Volume 7, Number 4, 2007 ISSN 1562-8353

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys

Volume 7 Number 4 2007

CONTENTS

Positive Invariance and Differential Inclusions with Periodic Right-Hand Side 339
I. Benedetti and E.A. Panasenko

An LMI Approach to H_∞ Filtering for Linear Parameter-Varying Systems with Delayed States and Outputs 351
H.R. Karimi, B. Lohmann and C. Buskens

Robust Stability for Nonlinear Uncertain Neural Networks with Delay 369
X.Y. Lou and B.T. Cui

Compact Explicit MPC Law with Guarantees of Feasibility for Reference Tracking 379
Sorin Olaru and Didier Dumur

Observer Design for a Class of Nonlinear Systems with Non-Full Relative Degree 399
K. Röbenack

Third-Body Perturbation Using Single Averaged Model: Application to Lunisolar Perturbations 409
C.R.H. Solórzano and A.F.B.A. Prado

On the Non-Oscillation of Solutions of Some Nonlinear Differential Equations of Third Order 419
Cemil Tunç

Generalized Monotone Iterative Technique for Functional Differential Equations with Retardation and Anticipation 431
A.S. Vatsala

Contents of Volume 7, 2007 439

NONLINEAR DYNAMICS & SYSTEMS THEORY

Volume 7, No. 4, 2007

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

 ISSN 1562-8353 Printed in Ukraine
To receive contents and abstracts by e-mail, visit our Website at: **http://www.e-ndst.kiev.ua**

Academic Periodical Press

Nonlinear Dynamics and Systems Theory
An International Journal of Research and Surveys

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys

INSTRUCTIONS FOR CONTRIBUTORS

(1) General. The Journal will publish original carefully refereed papers, brief notes and reviews on a wide range of nonlinear dynamics and systems theory problems. Contributions will be considered for publication in ND&ST if they have not been published previously. Before preparing your submission, it is essential that you consult our style guide; please visit our website: http://www.e-ndst.kiev.ua

(2) Manuscript and Correspondence. Contributions are welcome from all countries and should be written in English. Two copies of the manuscript, double spaced one column format, and the electronic version by AMSTEX, TEX or LATEX program (on diskette) should be sent directly to

Professor A.A. Martynyuk
Institute of Mechanics,
Nesterov str.3, 03057, MSP 680
Kiev-57, Ukraine
(e-mail: anmart@stability.kiev.ua
e-mail: amartynyuk@voliacable.com)

or to one of the Editors or to a member of the Editorial Board.

The title of the article must include: author(s) name, name of institution, department, address, FAX, and e-mail; an Abstract of 50-100 words should not include any formulas and citations; key words, and AMS subject classifications number(s). The size for regular paper should be 10-14 pages, survey (up to 24 pages), short papers, letter to the editor and book reviews (2-3 pages).

(3) Tables, Graphs and Illustrations. All figures must be suitable for reproduction without being retouched or redrawn and must include a title. Line drawings should include all relevant details and should be drawn in black ink on plain white drawing paper. In addition to a hard copy of the artwork, it is necessary to attach a PC diskette with files of the artwork (preferably in PCX format).

(4) References. Each entry must be cited in the text by author(s) and number or by number alone. All references should be listed in their alphabetic order. Use please the following style:

Journal: [1] Poincaré, H. Title of the article. *Title of the Journal* **Vol.1**(No.1) (year) pages. [Language].

Book: [2] Liapunov, A.M. *Title of the book.* Name of the Publishers, Town, year.

Proceeding: [3] Bellman, R. Title of the article. In: *Title of the book.* (Eds.). Name of the Publishers, Town, year, pages. [Language].

(5) Proofs and Reprints. Proofs sent to authors should be returned to the Editor with corrections within three days after receipt. Acceptance of the paper entitles the author to 10 free reprints.

(6) Editorial Policy. Every paper is reviewed by the regional editor, and/or a referee, and it may be returned for revision or rejected if considered unsuitable for publication.

(7) Copyright Assignment. When a paper is accepted for publication, author(s) will be requested to sign a form assigning copyright to Informath Publishing Group. Failure to do it promptly may delay the publication.

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys
Published since 2001

Volume 7 Number 4 2007

CONTENTS

Positive Invariance and Differential Inclusions with Periodic Right-Hand Side 339
I. Benedetti and E.A. Panasenko

An LMI Approach to H_∞ Filtering for Linear Parameter-Varying Systems with Delayed States and Outputs 351
H.R. Karimi, B. Lohmann and C. Buskens

Robust Stability for Nonlinear Uncertain Neural Networks with Delay 369
X.Y. Lou and B.T. Cui

Compact Explicit MPC Law with Guarantees of Feasibility for Reference Tracking 379
Sorin Olaru and Didier Dumur

Observer Design for a Class of Nonlinear Systems with Non-Full Relative Degree 399
K. Röbenack

Third-Body Perturbation Using Single Averaged Model: Application to Lunisolar Perturbations 409
C.R.H. Solórzano and A.F.B.A. Prado

On the Non-Oscillation of Solutions of Some Nonlinear Differential Equations of Third Order 419
Cemil Tunç

Generalized Monotone Iterative Technique for Functional Differential Equations with Retardation and Anticipation 431
A.S. Vatsala

Contents of Volume 7, 2007 439

Founded by A.A.Martynyuk in 2001.
Registered in Ukraine Number: KB №5267 / 04.07.2001.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Nonlinear Dynamics and Systems Theory (ISSN 1562-8353 (Print), ISSN 1813-7385 (Online)) is an international journal published under the auspices of the S.P.Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and the Laboratory for Industrial and Applied Mathematics (LIAM) at York University (Toronto, Canada). It is aimed at publishing high quality original scientific papers and surveys in area of nonlinear dynamics and systems theory and technical reports on solving practical problems. The scope of the journal is very broad covering:

SCOPE OF THE JOURNAL

Analysis of uncertain systems • Bifurcations and instability in dynamical behaviors • Celestial mechanics, variable mass processes, rockets • Control of chaotic systems • Controllability, observability, and structural properties • Deterministic and random vibrations • Differential games • Dynamical systems on manifolds • Dynamics of systems of particles • Hamilton and Lagrange equations • Hysteresis • Identification and adaptive control of stochastic systems • Modeling of real phenomena by ODE, FDE and PDE • Nonlinear boundary problems • Nonlinear control systems, guided systems • Nonlinear dynamics in biological systems • Nonlinear fluid dynamics • Nonlinear oscillations and waves • Nonlinear stability in continuum mechanics • Non-smooth dynamical systems with impacts or discontinuities • Numerical methods and simulation • Optimal control and applications • Qualitative analysis of systems with aftereffect • Robustness, sensitivity and disturbance rejection • Soft computing: artificial intelligence, neural networks, fuzzy logic, genetic algorithms, etc. • Stability of discrete systems • Stability of impulsive systems • Stability of large-scale power systems • Stability of linear and nonlinear control systems • Stochastic approximation and optimization • Symmetries and conservation laws

PUBLICATION AND SUBSCRIPTION INFORMATION

The **Nonlinear Dynamics and Systems Theory** is published four times per year in 2007. Base list subscription price per volume: US$ 149.00. This price is available only to individuals whose library subscribes to the journal OR who warrant that the Journal is for their own use and provide a home address for mailing. Separate rates apply to academic and corporate/government institutions. Our charge includes postage, packing, handling and airmail delivery of all issues. Mail order and inquires to: Department of Processes Stability, S.P.Timoshenko Institute of Mechanics NAS of Ukraine, Nesterov str., 3, 03057, Kiev-57, MSP 680, Ukraine, Tel: ++38-044-456-6140, Fax: ++38-044-456-0319, E-mail: anchern@stability.kiev.ua, http://www.sciencearea.com.ua; http://www.e-ndst.kiev.ua

ABSTRACTING AND INDEXING SERVICES

EBSCO Databases, Swets Information Services, Mathematical Reviews/MathSciNet, Zentralblatt MATH/Mathematics Abstracts.

Nonlinear Dynamics and Systems Theory, **7** (4) (2007) 339–349

Positive Invariance and Differential Inclusions with Periodic Right-Hand Side

I. Benedetti [1] and E.A. Panasenko [2*]

[1] *Dipartimento di Energetica "S.Stecco", Università degli Studi di Firenze, Via di S.Marta, 3-1-50139, Firenze, Italia*
[2] *Department of Mathematics, Physics, and Computer Science, Tambov State University, Internazional'naya 33, 392622, Tambov, Russia*

Received: September 08, 2006; Revised: August 19, 2007

Abstract: The paper is concerned with the non-autonomous ordinary differential inclusion in finite dimensional space with periodic, compact, but not necessarily convex valued right-hand side. The existence of periodic solution for such an inclusion which stays in a strongly positively invariant (under inclusion) set continuously depending on the time parameter is proved. The connection between the density principle and stability of the set of all periodic solutions on positively invariant sets with respect to internal and external perturbations of the inclusion is derived. The special attention is paid to the property of strong positive invariance which is studied here in terms of Lyapunov functions.

Keywords: *Differential inclusion; periodic solution; positively invariant set; Lyapunov function; stability of periodic solutions set; density principle.*

Mathematics Subject Classification (2000): 34A60, 34C25, 49K24, 37B25, 34D99.

* Corresponding author: panlena_t@mail.ru
[1] The author is partially supported by NATO Grant ICS.NR.CLG 981757.
[2] The author is partially supported by RFBR Grant 07-01-00305.

1 Introduction

We deal with the ordinary differential inclusion

$$\dot{x} \in F(t, x), \tag{1}$$

where $F\colon \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a multivalued map with nonempty compact (but not necessarily convex) values and periodic on t for every fixed x with some period $T > 0$. We investigate the problem of existence of a periodic solution for (1) which stays in a strongly positively invariant set under inclusion (1), as well as some properties of the set of all such solutions. We call the set $\mathfrak{M} \subset \mathbb{R} \times \mathbb{R}^n$ strongly positively invariant under differential inclusion (1) (note, that the map F needs not to be periodic here) if for every point $z_0 = (t_0, x_0) \in \mathfrak{M}$ and any solution $t \to x(t, z_0)$ of the Cauchy problem for (1) with initial condition $x(t_0) = x_0$ we have $(t, x(t, z_0)) \in \mathfrak{M}$ for $t \geq t_0$. In other words, if any solution of (1) enters the set $\mathfrak{M}$ at some point (t_0, x_0) it stays in $\mathfrak{M}$ after the time moment t_0. If we can find at least one solution which possesses such a property, then the set $\mathfrak{M}$ is considered to be weakly positively invariant.

The concepts of weak and strong positive invariance (sometimes named differently) and conditions of existence of positively invariant sets, both for autonomous and non-autonomous differential inclusions, can be found by now in a wide range of works. One may refer, e.g., to those by Aubin [1], Clarke (and others) [2], Deimling [3], etc. The approach followed by these authors in studying the positive invariance property is based on contingent (Bouligand's) cones conditions which sometimes appear to be quite difficult to verify. We apply, for the same purpose, the so-called Lyapunov functions, i.e., continuous functions $V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that $V(t, x) = 0$ for any point (t, x) which lays on the boundary of the set $\mathfrak{M}$ and $V(t, x) > 0$ for any point outside of $\mathfrak{M}$. The idea to use functions with similar properties was first introduced at the end of the 19th century by A.M. Lyapunov in order to investigate the problems of stability of the systems of ordinary differential equations. Since then the method of Lyapunov functions has been successfully developed in the works of N.N. Krasovskii, V.M. Matrosov, V. Lakshmikantham, A.A. Martynuyk, and many other authors (see, e.g. [4, 5, 6, 7]). We use this effective tool to get the necessary and sufficient conditions for the set $\mathfrak{M}$ to be strongly positively invariant under inclusion (1), and we consider a situation when the sections of the set $\mathfrak{M}$ can change continuously with time.

The problem of existence of periodic solution for inclusion (1) on invariant sets under different conditions on the map F has been studied very closely lately. We mention here some works we are aware of, e.g., recent papers [8, 9, 10]. The hypotheses we use in this article first of all do not include the convexity of the map F as well as convexity of the set $\mathfrak{M}$. To prove the existence of periodic solution in our case we apply the classical Brouwer fixed point theorem and some properties of the integral funnel, i.e., the existence of its selection continuously depending on the initial data.

In this paper we also continue to study the connection between the density principle (also known as relaxation theorem) and stability of the solutions set with respect to the different kinds of internal and external perturbations of the inclusion. We follow the earlier research in [11, 12, 13] and get the necessary and sufficient condition for the set of all periodic solutions to be stable on strongly invariant sets under internal and external perturbations.

2 Preliminaries

We start with recalling some notation and definitions (see, e.g., [3, 14, 15, 16, 17]).

Let $\mathbb{R}^n$ be Euclidian space with the scalar product $\langle x, y\rangle$, $x, y \in \mathbb{R}^n$, usual norm $|x| = \sqrt{\langle x, y\rangle}$, and metric $\rho(x, y) = |x - y|$. We denote by $\Omega(\mathbb{R}^n)$, $\mathrm{bd}(\mathbb{R}^n)$, $\mathrm{cl}(\mathbb{R}^n)$, $\mathrm{comp}(\mathbb{R}^n)$ the sets of all nonempty, nonempty and bounded, nonempty and closed, nonempty and compact subsets of $\mathbb{R}^n$, respectively. If $M \in \Omega(\mathbb{R}^n)$, then $\overline{M}$ stands for the closure of M, ∂M for the boundary of M, and $\mathrm{co}M$ for the convex hull of M.

By the relation

$$d(A, B) \doteq \sup_{a\in A} \rho(a, B),$$

where $\rho(a, B) \doteq \inf_{b\in B} \rho(a, b)$, we denote the deviation of set A from set B. Then the function $\mathrm{dist}\colon \mathrm{bd}(\mathbb{R}^n) \times \mathrm{bd}(\mathbb{R}^n) \to \mathbb{R}$

$$\mathrm{dist}(A, B) \doteq \max\{d(A, B),\, d(B, A)\}$$

defines the *Hausdorff pseudo-metric* in $\mathrm{bd}(\mathbb{R}^n)$. On bounded and closed subsets of $\mathbb{R}^n$ the function $\mathrm{dist}(\cdot, \cdot)$ defines a metric (Hausdorff metric).

We denote

$$\mathcal{O}_\delta(x_0) \doteq \{x \in \mathbb{R}^n\colon\ \rho(x, x_0) \le \delta\}, pO_\delta(0) \doteq \mathcal{O}_\delta,$$

and for any set $M \subset \mathbb{R}^n$ let $M^\varepsilon \doteq \{y \in \mathbb{R}^n\colon\ \rho(y, M) \le \varepsilon\}$ stand for a closed ε-neighborhood of M.

Definition 2.1 A map $F\colon \mathbb{R}^m \to \mathrm{comp}(\mathbb{R}^n)$ is called upper semicontinuous in Hausdorff metric (u.s.c.) at the point x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x) \subset (F(x_0))^\varepsilon$ for all $x \in \mathcal{O}_\delta(x_0)$. A map which is u.s.c. at every point x of the set $Z \subseteq \mathbb{R}^n$ is called u.s.c. on Z.

Definition 2.2 A map $F\colon \mathbb{R}^m \to \mathrm{comp}(\mathbb{R}^n)$ is called lower semicontinuous in Hausdorff metric (l.s.c.) at the point x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(x_0) \subset (F(x))^\varepsilon$ for all $x \in \mathcal{O}_\delta(x_0)$. A map which is l.s.c. at every point x of the set $Z \subseteq \mathbb{R}^n$ is called l.s.c. on Z.

Definition 2.3 A map $F\colon \mathbb{R}^m \to \mathrm{comp}(\mathbb{R}^n)$ which is both u.s.c. and l.s.c. (at the point x_0 or on the set Z) is called continuous.

Definition 2.4 A map $t \to M(t) \in \Omega(\mathbb{R}^n)$ is called continuous at the point t_0 if for every $r > 0$ the map $t \to \overline{M(t)} \cap \mathcal{O}_r$ is continuous in Hausdorff metric at point t_0. A map $t \to M(t)$ is continuous on the interval I if it is continuous at each point of the interval.

Definition 2.5 A single-valued map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is said to be a selection of a map $F\colon \mathbb{R}^m \to \Omega(\mathbb{R}^n)$ if

$$f(x) \in F(x)$$

for all $x \in \mathbb{R}^m$.

Definition 2.6 A map $F : \mathbb{R} \to \mathrm{comp}(\mathbb{R}^n)$ is called measurable, if there exists a countable set $\{q_i(t)\}_{i=1}^\infty$ of measurable selections approximating $F(t)$ for a.e. t (i.e., $F(t) = \overline{\bigcup_{i=1}^{\infty} q_i(t)}$ for a.e. t).

Definition 2.7 We say that a map $F : \mathbb{R} \times \mathbb{R}^n \to \text{comp}(\mathbb{R}^n)$ satisfies the Caratheodory conditions if

(i) F is measurable on t for every fixed x;

(ii) F is continuous on x for a.e. t;

(iii) for every $r > 0$ there exists a locally integrable function $k_r : \mathbb{R} \to \mathbb{R}_+$ such that $|F(t,x)| \leq k_r(t)$ for every point $(t,x) \in \mathbb{R} \times \mathcal{O}_r^m$, where $|F| = \max_{q \in F} |q|$.

By $C([t_0,t_1],\mathbb{R}^n)$ we denote the space of all continuous functions $x\colon [t_0,t_1] \to \mathbb{R}^n$ with the usual norm $\|x\|_C = \max_{t \in [t_0,t_1]} |x(t)|$, and by $AC([t_0,t_1],\mathbb{R}^n)$ the space of all absolutely continuous functions $x\colon [t_0,t_1] \to \mathbb{R}^n$ with the norm

$$\|x\|_{AC} = |x(t_0)| + \int_{t_0}^{t_1} |\dot{x}(t)|\, dt.$$

As a *solution of differential inclusion* (1) on the interval $I \subset \mathbb{R}$ we consider a function $x \in AC(I,\mathbb{R}^n)$ satisfying inclusion (1) for a.e. $t \in I$, so we deal with the Caratheodory type solutions.

Definition 2.8 For any set $Q \in \text{comp}(\mathbb{R}^n)$ the function $c\colon \mathbb{R}^n \to \mathbb{R}$, defined as

$$c(h) = c(h,Q) \doteq \max_{y \in Q} \langle y, h \rangle,$$

is called a support function of the set Q.

We also recall that support function is positively homogeneous (i.e., $c(\lambda h, Q) = \lambda c(h,Q)$ if $\lambda \geq 0$), and for any $h \in \mathbb{R}^n$ the inclusion $Q_1 \subset Q_2$ implies the inequality $c(h,Q_1) \leq c(h,Q_2)$.

3 Invariant Sets

Let us have a continuous map $M\colon \mathbb{R} \to \text{cl}(\mathbb{R}^n)$ and consider the set

$$\mathfrak{M} \doteq \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \colon\ x \in M(t)\}, \tag{2}$$

which represents the graph of M. Let us also have a map $F : \mathbb{R} \times \mathbb{R}^n \to \text{comp}(\mathbb{R}^n)$ satisfying the Caratheodory conditions.

Definition 3.1 The set $\mathfrak{M}$ is called strongly positively invariant (under inclusion (1)) if for every point $z_0 = (t_0, x_0) \in \mathfrak{M}$ any solution $t \to x(t,z_0)$ of the Cauchy problem

$$\dot{x} \in F(t,x), \quad x(t_0) = x_0, \tag{3}$$

satisfies the inclusion $(t, x(t,z_0)) \in \mathfrak{M}$ for every $t \geq t_0$.

Let $S(z_0)$ denote the set of all solutions (the integral funnel) for problem (3). We also set $S(t,z_0) \doteq \{x(t) \in \mathbb{R}^n \colon\ x(\cdot) \in S(z_0)\}$ to denote a section of $S(z_0)$ at the time moment t. It is quite obvious that $\mathfrak{M}$ is strongly positively invariant if and only if $S(t,z_0) \subset M(t)$ for all $z_0 = (t_0,x_0) \in \mathfrak{M}$ and $t \geq t_0$ (for which $S(t,z_0)$ exists). Moreover, if the set $M(t)$ is compact for every t and $\mathfrak{M}$ is strongly positively invariant, then for each point $z_0 \in \mathfrak{M}$ any solution for problem (3) is defined for every $t \geq t_0$.

Remark 3.1 Note that one can replace the Caratheodory conditions put on the map F with any other conditions which guarantee the existence of a local solution for problem (3).

We consider now a continuous function $V\colon \mathfrak{M}^r \to \mathbb{R}$, where $r > 0$, and

$$\mathfrak{M}^r \doteq \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \colon\ x \in M^r(t)\}. \tag{4}$$

Definition 3.2 We say that the function V is a Lyapunov function (with respect to the set $\mathfrak{M}$) if $V(t,x) = 0$ for $(t,x) \in \partial\mathfrak{M}$ and $V(t,x) > 0$ for $(t,x) \in \mathfrak{M}^r \backslash \mathfrak{M}$.

We give here some examples (the most natural ones) of such functions V.

Example 3.1 Let $\rho(x, M(t)) \doteq \min\limits_{y\in M(t)} |x-y|$, then the function $V(t,x) \doteq \rho(x, M(t))$, which is continuous (if M is continuous), can serve as a Lyapunov function.

Example 3.2 The function $V(t,x) \doteq \min\limits_{y\in M(t)} |x-y|^2$ as well as the function $V(t,x) \doteq \Big(\min\limits_{y\in M(t)} |x-y|\Big)^2$ is continuous, and both of them can be used as Lyapunov functions. Moreover, they are continuously differentiable if the set $M(t)$ has a smooth boundary and is strictly convex.

Example 3.3 Let the set $M(t)$ be defined as

$$M(t) \doteq \{x \in \mathbb{R}^n \colon \quad a(t,x) \le 0\},$$

where $a(t,x)$ is continuous (or even continuously differentiable) scalar function. Then as the function $V(t,x)$ we can take the very function $a(t,x)$. A large number of sets which appear in different applications can be described as the intersection of the sets $M_i(t) \doteq \{x \in \mathbb{R}^n \colon\ a_i(t,x) \le 0\},\ i = 1,\dots,n.$

Now let us have differential inclusion (1), a continuous map $M\colon\ \mathbb{R} \to \mathrm{cl}(\mathbb{R}^n)$, and a Lyapunov function $V(t,x)$ defined on $\mathfrak{M}^r$ (see (4)). Let the function V be also locally Lipschitz, so for every compact set $P \subset \mathfrak{M}^r$ there exists a constant l_P such that for any $(t_1,x_1),\ (t_2,x_2) \in P$ the inequality

$$|V(t_1,x_1) - V(t_2,x_2)| \le l_P(|t_1-t_2| + |x_1-x_2|) \tag{5}$$

holds. Then we can consider the generalized Clarke derivative (see[2]) for the function V at the point (t,x) in the direction $(1,h) \in \mathbb{R} \times \mathbb{R}^n$ which is defined as follows:

$$V^o(t,x;h) \doteq \limsup_{\substack{(\vartheta,y)\to(t,x)\\ \delta\to 0+}} \frac{V(\vartheta+\delta, y+\delta h) - V(\vartheta,y)}{\delta}. \tag{6}$$

We will call

$$V^o_F(t,x) \doteq \max_{h\in F(t,x)} V^o(t,x;h) \tag{7}$$

the derivative of function V with respect to inclusion (1).

For every $\varepsilon \in (0,r]$ we construct now the closed set

$$\mathfrak{N}^\varepsilon \doteq \overline{\mathfrak{M}^\varepsilon \backslash \mathfrak{M}}.$$

Then the following sufficient condition for the set $\mathfrak{M}$ to be strongly positively invariant takes place.

Theorem 3.1 *Let us have a Lyapunov function* $(t,x) \to V(t,x)$, $(t,x) \in \mathfrak{M}^r$, *which is locally Lipschitz. If for some* $\varepsilon \in (0,r]$ *the inequality* $V_F^o(t,x) \le 0$ *holds for any* $(t,x) \in \mathfrak{N}^\varepsilon$, *then the set* $\mathfrak{M}$ *(see* (2)*) is strongly positively invariant.*

Proof Let $\varepsilon \in (0,r]$ and $x(\cdot)$ be some solution for (3) such that $x(t) \in M^\varepsilon(t)$ on some finite time interval I. Then the function $v(t) = V(t,x(t))$ is absolutely continuous on I as a composition of a locally Lipschitz function with an absolutely continuous one.

Suppose now that solution $x(\cdot)$ reaches the boundary of $\mathfrak{M}$ at some moment $t_0 \in I$. This means that $x(t) \in M(t)$ for $t < t_0$ and $x(t_0) \doteq x_0 \in \partial M(t_0)$ (it may also happen that $x(t) \in \partial M(t)$ for some $t < t_0$). We will say that x_0 is the exit point of solution $x(\cdot)$ if there exists a sequence $\{t_k\}_{k=1}^\infty$ such that $t_k > t_0,\ t_k \to t_0$ and $x(t_k) \notin M(t_k)$. Let x_0 be the exit point of $x(\cdot)$. Then for the function $v(t) = V(t,x(t))$ we have relations $v(t_k) > 0$ and $v(t_0) = 0$. Fix large k and let τ_k be the closest (from the left) point to t_k such that $v(\tau_k) = 0$. Then $t_0 \le \tau_k,\ v(\tau_k) = 0,\ v(t) > 0$ for $t \in (\tau_k, t_k]$.

Since $x(\cdot)$ is absolutely continuous, we have for a.e. t

$$x(t+\delta) = x(t) + \delta \dot{x}(t) + r(\delta),$$

where $\lim\limits_{\delta \to 0+} \dfrac{r(\delta)}{\delta} = 0$. We denote $y(t,\delta) \doteq x(t) + r(\delta)$. Then

$$\begin{aligned} v(t+\delta) - v(t) &= V(t+\delta, x(t) + \delta\dot{x}(t) + r(\delta)) - V(t,x(t)) \\ &= V(t+\delta, y(t,\delta) + \delta\dot{x}(t)) - V(t, y(t,\delta)) + V(t,y(t,\delta)) - V(t,x(t)). \end{aligned}$$

Since V is locally Lipschitz, the inequality $|V(t,y(t,\delta)) - V(t,x(t))| \le l_P |r(\delta)|$ holds and hence

$$\lim_{\delta \to 0+} \frac{V(t,y(t,\delta)) - V(t,x(t))}{\delta} \le \lim_{\delta \to 0+} \frac{|V(t,y(t,\delta)) - V(t,x(t))|}{\delta} = 0.$$

So for a.e. $t \in (\tau_k, t_k]$ we have

$$\begin{aligned} \dot{v}(t) &= \lim_{\delta \to 0+} \frac{v(t+\delta) - v(t)}{\delta} \\ &\le \limsup_{\delta \to 0+} \frac{V(t+\delta, y(t,\delta) + \delta\dot{x}(t)) - V(t, y(t,\delta))}{\delta} \\ &\le \limsup_{\substack{\vartheta \to t \\ \delta \to 0+}} \frac{V(\vartheta+\delta, y(t,\delta) + \delta\dot{x}(t)) - V(\vartheta, y(t,\delta))}{\delta} \\ &\le V^o(t,x(t);\dot{x}(t)) \le V_F^o(t,x(t)) \le 0. \end{aligned}$$

From this estimation it follows that $v(t) = v(\tau_k) + \int\limits_{\tau_k}^{t} \dot{v}(s)ds \le 0$ for any $t \in (\tau_k, t_k]$ which contradicts the inequality $v(t) > 0$. $\square$

Example 3.4 Let us consider the differential equation

$$\ddot{y} + u(t,y,\dot{y})\dot{y} + p(y) = 0, \quad u(t,y,z) \in \{\alpha(t,y,z), \beta(t,y,z)\}, \tag{8}$$

of the oscillations of a mass point on a line under the driving force $p(y)$ and in the presence of friction $u(t,y,z)z$.

Using the standard procedure, one can replace this equation with the following differential inclusion

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &\in -p(x_1) - \{\alpha(t,x), \beta(t,x)\}x_2, \end{aligned} \tag{9}$$

where $x = (x_1, x_2)$ and functions $p(x_1)$, $\alpha(t,x)$, $\beta(t,x)$ take values in $\mathbb{R}$. In addition, we suppose that functions α, β are integrable and T-periodic on the first argument for every x, locally Lipschitz on the second argument, and satisfy the inequality $\alpha(t,x) \le \beta(t,x)$. The function p is locally Lipschitz and such that for some constant $\gamma > 0$ the relation $x_1 p(x_1) \ge 0$ takes place for all $|x_1| \ge \gamma$.

Next, we define the function $q(x) = \dfrac{(x_2)^2}{2} + \displaystyle\int_0^{x_1} p(z)dz$ and the set

$$\mathfrak{M}_\gamma \doteq \{(t,x) \in \mathbb{R} \times \mathbb{R}^2 : \; q(x) \le \gamma\}. \tag{10}$$

Then the function $V(x) = q(x) - \gamma$ can be taken as Lyapunov function with respect to the set $\mathfrak{M}_\gamma$. Since V does not depend on t, the derivative of V at the point x in the direction $h = (h_1, h_2)$ takes the form $V^o(x;h) = p(x_1)h_1 + x_2 h_2$, and the derivative $V^o(x)$ of the function V with respect to inclusion (9) can be written as:

$$V^o(x) = \max_{u \in \{\alpha(t,x), \beta(t,x)\}} \big(p(x_1)x_2 - x_2 p(x_1) - (x_2)^2 u\big) = -(x_2)^2 \alpha(t,x). \tag{11}$$

If $\alpha(t,x) \ge 0$ for all (t,x) outside of the set $\mathfrak{M}_\gamma$, then for such (t,x) the inequality $V^o(x) \le 0$ takes place. So, all the conditions of Theorem 3.1 are satisfied, and, hence, the set $\mathfrak{M}_\gamma$, defined by (10), is strongly positively invariant with respect to inclusion (9).

Consider now the case when the Lyapunov function V has more regularity, in particular, let it have continuous partial derivatives $V_t = \dfrac{\partial V}{\partial t}$ and $V_x = \dfrac{\partial V}{\partial x}$. Then the derivative of V with respect to inclusion can be written as

$$V_F^o(t,x) = \dot{V}_F(t,x) \doteq V_t(t,x) + c(V_x(t,x), F(t,x)). \tag{12}$$

In fact,

$$\begin{aligned} V^o(t,x;h) = &\limsup_{\substack{(\vartheta,y)\to(t,x) \\ \delta\to 0+}} \frac{V(\vartheta+\delta, y+\delta h) - V(\vartheta, y+\delta h)}{\delta} \\ &+ \limsup_{\substack{(\vartheta,y)\to(t,x) \\ \delta\to 0+}} \frac{V(\vartheta, y+\delta h) - V(\vartheta, y)}{\delta} = V_t(t,x) + \langle V_x(t,x), h\rangle. \end{aligned}$$

So when V is differentiable Theorem 3.1 can be reformulated in the following way:

Theorem 3.2 *Let $(t,x) \to V(t,x)$, $(t,x) \in \mathfrak{M}^r$, be continuously differentiable Lyapunov function. If for some $\varepsilon \in (0,r]$ and every $(t,x) \in \mathfrak{N}^\varepsilon$ the inequality $\dot{V}_F(t,x) \le 0$ holds, then the set $\mathfrak{M}$ (see (2)) is strongly positively invariant.*

Remark 3.2 It is obvious that the differentiability condition put on the function $V(t,x)$ is rather strict. It does not hold, for example, when the distance function is taken as the function V and $M(t)$ does not have enough regularity on the boundary. Meanwhile, the stated above result appears to be quite convenient in a number of applications.

Example 3.5 One may consider an ordinary differential inclusion on a smooth manifold of dimension n laying in $\mathbb{R}^{1+n}$, and this leads to the studying of strong positive invariance of the set

$$\mathfrak{M} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : a(t,x) = 0\},$$

where $a\colon\ \mathbb{R} \times \mathbb{R}^n \to \mathbb{R},\ a(t,x) \geq 0$ is a given continuously differentiable function. The function $a(t,x)$ can be taken as a Lyapunov function $V(t,x)$, and its derivative with respect to the corresponding inclusion can be calculated at any point $(t,x) \in \mathfrak{N}^\varepsilon$, where

$$\mathfrak{N}^\varepsilon = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n\colon\ 0 < a(t,x) \leq \varepsilon\}$$

for some $\varepsilon > 0$.

