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Positive Invariance and Differential Inclusions with Periodic Right-Hand Side

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Abstract: The paper is concerned with the non-autonomous ordinary differential inclusion in finite dimensional space with periodic, compact, but not necessarily convex valued right-hand side. The existence of periodic solution for such an inclusion which stays in a strongly positively invariant (under inclusion) set continuously depending on the time parameter is proved. The connection between the density principle and stability of the set of all periodic solutions on positively invariant sets with respect to internal and external perturbations of the inclusion is derived. The special attention is paid to the property of strong positive invariance which is studied here in terms of Lyapunov functions.

Keywords: *Differential inclusion; periodic solution; positively invariant set; Lyapunov function; stability of periodic solutions set; density principle.*

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1 Introduction

We deal with the ordinary differential inclusion

$$\dot{x} \in F(t, x), \tag{1}$$

where $F\colon \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a multivalued map with nonempty compact (but not necessarily convex) values and periodic on t for every fixed x with some period $T > 0$. We investigate the problem of existence of a periodic solution for (1) which stays in a strongly positively invariant set under inclusion (1), as well as some properties of the set of all such solutions. We call the set $\mathfrak{M} \subset \mathbb{R} \times \mathbb{R}^n$ strongly positively invariant under differential inclusion (1) (note, that the map F needs not to be periodic here) if for every point $z_0 = (t_0, x_0) \in \mathfrak{M}$ and any solution $t \to x(t, z_0)$ of the Cauchy problem for (1) with initial condition $x(t_0) = x_0$ we have $(t, x(t, z_0)) \in \mathfrak{M}$ for $t \geq t_0$. In other words, if any solution of (1) enters the set $\mathfrak{M}$ at some point (t_0, x_0) it stays in $\mathfrak{M}$ after the time moment t_0. If we can find at least one solution which possesses such a property, then the set $\mathfrak{M}$ is considered to be weakly positively invariant.

The concepts of weak and strong positive invariance (sometimes named differently) and conditions of existence of positively invariant sets, both for autonomous and non-autonomous differential inclusions, can be found by now in a wide range of works. One may refer, e.g., to those by Aubin [1], Clarke (and others) [2], Deimling [3], etc. The approach followed by these authors in studying the positive invariance property is based on contingent (Bouligand's) cones conditions which sometimes appear to be quite difficult to verify. We apply, for the same purpose, the so-called Lyapunov functions, i.e., continuous functions $V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $V(t, x) = 0$ for any point (t, x) which lays on the boundary of the set $\mathfrak{M}$ and $V(t, x) > 0$ for any point outside of $\mathfrak{M}$. The idea to use functions with similar properties was first introduced at the end of the 19th century by A.M. Lyapunov in order to investigate the problems of stability of the systems of ordinary differential equations. Since then the method of Lyapunov functions has been successfully developed in the works of N.N. Krasovskii, V.M. Matrosov, V. Lakshmikantham, A.A. Martynuyk, and many other authors (see, e.g. [4, 5, 6, 7]). We use this effective tool to get the necessary and sufficient conditions for the set $\mathfrak{M}$ to be strongly positively invariant under inclusion (1), and we consider a situation when the sections of the set $\mathfrak{M}$ can change continuously with time.

The problem of existence of periodic solution for inclusion (1) on invariant sets under different conditions on the map F has been studied very closely lately. We mention here some works we are aware of, e.g., recent papers [8, 9, 10]. The hypotheses we use in this article first of all do not include the convexity of the map F as well as convexity of the set $\mathfrak{M}$. To prove the existence of periodic solution in our case we apply the classical Brouwer fixed point theorem and some properties of the integral funnel, i.e., the existence of its selection continuously depending on the initial data.

In this paper we also continue to study the connection between the density principle (also known as relaxation theorem) and stability of the solutions set with respect to the different kinds of internal and external perturbations of the inclusion. We follow the earlier research in [11, 12, 13] and get the necessary and sufficient condition for the set of all periodic solutions to be stable on strongly invariant sets under internal and external perturbations.

2 Preliminaries

We start with recalling some notation and definitions (see, e.g., [3, 14, 15, 16, 17]).

