system (4.1) be of the form (4.8) and suppose that the function (4.10) satisfies the estimate (4.11). Then, if there exists $\mu^* \in M$ such that the matrix $D_1(t, \mu(t))$ in the expression*

$$v_+^\Delta(t, x(t))|_{(4.8)} = x^T(t)D_1(t, \mu)x(t), \tag{4.12}$$

where

$$D_1(t, \mu) = (I + \mu A^T(t))H^\Delta(t)(I + \mu A(t)) + A^T(t)H(t) + H(t)A(t) + \mu A^T(t)H(t)A(t), \tag{4.13}$$

is negative semidefinite (negative definite) for all $0 < \mu(t) \leq \mu^$, then the state $x = 0$ of system (4.8) is uniformly stable (uniformly asymptotically stable), respectively.*

Proof The statements of this theorem follow from Theorem 4.2. \square

Remark 4.1 If in the expression (4.13) the Δ -derivative of the matrix $H(t)$ satisfies $H^\Delta(t) \equiv 0$ for all $t \in \mathbb{T}^\kappa$, then the analysis of $v_+^\Delta(t, x(t))|_{(4.8)}$ being of definite sign is simplified.

Now we assume that there exists a positive definite constant matrix Q , $Q = Q^T$, such that

$$A^T(t)H(t) + H(t)A(t) + \mu(t)A^T(t)H(t)A(t) = -Q. \tag{4.14}$$

Then the expression (4.12) becomes

$$v_+^\Delta(t, x(t))|_{(4.8)} = x^T(t)[(I + \mu(t)A^T(t))H^\Delta(t)(I + \mu(t)A(t)) - Q]x(t), \quad t \in \mathbb{T}^\kappa.$$

By the equation

$$(I + \mu(t)A^T(t))H^\Delta(t)(I + \mu(t)A(t)) - Q = 0$$

we define $\mu_{\max} = \max\{\mu(t) : t \in \mathbb{T}^\kappa\} \in M$.

Theorem 4.7 *Let system (4.1) be of form (4.8) and suppose that the function (4.10) satisfies condition (4.11). If for $0 < \mu(t) < \mu_{\max}$,*

$$A(t)H(t) + H(t)A(t) + \mu(t)A^T(t)H(t)A(t) \leq -Q,$$

then the state $x = 0$ of system (4.8) is uniformly asymptotically stable.

Proof All conditions of Theorem 4.2 from Section 4.1 are satisfied and thus the state $x = 0$ of system (4.8) is uniformly asymptotically stable. \square

Remark 4.2 The matrix equation (4.14) is a generalization of the known matrix Liapunov equation [35]

$$A^T H + H A = -Q \tag{4.15}$$

for a stable linear autonomous system, whose solution is known in the form

$$H = \int_0^\infty \exp(A^T s) Q \exp(As) ds.$$

The matrix A in equation (4.15) is constant and stable.

In order to construct the solution H for equation (4.14) on \mathbb{T}^κ , we use the following result from [2].

Lemma 4.3 *Let be given $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n})$ and $C : \mathbb{T} \rightarrow \mathbb{R}^{n \times n}$. If the matrix-valued function C is differentiable and is a solution of the dynamic matrix equation*

$$C^\Delta(\tau) = A(\tau)C(\tau) - C(\sigma(\tau))A(\tau),$$

then

$$C(\tau)e_A(\tau, s) = e_A(\tau, s)C(s).$$

Corollary 4.4 *Let $A \in \mathcal{R}$. If the constant matrix C commutes with $A(t)$, then C commutes with $e_A(t)$. In particular, if A is a constant matrix, then A commutes with $e_A(t)$.*

Using Lemma 4.3 and Corollary 4.4, the solution of equation (4.14) is obtained in [4] in the following form.