We may also formulate the necessary condition for strong positive invariance, but with some additional assumptions on the map F such as continuity (by both arguments).

Theorem 3.3 *If the set $\mathfrak{M}$ (see (2)) is strongly positively invariant and the map F is continuous, then for any continuously differentiable Lyapunov function $(t,x) \to V(t,x)$, $(t,x) \in \mathfrak{M}^r$, and any point (t_0,x_0) such that $x_0 \in \partial M(t_0)$ the inequality $\dot{V}_F(t_0,x_0) \leq 0$ holds.*

Proof Suppose there exist a Lyapunov function $V(t,x)$ and a point (t_0,x_0) such that $x_0 \in \partial M(t_0)$ and $\dot{V}_F(t_0,x_0) > 0$. Taking into account the definition of a support function and compactness of the set $F(t_0,x_0)$, we can find a vector $h \in F(t_0,x_0)$ such that $c(V_x(t_0,x_0), F(t_0,x_0)) = \langle V_x(t_0,x_0), h\rangle$. Then, from the continuity of F, it follows that there exists a solution $x(\cdot)$ for the problem (3) such that $\dot{x}(t_0) = h$ and $\dot{x}(\cdot)$ is continuous from the right at the point t_0 (see, e.g., [18]). So we get that the function $v(t) = V(t,x(t))$ is differentiable in the neighborhood of the point $t = t_0$ and

$$\begin{aligned} \dot{v}(t_0) &= V_t(t_0,x_0) + \langle V_x(t_0,x_0), \dot{x}(t_0)\rangle \\ &= V_t(t_0,x_0) + \langle V_x(t_0,x_0), h\rangle = \dot{V}_F(t_0,x_0) > 0. \end{aligned}$$

This means that there exists $t_1 > t_0$ such that $\dot{v}(t) > 0$ for $t \in (t_0,t_1)$. Since $v(t_0) = 0$ and $v(t) = \displaystyle\int_{t_0}^{t} \dot{v}(s)ds$, we have that for every $t \in (t_0,t_1)$ the inequality $\dot{v}(t_0) > 0$ implies $v(t) > 0$. Then, due to the definition of the function V, the solution $x(t)$ leaves the set $M(t)$ for $t > t_0$, and this contradicts our assumption that $\mathfrak{M}$ is strongly positively invariant. □

4 Periodic Solutions

We consider the ordinary differential inclusion

$$\dot{x} \in F(t,x), \quad F(t+T,x) = F(t,x), \tag{13}$$

under the following assumptions:

(P1) $F\colon\ \mathbb{R} \times \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^n)$ satisfies the Caratheodory conditions;

(P2) there exists a continuous, T-periodic map $M\colon\ \mathbb{R} \to \mathrm{comp}(\mathbb{R}^n)$ such that $M(0)$ is convex and the corresponding set $\mathfrak{M}$ (see (2)) is strongly positively invariant under inclusion (13);

(P3) there exists an integrable function $k\colon\ \mathbb{R} \to \mathbb{R}_+$ such that for a.e. $t \in \mathbb{R}$ and each $x, y \in M(t)$
$$\operatorname{dist}(F(t,x), F(t,y)) \le k(t)|x-y|.$$

We denote by P_T the set of all continuous and T-periodic functions $t \to z(t) \in M(t)$ and by $S(P_T) \subset P_T$ the set of all T-periodic solutions $t \to x(t)$ for problem (13) such that $x(t) \in M(t)$ for every t.

Theorem 4.1 *Let the maps F and M satisfy conditions* (P1)–(P3). *Then the set $S(P_T)$ is not empty and relatively compact in the space of continuous functions.*

Proof For every $x_0 \in M(0)$ we denote by $\varphi(\cdot, x_0)$ a solution for (13) such that $\varphi(0, x_0) = x_0$ and $\varphi(\cdot, x_0)$ continuously depends on x_0 (according to [19] such a solution does exist). We consider $\varphi(\cdot, x_0)$ on the interval $[0, T]$. Since the set $\mathfrak{M}$ is strongly positively invariant, $\varphi(t, x_0) \in M(t)$ for every $t \in [0, T]$. So we can define a Poincarè map $p\colon\ M(0) \to M(0)$
$$p(x_0) \doteq \varphi(T, x_0)$$
which is continuous and acts in the convex set. Hence, according to the classical Brouwer's theorem, there exists a fixed point, say $\hat{x} \in M(0)$, for the map p, and this leads to the existence of a periodic solution $\varphi(\cdot, \hat{x})$ of problem (13).

The relative compactness of $S(P_T)$ in $C([0, T], \mathbb{R}^n)$ follows from the Arzela-Ascoli criterion. □

Now, in addition to problem (13), and under the same assumptions on the map F, we consider the convexified (relaxed) differential inclusion
$$\dot{x} \in \mathrm{co}F(t, x), \tag{14}$$
where the map co$F$ for every pair $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ is defined as $(\mathrm{co}F)(t, x) = \mathrm{co}(F(t, x))$, and the differential inclusion with *internal and external perturbations,* i.e., the inclusion of the type
$$\dot{x} \in F^{\varepsilon}(t, \mathcal{O}_\delta(x)) \tag{15}$$
the right-hand side of which for any $\varepsilon, \delta \ge 0$ and each $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ is a closed ε-neighborhood of the set
$$F\left(t, \mathcal{O}_\delta(x)\right) = \bigcup_{y \in \mathcal{O}_\delta(x)} F(t, y)$$
in the space $\mathbb{R}^n$. The constants $\delta, \varepsilon \ge 0$ define the radii of internal and external perturbations, correspondingly. Note that the radii of perturbations may also depend on time variable t (see [11, 12, 13]) or even on both time and phase variable x. So inclusion (15) represents the simplest model of an inclusion with internal and external perturbations.

Every solution (of Caratheodory type) for problem (15) for some fixed $\delta, \varepsilon \ge 0$ we call *an approximate solution* of problem (13). Let $\delta, \varepsilon \ge 0$. We denote by $S_{\mathrm{co}}(P_T)$ and $S_{\delta,\varepsilon}(P_T)$ the sets of all T-periodic solutions $t \to x(t)$ for inclusions (14) and (15), correspondingly, such that $x(t) \in M(t)$ for every t. It is obvious that, if the set $S(P_T)$ of periodic solutions for problem (13) on the set $\mathfrak{M}$ is not empty, then the sets $S_{\mathrm{co}}(P_T)$ and $S_{\delta,\varepsilon}(P_T)$ are not empty as well (but not vice versa). Moreover, the set $S_{\mathrm{co}}(P_T)$ will not only be relatively compact, but also closed (due to the convexity of the right-hand side of inclusion (14)) and hence compact in the space of continuous functions.

We are interested in connections between the sets $S(P_T)$, $S_{\mathrm{co}}(P_T)$, and $S_{\delta,\,\varepsilon}(P_T)$. First of all, it is easy to see that we have the relation

$$\lim_{\delta,\,\varepsilon\to 0+} \operatorname{dist}\left(F^{\varepsilon}(t,\mathcal{O}_\delta(x)), F(t,x)\right) = 0$$

for a.e. $t \in [0,T]$ and every $x \in M(t)$. But does this relation guarantee the equality

$$\overline{S(P_T)} = \bigcap_{\delta,\,\varepsilon > 0} \overline{S_{\delta,\,\varepsilon}(P_T)}, \tag{16}$$

where the closures of the solutions sets are taken in the space of continuous functions?

We say that inclusion (13) *is stable on the set* $\mathfrak{M}$ (*see* (2)) *under internal and external perturbations* if the equality (16) takes place.

As it was discussed in previous works [11, 12, 13], such a stability of the solutions set (and we can speak here also for a set of all solutions for a Cauchy problem, or a set of mild solutions for a semilinear differential inclusion) takes place only for inclusions with good enough right-hand side, for example, convex valued.

Theorem 4.2 *Let the maps F and M satisfy conditions* $(P1), (P2)$. *Then the equality*

$$S_{\mathrm{co}}(P_T) = \bigcap_{\delta,\,\varepsilon > 0} \overline{S_{\delta,\,\varepsilon}(P_T)} \tag{17}$$

takes place.

Proof Since the set $M(t)$ is compact for every t, the set $\mathfrak{M}$ is strongly positively invariant under inclusion (13), and the radius δ of internal perturbations is strictly greater than zero, we can apply the corresponding result in [12] to get relation (17). □

Remark 4.1 Note that under conditions of Theorem 4.2 it may happen that the set $S_{\mathrm{co}}(P_T)$, and hence set $\overline{S_{\delta,\,\varepsilon}(P_T)}$ $(\delta, \varepsilon > 0)$, is empty. One can avoid this situation requiring that the set $M(t)$ should be convex for every t (see [3]).

Next, we say that for inclusion (13) *on the set* $\mathfrak{M}$ (*see* (2)) *the density principle holds* if there holds the equality

$$\overline{S(P_T)} = S_{\mathrm{co}}(P_T). \tag{18}$$

The conditions for the density principle (or relaxation theorem) to be true for the sets of periodic solutions for differential inclusions can be found, e.g., in [20].

The following statement is straightforward.

Theorem 4.3 *Let the maps F and M satisfy conditions* $(P1)$, $(P2)$. *Then differential inclusion* (13) *is stable on the strongly positively invariant set* $\mathfrak{M}$ *under internal and external perturbations if and only if the density principle for* (13) *holds on* $\mathfrak{M}$.

Remark 4.2 Note that from Theorem 4.3 it follows that inclusion (13) is stable on the set $\mathfrak{M}$ under internal and external perturbations only if the map F has convex values or if the density principle holds on $\mathfrak{M}$.

Acknowledgments

The second author is grateful to University of Florence, the CIME Foundation, and personally to professor P. Zecca for an opportunity to work on this paper in Italy.

References

[1] Aubin, J.-P. *Viability Theory.* Boston, Birkhauser, 1991.

[2] Clarke, F. H., Ledyaev, Yu. S., Stern, R. J., and Wolenski, P. R. *Nonsmooth Analysis and Control Theory.* New York, Springer-Verlag, 1998.

[3] Deimling, K. *Multivalued Differential Equations.* De Gruyter Ser. Nonlinear Anal. Appl. 1. Berlin–New York, Walter de Gruyter, 1992.

[4] Matrosov, V. M. The comparison principle with a vector-valued Lyapunov function. I, II, III, IV. *Differentsial'nye Uravneniya* **4** (1968) 1374–1386; **4** (1968) 1739–1752; **5** (1969) 1171–1185; **5** (1969) 2129–2143. [Russian]

[5] Martynuyk, A. A. *Stability of Motion of Complex Systems.* Kiev, Naukova Dumka, 1975. [Russian]

[6] D'Anna, A. and Fiore, G. Global stability properties for a class of dissipative phenomena via one or several Liapunov functionals. *Nonlinear Dynamics and Systems Theory* **5**(1) (2005) 9–38.

[7] Aleksandrov, A. Yu. and Platonov, A. V. Construction of Lyapunov's functions for a class of nonlinear systems. *Nonlinear Dynamics and Systems Theory* **6**(1) (2006) 17–29.

[8] Bader, R. and Kryszewski, W. On the solution sets of constrained differential inclusions with applications. *Set-Valued Anal.* **9**(3) (2001) 289–313.

[9] Kornev, S. V. and Obukhovskii, V. V. On nonsmooth multivalent guiding functions. *Differentsial'nye Uravneniya* **39**(11) (2003) 1497–1502; translated in *Differential Equations* **39**(11) (2003) 1578–1584.

[10] Gabor, G. Periodic solutions for nonautonomous differential equations and inclusions in tubes. *Abstr. Appl. Anal.* **12** (2004) 1057–1079.

[11] Bulgakov, A. I., Efremov, A. A., and Panasenko, E. A. Ordinary differential inclusions with internal and external perturbations. *Differentsial'nye Uravneniya* **36**(12) (2000) 1587–1598; translated in *Differential Equations* **36**(12) (2000) 1741–1753.

[12] Panasenko, E. A. Density principle and stability of sets of periodic solutions for differential inclusion. *Vestnik Udmurt Univer.* **1** (2005) 139–154. [Russian]

[13] Benedetti, I. and Panasenko, E. Representation of the set of mild solutions for the relaxed semilinear differential inclusion. *Discuss. Math. Differ. Incl. Control Optim.* (to appear).

[14] Borisovich, Yu. G., Gelman, B. D., Myshkis, A. D. and Obukhovskii, V. V. *Introduction to the Theory of Multivalued Maps and Differential Inclusions.* Moscow, Editorial URSS, 2005. [Russian]

[15] Kamenskii, M., Obukhovskii, V. and Zecca, P. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces.* Berlin–New York, Walter de Gruyter, 2001.

[16] Aubin, J.-P. and Cellina, A. *Differential Inclusions. Set-Valued Maps and Viability Theory.* Berlin–Heidelberg–New York–Tokyo, Springer-Verlag, 1984.

[17] Tolstonogov, A. A. *Differential Inclusions in a Banach Space.* Dordrecht, Kluwer Acad. Publishers, 2000.

[18] Blagodatskikh, V. I. and Filippov, A. F. Differential inclusions and optimal control. *Proc. Steklov Inst. Math.* **169** (1986) 199–259.

[19] Cellina, A. and Ornelas, A. Representation of the attainable set for Lipschitzian differential inclusions. *Rocky Mountain Journal of Mathematics.* **22**(1) (1992) 117–124.

[20] Irisov, A. E. and Tonkov, E. L. On the closure of the set of periodic solutions of a differential inclusion. *Gorkov. Gos. Univ., Gorkii* **134** (1983) 32–38. [Russian]

Nonlinear Dynamics and Systems Theory, **7** (4) (2007) 351–368

An LMI Approach to H_∞ Filtering for Linear Parameter-Varying Systems with Delayed States and Outputs

H.R. Karimi [1*], B. Lohmann [2] and C. Buskens [3]

[1] *Control and Intelligent Processing, Center of Excellence, Department of Electrical and Computer Engineering, Faculty of Engineering, University of Tehran, Tehran, Iran*
[2] *Institute of Automatic Control, Technical University of Munich, D-85748 Garching-Munich, Germany*
[3] *Centre of Industrial Mathematics, Faculty of Mathematics & Computer Science, University of Bremen, D-28359 Bremen, Germany*

Received: August 13, 2006; Revised: August 12, 2007

Abstract: This paper considers the problem of delay-dependent robust H_∞ filtering for linear parameter-varying (LPV) systems with time-invariant delay in the states and outputs. It is assumed that the state-space matrices affinely depend on parameters that are measurable in real-time. By taking the relationship between the terms in the Leibniz-Newton formula and a suitable change of variables into account, some new parameter-dependent delay-dependent stability conditions are established in terms of linear matrix inequalities so that the filtering process remains asymptotically stable and satisfies a prescribed H_∞ performance level. Using polynomially parameter-dependent quadratic functions and some multiplier matrices, we establish the parameter-independent delay-dependent conditions with high precision under which the desired robust H_∞ filters exist and derive the explicit expression of these filters. A numerical example is provided to demonstrate the validity of the proposed design approach.

Keywords: *LPV systems; H_∞ filtering; delay; LMI; polynomially parameter-dependent quadratic functions.*

Mathematics Subject Classification (2000): 34D20, 93A30, 93E11.

* Corresponding author: hrkarimi@ut.ac.ir

The authors would like to thank the Alexander-von-Humboldt-Stiftung for providing the financial assistance to make this research possible.

1 Introduction

One of the problems with optimal Kalman filters, which has now been well recognized, is that they can be sensitive to the system data and the spectral densities of noise processes, or in other words, they may lack robustness [1]. Therefore, in the past decade, a number of papers have attempted to develop robust filters that are capable of guaranteeing satisfactory estimation in the presence of modeling errors and unknown signal statistics [35]. Concerning the energy bounded deterministic noise inputs, the H_∞ filtering theory has been developed which minimizes (or, in the suboptimal case, bound) the worst-case energy gain from the energy-bounded disturbances (without the need for knowledge of noise statistics) to the estimation errors [16]. Furthermore, the robust H_∞ filtering problem has recently received considerable attention. The aim of this problem is to pursue the enforcement of the upper bound constraint on the H_∞ norm where the system is affected by parameter uncertainties (see for instance [22], and the references therein).

The stability analysis and control design of linear parameter-varying (LPV) systems where the state-space matrices depend affinely on parameter vector, whose values are not known a priori, but can be measured online for control process, have received considerable attention recently (see for instance [3, 5, 17, 21, 31] and the references therein). Establishing stability via the use of classical quadratic Lyapunov function is conservative for the LPV systems. To investigate the stability of LPV systems one needs to resort the use of parameter-dependent Lyapunov functions to achieve necessary and sufficient conditions of system stability, see [4, 6, 7, 10, 13, 19, 23]. However, Bliman in [7] proposed robust stability analysis for LPV systems with polytopic uncertain parameters. He also developed some conditions for robust stability in terms of solvability of some linear matrix inequalities (LMIs) without conservatism. Concerning unknown parameter vector, an adaptive method has been presented in [21] for robust stabilization with H_∞ performance of LPV systems. Moreover, the existence of a polynomially parameter-dependent quadratic (PPDQ) Lyapunov function for parameter-dependent systems, which are robustly stable, is stated in [8]. Recently, sufficient conditions for robust stability of the linear state-space models affected by polytopic uncertainty have been provided in [9] using homogeneous PPDQ Lyapunov functions, which are formulated in terms of LMI feasibility tests.

On the other hand, in addition to the system uncertainties, it is well known that the time-delay is also often the main cause of instability and poor performance of dynamical systems [11, 12, 25, 37]. Stability criteria for time-delay systems can be classified into two categories: delay-dependent and delay-independent criteria. The stability and the performance issues of the LPV state-delayed systems are then both theoretically and practically important and are a field of intense research. Recently, some appreciable works have been performed to analyze and synthesize LPV time-delay systems (e.g. see [18, 20, 32, 34, 36, 38]). It is known that the conservatism of the delay-dependent stability conditions stems from two causes: one is the model transformation used and the other is the inequality bounding technique employed for some cross terms encountered in the analysis. Considering these, in [38], a model which is equivalent to the original delay system was proposed and the bounding technique in [26, 28] was used. However, conservatism still remains in these results, which motivates the present study.

The filter design problems of uncertain time-delay systems have received much less attention although they are important in control design and signal processing applications (e.g. see [15]). Recently, Pila et al. [27] have considered the problem of filtering for linear time-varying system with time-delay measurements. Moreover, the robust filtering problem for uncertain linear systems with delayed sates and outputs for both time-invariant and time-varying cases were investigated in [35]. It is also worth citing that few studies have been done for the design of robust H_∞ filters for LPV systems [24, 33]. However, the robust H_∞ filtering problem for LPV systems with delayed states and outputs has not been fully investigated and remains to be important and challenging.

In this work, we are concerned with the delay-dependent robust H_∞ filtering problem for a class of LPV systems with time-invariant delay in the states and outputs. It is assumed that the state-space data affinely depend on parameter vector that are measurable in real-time. Some new delay dependent stability conditions are established based on a new method with some interesting features. First, it is obtained without resorting to any model transformations and bounding techniques for some cross terms, thus reducing the conservatism in the derivation of the stability condition. Second, some free weighting matrices are employed to express the influence of the terms in the Leibniz-Newton formula which are determined by solving LMIs. Third, using a suitable change of variables the delay-dependent stability conditions are formulated in terms of LMIs such that the filtering process remains asymptotically stable and satisfies a prescribed H_∞ performance level. Forth, using polynomially parameter-dependent quadratic (PPDQ) functions and some multiplier matrices, the parameter-dependent delay-dependent conditions are relaxed to the parameter-independent delay-dependent conditions with high precision under which the desired robust H_∞ filters exist and the explicit expression of these filters is derived. Accordingly, the designed filters have the ability to track the plant states in the presence of external disturbances. Eventually, an illustrative example is given to show the qualification of our design methodology.

Notations. The symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix. Also, the symbol $\otimes$ denotes Kronecker product, the power of Kronecker products being used with the natural meaning $M^{0\otimes} = 1$, $M^{p\otimes} := M^{(p-1)\otimes} \otimes M$. Let $\{\hat{J}_k, \tilde{J}_k\} \in \Re^{k\times(k+1)}$, and $v^{[k]}$ be defined by $\hat{J}_k := [I_k, \quad 0_{k\times1}]$, $\tilde{J}_k := [0_{k\times1}, \quad I_k]$ and $v^{[k]} = \mathrm{col}\{1, v, \dots, v^{k-1}\}$, respectively, which have essential roles for polynomial manipulations [7].

2 Problem Description

Consider a class of LPV systems with delayed states and outputs as

$$\begin{aligned}
\dot{x}(t) &= A(\rho)x(t) + A_d(\rho)x(t-h) + E_1(\rho)w(t),\\
x(t) &= \phi(t), \quad t \in [-h, 0],\\
z(t) &= L(\rho)x(t) + L_d(\rho)x(t-h) + E_3(\rho)w(t),\\
y(t) &= C(\rho)x(t) + C_d(\rho)x(t-h) + E_2(\rho)w(t),
\end{aligned} \tag{1}$$

where $x(t) \in \Re^n$, $w(t) \in L_2[0,\infty)$, $z(t) \in \Re^z$ and $y(t) \in \Re^p$ are state vector, disturbance input, estimated output and measured output, respectively. $\phi(t)$ is continuous vector valued initial function. Moreover, the parameter $h > 0$ is the constant time-delay and the vector $\rho = \mathrm{col}\{\rho_1, \rho_2, \dots, \rho_m\} \in \zeta \subset \Re^m$ is uncertain but the parameters ρ_i are measurable in real-time with ζ being a compact set. In (1), the parameter-dependent matrices are unknown real continuous matrix functions, which affinely depend on the vector ρ, that are

$$\begin{bmatrix} A(\rho) & A_d(\rho) & E_1(\rho) \\ L(\rho) & L_d(\rho) & E_3(\rho) \\ C(\rho) & C_d(\rho) & E_2(\rho) \end{bmatrix} = \begin{bmatrix} A_0 & A_{0d} & E_{01} \\ L_0 & L_{0d} & E_{03} \\ C_0 & C_{0d} & E_{02} \end{bmatrix} + \sum_{j=1}^{m} \rho_j \begin{bmatrix} A_j & A_{jd} & E_{j1} \\ L_j & L_{jd} & E_{j3} \\ C_j & C_{jd} & E_{j2} \end{bmatrix}. \tag{2}$$

In this paper, we focus on the design of an n-th order H_∞ filter with delayed states and outputs with the following equations

$$\begin{aligned} \dot{\hat{x}}(t) &= F(\rho)\hat{x}(t) + F_d(\rho)\hat{x}(t-h) + G(\rho)y(t), \\ \hat{x}(t) &= 0, \quad t \in [-h, 0], \\ \hat{z}(t) &= L(\rho)\hat{x}(t) + L_d(\rho)\hat{x}(t-h) + E_3(\rho)w(t), \end{aligned} \tag{3}$$

where the state-space parameter-dependent matrices $F(\rho)$, $F_d(\rho)$ and $G(\rho)$ of the appropriate dimensions are the filter design objectives to be determined. In (3), it is assumed that $\hat{x}(t) \in \Re^n$ is the estimation of the plant's state. By defining $e(t) = x(t) - \hat{x}(t)$ as the estimation error, then we obtain the following state-space model:

$$\begin{aligned} \dot{x}_f(t) &= A_{f\rho}x_f(t) + A_{df\rho}x_f(t-h) + E_{f\rho}w(t), \\ z(t) - \hat{z}(t) &= L_{f\rho}x_f(t) + L_{df\rho}x_f(t-h), \end{aligned} \tag{4}$$

where $x_f(t) = \mathrm{col}\,\{x(t), e(t)\}$ and $L_{f\rho} := L_f(\rho) = [0,\ L(\rho)]$, $L_{df\rho} := L_{df}(\rho) = [0,\ L_d(\rho)]$, and

$$\begin{aligned} A_{f\rho} &:= A_f(\rho) = \begin{bmatrix} A(\rho) & 0 \\ A(\rho) - F(\rho) - G(\rho)C(\rho) & F(\rho) \end{bmatrix}, \\ A_{df\rho} &:= A_{df}(\rho) = \begin{bmatrix} A_d(\rho) & 0 \\ A_d(\rho) - F_d(\rho) - G(\rho)C_d(\rho) & F_d(\rho) \end{bmatrix}, \\ E_{f\rho} &:= E_f(\rho) = \begin{bmatrix} E_1(\rho) \\ E_1(\rho) - G(\rho)E_2(\rho) \end{bmatrix}. \end{aligned}$$

Remark 2.1 In the case of a free-delay filter, the delay-dependent filter (3) is written in the form

$$\begin{aligned} \dot{\hat{x}}(t) &= F(\rho)\hat{x}(t) + G(\rho)y(t), \\ \hat{x}(t) &= 0, \quad t \in [-h, 0], \\ \hat{z}(t) &= L(\rho)\hat{x}(t) + E_3(\rho)w(t). \end{aligned}$$

Definition 2.1 The delay-dependent robust H_∞ filter of the type (3) is said to *guarantee robust disturbance attenuation* if under zero initial condition

$$\limsup_{\rho \in \zeta} \limsup_{\|w\|_2 \neq 0} \frac{\|z(t) - \hat{z}(t)\|}{\|w(t)\|_2} \leq \gamma$$

for all bounded energy disturbances and a prescribed positive value γ.

Therefore, the main objective of the paper is to seek the state-space parameter-dependent matrices of the delay-dependent robust H_∞ filter (3) guarantees a prescribed H_∞ performance for the augmented system (4). To investigate the Lyapunov-based stability of the augmented system, one important role will be played by the search for PPDQ Lyapunov functions chosen within the following class.

Definition 2.2 We call a *polynomial parameter-dependent quadratic (PPDQ) function* any quadratic function $x^T S(\rho)x(t)$ such that

$$S(\rho) := (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T S_k (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)$$

for every $x(t) \in \Re^n$ and a certain $S_k \in \Re^{k^m n \times k^m n}$. The integer $k-1$ is called *the degree of the PPDQ function* $S(\rho)$ [7].

3 Delay-Dependent Robust H_∞ Filtering

In the following, it will be assumed that the delay-dependent robust H_∞ filter (3) is known and the delay-dependent stability conditions will be investigated under which the augmented system (4) is stable and satisfies the prescribed H_∞ performance for all admissible vectors $\rho \in \zeta$.

The approach employed here is to investigate the delay-dependent stability analysis of the augmented system (4) in the presence of the disturbance (or exogenous input). In the literature, extensions of the quadratic Lyapunov functions to the quadratic Lyapunov-Krasovskii functionals have been proposed for time-delayed systems [11, 12]. Now, we choose a Lyapunov-Krasovskii functional candidate for the LPV system with delayed states and outputs as

$$V(x_f(t)) = x_f(t)^T P_\rho x_f(t) + \int_{t-h}^{t} x_f(\sigma)^T Q_\rho x_f(\sigma)\, d\sigma + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}_f(\sigma)^T Z_\rho \dot{x}_f(\sigma)\, d\sigma\, d\theta \tag{5}$$

with the positive definite matrices

$$P_\rho := P(\rho) = \begin{bmatrix} P_{1\rho} & 0 \\ * & P_{2\rho} \end{bmatrix} \in \Re^{2n \times 2n}, \tag{6}$$

$$Q_\rho := Q(\rho) = \begin{bmatrix} Q_{11\rho} & Q_{12\rho} \\ * & Q_{22\rho} \end{bmatrix} \in \Re^{2n \times 2n}, \tag{7}$$

$$Z_\rho := Z(\rho) = \begin{bmatrix} Z_{11\rho} & 0 \\ * & P_{2\rho} \end{bmatrix} \in \Re^{2n \times 2n}, \tag{8}$$

where the PPDQ functions $P_{1\rho}$, $P_{2\rho}$, $Q_{11\rho}$, $Q_{22\rho}$, $Q_{12\rho}$ and $Z_{11\rho}$ satisfying the following representation forms:

$$\begin{aligned}
P_{1\rho} &:= P_1(\rho) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T P_{1,k} (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n),\\
P_{2\rho} &:= P_2(\rho) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T P_{2,k} (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n),\\
Q_{11\rho} &:= Q_{11}(\rho) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T Q_{11,k} (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n),\\
Q_{22\rho} &:= Q_{22}(\rho) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T Q_{22,k} (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n),\\
Q_{12\rho} &:= Q_{12}(\rho) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T Q_{12,k} (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n),\\
Z_{11\rho} &:= Z_{11}(\rho) = (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T Z_{11,k} (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)
\end{aligned} \tag{9}$$

with parameter-independent positive definite matrices
$\{P_{1,k}, P_{2,k}, Q_{11,k}, Q_{22,k}, Q_{12,k}, Z_{11,k}\} \in \Re^{k^m n \times k^m n}$ of the order $k-1$. Now, let us define a Hamiltonian function $H(x_f, w, \rho)$ as:

$$H(x_f, w, \rho) = \frac{d}{dt} V(x_f) + (z - \hat{z})^T (z - \hat{z}) - \gamma^2 w^T w. \tag{10}$$

It is known that the inequity

$$H(x_f, w, \rho) < 0 \tag{11}$$

implies the following inequality

$$\int_0^T (z-\hat{z})^T(z-\hat{z})\,dt < \gamma^2 \int_0^T w^T w\,dt + V(x_f(0)) - V(x_f(T)) < \gamma^2 \int_0^T w^T w\,dt,$$
$$\forall\, T > 0,\ \forall\, w$$

that is identical to the performance specification in Definition 2.1.

Using the Leibniz-Newton formula, we write

$$x_f(t-h) = x_f(t) - \int_{t-h}^{t} \dot{x}_f(\sigma)\,d\sigma,$$

then, for any appropriately dimensioned matrices Y_ρ, T_ρ and S_ρ, we have

$$2(x_f(t)^T Y_\rho + x_{fh}(t)^T T_\rho + w(t)^T S_\rho)\left(x_f(t) - x_f(t-h) - \int_{t-h}^{t} \dot{x}_f(\sigma)\,d\sigma\right) = 0, \tag{12}$$

which is added to the Hamiltonian function $H(x_f, w, \rho)$. On the other hand, for any semi-positive definite matrix

$$X_\rho = \begin{bmatrix} X_{11\rho} & X_{12\rho} & X_{13\rho} \\ * & X_{22\rho} & X_{23\rho} \\ * & * & X_{33\rho} \end{bmatrix} \geq 0, \tag{13}$$

the following holds

$$h\xi(t)^T X_\rho \xi(t) - \int_{t-h}^{t} \xi(t)^T X_\rho \xi(t)\,d\sigma = 0, \tag{14}$$

where $\xi(t) = \mathrm{col}\,\{x_f(t),\, x_f(t-h),\, w(t)\}$.