Let $\mathbb{R}^n$ be Euclidian space with the scalar product $\langle x, y\rangle$, $x, y \in \mathbb{R}^n$, usual norm $|x| = \sqrt{\langle x, y\rangle}$, and metric $\rho(x, y) = |x - y|$. We denote by $\Omega(\mathbb{R}^n)$, $\mathrm{bd}(\mathbb{R}^n)$, $\mathrm{cl}(\mathbb{R}^n)$, $\mathrm{comp}(\mathbb{R}^n)$ the sets of all nonempty, nonempty and bounded, nonempty and closed, nonempty and compact subsets of $\mathbb{R}^n$, respectively. If $M \in \Omega(\mathbb{R}^n)$, then $\overline{M}$ stands for the closure of M, ∂M for the boundary of M, and $\mathrm{co}M$ for the convex hull of M.

By the relation

$$d(A, B) \doteq \sup_{a\in A} \rho(a, B),$$

where $\rho(a, B) \doteq \inf\limits_{b\in B} \rho(a, b)$, we denote the deviation of set A from set B. Then the function $\mathrm{dist}\colon \mathrm{bd}(\mathbb{R}^n) \times \mathrm{bd}(\mathbb{R}^n) \to \mathbb{R}$

$$\mathrm{dist}(A, B) \doteq \max\{d(A, B),\, d(B, A)\}$$

defines the *Hausdorff pseudo-metric* in $\mathrm{bd}(\mathbb{R}^n)$. On bounded and closed subsets of $\mathbb{R}^n$ the function $\mathrm{dist}(\cdot, \cdot)$ defines a metric (Hausdorff metric).

We denote

$$\mathcal{O}_\delta(x_0) \doteq \{x \in \mathbb{R}^n\colon\ \rho(x, x_0) \le \delta\}, pO_\delta(0) \doteq \mathcal{O}_\delta,$$

and for any set $M \subset \mathbb{R}^n$ let $M^\varepsilon \doteq \{y \in \mathbb{R}^n\colon\ \rho(y, M) \le \varepsilon\}$ stand for a closed ε-neighborhood of M.

Definition 2.1 A map $F\colon \mathbb{R}^m \to \mathrm{comp}(\mathbb{R}^n)$ is called upper semicontinuous in Hausdorff metric (u.s.c.) at the point x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x) \subset (F(x_0))^\varepsilon$ for all $x \in \mathcal{O}_\delta(x_0)$. A map which is u.s.c. at every point x of the set $Z \subseteq \mathbb{R}^n$ is called u.s.c. on Z.

Definition 2.2 A map $F\colon \mathbb{R}^m \to \mathrm{comp}(\mathbb{R}^n)$ is called lower semicontinuous in Hausdorff metric (l.s.c.) at the point x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x_0) \subset (F(x))^\varepsilon$ for all $x \in \mathcal{O}_\delta(x_0)$. A map which is l.s.c. at every point x of the set $Z \subseteq \mathbb{R}^n$ is called l.s.c. on Z.

Definition 2.3 A map $F\colon \mathbb{R}^m \to \mathrm{comp}(\mathbb{R}^n)$ which is both u.s.c. and l.s.c. (at the point x_0 or on the set Z) is called continuous.

Definition 2.4 A map $t \to M(t) \in \Omega(\mathbb{R}^n)$ is called continuous at the point t_0 if for every $r > 0$ the map $t \to \overline{M(t)} \cap \mathcal{O}_r$ is continuous in Hausdorff metric at point t_0. A map $t \to M(t)$ is continuous on the interval I if it is continuous at each point of the interval.

Definition 2.5 A single-valued map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is said to be a selection of a map $F\colon \mathbb{R}^m \to \Omega(\mathbb{R}^n)$ if

$$f(x) \in F(x)$$

for all $x \in \mathbb{R}^m$.

Definition 2.6 A map $F : \mathbb{R} \to \mathrm{comp}(\mathbb{R}^n)$ is called measurable, if there exists a countable set $\{q_i(t)\}_{i=1}^\infty$ of measurable selections approximating $F(t)$ for a.e. t (i.e., $F(t) = \overline{\bigcup\limits_{i=1}^{\infty} q_i(t)}$ for a.e. t).

Definition 2.7 We say that a map $F : \mathbb{R} \times \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^n)$ satisfies the Caratheodory conditions if

(i) F is measurable on t for every fixed x;

(ii) F is continuous on x for a.e. t;

(iii) for every $r > 0$ there exists a locally integrable function $k_r : \mathbb{R} \to \mathbb{R}_+$ such that $|F(t,x)| \leq k_r(t)$ for every point $(t,x) \in \mathbb{R} \times \mathcal{O}_r^m$, where $|F| = \max\limits_{q \in F} |q|$.