Theorem 4.8 *Assume that system (4.8) is such that all eigenvalues of the $n \times n$ matrix-valued function A are in the Hilger circle, i.e., $\{z \in \mathbb{C} : |z + \frac{1}{h}| = \frac{1}{h}\}$, $h > 0$ for all $t \geq t_0$. Then for every $t \in \mathbb{T}$ there exists a time scale \mathbb{S} such that the integration on $\mathbb{T}_\mathbb{S} = [0, \infty)$ enables one to find the solution of equation (4.14) in the form*

$$H(t) = \int_{\mathbb{T}_\mathbb{S}} e_{A^T}(s, 0) Q e_A(s, 0) \Delta s. \tag{4.16}$$

Besides, if the matrix Q is positive definite, then the matrix $H(t)$ is also positive definite for all $t \geq t_0$.

Proof This assertion is proved by direct substitution of expression (4.16) into the left-hand part of equation (4.14). Moreover, when $\mu(t) > 0$, then $\mathbb{S} = \mu(t)\mathbb{N}_0$, and when $\mu(t) = 0$, then $\mathbb{S} = \mathbb{R}$. \square

Theorem 4.9 *Let system (4.1) be of form (4.8) and suppose the function (4.10) satisfies the estimate $\alpha\|x(t)\|^2 \leq x^T H(t)x$, where $\alpha > 0$ is constant, for all $(t, x) \in \mathbb{T} \times \mathbb{R}^n$, $H \in C_{\text{rd}}(\mathbb{T}, \mathbb{R}^{n \times n})$. If there exists a value $0 < \mu^* \in M$ such that for at least one value of $t^* \in \mathbb{T}$, the matrix $D_1(t^*, \mu(t^*))$ in (4.12) is positive semidefinite (positive definite), then the state $x = 0$ of system (4.8) is unstable (strongly unstable).*

Strong instability is understood as exponential growth of solutions x on \mathbb{T} of system (4.8).

In the end of this section we note that in [8] there is a result on the existence of a Liapunov function in the case of uniform exponential stability of the zero solution of system (4.8) in the form

$$v(t, x) = \sup_{\tau \in A_t} \|x(t + \tau; t, x)\|e^{c\tau}, \quad (4.17)$$

where $A_t = \{\tau \in [0, \infty) : t + \tau \in \mathbb{T}\}$. Conversion theorems with functions of type (4.17) for continuous systems are proved in [35, 36].

5 Concluding Remarks and Bibliography

The proofs of all assertions set out in Section 2 are found in [2, 3] (see also [5, 6]). The sufficient conditions of stability, uniform stability asymptotic stability and instability presented in the paper are obtained in terms of two general approaches set out in this paper. Namely, in Section 3, an approach is presented based on the application of integral inequalities on time scales. For stability analysis of the unperturbed motion of the quasilinear system (3.1), the known Gronwall inequality [2] and the nonlinear Stachurska inequality on time scales are applied, the latter being first established in this paper. This inequality is proved for the case of $m \in \mathbb{N} \setminus \{1\}$ in inequality (3.6).

In Section 4, stability analysis of system (4.1) is carried out in terms of the generalized direct Liapunov method. This generalization is associated with the application of a matrix-valued function for dynamic equations on time scales. Such investigations were undertaken in [29]. The application of matrix-valued functions for dynamic equations on time scales allows the construction of a heterogeneous Liapunov function [25], i.e., the functions consisting of continuous and discrete components, which is impossible to do in the framework of scalar Liapunov functions. Some concretization was made for the choice of Liapunov function in the investigation of linear dynamic equations on time scales.

In [1], the authors found new conditions on the coefficient matrix for certain perturbed linear dynamic equations (4.8) on time scales ensuring that there exists a bounded solution (which is explicitly given) to which all other solution converge, and similar conditions ensuring the existence of a bounded solution from which all other solutions diverge. In that paper, also periodic time scales and corresponding linear dynamic equations with periodic coefficients are considered and similar statements about periodic solutions to which all other solutions converge or from which all other solutions diverge are proved.

We note that in [8], the authors found conditions for the existence of a Liapunov function for the linear system (4.8) in the case of exponential stability of the state $x = 0$ on time scales. Thus, the versatility of the direct Liapunov method for dynamic equation on time scales was demonstrated.

We also remark that the construction of a general stability theory for dynamic equations on time scales is an *open problem* in the theory of this class of equations. The

extension of the proposed approaches to the analysis of oscillatory systems [27, 28] as well as hybrid systems [32] containing continuous and discrete components is of undoubt interest for applications.

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