Calculating the time derivative of $V(x_f(t))$ along the trajectory of the augmented system (4) and replacing in Eq. (10), results in

$$H(x_f, w, \rho) = \xi(t)^T \Xi_\rho \xi(t) - \int_{t-h}^{t} \xi(t,\sigma)^T \Omega_\rho \xi(t,\sigma)\,d\sigma, \tag{15}$$

where $\xi(t,\sigma) = \mathrm{col}\,\{x_f(t),\, x_f(t-h),\, w(t),\, \dot{x}_f(\sigma)\}$,

$$\Omega_\rho = \begin{bmatrix} X_{11\rho} & X_{12\rho} & X_{13\rho} & Y_\rho \\ * & X_{22\rho} & X_{23\rho} & T_\rho \\ * & * & X_{33\rho} & S_\rho \\ * & * & * & Z_\rho \end{bmatrix},$$

and

$$\Xi_\rho = \begin{bmatrix} \Delta_{11} & \Delta_{12} & P_\rho E_{f\rho} + S_\rho^T + hX_{13\rho} + hA_{f\rho}^T Z_\rho E_{f\rho} \\ * & \Delta_{22} & -S_\rho^T + hX_{23\rho} + hA_{df\rho}^T Z_\rho E_{f\rho} \\ * & * & -\gamma^2 I_s + hX_{33\rho} + hE_{f\rho}^T Z_\rho E_{f\rho} \end{bmatrix},$$

with

$$\Delta_{11} = P_\rho A_{f\rho} + A_{f\rho}^T P_\rho + Y_\rho + Y_\rho^T + Q_\rho + hX_{11\rho} + hA_{f\rho}^T Z_\rho A_{f\rho} + L_{f\rho}^T L_{f\rho},$$
$$\Delta_{12} = P_\rho A_{df\rho} - Y_\rho + hX_{12\rho} + hA_{f\rho}^T Z_\rho A_{df\rho} + L_{f\rho}^T L_{df\rho},$$
$$\Delta_{22} = -T_\rho - T_\rho^T - Q_\rho + hX_{22\rho} + hA_{df\rho}^T Z_\rho A_{df\rho} + L_{df\rho}^T L_{df\rho}.$$

According to partitioning the existing matrices if $\Xi_\rho < 0$ and $\Omega_\rho \geq 0$, then $H(x_f, w, \rho) < 0$ for any $\xi(t) \neq 0$. Applying the Schur complement Lemma shows that inequality $\Xi_\rho < 0$ implies

$$\hat{\Pi}_\rho = \begin{bmatrix} \hat{\Delta}_{11} & \hat{\Delta}_{12} & P_\rho E_{f\rho} + S_\rho^T + hX_{13\rho} & hA_{f\rho}^T Z_\rho \\ * & \hat{\Delta}_{22} & -S_\rho^T + hX_{23\rho} & hA_{df\rho}^T Z_\rho \\ * & * & -\gamma^2 I_s + hX_{33\rho} & hE_{f\rho}^T Z_\rho \\ * & * & * & -hZ_\rho \end{bmatrix} < 0, \tag{16}$$

with $\hat{\Delta}_{11} = P_\rho A_{f\rho} + A_{f\rho}^T P_\rho + Y_\rho + Y_\rho^T + Q_\rho + hX_{11\rho} + L_{f\rho}^T L_{f\rho}$, $\hat{\Delta}_{12} = P_\rho A_{df\rho} - Y_\rho + hX_{12\rho} + L_{f\rho}^T L_{df\rho}$ and $\hat{\Delta}_{22} = -T_\rho - T_\rho^T - Q_\rho + hX_{22\rho} + L_{df\rho}^T L_{df\rho}$.

Notice that the matrix inequality (16) includes multiplication of filter matrices and Lyapunov matrices. In the literature, more attention has been paid to the problems having this nature, which called bilinear matrix inequality (BMI) problems [29]. In the sequel, it is shown that, by a suitable change of variables, the robust H_∞ filtering problem can be converted into convex programming problems written in terms of LMIs.

Remark 3.1 Considering the parameter-dependent BMI (16) in addition to partitioning the existing matrices P_ρ, Q_ρ and Z_ρ and assuming

$$\begin{bmatrix} W_{1\rho} & W_{2\rho} & W_{3\rho} \end{bmatrix} = P_{2\rho} \begin{bmatrix} F_\rho & G_\rho & F_{d\rho} \end{bmatrix}, \tag{17}$$

where $\{W_{1\rho}, W_{3\rho}\} \in \Re^{n\times n}$ and $W_{2\rho} \in \Re^{n\times p}$ leads to

$$\Pi_\rho = \begin{bmatrix} \bar{\Delta}_{11} & \bar{\Delta}_{12} & \bar{\Delta}_{13} & \bar{\Delta}_{14} & \bar{\Delta}_{15} & hA_\rho^T Z_{11\rho} & \bar{\Delta}_{17} \\ * & \bar{\Delta}_{22} & \bar{\Delta}_{23} & \bar{\Delta}_{24} & \bar{\Delta}_{25} & 0 & hW_{1\rho}^T \\ * & * & \bar{\Delta}_{33} & \bar{\Delta}_{34} & -S_{11\rho}^T + hX_{23,11\rho} & hA_{d\rho}^T Z_{11\rho} & \bar{\Delta}_{37} \\ * & * & * & \bar{\Delta}_{44} & -S_{12\rho}^T + hX_{23,21\rho} & 0 & hW_{3\rho}^T \\ * & * & * & * & -\gamma^2 I_s + hX_{33\rho} & hE_{1\rho}^T Z_{11\rho} & \bar{\Delta}_{57} \\ * & * & * & * & * & -hZ_{11\rho} & 0 \\ * & * & * & * & * & * & -hP_{2\rho} \end{bmatrix} < 0. \tag{18}$$

In (18)

$$\begin{aligned}
&\bar{\Delta}_{11} = A_\rho^T P_{1\rho} + P_{1\rho} A_\rho + Q_{11\rho} + Y_{11\rho} + Y_{11\rho}^T + hX_{11,11\rho},\\
&\bar{\Delta}_{12} = A_\rho^T P_{2\rho} - W_{1\rho}^T - C_\rho^T W_{2\rho}^T + Y_{12\rho} + Y_{21\rho}^T + Q_{12\rho} + hX_{11,12\rho},\\
&\bar{\Delta}_{13} = P_{1\rho} A_{d\rho} - Y_{11\rho} + hX_{12,11\rho}, \quad \bar{\Delta}_{14} = -Y_{12\rho} + hX_{12,12\rho},\\
&\bar{\Delta}_{15} = P_{1\rho} E_{1\rho} + S_{11\rho}^T + hX_{13,11\rho}, \quad \bar{\Delta}_{17} = h(A_\rho^T P_{2\rho} - W_{1\rho}^T - C_\rho^T W_{2\rho}^T),\\
&\bar{\Delta}_{22} = W_{1\rho} + W_{1\rho}^T + L_\rho^T L_\rho + Q_{22\rho} + Y_{22\rho} + Y_{22\rho}^T + hX_{11,22\rho},\\
&\bar{\Delta}_{23} = P_{2\rho} A_{d\rho} - W_{3\rho} - W_{2\rho} C_{d\rho}, \quad \bar{\Delta}_{24} = W_{3\rho} + L_\rho^T L_{d\rho} - Y_{22\rho} + hX_{12,22\rho},\\
&\bar{\Delta}_{25} = P_{2\rho} E_{1\rho} - W_{2\rho} E_{2\rho} + S_{12\rho}^T + hX_{13,21\rho},\\
&\bar{\Delta}_{33} = -Q_{11\rho} - T_{11\rho} - T_{11\rho}^T + hX_{22,11\rho},\\
&\bar{\Delta}_{34} = -Q_{12\rho} - T_{12\rho} - T_{21\rho}^T + hX_{22,12\rho}, \quad \bar{\Delta}_{37} = h(A_{d\rho}^T P_{2\rho} - W_{3\rho}^T - C_{d\rho}^T W_{2\rho}),\\
&\bar{\Delta}_{44} = -Q_{22\rho} - T_{22\rho} - T_{22\rho}^T + L_{d\rho}^T L_{d\rho} + hX_{22,22\rho},\\
&\bar{\Delta}_{57} = h(E_{1\rho}^T P_{2\rho} - E_{2\rho}^T W_{2\rho}^T),
\end{aligned}$$

where

$$Y_\rho = \begin{bmatrix} Y_{11\rho} & Y_{12\rho} \\ Y_{21\rho} & Y_{22\rho} \end{bmatrix}, \quad T_\rho = \begin{bmatrix} T_{11\rho} & T_{12\rho} \\ T_{21\rho} & T_{22\rho} \end{bmatrix}, \quad S_\rho = \begin{bmatrix} S_{11\rho} & S_{12\rho} \end{bmatrix},$$

$$X_{ij\rho} = \begin{bmatrix} X_{ij,11\rho} & X_{ij,12\rho} \\ * & X_{ij,22\rho} \end{bmatrix} \quad \text{and} \quad X_{i3\rho} = \begin{bmatrix} X_{i3,11\rho} \\ X_{i3,21\rho} \end{bmatrix} \quad \text{for} \quad i, j = 1, 2.$$

Now, our results are summarized in the following Theorem.

Theorem 3.1 *The augmented system (4) obtained from the interconnection of the plant (1) and the filter (3) is stable and achieves the H_∞ performance for a given performance bound γ in the sense of Definition 2.1 if there exist the parameter-dependent positive definite matrices $P_\rho = P_\rho^T > 0$, $Q_\rho = Q_\rho^T > 0$ and $Z_\rho = Z_\rho^T > 0$, a symmetric semi-positive definite matrix $X_\rho \geq 0$, the parameter-dependent matrixes $W_{1\rho}$, $W_{2\rho}$, $W_{3\rho}$ and any appropriately dimensioned matrices Y_ρ, T_ρ and S_ρ such that the parameter-dependent LMIs $\Pi_\rho < 0$ and $\Omega_\rho \geq 0$ are satisfied, respectively.*

Remark 3.2 In the LPV system (1) if an uncertain time-invariant delay lies in $[0, \bar{h}]$, i.e., $h \in [0, \bar{h}]$, then according to the procedure of above it can be shown that the following inequality, which is delay-dependent H_∞ filter design criterion is concluded instead of (16):

$$\hat{\Pi}_\rho = \begin{bmatrix} \tilde{\Delta}_{11} & \tilde{\Delta}_{12} & P_\rho E_{f\rho} + S_\rho^T + \bar{h}X_{13\rho} & \bar{h}A_{f\rho}^T Z_\rho \\ * & \tilde{\Delta}_{22} & -S_\rho^T + \bar{h}X_{23\rho} & \bar{h}A_{df\rho}^T Z_\rho \\ * & * & -\gamma^2 I_s + \bar{h}X_{33\rho} & \bar{h}E_{f\rho}^T Z_\rho \\ * & * & * & -\bar{h}Z_\rho \end{bmatrix} < 0,$$

where $\tilde{\Delta}_{11} = P_\rho A_{f\rho} + A_{f\rho}^T P_\rho + Y_\rho + Y_\rho^T + Q_\rho + \bar{h}X_{11\rho} + L_{f\rho}^T L_{f\rho}$, $\tilde{\Delta}_{12} = P_\rho A_{df\rho} - Y_\rho + \bar{h}X_{12\rho} + L_{f\rho}^T L_{df\rho}$ and $\tilde{\Delta}_{22} = -T_\rho - T_\rho^T - Q_\rho + \bar{h}X_{22\rho} + L_{df\rho}^T L_{df\rho}$.

Remark 3.3 If the matrices X_ρ, Y_ρ, T_ρ and S_ρ in the matrix Ω_ρ are set to zero, and $Z_{11\rho} = \epsilon I_n$ (ϵ is a sufficiently small positive scalar), then Theorem 1 is identical to the delay-independent H_∞ filter design criterion such that the inequality (16) is stated in the following delay-independent form:

$$\begin{bmatrix} P_\rho A_{f\rho} + A_{f\rho}^T P_\rho + Q_\rho + L_{f\rho}^T L_{f\rho} & P_\rho A_{df\rho} + L_{f\rho}^T L_{df\rho} & P_\rho E_{f\rho} \\ * & -Q_\rho + L_{df\rho}^T L_{df\rho} & 0 \\ * & * & -\gamma^2 I_s \end{bmatrix} < 0.$$

In the next Section, a new framework for relaxing parameter-dependent matrix inequalities into conventional LMI problems is stated using the PPDQ functions.

4 Parameter-Dependent LMI Relaxations

This section is devoted to solve the parameter-dependent LMIs to finding the parameter-dependent state-space matrices F_ρ, $F_{d\rho}$ and G_ρ. These parameter-dependent LMIs are corresponded to infinite-dimensional convex problems. In the literature, there are some attempts to obtain a finite-dimensional optimization problem such the parameter-dependent Lyapunov functions are approximated using a finite set of basis functions [2, 30, 32, 36].

The main approach employed here is using the PPDQ functions as the basis functions to relax parameter-dependent LMIs into parameter-independent LMI forms by utilizing some multiplier matrices.

Lemma 4.1 *Let the degree of the PPDQ function $P_{1\rho}$ be $k-1$. A PPDQ function of degree k for parameter-dependent matrix $P_{1\rho}T_\rho$ is given by*

$$P_{1\rho}T_\rho := (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T S_k (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_q),$$

where $T_\rho = T_0 + \sum_{i=1}^{m} \rho_i T_i$ and $T_i \in \Re^{n\times q}$, then the parameter-independent matrix $S_k \in \Re^{(k+1)^m n\times(k+1)^m q}$ which depends on the parameter-independent matrix $P_{1,k}$ linearly is defined as

$$S_k = (\hat{J}_k^{m\otimes} \otimes I_n)^T P_{1,k} (\hat{J}_k^{m\otimes} \otimes T_0 + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes T_i).$$

According to Lemma 4.1 for the parameter-dependent matrices $E_{1\rho}$, A_ρ and $A_{d\rho}$, we obtain

$$\begin{aligned}
P_{1\rho}E_{1\rho} &= (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T \Xi_{1,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_s),\\
P_{2\rho}E_{1\rho} &= (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T \Xi_{2,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_s),\\
Z_{11\rho}E_{1\rho} &= (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T \Xi_{3,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_s),\\
P_{2\rho}A_\rho &= (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T S_{1,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n),\\
Z_{11\rho}A_\rho &= (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T S_{2,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n),\\
P_{1\rho}A_{d\rho} &= (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T S_{1d,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n),\\
P_{2\rho}A_{d\rho} &= (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T S_{2d,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n),\\
Z_{11\rho}A_{d\rho} &= (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T S_{3d,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n),
\end{aligned} \tag{19}$$

where the parameter-independent matrices $\Xi_{1,k}$, $\Xi_{2,k}$, $\Xi_{3,k}$, $S_{1,k}$, $S_{2,k}$, $S_{1d,k}$, $S_{2d,k}$ and $S_{3d,k}$ are represented in the following forms:

$$\begin{aligned}
\Xi_{1,k} &= (\hat{J}_k^{m\otimes} \otimes I_n)^T P_{1,k} \left(\hat{J}_k^{m\otimes} \otimes E_{01} + \sum_{j=1}^{m} \hat{J}_k^{(m-j)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(j-1)\otimes} \otimes E_{j1} \right),\\
\Xi_{2,k} &= (\hat{J}_k^{m\otimes} \otimes I_n)^T P_{2,k} \left(\hat{J}_k^{m\otimes} \otimes E_{01} + \sum_{j=1}^{m} \hat{J}_k^{(m-j)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(j-1)\otimes} \otimes E_{j1} \right),\\
\Xi_{3,k} &= (\hat{J}_k^{m\otimes} \otimes I_n)^T Z_{11,k} \left(\hat{J}_k^{m\otimes} \otimes E_{01} + \sum_{j=1}^{m} \hat{J}_k^{(m-j)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(j-1)\otimes} \otimes E_{j1} \right),\\
S_{1,k} &= (\hat{J}_k^{m\otimes} \otimes I_n)^T P_{2,k} \left(\hat{J}_k^{m\otimes} \otimes A_0 + \sum_{j=1}^{m} \hat{J}_k^{(m-j)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(j-1)\otimes} \otimes A_j \right),\\
S_{2,k} &= (\hat{J}_k^{m\otimes} \otimes I_n)^T Z_{11,k} \left(\hat{J}_k^{m\otimes} \otimes A_0 + \sum_{j=1}^{m} \hat{J}_k^{(m-j)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(j-1)\otimes} \otimes A_j \right),\\
S_{1d,k} &= (\hat{J}_k^{m\otimes} \otimes I_n)^T P_{1,k} \left(\hat{J}_k^{m\otimes} \otimes A_{0d} + \sum_{j=1}^{m} \hat{J}_k^{(m-j)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(j-1)\otimes} \otimes A_{jd} \right),\\
S_{2d,k} &= (\hat{J}_k^{m\otimes} \otimes I_n)^T P_{2,k} \left(\hat{J}_k^{m\otimes} \otimes A_{0d} + \sum_{j=1}^{m} \hat{J}_k^{(m-j)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(j-1)\otimes} \otimes A_{jd} \right),
\end{aligned} \tag{20}$$

$$S_{3d,k} = (\hat{J}_k^{m\otimes} \otimes I_n)^T Z_{11,k}\left(\hat{J}_k^{m\otimes} \otimes A_{0d} + \sum_{j=1}^{m} \hat{J}_k^{(m-j)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(j-1)\otimes} \otimes A_{jd}\right).$$

Remark 4.1 For the parameter-dependent matrix $R_{1\rho} := A_\rho^T P_{1\rho} + P_{1\rho} A_\rho$ the PPDQ function of degree k is given by

$$R_{1\rho} = (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T R_{1,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n),$$

and from Lemma 4.1, the parameter-independent positive definite matrices $R_{1,k} \in \Re^{(k+1)^m n \times (k+1)^m n}$ which depends on the parameter-independent matrix $P_{1,k}$ linearly is obtained as follows:

$$\begin{aligned} R_{1,k} = {} & (\hat{J}_k^{m\otimes} \otimes I_n)^T P_{1,k}\left(\hat{J}_k^{m\otimes} \otimes A_0 + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i\right) \\ & + \left(\hat{J}_k^{m\otimes} \otimes A_0 + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i\right)^T P_{1,k} (\hat{J}_k^{m\otimes} \otimes I_n). \end{aligned} \tag{21}$$

The parameter-dependent matrices $W_{1\rho}$, $W_{2\rho}$ and $W_{3\rho}$ in (17) can be expressed in the forms

$$\begin{aligned} W_{1\rho} &= (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T W_{1,k} (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n), \\ W_{2\rho} &= (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T W_{2,k} (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_p), \\ W_{3\rho} &= (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n)^T W_{3,k} (\rho_m^{[k]} \otimes \ldots \otimes \rho_1^{[k]} \otimes I_n) \end{aligned} \tag{22}$$

with parameter-independent matrices $\{W_{1,k},\, W_{3,k}\} \in \Re^{k^m n \times k^m n}$ and $\{\bar{W}_{2,k},\, \bar{W}_{2d,k},\, \tilde{W}_{2,k}\} \in \Re^{(k+1)^m n \times (k+1)^m p}$. Then, the following relations can be concluded

$$\begin{aligned} W_{2\rho} C_\rho &= (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T \bar{W}_{2,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_p), \\ W_{2\rho} C_{d\rho} &= (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T \bar{W}_{2d,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_p), \\ W_{2\rho} E_{2\rho} &= (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n)^T \tilde{W}_{2,k} (\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_p), \end{aligned} \tag{23}$$

where the parameter-independent matrices $\{W_{1,k}, W_{3,k}\} \in \Re^{k^m n \times k^m n}$ are defined, respectively, as

$$\begin{aligned} \bar{W}_{2,k} &= (\hat{J}_k^{m\otimes} \otimes I_n)^T W_{2,k}\left(\hat{J}_k^{m\otimes} \otimes C_0 + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes C_i\right), \\ \bar{W}_{2d,k} &= (\hat{J}_k^{m\otimes} \otimes I_n)^T W_{2d,k}\left(\hat{J}_k^{m\otimes} \otimes C_{0d} + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes C_{id}\right), \\ \tilde{W}_{2,k} &= (\hat{J}_k^{m\otimes} \otimes I_n)^T W_{2,k}\left(\hat{J}_k^{m\otimes} \otimes E_{02} + \sum_{i=1}^{m} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes E_{i2}\right). \end{aligned} \tag{24}$$

Similarly, the parameter-dependent matrices $L_\rho^T L_\rho$ and $L_{d\rho}^T L_{d\rho}$ and the parameter-independent matrix I_s can also be represented, respectively, by

$$\begin{aligned} L_\rho^T L_\rho &= \left(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n\right)^T \left(\hat{J}_k^{m\otimes} \otimes I_n\right)^T \bar{L}_k \\ &\quad \times \left(\hat{J}_k^{m\otimes} \otimes I_n\right) \left(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n\right), \\ L_{d\rho}^T L_{d\rho} &= \left(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n\right)^T \left(\hat{J}_k^{m\otimes} \otimes I_n\right)^T \bar{L}_{dk} \\ &\quad \times \left(\hat{J}_k^{m\otimes} \otimes I_n\right) \left(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_n\right), \\ I_s &= \left(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_s\right)^T \left(\hat{J}_k^{m\otimes} \otimes I_s\right)^T \bar{I}_k \\ &\quad \times \left(\hat{J}_k^{m\otimes} \otimes I_s\right) \left(\rho_m^{[k+1]} \otimes \ldots \otimes \rho_1^{[k+1]} \otimes I_s\right), \end{aligned} \tag{25}$$

where the certain parameter-independent matrices $\bar{L}_k \in \Re^{k^m n \times k^m n}$, $\bar{L}_{dk} \in \Re^{k^m n \times k^m n}$ and $\bar{I}_k \in \Re^{k^m s \times k^m s}$ are given by

$$\begin{aligned} \bar{L}_k &= \text{Block diagonal} \left(\begin{bmatrix} L_0^T \\ \vdots \\ L_m^T \end{bmatrix} \begin{bmatrix} L_0 & \ldots & L_m \end{bmatrix}, \underbrace{0_n, \ldots, 0_n}_{(k^m - m - 1) \text{ elements}} \right), \\ \bar{L}_{dk} &= \text{Block diagonal} \left(\begin{bmatrix} L_{0d}^T \\ \vdots \\ L_{md}^T \end{bmatrix} \begin{bmatrix} L_{0d} & \ldots & L_{md} \end{bmatrix}, \underbrace{0_n, \ldots, 0_n}_{(k^m - m - 1) \text{ elements}} \right), \\ \bar{I}_k &= \text{Block diagonal} \left(I_s, \underbrace{0_s, \ldots, 0_s}_{(k^m - 1) \text{ elements}} \right). \end{aligned} \tag{26}$$

We are now in the position to state our main results on parameter-dependent robust H_∞ filter design based on LMI approach in the following theorem.

Theorem 4.1 *Let the positive integer $k-1$ as the degree of the PPDQ functions be given. Consider the LPV system (1) with the known time-delay parameter h. For a given performance bound γ, if there exist the set of parameter-independent matrices $\{W_{1,k}, W_{2,k}, W_{3,k}, X_{11,11k}, X_{12,11k}, X_{12,21k}, X_{12,12k}, X_{12,22k}, X_{22,12k}, X_{13,11k}, X_{13,21k}, X_{23,11k}, X_{23,21k}, Y_{11,k}, Y_{12,k}, Y_{21,k}, Y_{22,k}, T_{11,k}, T_{12,k}, T_{21,k}, T_{22,k}, S_{11,k}, S_{12,k}\}$, the set of parameter-independent positive definite matrices $\{P_{1,k}, P_{2,k}, Q_{11,k}, Q_{22,k}, Z_{11,k}, X_{11,11k}, X_{11,22k}, X_{22,11k}, X_{22,22k}, X_{33,k}\}$ and the set of positive definite multipliers $\{\hat{Q}_{i,k}^{(1)}, \ldots, \hat{Q}_{i,k}^{(7)}, \tilde{Q}_{i,k}^{(1)}, \ldots, \tilde{Q}_{i,k}^{(7)}, \bar{Q}_{i,k}^{(1)}, \bar{Q}_{i,k}^{(2)}\}$ for $i = 1, 2, \ldots, m$ to the following LMIs,*

$$\Omega_{m,k} = \begin{bmatrix} \Psi_{11} & X_{11,12k} & X_{12,11k} & X_{12,12k} & X_{13,11k} & Y_{11,k} & Y_{12,k} \\ * & \Psi_{22} & X_{12,21k} & X_{12,22k} & X_{13,21k} & Y_{21,k} & Y_{22,k} \\ * & * & \Psi_{33} & X_{22,12k} & X_{23,11k} & T_{11,k} & T_{12,k} \\ * & * & * & \Psi_{44} & X_{23,21k} & T_{21,k} & T_{22,k} \\ * & * & * & * & \Psi_{55} & S_{11,k} & S_{12,k} \\ * & * & * & * & * & \Psi_{66} & 0 \\ * & * & * & * & * & * & \Psi_{77} \end{bmatrix} \geq 0, \tag{27}$$

$$\Pi_{m,k} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} & hS_{2,k}^T & \Sigma_{17} \\ * & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} & 0 & \Sigma_{27} \\ * & * & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} & hS_{3d,k}^T & \Sigma_{37} \\ * & * & * & \Sigma_{44} & \Sigma_{45} & 0 & \Sigma_{47} \\ * & * & * & * & \Sigma_{55} & h\Xi_{3,k}^T & h\left(\Xi_{2,k}^T - \tilde{W}_{2,k}^T\right) \\ * & * & * & * & * & \Sigma_{66} & 0 \\ \star & * & * & * & * & * & \Sigma_{77} \end{bmatrix} < 0, \tag{28}$$

$$\Phi_{m,k} = \begin{bmatrix} \Lambda_{11} & Q_{12,k} \\ * & \Lambda_{22} \end{bmatrix} > 0, \tag{29}$$

where

$$\Psi_{11} = X_{11,11k} - \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(1)} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)$$
$$+ \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(1)} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),$$

$$\Psi_{22} = X_{11,22k} - \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(2)} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)$$
$$+ \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(2)} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),$$

$$\Psi_{33} = X_{22,11k} - \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(3)} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)$$
$$+ \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(3)} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),$$

$$\Psi_{44} = X_{22,22k} - \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(4)} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)$$
$$+ \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(4)} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),$$

$$\Psi_{55} = X_{33,k} - \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(5)} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)$$
$$+ \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(5)} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),$$

$$\Psi_{66} = Z_{11,k} - \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(6)} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)$$
$$+ \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(6)} \left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),$$

$$\Psi_{77} = P_{2,k} - \sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \hat{Q}_{i,k}^{(7)} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)$$

$$+\sum_{i=1}^{m}\left(\hat{J}_k^{(m-i)\otimes}\otimes\tilde{J}_k\otimes I_{(k+1)^{i-1}n}\right)^T\hat{Q}_{i,k}^{(7)}\left(\hat{J}_k^{(m-i)\otimes}\otimes\tilde{J}_k\otimes I_{(k+1)^{i-1}n}\right),$$

$$\begin{aligned}
\Sigma_{11} &= R_{1,k}+\left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T\left(Q_{1,k}+Y_{11,k}+Y_{11,k}^T+hX_{11,11k}\right)\left(\hat{J}_k^{m\otimes}\otimes I_n\right)\\
&\quad+\sum_{i=1}^{m}\left(\hat{J}_k^{(m-i+1)\otimes}\otimes I_{(k+1)^{i-1}n}\right)^T\tilde{Q}_{i,k}^{(1)}\left(\hat{J}_k^{(m-i+1)\otimes}\otimes I_{(k+1)^{i-1}n}\right)\\
&\quad-\sum_{i=1}^{m}\left(\hat{J}_k^{(m-i)\otimes}\otimes\tilde{J}_k\otimes I_{(k+1)^{i-1}n}\right)^T\tilde{Q}_{i,k}^{(1)}\left(\hat{J}_k^{(m-i)\otimes}\otimes\tilde{J}_k\otimes I_{(k+1)^{i-1}n}\right),\\
\Sigma_{12} &= S_{1,k}^T-\bar{W}_{2,k}^T\\
&\quad+\left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T\left(-W_{1,k}^T+Y_{12,k}+Y_{21,k}^T+Q_{12,k}+hX_{11,12k}\right)\left(\hat{J}_k^{m\otimes}\otimes I_n\right),\\
\Sigma_{13} &= S_{1d,k}+\left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T\left(-Y_{11,k}+hX_{12,11k}\right)\left(\hat{J}_k^{m\otimes}\otimes I_n\right),\\
\Sigma_{14} &= \left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T\left(-Y_{12,k}+hX_{12,12k}\right)\left(\hat{J}_k^{m\otimes}\otimes I_n\right),\\
\Sigma_{15} &= \Xi_{1,k}+\left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T\left(S_{11,k}^T+hX_{13,11k}\right)\left(\hat{J}_k^{m\otimes}\otimes I_s\right),\\
\Sigma_{17} &= h\left(S_{1,k}^T-\bar{W}_{1,k}^T-\left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T W_{1,k}^T\left(\hat{J}_k^{m\otimes}\otimes I_n\right)\right),\\
\Sigma_{22} &= \left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T\Big(W_{1,k}+W_{1,k}^T+\bar{L}_k+Q_{22,k}\\
&\quad+Y_{22,k}+Q_{22,k}^T+hX_{11,22k}\Big)\left(\hat{J}_k^{m\otimes}\otimes I_n\right)\\
&\quad+\sum_{i=1}^{m}\left(\hat{J}_k^{(m-i+1)\otimes}\otimes I_{(k+1)^{i-1}n}\right)^T\tilde{Q}_{i,k}^{(2)}\left(\hat{J}_k^{(m-i+1)\otimes}\otimes I_{(k+1)^{i-1}n}\right)\\
&\quad-\sum_{i=1}^{m}\left(\hat{J}_k^{(m-i)\otimes}\otimes\tilde{J}_k\otimes I_{(k+1)^{i-1}n}\right)^T\tilde{Q}_{i,k}^{(2)}\left(\hat{J}_k^{(m-i)\otimes}\otimes\tilde{J}_k\otimes I_{(k+1)^{i-1}n}\right),\\
\Sigma_{23} &= S_{2d,k}-\bar{W}_{2d,k}^T-\left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T W_{3,k}\left(\hat{J}_k^{m\otimes}\otimes I_n\right),\\
\Sigma_{24} &= \left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T\left(W_{3,k}+\bar{L}_k-Y_{22,k}+hX_{12,22k}\right)\left(\hat{J}_k^{m\otimes}\otimes I_n\right),\\
\Sigma_{25} &= \Xi_{2,k}-\tilde{W}_{2,k}+\left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T\left(S_{12,k}^T+hX_{13,21k}\right)\left(\hat{J}_k^{m\otimes}\otimes I_s\right),\\
\Sigma_{27} &= h\left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T W_{1,k}^T\left(\hat{J}_k^{m\otimes}\otimes I_n\right),\\
\Sigma_{33} &= \left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T\left(-Q_{11,k}-T_{11,k}-T_{11,k}^T+hX_{22,11k}\right)\left(\hat{J}_k^{m\otimes}\otimes I_n\right)\\
&\quad+\sum_{i=1}^{m}\left(\hat{J}_k^{(m-i+1)\otimes}\otimes I_{(k+1)^{i-1}n}\right)^T\tilde{Q}_{i,k}^{(3)}\left(\hat{J}_k^{(m-i+1)\otimes}\otimes I_{(k+1)^{i-1}n}\right)\\
&\quad-\sum_{i=1}^{m}\left(\hat{J}_k^{(m-i)\otimes}\otimes\tilde{J}_k\otimes I_{(k+1)^{i-1}n}\right)^T\tilde{Q}_{i,k}^{(3)}\left(\hat{J}_k^{(m-i)\otimes}\otimes\tilde{J}_k\otimes I_{(k+1)^{i-1}n}\right),\\
\Sigma_{34} &= -\left(\hat{J}_k^{m\otimes}\otimes I_n\right)^T\left(Q_{12,k}+T_{12,k}+T_{12,k}^T-hX_{22,12k}\right)\left(\hat{J}_k^{m\otimes}\otimes I_n\right),
\end{aligned}$$

$$\Sigma_{35} = -\left(\hat{J}_k^{m\otimes} \otimes I_n\right)^T \left(S_{11,k}^T - hX_{23,11k}\right)\left(\hat{J}_k^{m\otimes} \otimes I_s\right),$$

$$\Sigma_{37} = h\left(S_{2d,k}^T - \bar{W}_{2d,k}^T - \left(\hat{J}_k^{m\otimes} \otimes I_n\right)^T W_{3,k}^T \left(\hat{J}_k^{m\otimes} \otimes I_n\right)\right),$$

$$\begin{aligned}\Sigma_{44} = &-\left(\hat{J}_k^{m\otimes} \otimes I_n\right)^T \left(Q_{22,k} + T_{22,k} + T_{22,k}^T - \bar{L}_{d,k} - hX_{22,22k}\right)\left(\hat{J}_k^{m\otimes} \otimes I_n\right)\\ &+ \sum_{i=1}^{m}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \tilde{Q}_{i,k}^{(4)}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)\\ &- \sum_{i=1}^{m}\left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \tilde{Q}_{i,k}^{(4)}\left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),\end{aligned}$$

$$\Sigma_{45} = -\left(\hat{J}_k^{m\otimes} \otimes I_n\right)^T \left(S_{12,k}^T - hX_{23,21k}\right)\left(\hat{J}_k^{m\otimes} \otimes I_s\right),$$

$$\Sigma_{47} = h\left(\hat{J}_k^{m\otimes} \otimes I_n\right)^T W_{3,k}^T \left(\hat{J}_k^{m\otimes} \otimes I_n\right),$$

$$\begin{aligned}\Sigma_{55} = &\left(\hat{J}_k^{m\otimes} \otimes I_s\right)^T \left(-\gamma^2 \bar{I}_k + hX_{33,k}\right)\left(\hat{J}_k^{m\otimes} \otimes I_s\right)\\ &+ \sum_{i=1}^{m}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \tilde{Q}_{i,k}^{(5)}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)\\ &- \sum_{i=1}^{m}\left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \tilde{Q}_{i,k}^{(5)}\left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),\end{aligned}$$

$$\begin{aligned}\Sigma_{66} = &-h\left(\hat{J}_k^{m\otimes} \otimes I_s\right)^T Z_{11,k}\left(\hat{J}_k^{m\otimes} \otimes I_s\right)\\ &+ \sum_{i=1}^{m}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \tilde{Q}_{i,k}^{(6)}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)\\ &- \sum_{i=1}^{m}\left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \tilde{Q}_{i,k}^{(6)}\left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),\end{aligned}$$

$$\begin{aligned}\Sigma_{77} = &-h\left(\hat{J}_k^{m\otimes} \otimes I_n\right)^T P_{2,k}\left(\hat{J}_k^{m\otimes} \otimes I_n\right)\\ &+ \sum_{i=1}^{m}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \tilde{Q}_{i,k}^{(7)}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)\\ &- \sum_{i=1}^{m}\left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \tilde{Q}_{i,k}^{(7)}\left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),\end{aligned}$$

$$\begin{aligned}\Lambda_{11} = &\,Q_{11,k} + \sum_{i=1}^{m}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \bar{Q}_{i,k}^{(1)}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)\\ &- \sum_{i=1}^{m}\left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right)^T \bar{Q}_{i,k}^{(1)}\left(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes I_{(k+1)^{i-1}n}\right),\end{aligned}$$

$$\Lambda_{22} = Q_{22,k} + \sum_{i=1}^{m}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)^T \bar{Q}_{i,k}^{(2)}\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)$$

$$-\sum_{i=1}^{m}\left(\hat{J}_k^{(m-i)\otimes}\otimes\tilde{J}_k\otimes I_{(k+1)^{i-1}n}\right)^T\bar{Q}_{i,k}^{(2)}\left(\hat{J}_k^{(m-i)\otimes}\otimes\tilde{J}_k\otimes I_{(k+1)^{i-1}n}\right).$$

Then the state-space parameter-dependent matrices for the delay-dependent robust H_∞ filter of the type (3) which achieve the asymptotic stability and H_∞ performance, simultaneously, in the sense of Definition 2.1 are given by

$$\begin{bmatrix}F_\rho & G_\rho & F_{d\rho}\end{bmatrix} = P_{2\rho}^{-1}\begin{bmatrix}W_{1\rho} & W_{2\rho} & W_{3\rho}\end{bmatrix}. \tag{30}$$

Notice that the conditions (27)–(29) are sufficient conditions to both asymptotic stability and H_∞ performance in the sense of Definition 2.1. Moreover, Theorem 4.1 gives a sub-optimal solution to the delay-dependent robust H_∞ filtering and this result can be reformulated as an optimal H_∞ filter by solving the following convex optimization problem

$$\min_{\text{subject to (27), (28), and (29) with } \lambda := \gamma^2} \lambda.$$

Remark 4.2 It is observed that the parameter-independent LMIs (27)–(29) are linear in the set of matrices which are calculated independently from the vector ρ.