By $C([t_0,t_1],\mathbb{R}^n)$ we denote the space of all continuous functions $x\colon [t_0,t_1] \to \mathbb{R}^n$ with the usual norm $\|x\|_C = \max\limits_{t\in[t_0,t_1]} |x(t)|$, and by $AC([t_0,t_1],\mathbb{R}^n)$ the space of all absolutely continuous functions $x\colon [t_0,t_1] \to \mathbb{R}^n$ with the norm

$$\|x\|_{AC} = |x(t_0)| + \int_{t_0}^{t_1} |\dot{x}(t)|\, dt.$$

As a *solution of differential inclusion* (1) on the interval $I \subset \mathbb{R}$ we consider a function $x \in AC(I,\mathbb{R}^n)$ satisfying inclusion (1) for a.e. $t \in I$, so we deal with the Caratheodory type solutions.

Definition 2.8 For any set $Q \in \mathrm{comp}(\mathbb{R}^n)$ the function $c\colon \mathbb{R}^n \to \mathbb{R}$, defined as

$$c(h) = c(h,Q) \doteq \max_{y\in Q} \langle y, h\rangle,$$

is called a support function of the set Q.

We also recall that support function is positively homogeneous (i.e., $c(\lambda h, Q) = \lambda c(h,Q)$ if $\lambda \geq 0$), and for any $h \in \mathbb{R}^n$ the inclusion $Q_1 \subset Q_2$ implies the inequality $c(h,Q_1) \leq c(h,Q_2)$.

3 Invariant Sets

Let us have a continuous map $M\colon \mathbb{R} \to \mathrm{cl}(\mathbb{R}^n)$ and consider the set

$$\mathfrak{M} \doteq \{(t,x) \in \mathbb{R} \times \mathbb{R}^n\colon\ x \in M(t)\}, \tag{2}$$

which represents the graph of M. Let us also have a map $F : \mathbb{R} \times \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^n)$ satisfying the Caratheodory conditions.

Definition 3.1 The set $\mathfrak{M}$ is called strongly positively invariant (under inclusion (1)) if for every point $z_0 = (t_0,x_0) \in \mathfrak{M}$ any solution $t \to x(t,z_0)$ of the Cauchy problem

$$\dot{x} \in F(t,x), \quad x(t_0) = x_0, \tag{3}$$

satisfies the inclusion $(t, x(t,z_0)) \in \mathfrak{M}$ for every $t \geq t_0$.

Let $S(z_0)$ denote the set of all solutions (the integral funnel) for problem (3). We also set $S(t,z_0) \doteq \{x(t) \in \mathbb{R}^n\colon\ x(\cdot) \in S(z_0)\}$ to denote a section of $S(z_0)$ at the time moment t. It is quite obvious that $\mathfrak{M}$ is strongly positively invariant if and only if $S(t,z_0) \subset M(t)$ for all $z_0 = (t_0,x_0) \in \mathfrak{M}$ and $t \geq t_0$ (for which $S(t,z_0)$ exists). Moreover, if the set $M(t)$ is compact for every t and $\mathfrak{M}$ is strongly positively invariant, then for each point $z_0 \in \mathfrak{M}$ any solution for problem (3) is defined for every $t \geq t_0$.

Remark 3.1 Note that one can replace the Caratheodory conditions put on the map F with any other conditions which guarantee the existence of a local solution for problem (3).

We consider now a continuous function $V\colon \mathfrak{M}^r \to \mathbb{R}$, where $r > 0$, and

$$\mathfrak{M}^r \doteq \{(t,x) \in \mathbb{R} \times \mathbb{R}^n\colon\ x \in M^r(t)\}. \tag{4}$$

Definition 3.2 We say that the function V is a Lyapunov function (with respect to the set $\mathfrak{M}$) if $V(t,x) = 0$ for $(t,x) \in \partial\mathfrak{M}$ and $V(t,x) > 0$ for $(t,x) \in \mathfrak{M}^r \backslash \mathfrak{M}$.

We give here some examples (the most natural ones) of such functions V.