Remark 4.3 A new set of matrices verifying $\Omega_{m,k+1} \geq 0$, $\Pi_{m,k+1} < 0$ and $\Phi_{m,k+1} > 0$ can be generated, with index $k+1$ instead of k in (27)–(29), respectively. In this case, the solvability of $\Omega_{m,k} \geq 0$, $\Pi_{m,k} < 0$ and $\Phi_{m,k} > 0$ implies the same property for the larger values of the index k.

5 Example

Consider the following state-space matrices for the LPV state-delayed system (case $m = 1$ and $r = 1$)

$$A_0 = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix},\quad A_1 = \begin{bmatrix}0 & 0.2\\ 0 & 0.1\end{bmatrix},\quad A_{0d} = \begin{bmatrix}0 & 0.1\\ -0.2 & -0.3\end{bmatrix},\quad A_{1d} = \begin{bmatrix}0.2 & 0\\ 0.1 & 0\end{bmatrix},$$

$$E_{01} = \begin{bmatrix}-0.2\\ -0.2\end{bmatrix},\quad C_0 = \begin{bmatrix}0 & 1\\ 0.5 & 0\end{bmatrix},\quad E_{02} = \begin{bmatrix}0\\ 1\end{bmatrix},\quad L_0 = I_2.$$

Assume that the compact set of the parameter ρ is $\zeta = [-1, 1]$. By considering $k = 2$ and the performance bound $\gamma = 0.9$, the delay-dependent robust H_∞ filter synthesis is solved using the Lmitool toolbox of the Matlab software [14]. By considering the parameter $\rho = 0.4$, the result of simulations for constant delay parameter $h = 0.1$ sec. and a unit step disturbance are shown in Figures 1 and 2. These figures show the plant and filter states trajectory. It is observed that the delay-dependent filter is doing well to estimate the plant states.

6 Conclusion

The delay-dependent robust H_∞ filtering problem for a class of LPV systems with constant delay in the states and outputs has been studied in this paper. By using the Leibniz-Newton formula and a suitable change of variables, some new parameter-dependent delay-dependent stability conditions are established in terms of LMIs such that the filtering process remains asymptotically stable and satisfies a prescribed H_∞

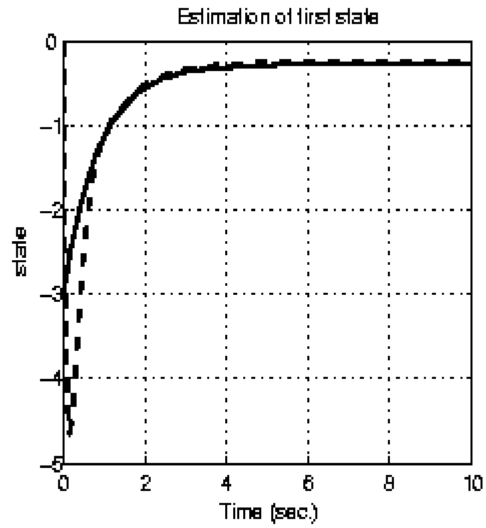

Figure 5.1: Estimation results of the first state (for $h = 0.1$ sec.): real state (solid line), and result of estimation with delay-dependent robust H_∞ filter (dashed line).

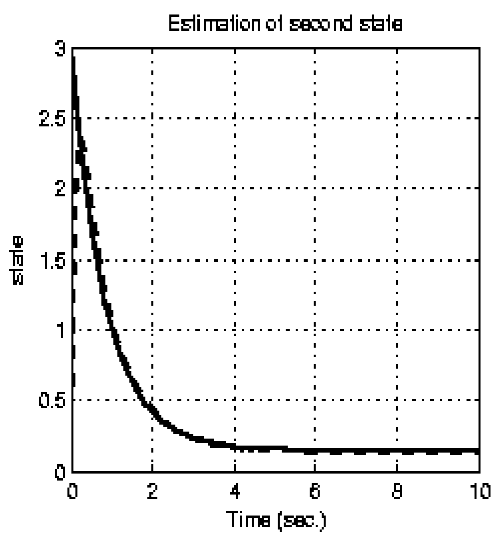

Figure 5.2: Estimation results of the second state (for $h = 0.1$ sec.): real state (solid line), and result of estimation with delay-dependent robust H_∞ filter (dashed line).

performance level. Moreover, using the PPDQ functions and some multiplier matrices, the parameter-independent delay-dependent conditions are developed with high precision under which the desired robust H_∞ filters exist and the explicit expression of these filters is derived. A numerical example has been provided to demonstrate the usefulness of the theory developed.

References

[1] Anderson, B.D.O. and Moore, J.B. *Optimal Filtering*. Englewood Cliffs, NJ, Prentice-Hall, 1979.

[2] Apkarian, P. and Adams, R.J. Advanced gain-scheduling techniques for uncertain systems. *IEEE Trans. Control Systems Technology* **6** (1998) 21–32.

[3] Apkarian, P. and Tuan, H.D. Parameterized LMIs in control theory. *SIAM J. Control Optim.* **38**(4) (2000) 1241–1264.

[4] Bara, G.I., Daafouz, J. and Kratz, F. Parameter-dependent control with γ-performance for affine LPV systems. *Proc. of the 40th CDC*. Florida USA, 2001, 2378–2379.

[5] Becker, G. and Packard, A. Robust performance of linear parametrically-varying systems using parametrically-dependent linear feedback. *Syst. Control Lett.* **23**(3) (1994) 205–512.

[6] Bliman, P.A. Stabilization of LPV systems. *Proc. 42nd IEEE CDC*. 2003, 6103–6108.

[7] Bliman, P.A. A convex approach to robust stability for linear systems with uncertain scalar parameters. *SIAM J. Control Optim* **42**(6) (2004) 2016–2042.

[8] Bliman, P.A. An existence result for polynomial solutions of parameter-dependent LMIs. *Systems & Control Letters* **51** (2004) 165–169.

[9] Chesi, G., Garulli, A., Tesi, A. and Vicino, A. Polynomially parameter-dependent Lyapunov functions for robust stability of polytopic systems: An LMI approach. *IEEE Trans. Automatic Control* **59**(3) (2005) 365–370.

[10] Feron, E., Apkarian, P. and Gahinet, P. Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions. *IEEE Trans. Automatic Control* **41**(7) (1996) 1041–1046.

[11] Fridman, E. and Shaked, U. A new H_∞ filter design for linear time delay systems. *IEEE Trans. Signal Processing* **49**(11) (2001) 2839–2843.

[12] Fridman, E. and Shaked, U. Delay-dependent stability and H_∞ control: Constant and time-varying delays. *Int. J. Control* **76** (2003) 48–60.

[13] Gahinet, P., Apkarian, P. and Chilali, M. Affine parameter-dependent Lyapunov functions and real parametric uncertainty. *IEEE Trans. Automatic Control* **41**(3) (1996) 436–442.

[14] Gahinet, P., Nemirovsky, A., Laub, A.J. and Chilali, M. *LMI control Toolbox: For use with Matlab*. Natik, MA, The MATH Works, Inc, 1995.

[15] Gao, H., Lam, J. and Wang, C. Robust H_∞ filtering for discrete stochastic time-delay systems with nonlinear disturbances. *Nonlinear Dynamics and Systems Theory: An International Journal of Research and Surveys* **4**(3) (2004) 285–301.

[16] Hassibi, B., Sayed, A.H. and Kailath, T. *Indefinite-quadratic estimation and control: A unified approach to H_2 and H_∞ theories*. SIAM Studies in Applied Mathematics, New York, 1999.

[17] Karimi, H.R. A successive approximation algorithm to optimal feedback control of time-varying LPV state-delayed systems. *Nonlinear Dynamics and Systems Theory* **7**(3) (2007) 289–301.

[18] Karimi, H.R. Robust dynamic parameter-dependent output feedback control of uncertain parameter-dependent state-delayed systems. *Nonlinear Dynamics and Systems Theory* **6**(2) (2006) 143–158.

[19] Karimi, H.R. Robust stabilization with H_∞ performance for a class of linear parameter-dependent systems. *Mathematical Problems in Engineering* **2006** (2006) Article ID 59867, 15 pages.

[20] Karimi, H.R., Jabedar Maralani, P., Lohmann, B. and Moshiri, B. H_∞ control of linear parameter-dependent state-delayed systems using polynomial parameter-dependent quadratic functions. *Int. J. Control* **78**(4) (2005) 254–263.

[21] Karimi, H.R., Moshiri, B., Jabedar Maralani, P. and Lohmann, B. Adaptive H_∞-control design for a class of LPV systems. *Proc. 44th IEEE CDC and ECC*, 2005, 7918–7923.

[22] Khargonekar, P.P., Rotea, M.A. and Bayens, E. Mixed H_2/H_∞ filtering. *Int. J. Robust Nonlinear Control* **6**(4) (1996) 313–330.

[23] Lu, B. and Wu, F. Control design of switched LPV systems using multiple parameter-dependent Lyapunov functions. *Proc. ACC* **4** (2004) 3875–3880.

[24] Mahmoud, M.S. and Boujarwah, A.S. Robust H_∞ filtering for a class of LPV systems. *IEEE Trans. Circuits and Systems–I: Fundamental Theory and Applications* **48** (2001) 1131–1138.

[25] Malek-Zavarei, M. and Jamshidi, M. *Time-delay systems: Analysis, optimisation and application*. Amsterdam, The Netherlands, North-Holland, 1987.

[26] Moon, Y.S., Park, P., Kwon, W.H. and Lee, Y.S. Delay-dependent robust stabilization of uncertain state-delayed systems. *Int. J. Control* **74** (2001) 1447–1455.

[27] Pila, A.W., Shaked, U. and de Souza, C.E. H_∞ filtering for continuous linear system with delay. *IEEE Trans. Automatic Control* **44** (1999) 1412–1417.

[28] Park, P. A delay-dependent stability criterion for systems with uncertain time-invariant delays. *IEEE Trans. Automatic Control* **44** (1999) 876–877.

[29] Safonov, M.G., Goh, K.C. and Ly, J.H. Control system synthesis via bilinear matrix inequalities. *Proc. ACC* **1** (1994) 45–49.

[30] Scherer, C.W. Mixed H_2/H_∞ control for time-varying and linear parametrically varying systems. *Int. J. Robust and Nonlinear Control* **6** (1996) 929–952.

[31] Shamma, J.S. and Athans, M. Guaranteed properties of gain-scheduled control of linear parameter-varying plants. *Automatica* **27** (1991) 559–564.

[32] Tan, K., Grigoriadis, K.M. and Wu, F. H_∞ and L_2-to-L_∞ gain control of linear parameter-varying systems with parameter-varying delays. *IEE Proc. Control Theory Appl.* **150** (2003) 509–517.

[33] Wang, J., Wang, C. and Gao, H. Robust H_∞ filtering for LPV discrete-time state-delayed systems. *J. of Nature and Science* **2**(2) (2004) 36–45.

[34] Wang, J., Wang, C. and Yuan, W. A novel H_∞ output feedback controller design for LPV systems with a state-delay. *J. of Nature and Science* **2**(1) (2004) 53–60.

[35] Wang Z. and Yang, F. Robust filtering for uncertain linear systems with delayed states and outputs. *IEEE Trans. Circuits and systems–I: Fundamental Theory and Applications* **49**(1) (2002) 125–130.

[36] Wu, F. and Grigoriadis, K.M. LPV systems with parameter-varying time delays: analysis and control. *Automatica* **37** (2001) 221–229.

[37] Wu, M., He, Y., She, J.H. and Liu, G.P. Delay-dependent criteria for robust stability of time-varying delay systems. *Automatica* **40** (2004) 1435–1439.

[38] Zhang, X.P., Tsiotras, P. and Knospe, C. Stability analysis of LPV time-delayed systems. *Int. J. Control* **75** (2002) 538–558.

Nonlinear Dynamics and Systems Theory, **7** (4) (2007) 369–378

Robust Stability for Nonlinear Uncertain Neural Networks with Delay

X.Y. Lou * and B.T. Cui

Research Center of Control Science and Engineering, Jiangnan University 1800 Lihu Rd.,Wuxi, Jiangsu 214122, P.R.China

Received: August 17, 2006; Revised: August 27, 2007

Abstract: The robust stability of nonlinear uncertain neural networks with constant or time-varying delays is studied. An approach combining the Lyapunov-Krasovskii functional with the linear matrix inequality is taken to study the problems. Some criteria for robust stability of neural networks with time delays are derived.

Keywords: *Nonlinear uncertain neural network; delay; robust stability; linear matrix inequality; Lyapunov-Krasovskii functional*

Mathematics Subject Classification (2000): 93C10, 92B20

1 Introduction

In applications of neural networks with or without delays to some practical problems, such as optimization solvers [1], pattern recognition, image compression [2], and quadratic programming problems [3, 4], the stability properties of system play an important role. The stability analysis for the neural network has received considerable attention in recent years. It is well known that the stability of neural network is prerequisite for the applications either as pattern recognition or as optimization solvers. There have been extensive results presented on the stability analysis of neural network and its applications. Moreover, parameter fluctuation in neural network implementation on very large scale integration (VLSI) chips is also unavoidable. This fact implies that a good neural network should have certain robustness which paves the way for introducing the theory of interval matrices and interval dynamics to investigate the global stability of interval neural networks. There exist several related results on robust stability, we refer to [5–9]. In recent years, the dynamics of neural network systems have been deeply investigated

* Corresponding author: louxuyang28945@163.com

and many important results on the global asymptotic stability and global exponential stability have been established(see for example [10–13, 19]).

In this paper, we shall investigate the problem of the robust stability analysis for nonlinear uncertain neural networks with constant or time-varying delays. Some sufficient conditions for the delay independent robust stability of the neural networks are developed. All of the results are presented in terms of linear matrix inequalities (LMIs).

The paper is organized as follows. In Section 2, the problem to be investigated is stated and some definitions and lemmas are listed. Based on the Lyapunov-Krasovskii stability theory and the LMI approach, some criteria are obtained in Section 3. Two cases of time delay, i.e. constant delays case and time-varying delays case are discussed. Then, an exponential stability criterion for the considered neural networks is provided. A numerical example is given in Section 4. Finally, some conclusions are drawn in Section 5.

2 System Description and Preliminaries

Consider a nonlinear uncertain neural network with time delay, which is described by a set of functional differential equations as follows

$$\begin{aligned}\dot{x}_i(t) = -a_i x_i(t) - a_{di} x_i(t-\tau) + \sum_{j=1}^{n} b_{ij} f_j[x_j(t)] \\ + \sum_{j=1}^{n} b_{dij} f_j[x_j(t-\tau)] + c_i g_i(x_i(t), x_i(t-\tau)), \quad i = 1, 2, \ldots, n,\end{aligned} \tag{1}$$

or, the considered neural networks can be represented in vector state space form as follows

$$\dot{x}(t) = -Ax(t) - A_1 x(t-\tau) + Bf[x(t)] + B_1 f[x(t-\tau)] + Cg(x(t), x(t-\tau)), \tag{2}$$

with initial values

$$x(t) = \phi_x(t), \quad t \in [-\tau, 0), \tag{3}$$

where $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$, $A_1 = \mathrm{diag}(a_{d1}, a_{d2}, \ldots, a_{dn})$, $B = (b_{ij})_{n\times n}$, $B_1 = (b_{dij})_{n\times n}$, $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$; $x(t) = (x_1(t), \ldots, x_n(t))^T$ is the state vector of the neural network; $x(t-\tau) = (x_1(t-\tau), \ldots, x_n(t-\tau))^T$ is the delayed state vector of the neural networks; $\tau > 0$ denotes the delay; $g(x(t), x(t-\tau))$ is the uncertain perturbation with the form of $g(x(t), x(t-\tau)) = [g_1(x_1(t), x_1(t-\tau)), \ldots, g_n(x_n(t), x_n(t-\tau))]^T$; the activation function is $f[x(t)] = \{f_1[x_1(t)], \ldots, f_n[x_n(t)]\}^T$.

Though out this paper, we assume that the activation function $f_i[x_i(t)]$ $(i = 1, 2, \ldots, n)$ and the nonlinear uncertain perturbation function $g(x(t), x(t-\tau))$ satisfy the following conditions:

(H_1) If there exist positive constants k_i, $i = 1, \ldots, n$, such that

$$0 < \frac{f_i(\xi_1) - f_i(\xi_2)}{\xi_1 - \xi_2} \le k_i$$

for all $\xi_1, \xi_2 \in R$, $\xi_1 \ne \xi_2$, $i = 1, \ldots, n$.

(H_2) There exist positive constant matrices M and M_1, such that

$$\|g(x(t), x(t-\tau))\| \le \|Mx(t)\| + \|M_1 x(t-\tau)\|.$$

In order to obtain our results, we need the following definitions and lemmas.

Definition 2.1 For any continuous function $V : R \to R$, Dini's time-derivative of $V(t)$ is defined as

$$D^+V(t) = \lim_{h\to 0^+} \sup \frac{V(t+h) - V(t)}{h}\,. \tag{4}$$

It is easy to see that if $V(t)$ is locally Lipschitz, then $|D^+V(t)| < \infty$.

Lemma 2.1 [14] (Lyapunov-Krasovskii stability theorem) *Consider the following functional-differential equation of the retarded type:*

$$\dot{x}(t) = f(t, x_t), \quad t \geq t_0, \quad x_{t_0} = \phi(\theta), \quad \forall\, \theta \in [-\tau, 0], \tag{5}$$

where $x_t(\cdot)$, for given $t \geq t_0$, denotes the restriction of $x(\cdot)$ to the interval $[t-\tau, t]$ translated to $[-\tau, 0]$, namely $x_t(\theta) = x(t+\theta)$, $\forall\, \theta \in [-\tau, 0]$.

Assume that there exists a continuous functional $V(t, \phi)$ such that

(i) $V_1(\|\phi(0)\|) \leq V(t,\phi) \leq V_2(\|\phi(\theta)\|)$;

(ii) $\dot{V}(t, x_t) \leq -V_3(\|x(t)\|)$,

where $V_1, V_2, V_3 :\ R_+ \to R_+$ are continuous nondecreasing functions, $V_1(s)$, $V_2(s)$ are positive for $s > 0$, and $V_1(0) - V_2(0) = 0$, $V_3(s) > 0$ for $s > 0$, and $\dot{V}(t, x_t)$ is Dini's time-derivative of $V(t, x_t)$ along the solution of equation (5). Then, the trivial solution of equation (5) is uniformly asymptotically stable.

Notice that condition (i) means that the function $V(t, \phi)$ is positive definite and has an infinitesimal upper limit.

Lemma 2.2 [15] *Given any real matrices A, B, C of appropriate dimensions and a scalar $\varepsilon > 0$ such that $0 < C = C^T$. Then, the following inequality holds:*

$$A^TB + B^TA \leq \varepsilon A^TCA + \varepsilon^{-1}B^TC^{-1}B, \tag{6}$$

where the superscript T means the transpose of a matrix.

Lemma 2.3 [15] (Schur complement) *Linear matrix inequality:*

$$\begin{pmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{pmatrix} > 0, \tag{7}$$

with $Q(x) = Q^T(x)$, $R(x) = R^T(x)$ is the same as

$$R(x) > 0, \quad Q(x) - S(x)R^{-1}(x)S^T(x) > 0.$$

3 Main Results

In this section, stability criteria for uncertain neural networks with time delay are given.

Theorem 3.1 *Consider the delayed neural networks with nonlinear perturbation (1), if there exist positive matrices $X > 0$, $W > 0$, positive diagonal matrices $S > 0$, $S_1 > 0$, and constants $\xi_1 > 0$, $\xi_2 > 0$, satisfying the LMI*

$$\Omega = \begin{bmatrix} \Omega_{11} & XK^T & XA_1^T & XM_1^T & XM^T \\ KX & -S - S_1 & 0 & 0 & 0 \\ A_1X & 0 & -W & 0 & 0 \\ M_1X & 0 & 0 & -\xi_2 I & 0 \\ MX & 0 & 0 & 0 & -\xi_1 I \end{bmatrix} < 0, \tag{8}$$

where

$$\Omega_{11} = -XA^T - AX + W + B_1S_1B_1^T + C(\xi_1+\xi_2)C^T,$$

then the system (1) is globally asymptotically stable. Here, $K = \mathrm{diag}\{k_1, \ldots, k_n\}$.

Proof Consider the Lyapunov functional

$$\begin{aligned} V(t) = x^T(t)Px(t) + \int_{t-\tau}^{t} f^T[x(s)]S_1^{-1}f[x(s)]\,ds \\ + \int_{t-\tau}^{t} x^T(s)A_1^TW^{-1}A_1x(s)\,ds + \xi_2^{-1}\int_{t-\tau}^{t} x^T(s)M_1^TM_1x(s)\,ds. \end{aligned} \tag{9}$$

It is easy to obtain

$$\begin{aligned} \lambda_m(P)\|x(t)\|^2 \le V(t) \le \Big\{\lambda_M(P) + \tau\Big[\lambda_M(K^TS_1^{-1}K) \\ + \lambda_M(A_1^TW^{-1}A_1) + \xi_2^{-1}\lambda_M(M_1^TM_1)\Big]\Big\}\|x(t)\|^2, \end{aligned}$$

where $\lambda_m(P)$ and $\lambda_M(P)$ denote the minimum and maximum eigenvalues of P, respectively.

Calculating the upper right derivative D^+V of (9) along the solution of (2), we have that

$$\begin{aligned} D^+V(t) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\ &\quad + f^T[x(t)]S_1^{-1}f[x(t)] - f^T[x(t-\tau)]S_1^{-1}f[x(t-\tau)] \\ &\quad + x^T(t)A_1^TW^{-1}A_1x(t) - x^T(t-\tau)A_1^TW^{-1}A_1x(t-\tau) \\ &\quad + \xi_2^{-1}x^T(t)M_1^TM_1x(t) - \xi_2^{-1}x^T(t-\tau)M_1^TM_1x(t-\tau) \\ &= \{-Ax(t) - A_1x(t-\tau) + Bf[x(t)] \\ &\quad + B_1f[x(t-\tau)] + Cg(x(t), x(t-\tau))\}^T Px(t) \\ &\quad + x^T(t)P\{-Ax(t) - A_1x(t-\tau) + Bf[x(t)] \\ &\quad + B_1f[x(t-\tau)] + Cg(x(t), x(t-\tau))\} \\ &\quad + f^T[x(t)]S_1^{-1}f[x(t)] - f^T[x(t-\tau)]S_1^{-1}f[x(t-\tau)] \\ &\quad + x^T(t)A_1^TW^{-1}A_1x(t) - x^T(t-\tau)A_1^TW^{-1}A_1x(t-\tau) \\ &\quad + \xi_2^{-1}x^T(t)M_1^TM_1x(t) - \xi_2^{-1}x^T(t-\tau)M_1^TM_1x(t-\tau). \end{aligned} \tag{10}$$

From Lemma 2.2 and (10), we have that

$$\begin{aligned} D^+V(t) \le{} & [-x^T(t)A^TPx(t) - x^T(t)PAx(t)] \\ & + x^T(t-\tau)A_1^TW^{-1}A_1x(t-\tau) + x^T(t)PWPx(t) \\ & + f^T[x(t)]S^{-1}f[x(t)] + x^T(t)PBSB^TPx(t) \\ & + f^T[x(t-\tau)]S_1^{-1}f[x(t-\tau)] + x^T(t)PB_1S_1B_1^TPx(t) \\ & + x^T(t)PC(\xi_1+\xi_2)C^TPx(t) \\ & + \xi_1^{-1}x^T(t)M^TMx(t) + \xi_2^{-1}x^T(t-\tau)M_1^TM_1x(t-\tau) \\ & + f^T[x(t)]S_1^{-1}f[x(t)] - f^T[x(t-\tau)]S_1^{-1}f[x(t-\tau)] \\ & + x^T(t)A_1^TW^{-1}A_1x(t) - x^T(t-\tau)A_1^TW^{-1}A_1x(t-\tau) \end{aligned}$$

$$
\begin{aligned}
&+ \xi_2^{-1} x^T(t) M_1^T M_1 x(t) - \xi_2^{-1} x^T(t-\tau) M_1^T M_1 x(t-\tau) \\
\le\, & [-x^T(t) A^T P x(t) - x^T(t) P A x(t)] + x^T(t) P W P x(t) \\
&+ f^T[x(t)] S^{-1} f[x(t)] + x^T(t) P B S B^T P x(t) + x^T(t) P B_1 S_1 B_1^T P x(t) \\
&+ x^T(t) P C(\xi_1 + \xi_2) C^T P x(t) + \xi_1^{-1} x^T(t) M^T M x(t) \\
&+ f^T[x(t)] S_1^{-1} f[x(t)] + x^T(t) A_1^T W^{-1} A_1 x(t) + \xi_2^{-1} x^T(t) M_1^T M_1 x(t) \\
\le\, & x^T(t) \Xi x(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\Xi = \big[& -A^T P - PA + PWP + K^T S^{-1} K + PBSB^T P \\
& + PB_1 S_1 B_1^T P + PC(\xi_1 + \xi_2) C^T P + \xi_1^{-1} M^T M \\
& + K^T S_1^{-1} K + A_1^T W^{-1} A_1 + \xi_2^{-1} M_1^T M_1 \big] .
\end{aligned}
\tag{11}
$$

Pre- and post-multiply (11) with $X = P^{-1}$. By the Schur complement, $\Xi < 0$ if and only if inequality (8) holds.

This completes the proof. □

Remark 3.1 Noting that the conditions of Theorem 3.1 do not include any information of the delay, that is, the theorem provides a delay-independent robust stability criterion for time-delayed neural networks with nonlinear perturbations in terms of LMIs. The results can be extended to time-varying delay case.

Consider the time-varying delay neural networks as follows

$$
\dot{x}(t) = -Ax(t) - A_1 x(t-\tau(t)) + Bf[x(t)] + B_1 f[x(t-\tau(t))] + Cg(x(t), x(t-\tau(t))), \tag{12}
$$

where τ is a function, $\tau\colon [0,+\infty) \to [0,+\infty]$. Furthermore, we assume that τ is differentiable and $\dot{\tau}(t) \le \tau^* < 1$.

We have the following result.

Theorem 3.2 *Consider the delayed neural networks with nonlinear perturbation (1), if there exist positive matrices $X > 0$, $W > 0$, positive diagonal matrices $S > 0$, $S_1 > 0$ satisfying the LMI*

$$
\Omega = \begin{bmatrix}
\Omega_{11} & XK^T & XA_1^T & XM_1^T & XM^T \\
KX & -S - S_1 & 0 & 0 & 0 \\
A_1 X & 0 & -W & 0 & 0 \\
M_1 X & 0 & 0 & -I & 0 \\
MX & 0 & 0 & 0 & -I
\end{bmatrix} < 0, \tag{13}
$$

where

$$
\Omega_{11} = -XA^T - AX + \frac{1}{1-\tau^*} W + \frac{1}{1-\tau^*} B_1 S_1 B_1^T + C\left(1 + \frac{1}{1-\tau^*}\right) C^T.
$$

Then the system (1) is globally asymptotically stable. Here, $K = \operatorname{diag}\{k_1, \dots, k_n\}$.

Proof Consider the Lyapunov functional

$$
\begin{aligned}
V(t) = x^T(t) P x(t) &+ \int_{t-\tau(t)}^{t} f^T[x(s)] S_1^{-1} f[x(s)]\, ds \\
&+ \int_{t-\tau(t)}^{t} x^T(s) A_1^T W^{-1} A_1 x(s)\, ds + \int_{t-\tau(t)}^{t} x^T(s) M_1^T M_1 x(s)\, ds.
\end{aligned}
\tag{14}
$$

It is easy to obtain

$$\lambda_m(P)\|x(t)\|^2 \le V(t) \le \left\{\lambda_M(P) + \tau\left[\lambda_M(K^T S_1^{-1} K) + \lambda_M(A_1^T W^{-1} A_1) + \lambda_M(M_1^T M_1)\right]\right\}\|x(t)\|^2,$$

where $\lambda_m(P)$ and $\lambda_M(P)$ denote the minimum and maximum eigenvalues of P, respectively.

Calculating the upper right derivative D^+V of (14) along the solution of (12), we obtain that

$$\begin{aligned}
D^+V(t) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\
&\quad + f^T[x(t)]S_1^{-1}f[x(t)] - (1-\tau^*)f^T[x(t-\tau(t))]S_1^{-1}f[x(t-\tau(t))] \\
&\quad + x^T(t)A_1^T W^{-1}A_1x(t) - (1-\tau^*)x^T(t-\tau(t))A_1^T W^{-1}A_1x(t-\tau(t)) \\
&\quad + x^T(t)M_1^T M_1x(t) - (1-\tau^*)x^T(t-\tau(t))M_1^T M_1x(t-\tau(t)) \\
&= \{-Ax(t) - A_1x(t-\tau(t)) + Bf[x(t)] \\
&\quad + B_1f[x(t-\tau(t))] + Cg(x(t), x(t-\tau(t)))\}^T(t)Px(t) \\
&\quad + x^T(t)P\{-Ax(t) - A_1x(t-\tau(t)) + Bf[x(t)] \\
&\quad + B_1f[x(t-\tau(t))] + Cg(x(t), x(t-\tau(t)))\} \\
&\quad + f^T[x(t)]S_1^{-1}f[x(t)] - (1-\tau^*)f^T[x(t-\tau(t))]S_1^{-1}f[x(t-\tau(t))] \\
&\quad + x^T(t)A_1^T W^{-1}A_1x(t) - (1-\tau^*)x^T(t-\tau(t))A_1^T W^{-1}A_1x(t-\tau(t)) \\
&\quad + x^T(t)M_1^T M_1x(t) - (1-\tau^*)x^T(t-\tau(t))M_1^T M_1x(t-\tau(t)).
\end{aligned} \tag{15}$$

From Lemma 2.2 and (15), it follows that

$$\begin{aligned}
D^+V(t) &\le [-x^T(t)A^TPx(t) - x^T(t)PAx(t)] \\
&\quad + (1-\tau^*)x^T(t-\tau(t))A_1^T W^{-1}A_1x(t-\tau(t)) \\
&\quad + f^T[x(t)]S^{-1}f[x(t)] + x^T(t)PBSB^TPx(t) \\
&\quad + (1-\tau^*)f^T[x(t-\tau(t))]S_1^{-1}f[x(t-\tau(t))] \\
&\quad + \frac{1}{1-\tau^*}x^T(t)PB_1S_1B_1^TPx(t) + \frac{1}{1-\tau^*}x^T(t)PWPx(t) \\
&\quad + x^T(t)PC\left(1 + \frac{1}{1-\tau^*}\right)C^TPx(t) \\
&\quad + x^T(t)M^TMx(t) + (1-\tau^*)x^T(t-\tau(t))M_1^T M_1x(t-\tau(t)) \\
&\quad + f^T[x(t)]S_1^{-1}f[x(t)] - (1-\tau^*)f^T[x(t-\tau(t))]S_1^{-1}f[x(t-\tau(t))] \\
&\quad + x^T(t)A_1^T W^{-1}A_1x(t) - (1-\tau^*)x^T(t-\tau(t))A_1^T W^{-1}A_1x(t-\tau(t)) \\
&\quad + x^T(t)M_1^T M_1x(t) - (1-\tau^*)x^T(t-\tau(t))M_1^T M_1x(t-\tau(t)) \\
&\le [-x^T(t)A^TPx(t) - x^T(t)PAx(t)] \\
&\quad + f^T[x(t)]S^{-1}f[x(t)] + x^T(t)PBSB^TPx(t) \\
&\quad + \frac{1}{1-\tau^*}x^T(t)PB_1S_1B_1^TPx(t) + \frac{1}{1-\tau^*}x^T(t)PWPx(t) \\
&\quad + x^T(t)PC\left(1 + \frac{1}{1-\tau^*}\right)C^TPx(t) + x^T(t)M^TMx(t)
\end{aligned}$$

$$
\begin{aligned}
&+ f^T[x(t)]S_1^{-1}f[x(t)] + x^T(t)A_1^T W^{-1}A_1x(t) + x^T(t)M_1^T M_1x(t)\\
&\le x^T(t)\Xi x(t),
\end{aligned}
$$

where

$$
\begin{aligned}
\Xi = \Big[&-A^TP - PA + \frac{1}{1-\tau^*}PWP + K^TS^{-1}K + PBSB^TP\\
&+ \frac{1}{1-\tau^*}PB_1S_1B_1^TP + PC\left(1+\frac{1}{1-\tau^*}\right)C^TP + \xi_1^{-1}M^TM\\
&+ K^TS_1^{-1}K + A_1^TW^{-1}A_1 + \xi_2^{-1}M_1^TM_1\Big].
\end{aligned}
$$

Pre- and post-multiply (16) with $X = P^{-1}$. By the Schur complement, $\Xi < 0$ if and only if inequality (13) holds.

This completes the proof. $\Box$

Remark 3.2 In Theorem 3.1 and Theorem 3.2, the delay-independent stability criteria are developed, however, no information on the state convergence degree of the neural networks is given. Here, we investigate the problem of exponential stability analysis for delayed neural networks.

Theorem 3.3 *Consider the delayed neural networks with nonlinear perturbation (1), if there exist positive matrices $X > 0$, $W > 0$, positive diagonal matrices $S > 0$, $S_1 > 0$, and constants $\xi_1 > 0, \xi_2 > 0, \alpha > 0$ satisfying the LMI*

$$
\Omega = \begin{bmatrix}
\Omega_{11} & e^{\alpha\tau}XK^T & e^{\alpha\tau}XA_1^T & e^{\alpha\tau}XM_1^T & e^{\alpha\tau}XM^T\\
e^{\alpha\tau}KX & -S-S_1 & 0 & 0 & 0\\
e^{\alpha\tau}A_1X & 0 & -W & 0 & 0\\
e^{\alpha\tau}M_1X & 0 & 0 & -\xi_2 I & 0\\
e^{\alpha\tau}MX & 0 & 0 & 0 & -\xi_1 I
\end{bmatrix} < 0, \tag{17}
$$

where

$$
\Omega_{11} = -XA^T - AX + W + B_1S_1B_1^T + 2\alpha X + C(\xi_1+\xi_2)C^T. \tag{18}
$$

Then the system (1) is exponential asymptotically stable. Here, $K = \mathrm{diag}\{k_1,\dots,k_n\}$.

Proof Let's introduce a transformation $x(t) = e^{-\alpha t}\eta(t)$, and define the Lyapunov functional as follows:

$$
\begin{aligned}
V(t) = \eta^T(t)P\eta(t) &+ \int_{t-\tau}^{t} f^T[\eta(s)]S_1^{-1}f[\eta(s)]\,ds\\
&+ \int_{t-\tau}^{t}\eta^T(s)A_1^TW^{-1}A_1\eta(s)\,ds + \xi_2^{-1}\int_{t-\tau}^{t}\eta^T(s)M_1^TM_1\eta(s)\,ds.
\end{aligned} \tag{19}
$$

Then follows the proof of Theorem 3.1, this theorem can be proved easily. $\Box$

Remark 3.3 As we can see, if $B \equiv 0$ and the uncertain perturbation $g(x(t), x(t-\tau)) \equiv 1$ in (2), then the neural network (1) or (2) represents the Hopfield's original neural network model and cellular neural networks [6, 7, 9, 10, 16, 17, 18].

4 An illustrative example

In this section, we present a numerical example to validate our results.

Example 4.1. We consider two-dimension nonlinear uncertain neural network (2) with time delay. The associated data are:

$$A = \begin{bmatrix} 1.8 & 0 \\ 0 & 1.8 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.01 & 0.02 \\ 0.03 & 1.08 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.32 & 0.45 \\ 0.30 & 0.50 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Suppose the activation function is described by $f_i(x) = \frac{1}{2}(|x+1| + |x-1|)$, $i = 1, 2$. Then we have $K = \mathrm{diag}(1, 1)$, and $f_i(x)$ satisfies (H_1). Now using the MATLAB LMI toolbox, we can obtain a feasible solution for LMI (8) as follows:

$$X = \begin{bmatrix} 56.5666 & 19.4526 \\ 19.4526 & 7.6137 \end{bmatrix} > 0, \quad W = \begin{bmatrix} 235.3002 & 178.2599 \\ 178.2599 & 146.4787 \end{bmatrix} > 0,$$
$$S = \begin{bmatrix} 983.9634 & 0 \\ 0 & 983.9634 \end{bmatrix} > 0, \quad S_1 = \begin{bmatrix} 802.8313 & 0 \\ 0 & 802.8313 \end{bmatrix} > 0, \quad \xi = 13.8307.$$

Then the conditions of Theorem 3.1 are satisfied. Therefore, the system (1) is globally asymptotically stable. Moreover, we can see from the behavior (see Figure 1) of the state variables, the solutions of system (1) converge upon the zero with the initial condition $\phi(s) = [0.1, -0.1]^T$.

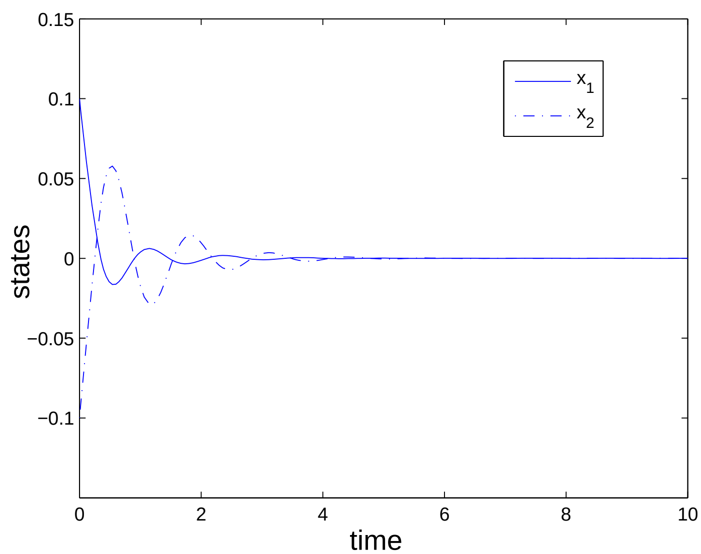

Figure 4.1: The time respond behavior of the system (1).

5 Conclusions

In this paper, the problem of robust stability analysis for uncertain neural networks with time delay is investigated. Based on Lyapunov stability theory, the robust stable criteria are given in terms of linear matrix inequalities. The proposed approach is more flexible in computation, and the results are more efficient then other existing results.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (No. 60574051), the Key Research Foundation of Science and Technology of the Ministry of Education of China (No. 107058), the National Natural Science Foundation of Jiangsu Province (No. BK2007016) and Program for Innovative Research Team of Jiangnan University.

References

[1] Bouzerman, A. and Pattison, T. Neural network for quadratic optimization with bound constraints. *IEEE Trans. Neural Networks* **4** (1993) 293–303.

[2] Venetianer, P. and Roska, T. Image compression by delayed CNNs. *IEEE Trans. Circ. Syst. I* **45** (1998) 205–215.

[3] Sudharsanan, S. and Sundareshan, M. Exponential stability and a systematic synthesis of a neural network for quadratic minimization. *Neural Networks* **4** (1991) 599–613.

[4] Chen, Y. and Fang, S. Neurocomputing with time delay analysis for solving convex quadratic programming problems. *IEEE Trans. Neural Networks* **11** (2000) 230–234.

[5] Ye, H. and Michel, A. N. Robust stability of nonlinear time-delay systems with applications to neural networks. *IEEE Trans. Circ. Syst. I* **43** (1996) 532–543.

[6] Liao, X. and Yu, J. Robust stability for interval Hopfield neural networks with time delay. *IEEE Trans. Neural Networks* **9** (1998) 1042–1045.

[7] Liao, X.F., Wang, K.W., Wu, Z.F. and Chen, G.R. Novel robust stability criteria for interval-delayed Hopfield neural networks. *IEEE Trans. Circ. Syst. I* **48** (2001) 1355–1359.

[8] Lou, X. Y. and Cui, B. T. On the global robust asymptotic stability of BAM neural networks with time-varying delays. *Neurocomputing* **70**(1–3) (2006) 273–279.

[9] Arik, S. Global robust stability of delayed neural networks. *IEEE Trans. Circ. Syst. I* **50**(1) (2003) 156–160.

[10] Fang, Y. and Kincaid, T. Stability analysis of dynamical neural networks. *IEEE Trans. Neural Networks* **7**(3) (1996) 996–1007

[11] Lou, X. Y. and Cui, B. T. Global asymptotic stability of delay BAM neural networks with impulses. *Chaos, Solitons and Fractals* **29**(4) (2006) 1023–1031.

[12] Lou, X. Y. and Cui, B. T. Absolute exponential stability analysis of delayed bi-directional associative memory neural networks. *Chaos, Solitons and Fractals* **31**(3) (2007) 695–701.

[13] Botelho, F. Dynamical features simulated by recurrent neural networks. *Neural Networks* **12** (1999) 609–615.

[14] Hale, J. K. and Lunel, S. M. V. *Introduction to the theory of functional differential equations.* Applied Mathematical Sciences, vol. 99. New York, Springer, 1991.

[15] Li, C. D., Liao, X. F. and Zhang, R. Delay-dependent exponential stability analysis of bi-directional associative memory neural networks with time delay: an LMI approach. *Chaos, Solitons and Fractals* **24** (2005) 1119–1134.

[16] Cao, J. D. On exponential stability and periodic solutions of CNNs with delays. *Physics Letters A* **267**(5–6) (2000) 312–318.

[17] Cao, J. D. Global stability conditions for delayed CNNs. *IEEE Trans. Circ. Syst. I* **48**(11) (2001) 1330–1333.

[18] Arik, S. An analysis of global asymptotic stability of delayed cellular neural networks. *IEEE Trans. Neural Networks* **13**(5) (2002) 1239–1242.

[19] Cao, J. D. and Song, Q. K. Stability in Cohen-Grossberg-type bidirectional associative memory neural networks with time-varying delays. *Nonlinearity* **19**(7) (2006) 1601–1617.

Nonlinear Dynamics and Systems Theory, **7** (4) (2007) 379–398

Compact Explicit MPC Law with Guarantees of Feasibility for Reference Tracking

Sorin Olaru * and Didier Dumur

Automatic Control Department, SUPELEC, Gif-sur-Yvette, 91192, France

Received: July 10, 2006; Revised: July 31, 2007

Abstract: The present paper deals with the constrained model predictive control for linear time invariant systems. Even if these techniques reached a considerable maturity in the last decade, the feasibility problems remain a sensitive point at least for applications which involve tracking of challenging reference signals, most often in conjunction with restrictive physical limitations. The main goal here is the adaptation (enlargement) of the set of feasible trajectories. Two strategies are discussed: the tuning of the predictive control parameters and the reference governor schemes. Specifically on the former direction it will be shown that a compact piecewise affine "feedback control law" with guarantees of feasibility can be constructed. This compactness is given by mixing the explicit formulations of the predictive law and those of the reference adjustment mechanism.

Keywords: *Optimal control; predictive control; multiparametric optimization.*

Mathematics Subject Classification (2000): 34H05, 49N05, 58E25, 93C83.

1 Introduction

Model Predictive Control (MPC) has imposed itself as a flexible optimization based technique with versatile constraints handling capabilities due to its time-domain formulation (see the up-to-date monographies [15], [6], [19], [9]). In the same time, the optimization fundament imposes the feasibility as a crucial demand as long as it represents the main ingredient for the stability of the entire closed loop [16].

For the regulation problem, the necessary and sufficient conditions of MPC feasibility are based on pseudo-infinite prediction horizons or, similarly, on terminal constraints [11] designed in concordance with *positive invariant sets* principles [4]. The reference

* Corresponding author: sorin.olaru@supelec.fr

tracking problem [5] is somehow more laborious but it can take advantage of the prediction capabilities of MPC and thus deliver excellent control performances. However, the infeasibility threatening becomes severe in this case [22], mainly when the reference contradicts with the imposed constraints or when the a priori information regarding the set-point is limited. The modification of the trajectory to be followed may induce the reduction of the feasible domain and even lead to infeasibility.

In order to deal with this phenomenon several strategies can be followed and a possible taxonomy can differentiate between: infeasibility avoidance and feasibility recovery. The first category contains the predictive laws which deal with the feasibility in terms of a robustness issue. This elegant approach implies that the exogenous reference signal can be described either statistically, either by deterministic models and by consequence included in the control law synthesis. The main criticisms are linked to the numerical difficulties in solving the new optimizations (for example the computational aspects involving min-max problems) and often to the conservative performances due to the worst case combination of set-points which have to be considered.

The feasibility recovery strategies follow a different philosophy. The idea is to design a control law which optimizes the performances in relation with an usual reference signal and to treat the eventual infeasible situation either through the enhancement of the feasible domain either by increasing the prediction horizon, either by the on-line adjustment of the trajectory to be followed (using a so-called "reference governor"). In the first case, the feasible domain is enlarged but remains limited, another disadvantage being the relative augmentation of the set of decision variables and related constraints. The "reference governor" method (see [3, 7] and the citations therein) replaces the set-points with the best admissible reference found as the solution of an optimization problem apart MPC.

The current paper revisits the main concepts related to the feasibility of MPC and extends their definitions to the reference tracking case. The structure of the feasible domain and the limitations on the reference signal are further considered together with their links to the MPC parameters. As infeasibility avoidance technique, this paper focuses on the reference governor schemes, the main goal being to obtain a control law with guarantees of feasibility. Due to the fact that both ingredients (MPC and reference governors) are represented by multiparametric optimization problems (mpOP) [2, 12], their explicit formulation is obtained and the result gathered in a compact form. The main contribution will be the construction of this piecewise affine control law, feasible over any initial set. The conservativeness and the compromise between the memory needs and the performance of the evaluation mechanism are some of the addressed issues.

In the following, Section 2 reminds the MPC problem, the explicit formulation and states some definitions related with the feasibility. Section 3 deals with the reference tracking problems, analyzing the feasibility limitations. In Section 4, the infeasibility avoidance mechanism is integrated in the predictive control scheme resulting in the compact MPC with guarantees of feasibility. Finally, Section 5 presents some study cases and section VI the conclusions.

2 Model Predictive Control

2.1 Constrained model predictive control

MPC implies the idea of minimizing a cost index based on the predicted plant evolution. For the regulation to origin, consider the discrete time LTI system in a state-space description:

$$\Sigma_P : \begin{cases} x_{t+1} = Ax_t + Bu_t, t \geq 0, x_0 = x^0 \in X_0, \\ y_t = Hx_t, \\ Cx_t + Du_t \leq \gamma, \end{cases} \tag{1}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $y_t \in \mathbb{R}^r$ are the state, input and output vectors and X_0 is the set of initial conditions. It is assumed throughout that the pair (A, B) is stabilizable. The inequality constraints, given by $\gamma \in \mathbb{R}^q$, $D \in \mathbb{R}^{q\times m}$, $C \in \mathbb{R}^{q\times n}$, describe a polyhedral region including the origin. At each sampling time, the current state, $x = x_t$ (assumed available), is used to find the optimal open-loop control sequence $\mathbf{k}_u^* = [u_{t|t}^T, \ldots, u_{t+N-1|t}^T]^T \in \mathbb{R}^{N\times m}$:

$$\begin{aligned} \mathbf{k}_u^* = \arg\min_{\mathbf{k}_u} x_{t+N|t}^T P x_{t+N|t} + \sum_{k=0}^{N-1} \left\{ x_{t+k|t}^T Q x_{t+k|t} + u_{t+k|t}^T R u_{t+k|t} \right\}, \\ \begin{cases} x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t}, \; k \geq 0, \\ Cx_{t+k|t} + Du_{t+k|t} \leq \gamma, \; 0 \leq k \leq N-1, \\ x_N \in X_N, \end{cases} \end{aligned} \tag{2}$$

where $Q = Q^T \geq 0$ and $R = R^T > 0$ are weighting matrices and the pair $(Q^{1/2}, A)$ is detectable. P is characterizing the terminal cost while X_N is the associated terminal set. The prediction horizon - N, together with the matrices P, Q and R are the knobs of this construction based on optimization.

The first part of the resulting optimal open-loop control sequence (2) - $u_{t|t}^*$, is effectively applied and the whole procedure is restarted following the "receding horizon principle" which provides the MPC law with all the advantages of a closed-loop control law.

2.2 Multiparametric optimization. Explicit solutions

The optimization problem (2) is tractable as it has mN decision variables and qN constraints. It can be rewritten after simple matrix manipulations as:

$$\begin{aligned} &\arg\min_{\mathbf{k}_u} \mathbf{k}_u^T H \mathbf{k}_u + \mathbf{k}_u^T F x + x^T G x, \\ &\text{subject to}: \quad A_{in}\mathbf{k}_u \leq b_{in} + B_{in}x. \end{aligned} \tag{3}$$

This is known in the literature as the multiparametric quadratic problem (mpQP) and its solution is represented by a piecewise linear and continuous function [2, 20, 17]:

$$\mathbf{k}_u^*(x) = K_i * x + \kappa_i, \text{ for } x \in D_i\,, \tag{4}$$

where D_i are convex polyhedral regions in $\mathbb{R}^n$. MPC uses only the first component of this optimal solution:

$$u^{MPC}(x) = K_i^{MPC} * x + \kappa_i^{MPC}, \text{with}\, i \,\text{such that}\, x \in D_i\,, \tag{5}$$

and $K_i^{MPC}, \kappa_i^{MPC}$ the first components of K_i, κ_i.

Lately, efficient algorithms are available [13] to develop these explicit solutions and thus the constrained predictive control policy can dispose of an additional design and analysis tool.

2.3 Feasibility within constrained MPC

Definition 2.1 The feasible set for the MPC law (2), is the set of all states for which a control sequence $\mathbf{k}_u^*(x)$ exists:

$$X_f = \bigcup_i D_i \,, \tag{6}$$

with D_i as in (4). X_f is a convex polyhedron and corresponds to the projection of the parameterized polyhedron [18] formed by the constraints in (3), onto the state space.

Definition 2.2 A set $X \in \mathbb{R}^n$ is positively invariant with respect to the dynamic $x_{t+1} = Ax_t$ if and only if $\forall x_t \in X \Rightarrow x_{t+1} \in X$.

In the literature, one can find exhaustive results regarding the feasibility of MPC laws [11]. In the following a classification of the infeasibility for MPC based on the relation with the set of initial conditions, X_0 is introduced:

- $\mathbf{X_0} \setminus \mathbf{X_f} \neq \emptyset$.

  The MPC law is infeasible w.r.t. X_0, because it exists at least one initial condition $x^0 \in X_0$ such that $u^{MPC}(x^0)$ is not well defined.

- $\mathbf{X_0} = \mathbf{X_f}$

  The MPC law is feasible w.r.t. X_0 if and only if X_f is positively invariant w.r.t. $x_{t+1} = Ax_t + Bu^{MPC}(x_t)$.

- $\mathbf{X_0} \subset \mathbf{X_f}$

  The MPC law is feasible w.r.t. X_0 if and only if it exists a set $\Omega \subset \mathbb{R}^n$, positively invariant w.r.t. $x_{t+1} = Ax_t + Bu^{MPC}(x_t)$, which satisfies $X_0 \subset \Omega \subset X_f$.

This classification does underline the role of the initial conditions and also the threatening represented by the inappropriate tuning of the MPC which may lead to self-generated infeasibility [21]. However, the set X_f depends on the MPC parameters and thus constructive methods do exist for designing feasible laws for the regulation problem (1-2).

Definition 2.3 [8] The maximal admissible set for (1) under a constant feedback law $u = Kx$ is represented by:

$$O_\infty = \left\{ x \in \mathbb{R}^n | (C(A+BK)^k + DK(A+BK)^{k-1})x \leq \gamma, \forall k \geq 0 \right\}. \tag{7}$$

Definition 2.4 A set $X \subset \mathbb{R}^n$ is called control invariant for the system $x_{t+1} = Ax_t + Bg(x_t)$ if there exists a function $g(x_t)$ such that X is positive invariant with respect to this dynamic. The maximal control invariant set - C_∞, is the control invariant set containing all other control invariant sets.

Remark 2.1 X is control invariant only if $X \subset C_\infty$.

Given these definitions, a design sketch for a feasible MPC law could be:

- Compute C_∞.
- If $X_0 \setminus C_\infty \neq \emptyset$ stop ; there is no MPC control law feasible for all $x \in X_0$.

- Choose a stabilizing control law $u = Kx$ and compute the maximal admissible set O_∞. Fix $X_N = O_\infty$.
- Find $N < \infty$ for the problem (2) such that $X_0 \subset X_f$.

Remark 2.2 This discussion was dedicated to the feasibility analysis as it represents the main issue of the current paper. It must be mentioned that the feasibility of the predictive control law is not guaranteeing the stability but is one of its major ingredients (see [16]). If all the signals of the system are directly or indirectly bounded, then a BIBO stability is achieved once the infeasibility is avoided.

3 MPC Feasibility For Tracking Systems

A case which is often studied in the literature [5], [14] is the one of "constant reference tracking". The necessary and sufficient condition in the unconstrained case for such reference signals is resumed by the following assumption.

Assumption 1: The pencil matrix:

$$A_c = \begin{bmatrix} A - I & B \\ H & 0 \end{bmatrix} \tag{8}$$

is invertible.

Given the discrete-time dynamical system (1) with the associated mixed state-inputs constraints it is worth to recall that the set of admissible constant references $\mathcal{Y}_r$ which the system will be able to track will be given by:

$$\mathcal{Y}_r = \left\{ y_r | \, [CH^T(H^TH)^{-1} - D(H(A-I)^{-1}B)^{-1}] y_r \leq \gamma \right\}. \tag{9}$$

Remark 3.1 If the polyhedron described by:

$$PC = \left\{ \begin{bmatrix} x_t \\ u_t \end{bmatrix} \middle| Cx_t + Du_t \leq \gamma \right\} \tag{10}$$

is bounded, then the admissible constant references $\mathcal{Y}_r$ will be also bounded.

In the following it is considered the reference tracking in the general case by relaxing this "piece-wise constant" assumption for the reference signal. However, the results obtained for this family of set-point can be found as particular cases of reference management.

3.1 Classification of general reference tracking problems

The problem of regulating the system state to origin using optimal control sequences over receding horizons can be extended to the reference tracking problems. A classification of these tracking problems [1] might be:

1) The model-following problem: The reference signal is the output of a known linear model:

$$\Sigma_R : \begin{cases} z_{t+1} = A_z z_t, z_0 \in Z_0, \\ r_t = H_z z_t \end{cases} \tag{11}$$

with $r_t \in \mathbb{R}^p$, A_z stable, the pair (A_z, H_z) observable and $Z_0 \in \mathbb{R}^{n_z}$ the set of initial conditions. The model predictive control is supposed to find the optimal control sequence

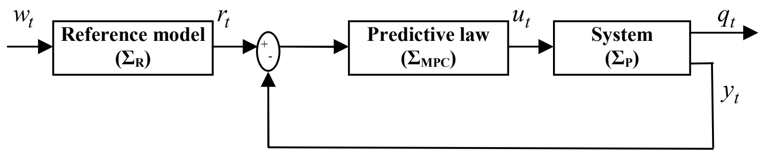

Figure 3.1: Classical MPC tracking scheme (q_t-the tracking quality).

for the system (1) such that the output y tracks the incoming reference r. By recasting this problem on the form:

$$\begin{aligned} &\tilde{x}_{t+1} = \begin{bmatrix} z_{t+1} \\ x_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A_z & 0 \\ 0 & A \end{bmatrix}}_{\tilde{A}} \tilde{x}_t + \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{\tilde{B}} u_t, \\ &\tilde{C}\tilde{x}_t + Du_t \leq \tilde{\gamma}, \\ &\tilde{C} = \begin{bmatrix} 0 & C \end{bmatrix}, \ \tilde{D} = D, \ \tilde{\gamma} = \gamma \end{aligned} \tag{12}$$

and putting $\tilde{Q} = \begin{bmatrix} H_z & -H \end{bmatrix} Q \begin{bmatrix} H_z & -H \end{bmatrix}^T$, the classical MPC design techniques for the problem (1-2) can be applied to assure the existence of a feasible law for all initial conditions $\tilde{x}_0 \in \tilde{X}_0 = \left\{ \begin{bmatrix} z_0 \\ x_0 \end{bmatrix} \middle| z_0 \in Z_0; x_0 \in X_0 \right\}$.

2) The tracking problem: The desired trajectory is the output of a known linear model (Figure 3.1) excited by an exogenous signal, partially known:

$$\Sigma_R : \begin{cases} z_{t+1} = A_z z_t + B_z w_t, \ z_0 \in Z_0, \\ r_t = H_z z_t \end{cases} \tag{13}$$

with $Z_0 \in \mathbb{R}^{n_z}$ and considering that the same assumptions as for the previous case are satisfied. Note that w_k can be the output of a high-order, time-varying system.

All the information on the exogenous signal w_t should be incorporated in the reference model such that the MPC law could improve the prediction accuracy. For example if the signal w_t is supposed to be known in advance over a horizon $N_w - \left\{ w_{t|t}, \ldots, w_{t+N_w-1|t} \right\}$, then the reference model to be used for the MPC design is:

$$A_z \leftarrow \left[\begin{array}{c|cccc} A_z & B_z & 0 & \cdots & 0 \\ \hline & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ & \vdots & \vdots & \ddots & 1 \\ & 0 & 0 & \cdots & 0 \end{array} \right], \quad B_z \leftarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$H_z \leftarrow \begin{bmatrix} H_z \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T, \quad z_t \leftarrow \begin{bmatrix} z_t \\ w_t \\ w_{t+1} \\ \vdots \\ w_{t+N_w-1} \end{bmatrix}, \quad w_t \leftarrow w_{t+N_w}.$$

With this reference description, one can construct the augmented model as in (12) and proceed to the MPC synthesis based on the minimization of a cost function weighting the tracking error and the control effort. Following developments will focus on the influence of the exogenous signal on the feasibility of the overall control scheme.

3.2 Infeasibility within reference tracking

Even if the MPC design is reduced to the classical case, the feasibility requirements must be fulfilled with respect to the set of initial conditions in the augmented space $\tilde{X}_0$ and not only for X_0. In this respect, the terminal set must be based on the model (12) - $\tilde{x}_N \in \tilde{X}_N$ and not only $x_N \in X_N$.

The MPC law will be characterized by a polyhedral set of feasible points in the augmented state space as in (6):

$$\tilde{X}_f = \{\tilde{x}|\Lambda\tilde{x} \leq \lambda\} = \left\{\begin{bmatrix} z^T & x^T \end{bmatrix}^T|\Lambda_z z + \Lambda_x x \leq \lambda\right\}. \tag{14}$$

The evolution of the reference model is independent of the chosen control action at time t:

$$z_{t+1} = A_z z_t + B_z w_t$$

and thus at time $t+1$ the infeasibility phenomenon entails that $x_{t+1} \notin X_f(z_{t+1}) = \{x|\Lambda_x x \leq \lambda - \Lambda_z z_{t+1}\}$.

But in the same time the MPC law is designed such that in the absence of exogenous signal ($w_t = 0$), the feasibility is preserved (a necessary condition for the constraint fulfilment over the prediction horizon). This means that $x_{t+1} \in X_f(\hat{z}_{t+1}) = \{x|\Lambda_x x \leq \lambda - \Lambda_z \hat{z}_{t+1}\}$ with $\hat{z}_{t+1} = A_z z_t$ (absence of exogenous signal). Based on this observation the infeasibility of the MPC tracking scheme at time t can be classified as:

- $\mathbf{B_z w_t = 0}$

  The MPC law is feasible because $\hat{z}_{t+1} = z_{t+1}$.

- $\mathbf{B_z w_t \neq 0 \wedge D = X_f(z_{t+1}) \cap X_f(\hat{z}_{t+1}) \neq \emptyset}$ (Figure 3.2 left)

  If $x_{t+1} \in D$ the MPC law is feasible.