Example 3.1 Let $\rho(x, M(t)) \doteq \min\limits_{y\in M(t)} |x-y|$, then the function $V(t,x) \doteq \rho(x, M(t))$, which is continuous (if M is continuous), can serve as a Lyapunov function.

Example 3.2 The function $V(t,x) \doteq \min\limits_{y\in M(t)} |x-y|^2$ as well as the function $V(t,x) \doteq \Big(\min\limits_{y\in M(t)} |x-y|\Big)^2$ is continuous, and both of them can be used as Lyapunov functions. Moreover, they are continuously differentiable if the set $M(t)$ has a smooth boundary and is strictly convex.

Example 3.3 Let the set $M(t)$ be defined as

$$M(t) \doteq \{x \in \mathbb{R}^n\colon \quad a(t,x) \le 0\},$$

where $a(t,x)$ is continuous (or even continuously differentiable) scalar function. Then as the function $V(t,x)$ we can take the very function $a(t,x)$. A large number of sets which appear in different applications can be described as the intersection of the sets $M_i(t) \doteq \{x \in \mathbb{R}^n\colon\ a_i(t,x) \le 0\},\ i = 1, \dots, n.$

Now let us have differential inclusion (1), a continuous map $M\colon\ \mathbb{R} \to \mathrm{cl}(\mathbb{R}^n)$, and a Lyapunov function $V(t,x)$ defined on $\mathfrak{M}^r$ (see (4)). Let the function V be also locally Lipschitz, so for every compact set $P \subset \mathfrak{M}^r$ there exists a constant l_P such that for any $(t_1,x_1),\ (t_2,x_2) \in P$ the inequality

$$|V(t_1,x_1) - V(t_2,x_2)| \le l_P(|t_1-t_2| + |x_1-x_2|) \tag{5}$$

holds. Then we can consider the generalized Clarke derivative (see[2]) for the function V at the point (t,x) in the direction $(1,h) \in \mathbb{R}\times\mathbb{R}^n$ which is defined as follows:

$$V^o(t,x;h) \doteq \limsup_{\substack{(\vartheta,y)\to(t,x)\\ \delta\to 0+}} \frac{V(\vartheta+\delta, y+\delta h) - V(\vartheta,y)}{\delta}. \tag{6}$$

We will call

$$V^o_F(t,x) \doteq \max_{h\in F(t,x)} V^o(t,x;h) \tag{7}$$

the derivative of function V with respect to inclusion (1).

For every $\varepsilon \in (0,r]$ we construct now the closed set

$$\mathfrak{N}^\varepsilon \doteq \overline{\mathfrak{M}^\varepsilon \backslash \mathfrak{M}}.$$

Then the following sufficient condition for the set $\mathfrak{M}$ to be strongly positively invariant takes place.

Theorem 3.1 *Let us have a Lyapunov function* $(t,x) \to V(t,x),\ (t,x) \in \mathfrak{M}^r$, *which is locally Lipschitz. If for some* $\varepsilon \in (0,r]$ *the inequality* $V_F^o(t,x) \le 0$ *holds for any* $(t,x) \in \mathfrak{N}^\varepsilon$, *then the set* $\mathfrak{M}$ *(see (2)) is strongly positively invariant.*

Proof Let $\varepsilon \in (0,r]$ and $x(\cdot)$ be some solution for (3) such that $x(t) \in M^\varepsilon(t)$ on some finite time interval I. Then the function $v(t) = V(t,x(t))$ is absolutely continuous on I as a composition of a locally Lipschitz function with an absolutely continuous one.

Suppose now that solution $x(\cdot)$ reaches the boundary of $\mathfrak{M}$ at some moment $t_0 \in I$. This means that $x(t) \in M(t)$ for $t < t_0$ and $x(t_0) \doteq x_0 \in \partial M(t_0)$ (it may also happen that $x(t) \in \partial M(t)$ for some $t < t_0$). We will say that x_0 is the exit point of solution $x(\cdot)$ if there exists a sequence $\{t_k\}_{k=1}^\infty$ such that $t_k > t_0,\ t_k \to t_0$ and $x(t_k) \notin M(t_k)$. Let x_0 be the exit point of $x(\cdot)$. Then for the function $v(t) = V(t,x(t))$ we have relations $v(t_k) > 0$ and $v(t_0) = 0$. Fix large k and let τ_k be the closest (from the left) point to t_k such that $v(\tau_k) = 0$. Then $t_0 \le \tau_k,\ v(\tau_k) = 0,\ v(t) > 0$ for $t \in (\tau_k, t_k]$.