  If $x_{t+1} \notin D$ the MPC law is infeasible due to the incompatibility between the current state and the exogenous signal entering the reference model.

- $\mathbf{B_z w_t \neq 0 \wedge D = X_f(z_{t+1}) \cap X_f(\hat{z}_{t+1}) = \emptyset}$ (Figure 3.2 right)

  Infeasibility due to a jump of the reference model state which overwhelms the closed loop tracking capabilities.

Remark 3.2 The only degree of freedom one can dispose within MPC to diminish the infeasibility risk is to enlarge the set $\tilde{X}_f$. This can be done by augmenting the prediction horizon. Unfortunately this manoeuvre is increasing the complexity of the resulting control law. Another limitation towards this augmentation is the fact that $\tilde{X}_f$ can never go beyond $\tilde{C}_\infty$.

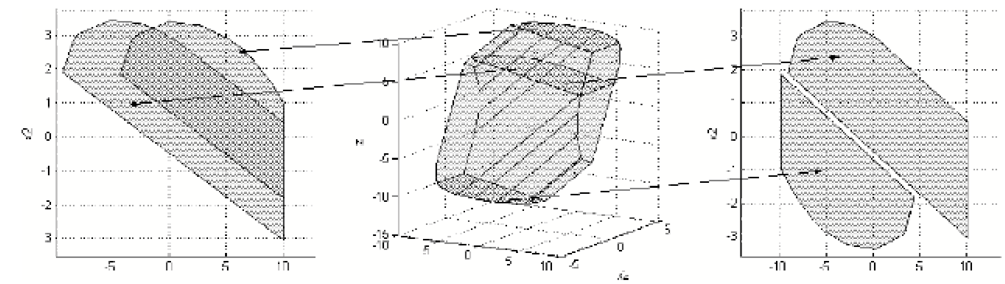

Figure 3.2: Example of feasible set $\tilde{X}_f$ (center). $\mathbf{D} = \mathbf{X_f}(\mathbf{z_{t+1}}) \cap \mathbf{X_f}(\hat{\mathbf{z}}_{\mathbf{t+1}}) \neq \emptyset$ (left). $\mathbf{D} = \mathbf{X_f}(\mathbf{z_{t+1}}) \cap \mathbf{X_f}(\hat{\mathbf{z}}_{\mathbf{t+1}}) = \emptyset$ (right).

3.3 Feasible bounds expressed as multiparametric optimization solutions

In the previous discussion the feasible set $\tilde{X}_f$ has been considered as the extended version of a parameterized polyhedron $X_f(z_t)$ with parameters given by the state of the reference model. If the goal is to describe the family of feasible references, then a dual approach has to be considered, with the characterization of the infeasibility by the fact that $z_{t+1} \notin Z_f(x_{t+1}) = \{z | \Lambda_z z \leq \lambda - \Lambda_x x_{t+1}\}$. Using this set, in the SISO case, the feasibility can be characterized based on the limits of the feasible reference signal r_t or of the feasible exogenous signal w_t as in Table 3.1.

$r_t^{\min}(x_t) \leq r_t \leq r_t^{\max}(x_t)$	$w_t^{\min}(x_t, z_t) \leq w_t \leq w_t^{\max}(x_t, z_t)$
$r_t^{\min}(x_t) = \min\limits_z H_z z$ s.t. $\Lambda_z z \leq \lambda - \Lambda_x x_t$	$w_t^{\min} = \min\limits_w w$ s.t. $\Lambda_z B_z w \leq \lambda - \Lambda_x x_{t+1} - \Lambda_z A_z z_t$
$r_t^{\max}(x_t) = \max\limits_z H_z z$ s.t. $\Lambda_z z \leq \lambda - \Lambda_x x_t$	$w_t^{\max} = \max\limits_w w$ s.t. $\Lambda_z B_z w \leq \lambda - \Lambda_x x_{t+1} - \Lambda_z A_z z_t$

Table 3.1: Feasibility conditions.

The feasible bounds $r_t^{\min}, r_t^{\max}, w_t^{\min}, w_t^{\max}$ are solutions of linear multiparametric optimization problems. Their expression provides a hint about the way the feasibility can be recovered by the adjustment of the reference to the corresponding feasible limitation once this is violated.

4 Feasibility recovery

The idea resumed in Table 3.1 is that at each sampling time, the feasibility of the MPC scheme depends on whether the reference signal is contained in a safe region (X_r):

$$r_t \in X_r(x_t, z_t) \Leftrightarrow H_z z_t \in X_r(x_t, z_t). \tag{15}$$

As long as the exogenous signal is given, and the state of the system is supposed to be known, the only degree of freedom available to force the feasibility of the control law is the adjustment of the reference, to follow the best feasible approximation of the reference

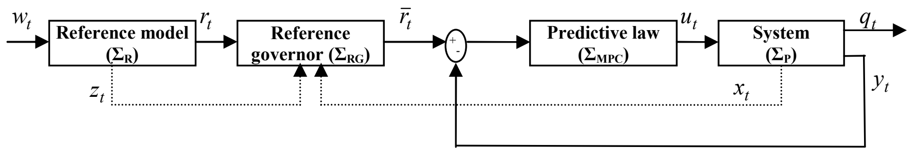

Figure 4.1: Reference governor scheme for MPC feasibility.

signal. This technique has already a wide experience, being known in the literature as the reference governor scheme.

4.1 Reference governor

A reference governor (RG) is based on the idea that for the reference tracking scheme in (Figure 3.1) a MPC law can be designed such that the closed loop performance, feasibility and stability requirements are fulfilled for the model-following problem (in the absence of the exogenous excitation). Once the MPC law description is available, the associate feasible set $\tilde{X}_f$ is also available. Based on this assumption, the goal of the RG is to replace the reference r_t by:

$$\bar{r}_t = \bar{r}_t(x_t, r_t) \tag{16}$$

such that $\bar{r}_t$ is the best approximation of r_t and the MPC law is well defined. Due to the fact that r_t is the output of the reference model, one can rewrite (16) as $\bar{r}_t(z_t, x_t)$. The best approximation must be judged with respect to a cost index. According to this principle the adjusted reference might be:

$$\begin{aligned} &\bar{r}_t = H_z \bar{z}_t^*, \\ &\bar{z}_t^*(z_t, x_t) = \arg\min_{\bar{z}_t} (H_z \bar{z}_t - H_z z_t)^T S (H_z \bar{z}_t - H_z z_t), \\ &\qquad\qquad \text{such that} \begin{bmatrix} \bar{z}_t \\ x_t \end{bmatrix} \in \tilde{X}_f, \end{aligned} \tag{17}$$

where the matrix S weights the deviation of the references. This mathematical formulation of the RG underlines the fact that the information needed for the optimization (17) is restricted to the current measurements (Figure 4.1).

4.2 Compact MPC law with guarantees of feasibility

An important detail regarding the optimization in (17) is the dependence of the argument $\bar{z}_t^*$ on the set of parameters $\{z_t, x_t\}$. The set of constraints depends exclusively on the vector x_t while the dependence on z_t comes exclusively from the cost index. If there is no other restriction added to (17), then the reference model state can be any value $z_t \in \mathbb{R}^{n_z}$. This aspect is decisive from the feasibility point of view because indirectly, all the constraints on z_t represent limitations on the reference or on the exogenous signal. As a first result, by applying the MPC law $u^{MPC}(\bar{z}_t, x_t)$ instead of $u^{MPC}(\tilde{z}_t, x_t)$, one can obtain a predictive law with guarantees of feasibility at each sampling time.

Despite this advantage the cascaded implementation of the reference governor with the MPC law is quite demanding on-line. A first step is to observe as in [12] that the RG

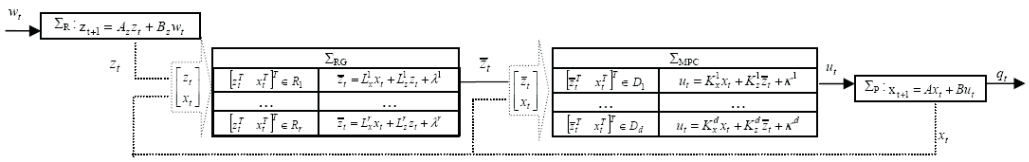

Figure 4.2: Explicit description of reference governor and MPC law.

implies in fact a multiparametric quadratic problem (17) for which the explicit solution does exist as in (4).

$$\bar{z}_t(z_t, x_t) = L_i * \begin{bmatrix} z_t \\ x_t \end{bmatrix} + l_i, \text{ for } \begin{bmatrix} z_t \\ x_t \end{bmatrix} \in R_i\,, \tag{18}$$

which can be further written componentwise:

$$\bar{z}_t(z_t, x_t) = L_{i_x} * x_t + L_{i_z} * z_t + l_i. \tag{19}$$

The union $R = \bigcup_i R_i$ represents the domain where the guarantees of feasibility are accomplished. The number of regions R_i is depending on the freedom allowed for the family of references. For diminishing the complexity of the explicit formulation (18), only a part of the cutting R can be retained, the one which is critical for the MPC feasibility.

Regarding the explicit implementation of the MPC law, as already mentioned (5), it is expressed as a piecewise linear continuous function:

$$u(\bar{z}_t, x_t) = K_i * \begin{bmatrix} \bar{z}_t \\ x_t \end{bmatrix} + \kappa_i, \quad \text{for } \begin{bmatrix} \bar{z}_t \\ x_t \end{bmatrix} \in D_i\,, \tag{20}$$

or written componentwise:

$$u_t(\bar{z}_t, x_t) = K_{i_x} * x_t + K_{i_z} * \bar{z}_t + \kappa_i. \tag{21}$$

Using these explicit formulations of the RG and MPC blocks, one can conclude that the real time implementation of the predictive scheme with guarantees of feasibility comes to a successive look-up table positioning for the evaluation of the control law at each sampling time (Figure 4.2).

Given these two piecewise linear (PWA) functions, and the fact that their evaluations depend on the depth of the binary search tree for each look-up table, the natural question is whether, the two functions can be compacted in a single control law including both the MPC and the RG mechanism (MPC-RG) and being "everywhere" feasible.

Proposition 4.1 *Let two piecewise linear and continuous functions:*

$$\begin{aligned} &f : R \to D,\ R = \bigcup_{i=1}^{r} R_i, \\ &\qquad f(x) = A_{f_i} x + b_{f_i}\ \forall x \in R_i, \\ &g : D \to F,\ \ D = \bigcup_{j=1}^{d} D_j, \\ &\qquad g(x) = A_{g_j} x + b_{g_j}\ \forall x \in D_j, \end{aligned} \tag{22}$$

then a composed piecewise affine and continuous function exists such that:

$$\begin{aligned} h : R \to F, \quad R = \bigcup_{k=1}^{n_h} DR_k, \\ h(x) = A_{h_k} x + b_{h_k} = g(f(x)), \ \forall x \in DR_k, \end{aligned} \tag{23}$$

with D, R, F, D_i, R_j, DR_k *convex sets.*

Proof For the existence of the cutting $R = \bigcup_{i=1}^{n_h} DR_i$ it is sufficient to construct for each $R_i, i = 1, .., r$ the subsets:

$$DR_{ij} = \{x | \, x \in R_i \text{ and } f(x) \in D_j\}, j = 1, .., d. \tag{24}$$

From hypothesis $f(R_i) \subset D$ and by retaining only the nonempty subsets of this construction on can obtain:

$$R_i = \bigcup_{j}^{DR_{ij} \neq \emptyset} DR_{ij}. \tag{25}$$

Now, by associating for each DR_{ij} :

$$A_{h_{ij}} \leftarrow A_{g_j} A_{f_i}, \quad b_{h_{ij}} \leftarrow A_{g_j} b_{f_i} + b_{g_j} \tag{26}$$

a finite n_h is obtained such that:

$$\begin{aligned} h : R \to F, \quad R = \bigcup_{k=1}^{n_h} DR_k \equiv \bigcup_{i,j}^{DR_{ij} \neq \emptyset} DR_{ij}, \\ h(x) = A_{h_k} x + b_{h_k} = g(f(x)) \ \forall x \in DR_k. \ \square \end{aligned} \tag{27}$$

Remark 4.1 For the resulting function $h(.)$, the number of subsets in the definition domain will satisfy $h_h \leq d * r$ and due to the fact that the evaluation mechanism for $f(.), g(.), h(.)$ is logarithmic in the number of partitions [23], it follows that the complexity of evaluation for $h(.)$ is inferior to the sequential evaluation of $f(.)$ and $g(.)$.

The previous result assures the existence of a compact law (MPC-RG), which inherits the qualities of the RG mechanism and the MPC performances (Figure 4.3). It can be noticed that the intermediary adjusted reference $\bar{r}_k$ and the associated governed reference model state $\bar{z}_k$ needs no evaluation, all this process being nested in the MPC-RG law (Figure 4.4).

Certainly, the on-line evaluation of the control action is optimized by this formulation but one may wonder about the price to be paid. As it was already observed in the literature related to the explicit formulations, the on-line computational gains are obtained by increasing the memory needs. Is the case also for the MPC-RG law which needs to store a much more complex look-up table than the original MPC law. On the contrary, the RG explicit formulation stores the entire explicit solution (19) and not only the first component as it is the case for the MPC. This fact burdens in some extent the complexity of the sequential scheme. The compact MPC-RG law does not suffer from this point of view and thus memory storage disadvantages are mitigated.

The following algorithm resumes the MPC-RG design.

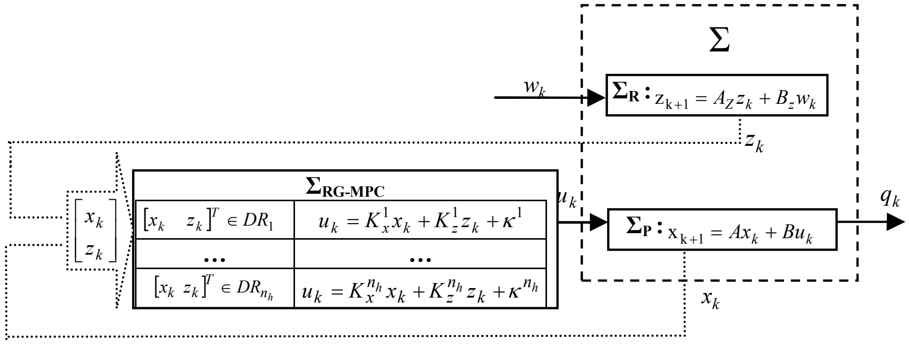

Figure 4.3: MPC with guarantees of feasibility.

Algorithm:

1. Find the explicit MPC for the model-following problem.

$$u = K_x^i x + K_z^i \bar{z} + k^i, \text{ for } \begin{bmatrix} x^T & \bar{z}^T \end{bmatrix}^T \in D_i, i = 1, .., d.$$

2. Determine the MPC feasible set $\tilde{X}_f = \bigcup_{i=1}^{d} D_i$.
3. Construct the explicit form of the RG:

$$\bar{z} = L_x^i x + L_z^i z + \lambda^i, \text{for } \begin{bmatrix} x^T & z^T \end{bmatrix}^T \in R_i, i = 1, .., r.$$

4. Build the compact MPC-RG law:
   For all $i = 1, .., r$
   For all $j = 1, .., d$
   Compute $DR_{ij} = \{x | \, x \in R_i \ et \ f(x) \in D_j\}$
   If DR_{ij} is nondegenerate store it together with:

$$u = \underbrace{(K_x^i + K_z^i L_x^j)}_{K_x^{ij}} x + \underbrace{K_z^i L_z^j}_{K_z^{ij}} z + \underbrace{(k^i + K_z^i \lambda^j)}_{k^{ij}}$$

   end
   end □

Proposition 4.2 *The compact MPC-RG law enjoys the following properties:*

i) *MPC-RG is a piecewise linear and continuous function of the extended state* $\tilde{x}_t = \begin{bmatrix} z_t^T & x_t^T \end{bmatrix}^T$.

ii) *If* $\tilde{x}_t \in \tilde{X}_f$ *then the MPC and MPC-RG are equivalent.*

iii) $y_k \to \hat{r}$ *if* $r_k \to r$ *where* $\hat{r}$ *is the best feasible approximation of* r *with respect to the criterium in (17).*

iv) *If the original MPC law was designed for zero steady error for constant references* r*, the MPC-RG law has a finite settling time to the best feasible approximation* $\hat{r}$.

Proof The property *i)* is a consequence of the fact that the composition of two piecewise linear and continuous functions inherits the same properties.

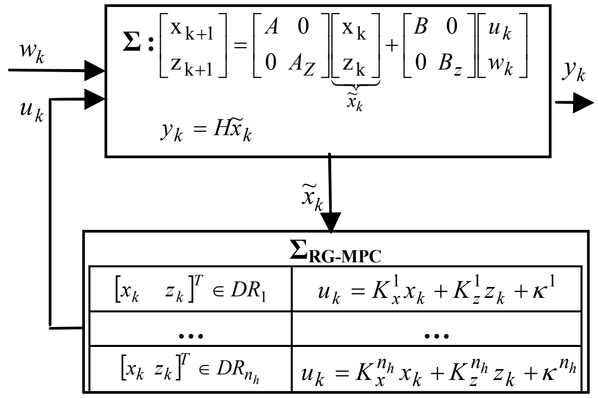

Figure 4.4: Compact MPC law with guarantees of feasibility.

For *ii)* it is sufficient to see that if the condition $\tilde{x}_t \in \tilde{X}_f$ is verified then (17) becomes an unconstrained optimization problem and thus $\bar{z}_t = z_t$ implying the equivalence of the MPC law with the MPC-RG version.

In order to demonstrate *iii*) the continuity of the RG must be taken into account to generate a sequence of references leading to a steady set-point $H_z\tilde{z}_k \to \hat{r}$. Further due to the stabilization properties of the MPC law this position will be regulated $y_k \to \hat{r}$. If the obtained steady output $\hat{r}$ is not corresponding to the best approximation then the optimality of the RG is denied leading to a contradiction.

The point *iv*) is a special case of the former problem. $\square$

5 Example

Consider the discrete version of the double integrator:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k, \\ y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_k. \end{aligned} \tag{28}$$

For the exemplification of the feasibility analysis, an autonomous reference model is considered at the beginning:

$$\begin{aligned} z_{k+1} &= 0.99 z_k, \\ r_k &= z_k. \end{aligned} \tag{29}$$

The system has to obey a set of constraints:

$$\begin{aligned} &-1 \le u_k \le 1, \\ &x_k \in \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} | -10 \le x_1, x_2 \le 10 \right\}. \end{aligned} \tag{30}$$

One can construct based on (28-30) the extended model and develop the MPC law for the regulation case. The first step is to find the optimal control law in the unconstraint case for a infinite cost index as in (2) with $Q = 1, R = 10$:

$$u_k = \begin{bmatrix} 0.23335 & 0.67756 \end{bmatrix} x_k - 0.22664 z_k$$

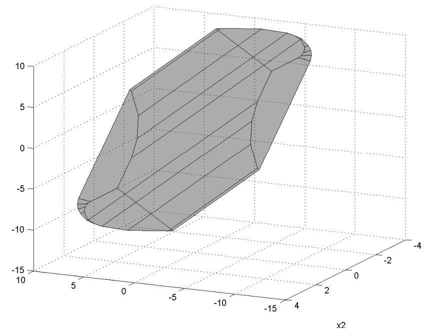

Figure 5.1: The maximal admissible set O_∞.

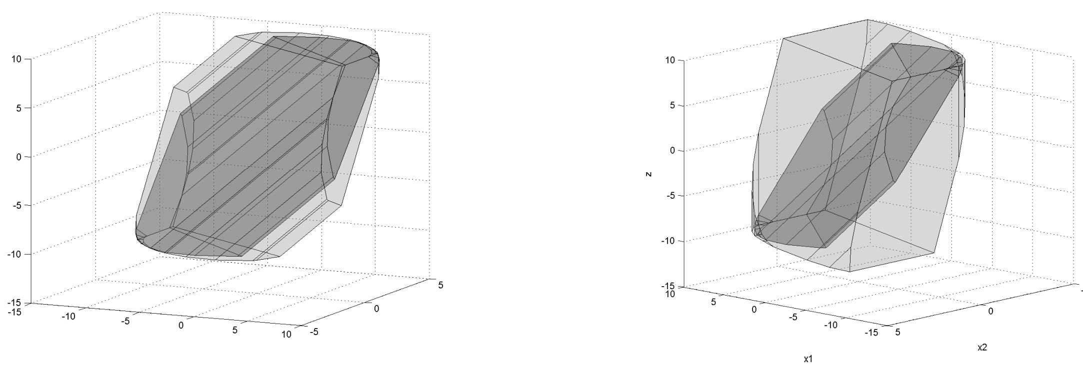

Figure 5.2: a) $\tilde{X}_f$ for $N = 1$, b) $\tilde{X}_f$ for $N = 4$.

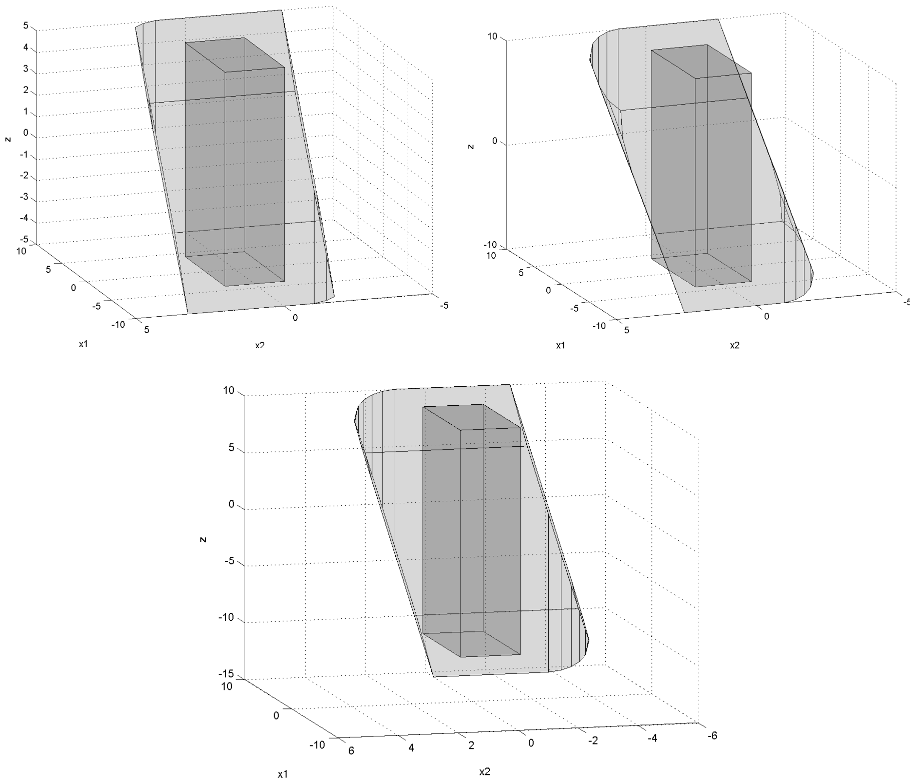

Figure 5.3: Intersection of $\tilde{X}_f$ with the set of initial conditions. a) $N = 4$ and $-5 \leq z \leq 5$ b) $N = 4$ and $-10 \leq z \leq 10$ c) $N = 5$ and $-10 \leq z \leq 10$.

and further find the maximal admissible set O_∞(Figure 5.1).

The predictive law can be synthesized using it as a terminal invariant set. As mentioned in Section 3, the prediction horizon has a decisive influence on the shape of the feasible set for the MPC law. In Figure 5.2, it is presented the feasible set $\tilde{X}_f$ for the MPC laws with prediction horizon $N = 1$ (Figure 5.2a) and $N = 4$ (Figure 5.2b).

The feasibility guarantee has to be given with respect to a set of initial conditions. Choosing:

$$z_0 \in Z_0 = \{ z \in \mathbb{R} | -5 \leq z \leq 5 \},$$
$$x_0 \in X_0 = \left\{ \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] \middle| -4 \leq x_1 \leq 4; -1 \leq x_2 \leq 1 \right\}$$

a MPC law with $N = 4$ is guaranteed to be feasible (Figure 5.3a). If the set of initial conditions for the reference model is changed to:

$$z_0 \in Z_0 = \{ z \in \mathbb{R} | -10 \leq z \leq 10 \},$$

the infeasibility is no longer guaranteed (Figure 5.3b). By augmenting the prediction horizon to $N = 5$, the feasibility is retrieved (Figure 5.3c).

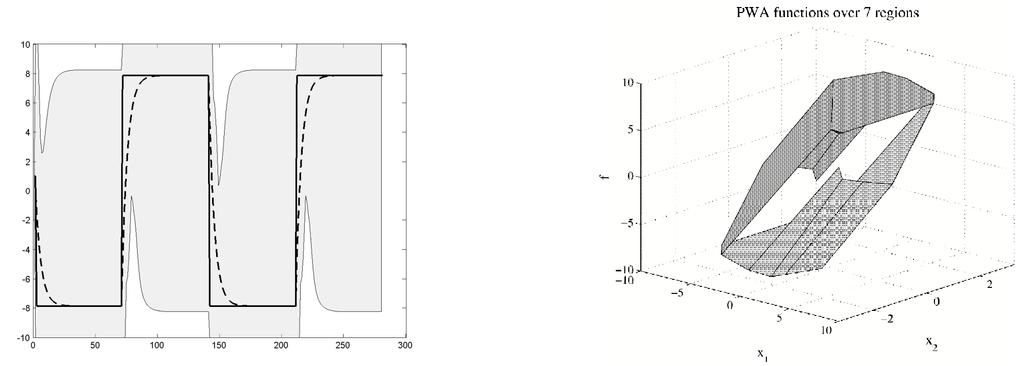

Figure 5.4: a) Time-domain simulation $w_k^{min} \le w_k \le w_k^{max}$ (dotted line — r_k); b) The explicit description of the reference limitations.

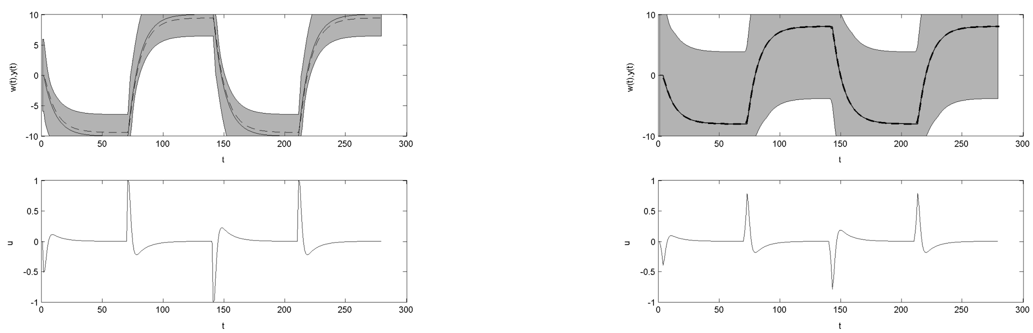

Figure 5.5: Feasible points along a given trajectory: a) $N = 1$, b) $N = 4$.

For the trajectory tracking problem, the following reference model is considered:

$$\begin{aligned} z_{k+1} &= 0.85 z_k + 0.15 w_k, \\ r_k &= z_k. \end{aligned} \tag{31}$$

The MPC law can be synthesized with respect to the set of constraints:

$$\begin{aligned} &-1 \le u_k \le 1, \\ &x_k \in \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| -10 \le x_1, x_2 \le 10 \right\}, \\ &-10 \le r_k \le 10. \end{aligned} \tag{32}$$

The exogenous signal will affect the evolution and using the formulations in Table 3.1, one can obtain a time domain simulation of the feasible limitations of w_k (Figure 5.4a) or the explicit solution for $r_k^{min}(x_k)$, $r_k^{max}(x_k)$ (Figure 5.4b).

The prediction horizon plays a decisive role in the feasibility limitations as it can be seen in Figure 5.5.

If the feasible set available is not satisfying the feasibility demands for the tracking problem, then an avoiding redundancy mechanism can be synthesized in terms of a reference governor (RG). In order to give a slight idea about the complexity of the explicit

N	NZ_{MPC}	NZ_{RG}	Combinations to be explored	NZ_{MPC-RG}
1	5	37	185	79
2	23	154	3542	597
3	99	627	62073	3979
4	421	2373	999033	25114

Table 5.1: Explicit formulations.

	ST_{RG+MPC}			ST_{MPC-RG}		
N	Nodes	Depth	Memory(bytes)	Nodes	Depth	Memory(bytes)
1	176+6	8+4	27616+1136	228	8	25402
2	1479+71	11+6	173928 + 8408	1756	12	212963

Table 5.2: Optimal search tree complexity.

solution, one can find in Table 5.1 the number of zones NZ_* for prediction horizons going from 1 to 4.

The column NZ_{MPC} contains the number of domains for the predictive control explicit solution. Their union will form $\tilde{X}_f$ which is further the base for the RG scheme. The column NZ_{RG} presents the number of zones exclusively for the reference avoidance scheme. If these two blocks function independently, then the number of combinations to be explored are represented by the product $NZ_{MPC} * NZ_{RG}$ which is reported in the fourth column of the table. It can be observed that for large prediction horizon this indicator becomes very large and the natural question is whether all of them are representing valid combinations of the extended (reference model + system) state. The answer is given by the compact MPC law with guarantees of feasibility, constructed by the composition of the MPC and RG descriptions. The fifth column in Table 5.1 contains the number of zones NZ_{MPC-RG} for its explicit formulation.

One conclusion appears: the compact $MPC-RG$ scheme is avoiding the exploration of useless combinations but in the same time its number of zones is larger than the number of zones needed for the sequential implementation:

$$NZ_{MPC} + NZ_{RG} \leq NZ_{MPC-RG} \leq NZ_{MPC} * NZ_{RG}.$$

From the point of view of the on-line evaluation, the second inequality is relevant. From the point of view of the memory needed to store the explicit solution, the first inequality mainly who reflects the comparison between the sequential implementation or the compact implementation.

In order to get a better insight on the on-line evaluation vs. memory used compromise, the search tree for the explicit solutions can be constructed and their complexity compared (Table 5.2).

Unfortunately, due to the huge number of zones to be explored, only the $N = 1$ and $N = 2$ are obtainable in reasonable time. However, the results are insightful, and for example for $N = 1$ one can observe that the depth of the search tree for the compact formulation is equivalent with the depth of the RG. This means that in the worst case the RG evaluation is equivalent with MPC-RG evaluation which on its turn is far less expensive that the sequential evaluation of MPC and RG (composed depth of the search

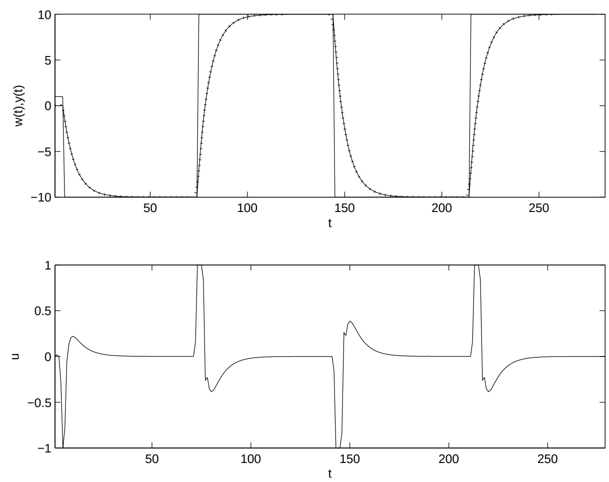

Figure 5.6: Simulation of the evolution under the MPC-RG law. In the upper part, the exogenous signal w, the effective reference signal r and the system output y (dotted). In the lower part the input signal.

tree 12). The price to be paid as already mentioned is transparent in the memory needs. But due to the fact that the RG scheme needs the storage of the complete explicit solution, for $N = 1$ this disadvantage is mitigated and it can be noticed that the sequential scheme is much memory involved that the compact scheme. For $N = 2$ the evaluation mechanism has to deal with a $12 vs. 17$ depth search tree. The memory used for the compact scheme is still comparable with the sequential case. For larger prediction horizons the differences from the memory point of view become more evident. As a conclusion the choice between the compact or the sequential scheme are dependent on the aspiration towards a small on-line evaluation time on one hand and the memory available on the other hand.

x_t^1	x_t^1	z_t	w_t	w_{t+1}	w_{t+2}	w_{t+3}
-9.843	0.078	-10	-10	10	10	10
-8.76	0.578	-10	10	10	10	10
-7.18	1.07	-7.14	10	10	10	10
9.842	-0.078	10	10	-10	-10	-10
8.763	-0.578	10	-10	-10	-10	-10
7.18	-1.07	7.14	-10	-10	-10	-10

Table 5.3: Infeasible combinations for Figure 5.6.

The fact that the infeasibility avoidance scheme is effective can be illustrated by a time domain simulation as the one in Figure 5.6.

The reference is adjusted for the combinations corresponding to the jumps on the reference like the ones enumerated in Table 5.3.

The compact MPC scheme with guarantees of feasibility can prove to be versatile even for extreme reference signals which are outside the operating zone of the constrained system. For example, in Figure 5.7, the reference is bringing the MPC law to infeasibility at almost every sample time. The RG can adjust it to the best admissible value proving the good convergence properties of the scheme.

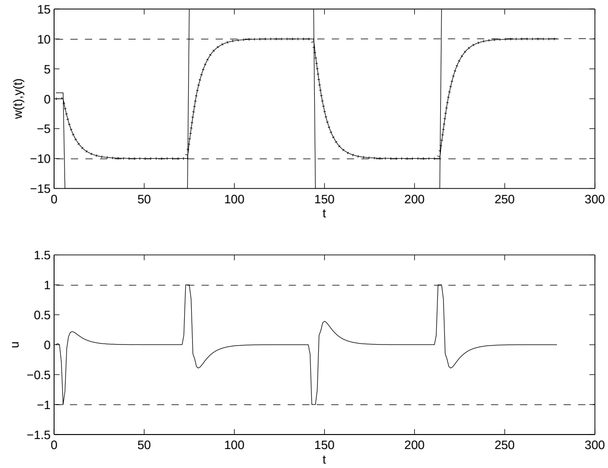

Figure 5.7: Simulation of the evolution under the MPC-RG law. In the upper part, the exogenous signal w, the effective reference signal r and the system output y (dotted). In the lower part the input signal.

6 Conclusions

The feasibility problem within the model predictive control framework was treated with a special attention to the tracking problems. Based on the feasibility results existing for the regulation case, the limitations of the feasible trajectories are established and the infeasible behavior classified.

In order to avoid the infeasibility, a reference adjustment mechanism based on the idea of a ”reference governor” was used. Based on the fact that both the MPC and the RG are in fact formulated as multiparametric optimization problems, their explicit formulation in terms of piecewise affine functions was proposed.

The independent implementation of the RG in conjunction MPC law was shown to be not optimal from the point of view of the evaluation mechanism. A compact predictive law with guarantees of feasibility was constructed, optimal from this point of view. Its application might be considered if the memory demands are not overwhelming.

References

[1] Anderson, B. and Moore, J. *Linear Optimal Control*. Prentice Hall, 1971.

[2] Bemporad, A., Morari, M., Dua, V. and Pistikopoulos, E. The explicit linear quadratic regulator for constrained systems. *Automatica* **38** (2002) 3–20.

[3] Bemporad, A. and Mosca, E. Fulfiling hard constraints in uncertain linear systems by reference managing. *Automatica* **34** (1996) 451–461.

[4] Blanchini, F. Set invariance in control. *Automatica* **35** (1999) 1747–1467.

[5] Blanchini, F. and Miani, S. Set based constant reference tracking for Continuous-time Constrained Systems. *Nonlinear Dynamics and Systems Theory* **1**(2) (2001) 121–131.

[6] Camacho, E.F. and Bordons, C. *Model Predictive Control*. Springer, London, 2003.

[7] Casavola, A. Mosca, E. and Papini, M. Control under Constraints: An application of the command governor approach to inverted pendulum. *IEEE Trans. Control System Technology* **12** (2004) 193–204.

[8] Gilbert, E.G. and Tan, K.T. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Trans. Automat. Control* **36** (1991) 1008–1020.

[9] Goodwin, G.C., Seron, M.M. and De Dona, J.A. *Constrained Control and Estimation*. Springer-Verlag, London, 2004.

[10] Kerrigan, E. *Robust Constraint Satisfaction: Invariant Sets and Predictive Control*. PhD thesis, University of Cambridge, 2000.

[11] Kerrigan, E.C. and Maciejowski, J.M. Invariant Sets for Constrained Nonlinear Discrete-time Systems with Application to Feasibility in Model Predictive Control. In *39th IEEE Conference on Decision and Control*, Sydney, 2000.

[12] Kogiso, K. and Hirata, K. A reference governor in a piecewise state affine function: The implementation and validation. In *42nd IEEE Conference on Decision and Control*, Maui, 2003.

[13] Kvasnica, M., Grieder, P. Baotic, M. and Morari, M. *Multi-parametric Toolbox (MPT)*. Available online at: http://control.ee.ethz.ch/ mpt/, 2004.

[14] Limon, D., Alvarado, I., Alamo, T. and Camacho, E.F. MPC for tracking of piece-wise constant references for constrained linear systems. *Proc. 16th IFAC World Congress*, Prague, 2005.

[15] Maciejowski, J.M. *Predictive Control with constraints*. Prentice Hall, Harlow, 2002.

[16] Mayne, D.Q., Rawlings, J.B., Rao, C.V.and Scokaert, P.O.M. Constrained model predictive control: Stability and optimality. *Automatica* **36** (2000) 789–814.

[17] Olaru, S. and Dumur, D. A Parameterized Polyhedra Approach for Explicit Constrained Predictive Control. In *43rd IEEE Conference on Decision and Control*, Bahamas, 2004.

[18] Olaru, S. and Dumur, D. Avoiding Constraints Redundancy in Predictive Control Optimization Routines. *IEEE Trans. Automat. Control* **50** (2005) 1459-1466.

[19] Rossiter, J.A. *Model-Based Predictive Control*. CRC Press, Boca Raton, 2003.

[20] Seron, M.M., Goodwin, G.C. and De Dona, J.A. Characterisation of Receding Horizon Control for Constrained Linear Systems. *Asian Journal of Control* **5** (2002) 271–286.

[21] Scockaert, P.O.M. and Clarke, D. Stability and feasibility in constrained predictive control. In: *Advances in model-based predictive control*. Oxford University Press, 1994a, 217-230.

[22] Scockaert, P.O.M. and Clarke, D. Stabilising properties of constrained predictive control. *IEE Proceedings Control Theory Applications*, **141** (1994b) 295-304.

[23] Tondel, P., Johansen, T.A. and Bemporad, A. Evaluation of piecewise affine control via binary search tree. *Automatica* **39** (2003) 945–950.

Nonlinear Dynamics and Systems Theory, **7** (4) (2007) 399–408

Observer Design for a Class of Nonlinear Systems with Non-Full Relative Degree

K. Röbenack *

Institut für Regelungs- und Stenerungstheorie,
Fakultät Elektrotechnik und Informationstechnik,
Technische Universität Dresden
01062 Dresden, Germany

Received: November 13, 2006; Revised: August 9, 2007

Abstract: The paper proposes a method for observer design for a class of nonlinear systems. We decompose the system using a weaker concept than the relative degree. We provide sufficient conditions for global asymptotic stability of the error dynamics. The observer design is carried out by means of a change of coordinates combined with a high gain technique. In particular, our approach results in an observer gain vector field which is extraordinary easy to compute.

Keywords: *Nonlinear system; coordinate change; observer design; Moore-Penrose inverse.*

Mathematics Subject Classification (2000): 93B17, 93B29, 93B50, 93C10.

1 Introduction

We consider the problem of observer design for nonlinear single-input single-output systems. A particularly interesting class of design methods use differential geometric concepts. These design methods are based on various normal forms. In [19, 3], the observer canonical form consisting of a linear output map and linear dynamics driven by a nonlinear output injection is used. The resulting observer has exactly linear error dynamics, i.e., nonlinearities are compensated exactly. The approaches suggested in [13, 9, 10, 5] rely on the observability canonical form, which has significantly weaker existence conditions than the observer canonical form. In the observability canonical form, the observer is designed by a high-gain technique with a constant observer gain, i.e., the nonlinearities are not compensated but dominated by a linear part. For an implementation of the

* Corresponding author: klaus@roebenack.de

observer in the original coordinates one gets a Luenberger-like observer with a possibly nonlinear gain vector field.

In the last decade, new approaches have been developed for nonlinear systems that are not uniformly observable. Several approaches use Kalman-like decompositions, see [1, 17, 18, 26]. For example, the observer design method suggested in [17] uses the Byrnes-Isidori normal form [6, 7] in almost the same way as the observability canonical form is used in [13, 9, 10]. Similarly, the partial observer canonical form used in [18] generalizes the design method given in [19].

For the observability canonical form, the change of coordinates is explicitly given in terms of iterated Lie derivatives of the system's output map. In contrast to that, the transformation into the Byrnes-Isidori normal form is not unique. Although this non-uniqueness offers some degrees of freedom that may be utilized by an experienced control engineer, it makes a symbolic implementation by means of computer algebra systems less straight forward.

In this work we use a weaker concept than the well-known relative degree [10]. The observer design uses a normal form similar to the Byrnes-Isidori normal form. Our work is strongly related to [17], but in contrast to [17] we exploit possible degrees of freedom in the change of coordinates to obtain an explicit expression of the observer gain. Similar as in [16], our approach may even be applicable for systems with ill-defined relative degree. This paper extends preliminary results presented in [26].

The paper is structured as follows. In Section 2 we suggest a decomposition of the system. Section 3 presents an observer and conditions for global asymptotic convergence of the error dynamics. The main contribution of the paper is presented in Section 4, where we suggest a new approach to compute the observer gain. The design method is demonstrated on an example in Section 5.

2 Preliminaries

Consider a nonlinear single-input single-output system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \tag{1}$$

with smooth maps $f, g\colon \Omega \to \mathbb{R}^n$ and $h\colon \Omega \to \mathbb{R}$ defined on an open and connected subset $\Omega \subseteq \mathbb{R}^n$. We assume that Ω is positively invariant under the dynamics of (1). The notation used in this paper is common in context of differential-geometric control theory (see [14, 22]). In particular, the Lie derivative of h along f is given by $L_f h(x) = \langle dh(x), f(x)\rangle$, where $dh = h'$ denotes the gradient of h and $\langle\cdot,\cdot\rangle$ is the inner product. Iterated Lie derivatives are defined by $L_f^{k+1}h(x) = L_f(L_f^k h(x))$ with $L_f^0 h(x) = h(x)$. The Lie bracket of two vector fields f and g is given by $[f,g](x) = g'(x)f(x) - f'(x)g(x)$. The Euclidean norm of a vector x is noted by $\|x\|$.

The decomposition of system (1) is based on the following assumption [10]:

A1 System (1) has an *observation relative degree* $r < n$ in Ω, i.e., $L_g L_f^k h(x) = 0\ \forall x \in \Omega$ and for $k = 0, \ldots, r-2$, and $\exists x \in \Omega$ with $L_g L_f^{r-1} h(x) \neq 0$. Moreover, the covector fields dh, $dL_f h$, $\ldots$, $dL_f^{r-1}h$ are linearly independent in Ω.

The concept of an observation relative degree is weaker than the well-known relative degree. In particular, assumption A1 may hold for systems with ill-defined relative degree. Clearly, if system (1) has an uniform relative degree r it also has the observation relative degree r. In this case, the covector fields occurring in A1 are linearly independent [14, Lemma 4.1.1].

Assumption A1 guarantees that for each $x_0 \in \Omega$ there exists a neighborhood $\mathcal{U} \subseteq \Omega$ and smooth functions $\phi_{r+1}, \ldots, \phi_n : \mathcal{U} \to \mathbb{R}$ such that the map $(z, \eta) = \Phi(x)$ defined by

$$\begin{aligned} z_i &= L_f^{i-1} h(x) \quad \text{for } i = 1, \ldots, r; \\ \eta_j &= \phi_{j+r}(x) \quad \text{for } j = 1, \ldots, n-r \end{aligned} \tag{2}$$

with $z = (z_1, \ldots, z_r)^T$ and $\eta = (\eta_1, \ldots, \eta_{n-r})^T$ is a local diffeomorphism. This diffeomorphism transforms system (1) into

$$\dot{z} = Az + b(\alpha(z,\eta) + \beta(z,\eta)u), \quad y = c^T z, \tag{3a}$$

$$\dot{\eta} = q(z,\eta) + p(z,\eta)u, \tag{3b}$$

with possibly nonlinear maps

$$\begin{aligned} \alpha(z,\eta) &= L_f^r h(\Phi^{-1}(z,\eta)), & \beta(z,\eta) &= L_g L_f^{r-1} h(\Phi^{-1}(z,\eta)), \\ q_i(z,\eta) &= L_f \phi_{r+i}(\Phi^{-1}(z,\eta)), & p_i(z,\eta) &= L_g \phi_{r+i}(\Phi^{-1}(z,\eta)) \end{aligned}$$

for $i = 1, \ldots, n-r$. The triple (A, b, c) is in Brunovsky form

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{r \times r}, \quad b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^r, \quad c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^r. \tag{4}$$

The functions $\phi_{r+1}, \ldots, \phi_n$ are not uniquely determined. If system (1) has a well-defined relative degree, these functions can be chosen such that

$$\forall x \in \mathcal{U}\colon \quad L_g \phi_i(x) = 0 \quad \text{for} \quad i = r+1, \ldots, n, \tag{5}$$

where in the normal form (3b) we have $p_i \equiv 0$ for $i = 1, \ldots, n-r$. In this special case, equation (3) becomes the Byrnes-Isidori normal form [7], because the second subsystem (3b) of (3) does not explicitly depend on the input u. In general, the choice of the functions ϕ_i in (5) is rather difficult (see [15]), and only in particular cases (e.g. textbook examples) the choice is easy.

3 Observer Setup

We propose an observer for system (1) based on the form (3). The first subsystem (3a) is observable since $z_1 = y$, $z_2 = \dot{y}$, $\ldots$, $z_r = y^{(r-1)}$. For this subsystem we design a high-gain observer [13, 9, 10]. Similar as in [17] we suggest an observer of the structure

$$\dot{\hat{z}} = A\hat{z} + b(\alpha(\hat{z},\hat{\eta}) + \beta(\hat{z},\hat{\eta})u) + k(y - c^T \hat{z}), \tag{6a}$$

$$\dot{\hat{\eta}} = q(\hat{z},\hat{\eta}) + p(\hat{z},\hat{\eta})u \tag{6b}$$

with the constant gain vector $k \in \mathbb{R}^r$. In the original coordinates the observer (6) has the form

$$\dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + l(\hat{x})(y - h(\hat{x})) \tag{7}$$

with the gain

$$l(\hat{x}) = (\Phi'(\hat{x}))^{-1} \begin{pmatrix} k \\ 0 \end{pmatrix}. \tag{8}$$

Because in (6) the observer gain interacts only with the first subsystem (6a), we augment in (8) the gain vector k by an $(n-r)$-dimensional zero vector.

The convergence analysis for system (1) and observer (7) is carried out in the normal form (3) and (6), respectively. The observation errors $\tilde{z} = z - \hat{z}$ and $\tilde{\eta} = \eta - \hat{\eta}$ are governed by the error dynamics

$$\dot{\tilde{z}} = (A - kc^T)\tilde{z} + b(\alpha(z,\eta) + \beta(z,\eta)u - \alpha(\hat{z},\hat{\eta}) - \beta(\hat{z},\hat{\eta})u), \tag{9a}$$

$$\dot{\tilde{\eta}} = q(z,\eta) + p(z,\eta)u - q(\hat{z},\hat{\eta}) - p(\hat{z},\hat{\eta})u. \tag{9b}$$

The dynamics of subsystem (9a) can be influenced by the gain vector $k = (k_1, \ldots, k_r)^T$. The linear part of this subsystem has the characteristic polynomial

$$\det(\lambda I - (A - kc^T)) = \lambda^r + k_1\lambda^{r-1} + \cdots + k_{r-1}\lambda + k_r. \tag{10}$$

We need the following assumptions:

A2 The map Φ given in (2) is defined on whole Ω and diffeomorphic onto its image $\Phi(\Omega)$, i.e., Φ is a global diffeomorphism.

A3 The maps α and β are globally Lipschitz, i.e., there exist constants $\gamma_1, \gamma_2 > 0$ such that

$$|\alpha(z,\eta) + \beta(z,\eta)u - \alpha(\hat{z},\hat{\eta}) - \beta(\hat{z},\hat{\eta})u| \le \gamma_1\|z - \hat{z}\| + \gamma_2\|\eta - \hat{\eta}\| \tag{11}$$

for all $(z,\eta), (\hat{z},\hat{\eta}) \in \Phi(\Omega)$ and bounded u.

A4 There exist a positive definite matrix $P_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ and constants $\gamma_3, \gamma_4 > 0$ such that for $V_2(\tilde{\eta}) = \tilde{\eta}^T P_2 \tilde{\eta}$ we have

$$\frac{\partial V_2(\tilde{\eta})}{\partial \tilde{\eta}} (q(z,\eta) + p(z,\eta)u - q(\hat{z},\hat{\eta}) - p(\hat{z},\hat{\eta})u) \le \gamma_3\|z - \hat{z}\|^2 - \gamma_4\|\eta - \hat{\eta}\|^2 \tag{12}$$

for all $(z,\eta), (\hat{z},\hat{\eta}) \in \Phi(\Omega)$ and any bounded u.

From A1 we already concluded that Φ is a local diffeomorphism. Conditions for Φ to be a global diffeomorphism as required in A2 are presented in [28, 4, 25]. A3 is a standard assumption in high gain design [24]. Assumption A4 means that the subsystem (3b) possesses a global steady state solution property [1]. If one considers the full state z of the observable subsystem (3a) as an output, the function V_2 becomes a global exponential-decay output-to-state stable (OSS) Lyapunov function [27]. This is a cruicial difference to assumption H3 of [17], where a classical (not OSS) Lyapunov function is required for the second subsystem. Note that the property A4 depends on the coordinate transformation (2), especially on the choice of the functions ϕ_i for $i = r+1, \ldots, n$. Similar as H3 in [17], this property is difficult to check.

Theorem 3.1 *Consider system* (1) *with the observer* (7)*. Assume that the input u is bounded and conditions A1-A4 hold, where r denotes the observations relative degree. Then, there exist a vector $k \in \mathbb{R}^r$ such that $\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0$ for all $x(0), \hat{x}(0) \in \Omega$.*

The proof of Theorem 3.1 is shown in the appendix. In [9, 10], the vector k is chosen such that the roots $\lambda_1, \dots, \lambda_r$ of (10) are placed at $\lambda_i = -\theta^i$ for $i = 1, \dots, r$ and sufficiently large $\theta > 0$. The technique suggested in [13] corresponds to the multiple root $\lambda_i = -\theta$ for $i = 1, \dots, r$. A general discussion about the computation of the constant observer gain for high-gain design can be found in [24].

4 Observer Gain Based on the Moore-Penrose Inverse

Up to now, one has to compute the $n-r$ gradients $d\phi_{r+1}, \dots, d\phi_n$. To obtain the Byrnes-Isidori normal form, these gradients must additionally satisfy (5). In the following, we consider a special choice of the nonlinear observer gain vector field (8), which can be computed directly, i.e., without an explicit knowledge of these gradients.

The Jacobian matrix of the transformation (2) is split up into two parts, where the first r rows consisting of gradients of Lie derivatives form a reduced observability matrix

$$Q(x) = \begin{pmatrix} dh(x) \\ \vdots \\ dL_f^{r-1}h(x) \end{pmatrix}. \tag{13}$$

There remaining $n-r$ rows are collected in a matrix

$$R(x) = \begin{pmatrix} d\phi_{r+1}(x) \\ \vdots \\ d\phi_n(x) \end{pmatrix}. \tag{14}$$

Matrix Q results directly from system (1), whereas the matrix R is not uniquely determined. However, the observer gain in (7) depends also on R:

$$l(x) = (\Phi'(x))^{-1} \begin{pmatrix} k \\ 0 \end{pmatrix} = \left(\frac{Q(x)}{R(x)} \right)^{-1} \begin{pmatrix} k \\ 0 \end{pmatrix}. \tag{15}$$

In the following we suggest an observer gain, in which the matrix R does not occur. In particular, if the rows of (13) and (14) are orthogonal to each other, that is

$$\forall\, x \in \Omega:\ R(x)Q^T(x) = 0, \tag{16}$$

the observer gain (15) becomes

$$l(x) = \left(\frac{Q(x)}{R(x)} \right)^{-1} \begin{pmatrix} k \\ 0 \end{pmatrix} = \Big(Q^+(x) \,|\, R^+(x) \Big) \begin{pmatrix} k \\ 0 \end{pmatrix} = Q^+(x)\, k, \tag{17}$$

which depends explicitly only on the reduced observability matrix Q, where Q^+ denotes the Moore-Penrose inverse of Q, see [21, 23].

Clearly, the gain (17) is a special case of (15). The crucial question is whether the functions $\phi_{r+1}, \dots, \phi_n$ in (14) can be chosen such that condition (16) holds. To formulate the existence conditions, we consider the matrix $Q^+(x)$. This matrix is well-defined and smooth on Ω because of A1. The columns of Q^+ are vector fields:

$$Q^+(x) = (\tau_1(x), \dots, \tau_r(x)). \tag{18}$$

We need the following assumptions:

A5 The distribution

$$\Delta(x) = \operatorname{span}\{\tau_1(x), \ldots, \tau_r(x)\} \tag{19}$$

spanned by the columns of $Q^+(x)$ is involutive, i.e., for every two vector fields $\tau_1, \tau_2 \in \Delta$ there holds $[\tau_1, \tau_2] \in \Delta$.

A6 The vector fields $\tau_1, \ldots, \tau_r$ are complete.

Theorem 4.1 *Suppose system* (1) *fulfills A1, A5 and A6. Then, there exists a global diffeomorphism of the form* (2)*, for which the observer gain* (15) *becomes* (17)*.*

Proof The rows of Q are linearly independent due to A1. Therefore, the distribution Δ is regular with rank r. Then, the distribution Δ is integrable by the Theorem of Frobenius [14, p. 23], i.e., for any $x_0 \in \Omega$ there exist a neighbourhood $\mathcal{U}$ and smooth functions $\phi_{r+1}, \ldots, \phi_n$ such that

$$\forall x \in \mathcal{U}: \quad \langle d\phi_i(x), \tau_j(x) \rangle = 0 \tag{20}$$

for $j = 1, \ldots, r$ and $i = r+1, \ldots, n$. In addition the covector fields $d\phi_{r+1}, \ldots, d\phi_n$ are linearly independent. Equation (20) is equivalent to $R(x)Q^+(x) = 0$ for all $x \in \mathcal{U}$. From $Q^+ = Q^T(QQ^T)^{-1}$ we get (16) on $\mathcal{U}$. Therefore, the observer gain (15) becomes (17) on $\mathcal{U}$.

Now, we want to address global aspects. The proof of the Theorem of Frobenius is constructive, see [14]. In particular, the construction of the maps $\phi_{r+1}, \ldots, \phi_n$ is based on the flows of the vector fields $\tau_1, \ldots, \tau_r$. These vector fields are complete by A6. Moreover, we can always augment $\Delta(x)$ to $\mathbb{R}^n$ using a basis of complete vector fields $\tau_{r+1}, \ldots, \tau_n$. The map $\Psi(z, \eta) = \varphi^{\tau_1}_{z_1} \circ \cdots \circ \varphi^{\tau_r}_{z_r} \circ \varphi^{\tau_{r+1}}_{\eta_1} \circ \cdots \circ \varphi^{\tau_n}_{\eta_{n-r}}(x_0)$ with arbitrary $x_0 \in \Omega$, in which $\varphi^{\tau_i}_t$ denotes the flow of a vector field τ_i, is a global diffeomorphism onto Ω due to the completeness of the vector fields $\tau_1, \ldots, \tau_n$, see [25]. This implies that $\Phi := \Psi^{-1}$ is also a global diffeomorphism. The maps $\phi_{r+1}, \ldots, \phi_n$ are the last $n - r$ components of Ψ^{-1}. Therefore, these maps are defined on whole Ω, and (20) holds globally.

Instead of (5), which can be written as $\langle d\phi_i, g \rangle = 0$, the additional functions $\phi_{r+1}, \ldots, \phi_n$ now satisfy (20). However, to obtain the observer gain (17) we neither have to compute the functions ϕ_i nor the gradients $d\phi_i$ for $i = r+1, \ldots, n$. If we also have $g \in \Delta$, the second subsystem does not directly depend on u, i.e., in this case the observer design is carried out in the Byrnes-Isidori form as in [17], but without an explicit computation the zero dynamics. Condition A5 always holds for $r \in \{1, n\}$, where for $r = n$ we also have $Q(x) = \Phi'(x)$ and $Q^+(x) = (\Phi'(x))^{-1}$, by which we obtain the observer gain given in [13, 9, 10].

Note that one could in principle design an observer gain vector field like (17) with an arbitrary generalized inverse [2, 8] of the reduced observability matrix (13) to project the correction term $k(y - h(\hat{x}))$ to observable dynamics of the first subsystem. However, the crucial contribution of Theorem 4.1 is the insight, that the Moore-Penrose inverse used in (17) is part of a change of coordinates (2). Our observer is similar to that in [20] and [11], but derived by a different framework.

Combining Theorem 3.1 and 4.1 results in the following conclusion.

Corollary 4.1 *Consider system* (1) *with the observer* (7) *and the observer gain* (17)*. Assume that the input u is bounded and conditions A1 and A3–A6 hold, where r denotes the observation relative degree. Then, there exist a vector $k \in \mathbb{R}^r$ such that $\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0$ for all $x(0)$, $\hat{x}(0) \in \Omega$.*

5 Example

Consider the system

$$\dot{x} = \begin{pmatrix} x_1x_2 - x_1^3 \\ x_1 \\ -x_3 \\ x_1^2 + x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 2+2x_3 \\ 1 \\ 0 \end{pmatrix} u, \quad y = x_4, \tag{21}$$

taken from [14, p. 146] on $\Omega = \mathbb{R}^4$. From $L_g h(x) \equiv 0$ and $L_g L_f h(x) = 2(1+x_3)$ we conclude that system (21) has the observation relative degree $r = 2$. System (21) also has relative degree $r = 2$ if $x_3 \neq -1$. The first two components of the transformation (2) are $\phi_1(x) = h(x) = x_4$ and $\phi_2(x) = L_f h(x) = x_2 + x_1^2$. First, we design the observer as in [17] based on the Byrnes-Isidori normal form. The components ϕ_3 and ϕ_4 must satisfy (5), that is

$$L_g\phi_i(x) = (2+2x_3)\frac{\partial \phi_i}{\partial x_2} + \frac{\partial \phi_i}{\partial x_3} = 0 \tag{22}$$

for $i = 3, 4$. Two independent choices are $\phi_3(x) = x_2 - 2x_3 - x_3^2$ and $\phi_4(x) = x_4$, from which we obtain the Jacobian matrix

$$\Phi'(x) = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 2x_1 & 1 & 0 & 0 \\ \hline 0 & 1 & -2-2x_3 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right). \tag{23}$$

This Jacobian is singular if $x_3 = -1$. The singularity occurs on the same set where the relative degree is not defined. As a consequence, the observer gain (15) given by

$$l(x) = \begin{pmatrix} 0 \\ k_2 \\ \dfrac{k_2}{2+2x_3} \\ k_1 \end{pmatrix} \tag{24}$$

has a pole for $x_3 = -1$, i.e., the gain (24) is not defined for all $x \in \Omega$.

Now, we consider the approach suggested in Sect. 4. We have

$$Q^+(x) = (\tau_1(x), \tau_2(x)) = \begin{pmatrix} 0 & \dfrac{2x_1}{4x_1^2+1} \\ 0 & \dfrac{1}{4x_1^2+1} \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{25}$$

where the vector fields τ_1 and τ_2 are the first and second column of Q^+, respectively. Condition A5 is fulfilled since $[\tau_1, \tau_2] \equiv 0$. Note that these vector fields are complete. The observer gain can be computed directly from (17). We get

$$l(x) = \begin{pmatrix} \dfrac{2k_2x_1}{4x_1^2+1} \\ \dfrac{k_2}{4x_1^2+1} \\ 0 \\ k_1 \end{pmatrix}. \tag{26}$$

In several simulation scenarios, the observers (7) with the gain vector fields (24) and (26) behave similarly. However, in contrast to (24) the new observer gain (26) is well-defined for all $x \in \Omega$.

6 Conclusion

We addressed the problem of observer design for the special class of nonlinear systems. Similar as in [1, 17, 18, 26], the approach is based on a decomposition of the system into an observable and a possibly unobservable subsystem. In contrast to previous work, this paper is dedicated to the actual computation of the observer gain vector field. We exploit degrees of freedom to get an observer gain, whose symbolic computation is straightforward. In particular, the observer gain is an immediate generalization of the gain vector used in [13, 9, 10].

Acknowledgment

The author would like to thank Prof. Kurt J. Reinschke for his long-standing support. Moreover, Prof. Michael Zeitz (Stuttgart) is thanked for helpful suggestions. This work was supported by Deutsche Forschungsgemeinschaft under research grant RO 2427/1-2.

References

[1] Amicucci, G. L. and Monaco, S. On nonlinear detectability. *J. Franklin Inst.* **335B**(6) (1998) 1105–1123.

[2] Ben-Israel, A. and Greville, T. N. E. *Generalized Inverses: Theory and Applications.* Wiley–Interscience, 1974.

[3] Bestle, D. and Zeitz, M. Canonical form observer design for non-linear time-variable systems. *Int. J. Control* **38**(2) (1983) 419–431.

[4] Boothby, W. M. Some comments on global linearization of nonlinear systems. *Systems & Control Letters* **4** (1984) 143–147.

[5] Busawon, K., Farza, M. and Hammouri, H. Observer design for a special class of nonlinear systems. *Int. J. Control* **71**(3) (1998) 405–410.

[6] Byrnes, C. I. and Isidori, A. New results and examples in nonlinear feedback stabilization. *Systems & Control Letters* **12** (1989) 437–442.

[7] Byrnes, C. I. and Isidori, A. Asymptotic stabilization of minimum phase nonlinear systems. *IEEE Trans. on Automatic Control* **36**(10) (1991) 1122–1137.

[8] Campbell, S. L. and Meyer, C. D. *Generalized Inverses of Linear Transformations.* Dover Publications, New York, 1979.

[9] Ciccarella, G., Mora, M. D. and Germani, A. A Luenberger-like observer for nonlinear systems. *Int. J. Control* **57**(3) (1993) 537–556.

[10] Dalla Mora, M., Germani, A. and Manes, C. A state observer for nonlinear dynamical systems. *Nonlinear Analysis, Theory, Methods & Applications* **30**(7) (1997) 4485–4496.

[11] Farza, M., M'Saad, M. and Rossignol, L. Observer design for a class of MIMO nonlinear systems. *Automatica* **40**(1) (2004) 135–143.

[12] Fliess, M. and Hazewinkel, M. (Eds) *Algebraic and Geometric Methods in Nonlinear Control.* Reidel, Dordrecht, 1986.

[13] Gauthier, J. P., Hammouri, H. and Othman, S. A simple observer for nonlinear systems — application to bioreactors. *IEEE Trans. on Automatic Control* **37**(6) (1992) 875–880.

[14] Isidori, A. *Nonlinear Control Systems: An Introduction*. Springer-Verlag, London, 3-d edition, 1995.

[15] de Jager, B. Symbolic calculation of zero dynamics for nonlinear control systems. In *Proc. of the 1991 International Symposium on Symbolic and Algebraic Computation*, 1991, 321–322.

[16] Jo, N. H. and Seo, J. H. A state observer for nonlinear systems and its application to ball and beam system. *IEEE Trans. on Automatic Control* **45**(5) (2000) 968–973.