Since $x(\cdot)$ is absolutely continuous, we have for a.e. t

$$x(t+\delta) = x(t) + \delta\dot{x}(t) + r(\delta),$$

where $\lim\limits_{\delta\to 0+} \dfrac{r(\delta)}{\delta} = 0$. We denote $y(t,\delta) \doteq x(t) + r(\delta)$. Then

$$\begin{aligned} v(t+\delta) - v(t) &= V(t+\delta, x(t) + \delta\dot{x}(t) + r(\delta)) - V(t,x(t)) \\ &= V(t+\delta, y(t,\delta) + \delta\dot{x}(t)) - V(t, y(t,\delta)) + V(t,y(t,\delta)) - V(t,x(t)). \end{aligned}$$

Since V is locally Lipschitz, the inequality $|V(t,y(t,\delta)) - V(t,x(t))| \le l_P |r(\delta)|$ holds and hence

$$\lim_{\delta\to 0+} \frac{V(t,y(t,\delta)) - V(t,x(t))}{\delta} \le \lim_{\delta\to 0+} \frac{|V(t,y(t,\delta)) - V(t,x(t))|}{\delta} = 0.$$

So for a.e. $t \in (\tau_k, t_k]$ we have

$$\begin{aligned} \dot{v}(t) &= \lim_{\delta\to 0+} \frac{v(t+\delta) - v(t)}{\delta} \\ &\le \limsup_{\delta\to 0+} \frac{V(t+\delta, y(t,\delta) + \delta\dot{x}(t)) - V(t,y(t,\delta))}{\delta} \\ &\le \limsup_{\substack{\vartheta\to t \\ \delta\to 0+}} \frac{V(\vartheta+\delta, y(t,\delta) + \delta\dot{x}(t)) - V(\vartheta, y(t,\delta))}{\delta} \\ &\le V^o(t,x(t);\dot{x}(t)) \le V_F^o(t,x(t)) \le 0. \end{aligned}$$

From this estimation it follows that $v(t) = v(\tau_k) + \int\limits_{\tau_k}^{t} \dot{v}(s)ds \le 0$ for any $t \in (\tau_k, t_k]$ which contradicts the inequality $v(t) > 0$. $\square$

Example 3.4 Let us consider the differential equation

$$\ddot{y} + u(t,y,\dot{y})\dot{y} + p(y) = 0, \quad u(t,y,z) \in \{\alpha(t,y,z), \beta(t,y,z)\}, \tag{8}$$

of the oscillations of a mass point on a line under the driving force $p(y)$ and in the presence of friction $u(t,y,z)z$.

Using the standard procedure, one can replace this equation with the following differential inclusion

$$\begin{aligned} &\dot{x}_1 = x_2, \\ &\dot{x}_2 \in -p(x_1) - \{\alpha(t,x), \beta(t,x)\}x_2, \end{aligned} \tag{9}$$

where $x = (x_1, x_2)$ and functions $p(x_1)$, $\alpha(t,x)$, $\beta(t,x)$ take values in $\mathbb{R}$. In addition, we suppose that functions α, β are integrable and T-periodic on the first argument for every x, locally Lipschitz on the second argument, and satisfy the inequality $\alpha(t,x) \le \beta(t,x)$. The function p is locally Lipschitz and such that for some constant $\gamma > 0$ the relation $x_1 p(x_1) \ge 0$ takes place for all $|x_1| \ge \gamma$.

Next, we define the function $q(x) = \dfrac{(x_2)^2}{2} + \displaystyle\int_0^{x_1} p(z)dz$ and the set

$$\mathfrak{M}_\gamma \doteq \{(t,x) \in \mathbb{R} \times \mathbb{R}^2 \colon\; q(x) \le \gamma\}. \tag{10}$$

Then the function $V(x) = q(x) - \gamma$ can be taken as Lyapunov function with respect to the set $\mathfrak{M}_\gamma$. Since V does not depend on t, the derivative of V at the point x in the direction $h = (h_1, h_2)$ takes the form $V^o(x;h) = p(x_1)h_1 + x_2 h_2$, and the derivative $V^o(x)$ of the function V with respect to inclusion (9) can be written as:

$$V^o(x) = \max_{u \in \{\alpha(t,x), \beta(t,x)\}} \big(p(x_1)x_2 - x_2 p(x_1) - (x_2)^2 u\big) = -(x_2)^2 \alpha(t,x). \tag{11}$$

If $\alpha(t,x) \ge 0$ for all (t,x) outside of the set $\mathfrak{M}_\gamma$, then for such (t,x) the inequality $V^o(x) \le 0$ takes place. So, all the conditions of Theorem 3.1 are satisfied, and, hence, the set $\mathfrak{M}_\gamma$, defined by (10), is strongly positively invariant with respect to inclusion (9).

Consider now the case when the Lyapunov function V has more regularity, in particular, let it have continuous partial derivatives $V_t = \dfrac{\partial V}{\partial t}$ and $V_x = \dfrac{\partial V}{\partial x}$. Then the derivative of V with respect to inclusion can be written as

$$V_F^o(t,x) = \dot{V}_F(t,x) \doteq V_t(t,x) + c(V_x(t,x), F(t,x)). \tag{12}$$

In fact,

$$\begin{aligned} V^o(t,x;h) = &\limsup_{\substack{(\vartheta,y)\to(t,x)\\ \delta\to 0+}} \frac{V(\vartheta+\delta, y+\delta h) - V(\vartheta, y+\delta h)}{\delta} \\ &+ \limsup_{\substack{(\vartheta,y)\to(t,x)\\ \delta\to 0+}} \frac{V(\vartheta, y+\delta h) - V(\vartheta, y)}{\delta} = V_t(t,x) + \langle V_x(t,x), h\rangle. \end{aligned}$$

So when V is differentiable Theorem 3.1 can be reformulated in the following way:

Theorem 3.2 *Let $(t,x) \to V(t,x)$, $(t,x) \in \mathfrak{M}^r$, be continuously differentiable Lyapunov function. If for some $\varepsilon \in (0,r]$ and every $(t,x) \in \mathfrak{N}^\varepsilon$ the inequality $\dot{V}_F(t,x) \le 0$ holds, then the set $\mathfrak{M}$ (see (2)) is strongly positively invariant.*

Remark 3.2 It is obvious that the differentiability condition put on the function $V(t,x)$ is rather strict. It does not hold, for example, when the distance function is taken as the function V and $M(t)$ does not have enough regularity on the boundary. Meanwhile, the stated above result appears to be quite convenient in a number of applications.

Example 3.5 One may consider an ordinary differential inclusion on a smooth manifold of dimension n laying in $\mathbb{R}^{1+n}$, and this leads to the studying of strong positive invariance of the set

$$\mathfrak{M} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : a(t,x) = 0\},$$

where $a\colon\ \mathbb{R} \times \mathbb{R}^n \to \mathbb{R},\ a(t,x) \geq 0$ is a given continuously differentiable function. The function $a(t,x)$ can be taken as a Lyapunov function $V(t,x)$, and its derivative with respect to the corresponding inclusion can be calculated at any point $(t,x) \in \mathfrak{N}^\varepsilon$, where

$$\mathfrak{N}^\varepsilon = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n\colon\ 0 < a(t,x) \leq \varepsilon\}$$

for some $\varepsilon > 0$.

We may also formulate the necessary condition for strong positive invariance, but with some additional assumptions on the map F such as continuity (by both arguments).

Theorem 3.3 *If the set $\mathfrak{M}$ (see (2)) is strongly positively invariant and the map F is continuous, then for any continuously differentiable Lyapunov function $(t,x) \to V(t,x)$, $(t,x) \in \mathfrak{M}^r$, and any point (t_0,x_0) such that $x_0 \in \partial M(t_0)$ the inequality $\dot{V}_F(t_0,x_0) \leq 0$ holds.*

Proof Suppose there exist a Lyapunov function $V(t,x)$ and a point (t_0,x_0) such that $x_0 \in \partial M(t_0)$ and $\dot{V}_F(t_0,x_0) > 0$. Taking into account the definition of a support function and compactness of the set $F(t_0,x_0)$, we can find a vector $h \in F(t_0,x_0)$ such that $c(V_x(t_0,x_0), F(t_0,x_0)) = \langle V_x(t_0,x_0), h\rangle$. Then, from the continuity of F, it follows that there exists a solution $x(\cdot)$ for the problem (3) such that $\dot{x}(t_0) = h$ and $\dot{x}(\cdot)$ is continuous from the right at the point t_0 (see, e.g., [18]). So we get that the function $v(t) = V(t,x(t))$ is differentiable in the neighborhood of the point $t = t_0$ and

$$\begin{aligned}\dot{v}(t_0) &= V_t(t_0,x_0) + \langle V_x(t_0,x_0), \dot{x}(t_0)\rangle \\ &= V_t(t_0,x_0) + \langle V_x(t_0,x_0), h\rangle = \dot{V}_F(t_0,x_0) > 0.\end{aligned}$$

This means that there exists $t_1 > t_0$ such that $\dot{v}(t) > 0$ for $t \in (t_0,t_1)$. Since $v(t_0) = 0$ and $v(t) = \displaystyle\int_{t_0}^{t} \dot{v}(s)ds$, we have that for every $t \in (t_0,t_1)$ the inequality $\dot{v}(t_0) > 0$ implies $v(t) > 0$. Then, due to the definition of the function V, the solution $x(t)$ leaves the set $M(t)$ for $t > t_0$, and this contradicts our assumption that $\mathfrak{M}$ is strongly positively invariant. □

4 Periodic Solutions

We consider the ordinary differential inclusion

$$\dot{x} \in F(t,x), \quad F(t+T,x) = F(t,x), \tag{13}$$

under the following assumptions:

(P1) $F\colon\ \mathbb{R} \times \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^n)$ satisfies the Caratheodory conditions;

(P2) there exists a continuous, T-periodic map $M\colon\ \mathbb{R} \to \mathrm{comp}(\mathbb{R}^n)$ such that $M(0)$ is convex and the corresponding set $\mathfrak{M}$ (see (2)) is strongly positively invariant under inclusion (13);

(P3) there exists an integrable function $k\colon \mathbb{R} \to \mathbb{R}_+$ such that for a.e. $t \in \mathbb{R}$ and each $x, y \in M(t)$
$$\operatorname{dist}(F(t,x), F(t,y)) \le k(t)|x-y|.$$

We denote by P_T the set of all continuous and T-periodic functions $t \to z(t) \in M(t)$ and by $S(P_T) \subset P_T$ the set of all T-periodic solutions $t \to x(t)$ for problem (13) such that $x(t) \in M(t)$ for every t.

Theorem 4.1 *Let the maps F and M satisfy conditions* (P1)–(P3)*. Then the set $S(P_T)$ is not empty and relatively compact in the space of continuous functions.*

Proof For every $x_0 \in M(0)$ we denote by $\varphi(\cdot, x_0)$ a solution for (13) such that $\varphi(0, x_0) = x_0$ and $\varphi(\cdot, x_0)$ continuously depends on x_0 (according to [19] such a solution does exist). We consider $\varphi(\cdot, x_0)$ on the interval $[0, T]$. Since the set $\mathfrak{M}$ is strongly positively invariant, $\varphi(t, x_0) \in M(t)$ for every $t \in [0, T]$. So we can define a Poincarè map $p\colon M(0) \to M(0)$
$$p(x_0) \doteq \varphi(T, x_0)$$
which is continuous and acts in the convex set. Hence, according to the classical Brouwer's theorem, there exists a fixed point, say $\hat{x} \in M(0)$, for the map p, and this leads to the existence of a periodic solution $\varphi(\cdot, \hat{x})$ of problem (13).

The relative compactness of $S(P_T)$ in $C([0,T], \mathbb{R}^n)$ follows from the Arzela-Ascoli criterion. □

Now, in addition to problem (13), and under the same assumptions on the map F, we consider the convexified (relaxed) differential inclusion
$$\dot{x} \in \mathrm{co}F(t,x), \tag{14}$$
where the map co$F$ for every pair $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ is defined as $(\mathrm{co}F)(t,x) = \mathrm{co}(F(t,x))$, and the differential inclusion with *internal and external perturbations,* i.e., the inclusion of the type
$$\dot{x} \in F^{\varepsilon}(t, \mathcal{O}_\delta(x)) \tag{15}$$
the right-hand side of which for any $\varepsilon, \delta \ge 0$ and each $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ is a closed ε-neighborhood of the set
$$F\left(t, \mathcal{O}_\delta(x)\right) = \bigcup_{y \in \mathcal{O}_\delta(x)} F(t,y)$$
in the space $\mathbb{R}^n$. The constants $\delta, \varepsilon \ge 0$ define the radii of internal and external perturbations, correspondingly. Note that the radii of perturbations may also depend on time variable t (see [11, 12, 13]) or even on both time and phase variable x. So inclusion (15) represents the simplest model of an inclusion with internal and external perturbations.

Every solution (of Caratheodory type) for problem (15) for some fixed $\delta, \varepsilon \ge 0$ we call *an approximate solution* of problem (13). Let $\delta, \varepsilon \ge 0$. We denote by $S_{\mathrm{co}}(P_T)$ and $S_{\delta,\varepsilon}(P_T)$ the sets of all T-periodic solutions $t \to x(t)$ for inclusions (14) and (15), correspondingly, such that $x(t) \in M(t)$ for every t. It is obvious that, if the set $S(P_T)$ of periodic solutions for problem (13) on the set $\mathfrak{M}$ is not empty, then the sets $S_{\mathrm{co}}(P_T)$ and $S_{\delta,\varepsilon}(P_T)$ are not empty as well (but not vice versa). Moreover, the set $S_{\mathrm{co}}(P_T)$ will not only be relatively compact, but also closed (due to the convexity of the right-hand side of inclusion (14)) and hence compact in the space of continuous functions.

We are interested in connections between the sets $S(P_T)$, $S_{\rm co}(P_T)$, and $S_{\delta,\,\varepsilon}(P_T)$. First of all, it is easy to see that we have the relation

$$\lim_{\delta,\,\varepsilon\to 0+} \operatorname{dist}\left(F^{\varepsilon}(t,\mathcal{O}_{\delta}(x)),F(t,x)\right)=0$$

for a.e. $t\in[0,T]$ and every $x\in M(t)$. But does this relation guarantee the equality

$$\overline{S(P_T)}=\bigcap_{\delta,\,\varepsilon>0}\overline{S_{\delta,\,\varepsilon}(P_T)}, \tag{16}$$

where the closures of the solutions sets are taken in the space of continuous functions?

We say that inclusion (13) *is stable on the set* $\mathfrak{M}$ (*see* (2)) *under internal and external perturbations* if the equality (16) takes place.

As it was discussed in previous works [11, 12, 13], such a stability of the solutions set (and we can speak here also for a set of all solutions for a Cauchy problem, or a set of mild solutions for a semilinear differential inclusion) takes place only for inclusions with good enough right-hand side, for example, convex valued.

Theorem 4.2 *Let the maps* F *and* M *satisfy conditions* $(P1),(P2)$. *Then the equality*

$$S_{\rm co}(P_T)=\bigcap_{\delta,\,\varepsilon>0}\overline{S_{\delta,\,\varepsilon}(P_T)} \tag{17}$$

takes place.

Proof Since the set $M(t)$ is compact for every t, the set $\mathfrak{M}$ is strongly positively invariant under inclusion (13), and the radius δ of internal perturbations is strictly greater than zero, we can apply the corresponding result in [12] to get relation (17). □

Remark 4.1 Note that under conditions of Theorem 4.2 it may happen that the set $S_{\rm co}(P_T)$, and hence set $\overline{S_{\delta,\,\varepsilon}(P_T)}$ $(\delta,\varepsilon>0)$, is empty. One can avoid this situation requiring that the set $M(t)$ should be convex for every t (see [3]).

Next, we say that for inclusion (13) *on the set* $\mathfrak{M}$ (*see* (2)) *the density principle holds* if there holds the equality

$$\overline{S(P_T)}=S_{\rm co}(P_T). \tag{18}$$

The conditions for the density principle (or relaxation theorem) to be true for the sets of periodic solutions for differential inclusions can be found, e.g., in [20].

The following statement is straightforward.

Theorem 4.3 *Let the maps* F *and* M *satisfy conditions* $(P1)$, $(P2)$. *Then differential inclusion* (13) *is stable on the strongly positively invariant set* $\mathfrak{M}$ *under internal and external perturbations if and only if the density principle for* (13) *holds on* $\mathfrak{M}$.

Remark 4.2 Note that from Theorem 4.3 it follows that inclusion (13) is stable on the set $\mathfrak{M}$ under internal and external perturbations only if the map F has convex values or if the density principle holds on $\mathfrak{M}$.

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