[17] Jo, N. H. and Seo, J. H. Input output linearization approach to state observer design for nonlinear system. *IEEE Trans. on Automatic Control* **45**(12) (2000) 2388–2393.

[18] Jo, N. H. and Seo, J. H. Observer design for non-linear systems that are not uniformly observable. *Int. J. Control* **75**(5) (2002) 369–380.

[19] Krener, A. J. and Isidori, A. Linearization by output injection and nonlinear observers. *Systems & Control Letters* **3** (1983) 47–52.

[20] Maggiore, M. and Passino, K. M. A separation principle for non-UCO systems. *IEEE Trans. on Automatic Control* **48**(7) (2003) 1122–1133.

[21] Moore, E. H. On the reciprocal of the general algebraic matrix. *Bull. Amer. Math. Soc.* **26** (1920) 394–395.

[22] Nijmeijer, H. and van der Schaft, A. J. *Nonlinear Dynamical Control systems*. Springer, New York, 1990.

[23] Penrose, R. A generalized inverse for matrices. *Proc. Cambridge Philosophical Society* **51** (1955) 406–413.

[24] Rajamani, R. Observers for Lipschitz nonlinear systems. *IEEE Trans. on Automatic Control* **43**(3) (1998) 397–401.

[25] Respondek, W. Global aspects of linearization, equivalence to polynomial forms and decomposition of nonlinear control systems. In: *Fliess and Hazewinkel* [12], 257–284.

[26] Röbenack, K. Zum High-Gain-Beobachterentwurf für eingangs-ausgangs-linearisierbare SISO-Systeme. *Automatisierungstechnik* **52**(10) (2004) 481–488.

[27] Sontag, E. D. and Wang, Y. Output-to-state stability and detectability of nonlinear system. *Systems & Control Letters* **29**(5) (1997) 279–290.

[28] Wu, F. F. and Desoer, C. A. Global inverse function theorem. *IEEE Trans. on Circuit Theory* **19** (1972) 199–201.

Appendix A. Proof of Theorem 3.1

The following lemma is a straight consequence of [10, Lemma 3.11]:

Lemma 1 *Given A, b, c defined in* (4) *and arbitrary constants $\nu, \rho > 0$. Then, there exist a vector $k \in \mathbb{R}^r$ and a positive definite matrix P such that*

$$(A - kc^T)^T P + P(A + kc^T) + \nu P b b^T P + \rho I < 0, \tag{27}$$

i.e., the matrix on the left hand side of (27) *is negative definite.*

Now, we prove Theorem 3.1.

We consider system (1) and observer (7) in the transformed coordinates, namely (3) and (6). We have to show that the equilibrium point $(\tilde{z}, \tilde{\eta}) = (0, 0)$ of the error dynamics (9) is asymptotically stable. We choose the candidate Lyapunov function $V(\tilde{z}, \tilde{\eta}) = \tilde{z}^T P \tilde{z} + \tilde{\eta}^T P_2 \tilde{\eta}$, where the positive definite matrix P will be specified later and the positive definite matrix P_2 is taken from A3. Then, we have

$$\begin{aligned}\dot{V}(\tilde{z},\tilde{\eta})\big|_{(9)} &= \tilde{z}^T\left[(A-kc^T)^TP+P(A+kc^T)\right]\tilde{z}+\gamma_3\|\tilde{z}\|^2-\gamma_4\|\tilde{\eta}\|^2\\ &\quad+2\,\tilde{z}^TPb\,[\alpha(z,\eta)+\beta(z,\eta)u-\alpha(\hat{z},\hat{\eta})-\beta(\hat{z},\hat{\eta})u]\end{aligned}\tag{28}$$

for all $(z,\eta),(\hat{z},\hat{\eta})\in\Phi(\Omega)$ and bounded u. Using (11), the inequality

$$\begin{aligned}\alpha(z,\eta)+\beta(z,\eta)u-\alpha(\hat{z},\hat{\eta})-\beta(\hat{z},\hat{\eta})u &\le 2\gamma_1|\tilde{z}^TPb|\cdot\|\tilde{z}\|+2\gamma_2|\tilde{z}^TPb|\cdot\|\tilde{\eta}\|\\ &\le\left(\gamma_1^2+\frac{\gamma_2^2}{\mu}\right)\tilde{z}Pbb^TP\tilde{z}+\tilde{z}^T\tilde{z}+\mu\,\tilde{\eta}^T\tilde{\eta}\end{aligned}$$

holds for arbitrary $\mu>0$, see [10, 24]. We set $\nu=\gamma_1^2+\gamma_2^2/\mu$, $\rho=\gamma_3+1$, where (28) becomes

$$\dot{V}(\tilde{z},\tilde{\eta})\big|_{(9)}\le\ \tilde{z}^T\left[(A-kc^T)^TP+P(A+kc^T)+\nu Pbb^TP+\rho I\right]\tilde{z}-(\gamma_4-\mu)\,\tilde{\eta}^T\tilde{\eta}.$$

This quadratic form is negative definite if we choose $\mu\in(0,\gamma_4)$ and take P and k from Lemma 6.

Nonlinear Dynamics and Systems Theory, **7** (4) (2007) 409–417

Third-Body Perturbation Using Single Averaged Model: Application to Lunisolar Perturbations

C.R.H. Solórzano and A.F.B.A. Prado *

Instituto Nacional de Pesquisas Espaciais (INPE)
São José dos Campos – 12227–010 – Brazil

Received: November 7, 2006; Revised: August 31, 2007

Abstract: In this paper, we considered the third-body perturbation using a single averaged model to study the effect of lunisolar perturbations on high-altitude Earth satellites. We combine two third-body perturbations. If no resonance occurs with the Moon or the Sun, short period terms are eliminated. In this way, we developed a semi-analytical study of the perturbation caused in a spacecraft by a third body with a single averaged model to eliminate the terms due to the short time periodic motion of the spacecraft. Several plots will show the time histories of the Keplerian elements.

Keywords: *Single averaged model; lunisolar perturbation; spacecraft.*

Mathematics Subject Classification (2000): 34C29, 65H05.

1 Introduction

The effects of the gravitational attractions of the Sun and the Moon in the orbits of an Earth's artificial satellites have been studied in several papers. Kozai [8] writes down the Lagrange's planetary equations and the disturbing function due to the Sun and to the Moon, including both secular and long periodic terms. Frick and Garber [4], using linear analysis, show that the result of the lunisolar attraction is a change of the orbital plane with small oscillations. Moreover, Musen [10] determines the long periodic disturbances caused by the Moon and the Sun in the motion of an artificial satellite. Kaula [6] derived general terms from the disturbing function for the lunisolar disturbance using equatorial elements for the Moon and the Sun.

* Corresponding author: prado@dem.inpe.br

Zee [16] studied the effects of the Sun and the Moon on a near-equatorial synchronous satellite, with particular attention to the trajectories of a geo-stationary synchronous satellite under the influence of the gravitational fields of an oblate Earth, the Sun and the Moon. Another work is the one by Kozai [9] that developed an alternative method for the calculation of the lunisolar disturbances. The disturbing function was expressed in terms of the orbital elements of the satellite and the geocentric coordinates of the Sun and the Moon.

Hough [5] used the Hamiltonian formed by a combination of the declination and the right ascension of the satellite, the Moon, and the Sun, and studied the periodic perigee motion for orbits near the critical inclinations 63.40^0 and 116.60^0. The theory predicts the existence of larger maximum fluctuations in eccentricity and faster oscillations near stable equilibrium points. Delhaise and Morbidelli [3] investigated the lunisolar effects of a geosynchronous artificial satellite orbiting near the critical inclination, analyzing each harmonic formed by a combination of the satellite and the Moon's longitude of the node. He demonstrated that the dynamics induced by these harmonics does not show resonance phenomena. Breiter [1] studied the effect in the resonance of apsides for satellites of low altitude, determining the resonant eccentricities between the secular motion of a satellite in terrestrial orbit and the longitudes of the Moon and the Sun. This study was made in hamiltonian form.

All these works present rich contributions and possess a sufficiently analytical approach, rich in derivations of equations. In the present work, an approach will be used to search numerical results, aiming to complement the existing literature. Papers more directed toward results and numerical comparisons had appeared recently, as the ones made by Broucke [2], Prado [11]. They all studied the disturbance of one third body on a satellite making an analytical and numerical study.

2 Mathematical Models

Our model can be formulated in a very similar way of the formulation of the planar restricted three-body problem. There are three bodies involved in the dynamics: one body with mass m_0, fixed in the origin of the reference system, a second massless body in orbit around m_0 and a third body (m') in a circular orbit around m_0 (see Figure 2.1). The motion of the spacecraft (the second massless body) is Keplerian and three-dimensional, with its orbital elements perturbed by the third body. The motion of the spacecraft is studied with the single averaged model, where the average is performed with respect to the true anomaly of the spacecraft (f). The disturbing function is then expanded in Legendre polynomials. The main body m_0 is fixed in the center of the reference system X-Y. The perturbing body m' is in a circular orbit with semi-major axis a' and mean motion n'. The spacecraft is in a three dimensional orbit, with orbital elements a, e, i, ω, Ω and mean motion n. In this situation, the disturbing potential that the spacecraft has from the action of the perturbing body is given by using the expansion in Legendre polynomials and assuming that $r' \gg r$.

$$\begin{aligned} R &= \mu' G(m_0 + m')/\sqrt{r^2 + r'^2 - 2rr'\cos(S)}, \\ R &= (\mu'(m_0 + m')/r') \sum_{n=2}^{\infty} (r/r')^n P_n \cos(S). \end{aligned} \tag{1}$$

The next step is to average all there terms of the disturbing function over the short period of the satellite. The definition for average used is:

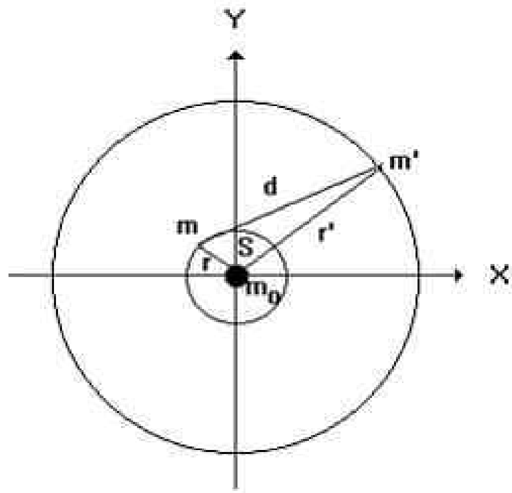

Figure 2.1: Illustration of the third body perturbation.

$$\langle G \rangle = (1/2)\int_0^{\infty} G dM. \tag{2}$$

Remember that M is the mean anomaly of the satellite and M' is the mean anomaly of the perturbing body. The results are made for the special case of circular orbits for the perturbing body and with the initial mean anomaly of the perturbing body equal to zero. The following relations are available (see [2]):

$$\alpha = \cos(\omega)\cos(\Omega - M') - \cos(i)\sin(\omega)\sin(\Omega - M'), \tag{3}$$
$$\beta = -\sin(\omega)\cos(\Omega - M') - \cos(i)\cos(\omega)\sin(\Omega - M'). \tag{4}$$

With those relations it is possible to relate the angle S with the positions of the perturbing and the perturbed bodies.

$$\cos(S) = \alpha\cos(f) + \beta\sin(f). \tag{5}$$

Substituting expression (5) into equation (1), and considering the equations (2)-(4), we have, after the averaged equations of motion of the spacecraft that are derived from the Lagrange's planetary equations, that they depend on the derivatives of the disturbing function [14]. It is noticed that the semi-major axis always remains constant. This occurs because, after the averaging, the disturbing function does not depend on M_0 (more details are available in [13]).

3 Results

We are now interested in the combined effects of the Moon and the Sun. For this, it is important to find the expansions of the disturbing function of the Sun and the Moon.

The term of second order due to the Sun is equivalent to the one of fourth order due to the effects of the Moon. In this section, simulations will consider the expansion made for the disturbing function of the Sun and the Moon in a combined form. The tests will be made considering satellites located in orbits with semi-major axis of 0.070 and 0.110 canonical units (26908 km and 42284 km).

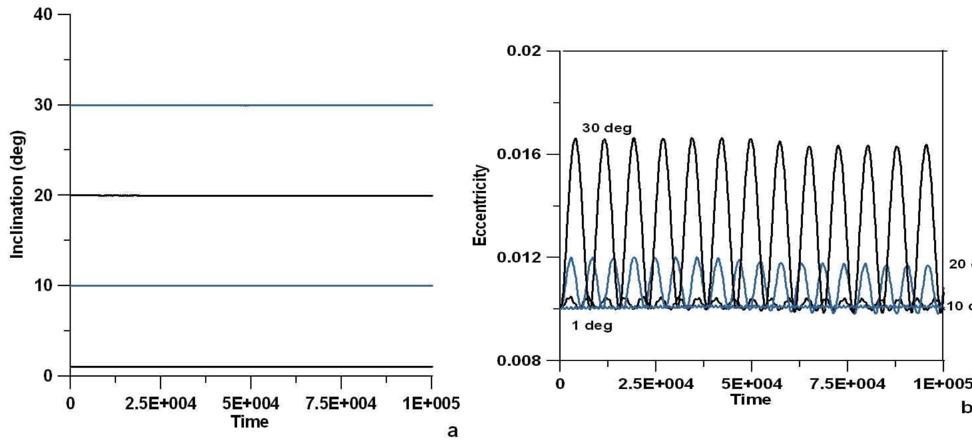

Figure 3.1: Behaviour of the orbital elements for values of the initial inclination below the critical value. (a) Inclination, (b) Eccentricity, both with a = 0.110.

The results for the lunisolar disturbance show a behavior similar to the ones obtained for the disturbance of the third body. Figure 3.1a shows the evolution of the inclination for initial values below the critical inclination. For the time scale used, the behavior of the inclination is constant. When analyzing the behavior of the eccentricity, it can be observed the several amplitudes reached with the increment of the initial inclination (Figure 3.1b). For an initial value of the inclination of 30 degrees, the eccentricity presents an amplitude around of 0.006, and for an initial inclination of 20 degrees, the eccentricity has an amplitude of 0.002. These small values do not change too much the orbit, since they still remain as almost-circular.

For larger values of the initial inclination (above the critical value), there is a typical behavior. It initiates in its initial value and soon it goes down until the critical value (Figure 3.2a) and then returns to the initial value. This oscillatory behavior presents the characteristic that, as the initial inclination increases, the amplitude of the inclination suffer increases. This fact reflects in the evolution of the eccentricity (Figure 3.2b), where the orbits with small eccentricities (almost circular) reach high values for the eccentricities, what affects the stability of the near-circular orbits. When the inclination reaches its minimum value, the eccentricity reaches its maximum value. This repetitive behavior is shown the Figures 3.2a and 3.2b, where the inclination starts in its initial value and, after a certain period, the inclination goes to its minimum value and the eccentricity reaches its maximum value.

For values of the inclination near the critical one (around 40°), we observe that, for values smaller than the critical inclination, there is an almost constant variation

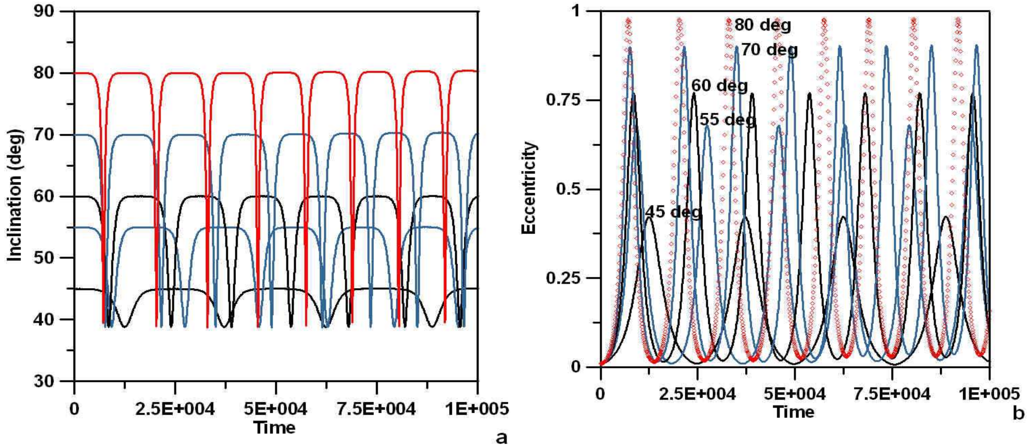

Figure 3.2: Behaviour of the orbital elements for values of the initial inclination above the critical value. (a) Inclination, (b) Eccentricity, both with a = 0.110.

(Figure 3.3a). As this value suffer increases, the inclination variations become larger. In the analysis of the eccentricity (Figure 3.3b), we observe that a zone exists where the eccentricity presents small oscillations. However, there are regions where it suffers large variations, making the almost circular orbit a very elliptical one. Figure 3.3c shows the evolution of the argument of the pericenter for values of the initial inclination near the critical value. The figures show the secular behavior of the argument of the pericenter for the time scale used. For larger times, it is observed the phenomenon called circulation. For an interval of shorter times, the secular curves are formed by small oscillations. Moreover, Figure 3.3d illustrates the retrograde behavior of the longitude of the node.

For an orbit with semi-major axis of 0.07 canonical units, one of the main characteristics is the small number of oscillations per unit of time. This brings, as a consequence, that when the satellite is located in a orbit with a semi-major axis of 0.110 canonical units, it reaches the critical inclination quickly. In Figure 3.5a we observe that, for initial inclination of 45 degrees, the satellite reaches its first critical value for a time of 25000 canonical units. In Figure 3.2a it is observed that the orbit reaches the critical inclination near 12000 canonical units of time. In the evolution of the inclination and the eccentricity, for the values of the initial inclination below the critical value, the inclination (Figure 3.4a) keeps its typical behavior, even that the eccentricity does not change significantly. As a consequence of that, the time necessary to reach the critical inclination is larger, and the number of oscillations in the evolution of the eccentricity is smaller (Figure 3.4b and Figure 3.5b). So, the analyzed behavior shows that, with high eccentricities, smaller inclinations are reached, as can be seen in the previous figures.

The evolution of the argument of the pericenter for initial inclinations larger than the critical value has secular and oscillatory behavior. It is an interesting point that, as the initial inclination increases, the satellite reaches higher values for the argument of the perigee. In the case of the longitude of the node, it presents the secular and

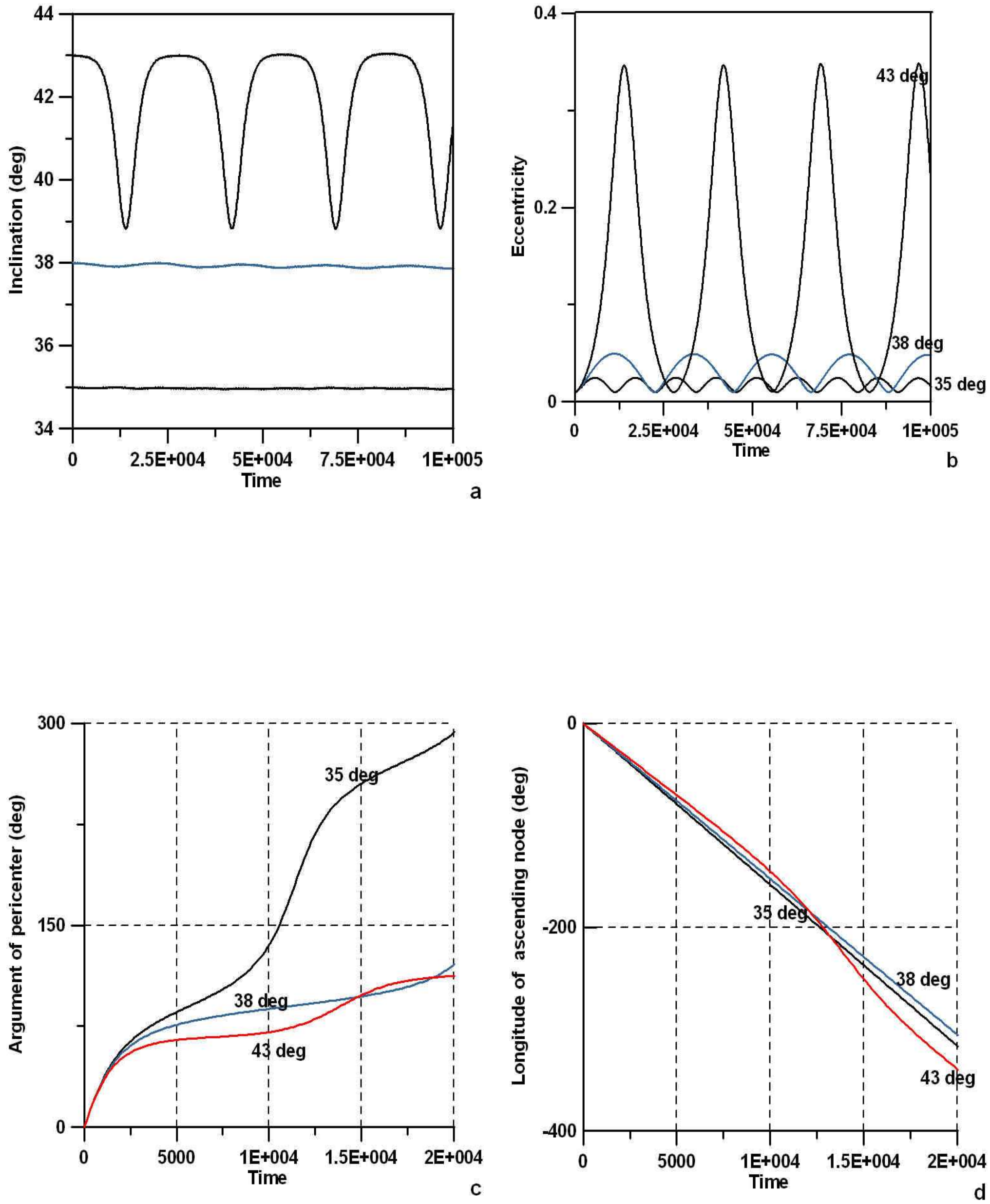

Figure 3.3: Behaviour of the orbital elements for values of the initial inclination near of critical value (a=0.110).

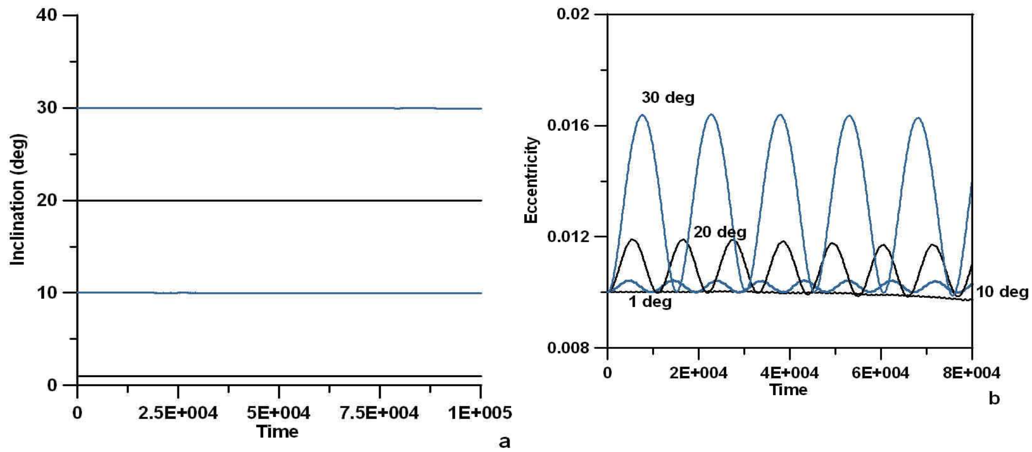

Figure 3.4: Behaviour of the orbital elements for values of the initial inclination below the critical value. (a) Inclination, (b) Eccentricity, both with a = 0.07.

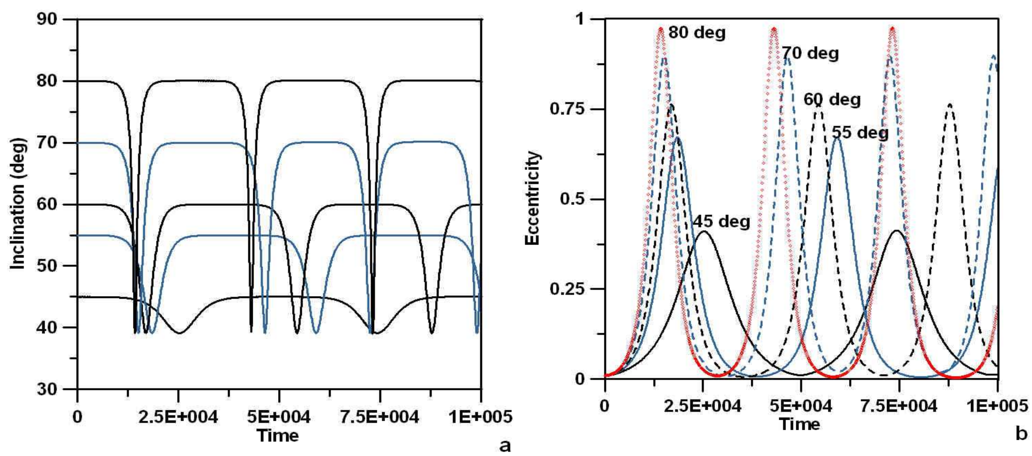

Figure 3.5: Behaviour of the orbital elements for values of the initial inclination above the critical value. (a) Inclination, (b) Eccentricity, both with a = 0.07.

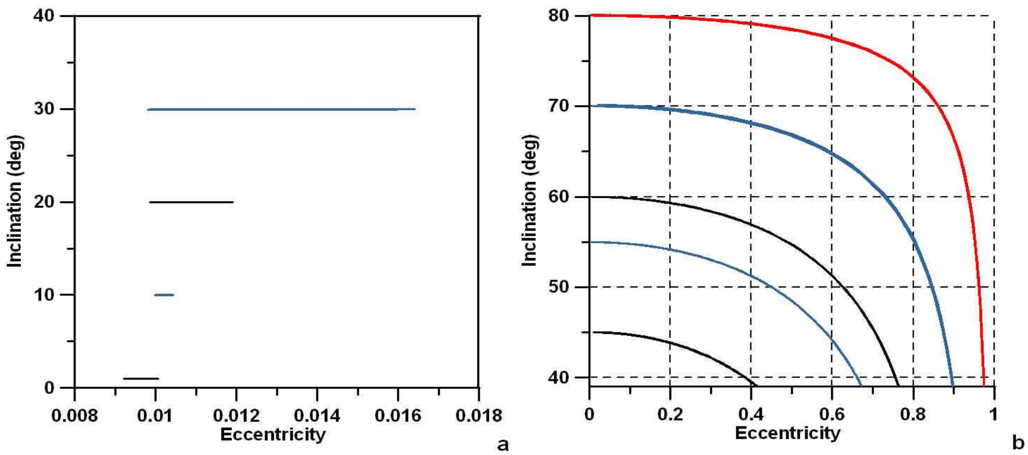

Figure 3.6: Eccentricity vs. inclination for several initial inclination. (a) Initial inclination below the critical value, (b) Initial inclination above the critical value, both with a = 0.07.

retrograde typical behavior. Figure 3.6 shows the behavior of the inclination and the eccentricity. For values of the inclination smaller than the critical value, there are small variations in the inclination and the eccentricity. These small oscillations allow that the almost circular orbits remain almost circular, but when the initial inclinations increase, the amplitude become larger.

4 Conclusions

Broucke [2] and Prado [11] used the approach of double-averaging technique to develop semi-analytical methods for the third-body perturbation. In another work, Solórzano et al [12] determined the effect of the disturbance of the third body by means of the single averaged model, being dedicated to the perturbative effects of the Moon in a spacecraft, but it did not considerer the case of lunisolar perturbation. When considering the disturbance of the third body by means of the single averaged model for the combined effect of the Sun and the Moon, in particular when the values of the initial inclination are below the critical value, the near-circular orbits remain near-circular. However, for values of the initial inclination above this value, the near-circular orbits become highly elliptical. This fact causes serious problems for the the stability of these orbits, being able to cause the collision of a spacecraft with the mother planet or the expulsion of the spacecraft of the orbit around the primary. All the results are a demonstration of the Kozai resonance. Our solution to the problem take terms of up to second-order in the expansion of the Sun's disturbing function and the fourth-order in the expansion of the Moon's disturbing function. The orbits of both (Sun and Moon) are considered as circular and coplanar. The critical angle of the disturbance of the third body appears of similar form to the critical angle obtained for the perturbative effects of the oblatenesses of the Earth on the spacecraft. However, other works as Kinoshita and Nakai [7] and

Yokoyama [15] make an estimate of the critical semi-major axis, as being the distance to which the effect of the oblatenesses and the solar disturbance are equivalent.

Acknowledgments

The authors are grateful to CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior), to the São Paulo State Science Foundation (FAPESP) for the research grant received under Contract 2003/03262-4 and to CNPq (Brazilian National Council for Scientific and Technological Development) for the contract 300221/95-9.

References

[1] Breiter, S. Lunisolar apsidal resonances at low satellite orbits. *Celestial Mechanics and Dynamical Astronomy* **74** (1999) 253–274.

[2] Broucke, R.A. Long-term third body effects via double averaging. *Journal of Guidance Control and Dynamics* **25**(1) (2003) 27–32.

[3] Delhaise, F. and Morbidelli, A. Lunisolar effects of Geosynchronous orbits at the critical inclination. *Celestial Mechanics and Dynamical Astronomy* **57** (1993) 155–173.

[4] Frick, R.H. and Garber, T.B. Perturbations of synchronous satellites. *Rand Corp. Report NASA* **399** (1962), 186 p.

[5] Hough, M.E. Lunisolar perturbations. *Celestial Mechanics and Dynamical Astronomy* **25** (1981) 111–136.

[6] Kaula, W.M. Development of the lunar and solar disturbing functions for a close satellite. *The Astronomical Journal* **67**(5) (1962) 300–303.

[7] Kinoshita, H. and Nakai, H. Secular perturbation of fictitious satellites of Uranus. *Celestial Mechanics and Dynamical Astronomy* **52** (1991) 293–303.

[8] Kozai, Y. On the effects of the Sun and Moon upon the motion of a close Earth satellite. *Smithsonian Inst. Astrophysical Observatory* **6** (1963), 47 p.

[9] Kozai, Y. A new method to compute lunisolar perturbations in satellite motions. *Smithsonian Inst. Astrophysical Observatory* **349** (1973) 27 p.

[10] Musen, P. On the long-period lunar and solar effect on the motion of an artificial satellite. *Journal of Geophysical Research* **66** (1961) 2797–2813.

[11] Prado, A.F.B.A. Third-body perturbation in orbits around natural satellites. *Journal of Guidance, Control and Dynamics* **26**(1) (2003) 33–40.

[12] Solórzano, C.R.H., Prado, A.F.B.A. and Kuga, H.K. Third-body perturbation using a single averaged model. *Proc. of the 16th Int. Symposium on Space Flight Dynamics*. Pasadena, California, EE.UU, 2001, 9 p.

[13] Solórzano, C.R.H. *Third-body perturbation using a single averaged model*. Master Dissertation, National Institute for Space Research (INPE), São José dos Campos, SP, Brazil, 2002.

[14] Taff, L.G. *Celestial Mechanics*. A Wiley – Interscience Publication (1985).

[15] Yokoyama, T. Dynamics of some fictitious satellites of Venus and Mars. *Planetary and SpaceScience* **47** (1999) 619–617.

[16] Zee, C. H. Effects of the Sun and the Moon on a near-equatorial synchronous satellite. *Astronautica Acta* **17** (1972) 891–906.

Nonlinear Dynamics and Systems Theory, **7** (4) (2007) 419–430

On the Non-Oscillation of Solutions of Some Nonlinear Differential Equations of Third Order

Cemil Tunç *

Department of Mathematics, Faculty of Arts and Sciences
Yüzüncü Yıl University, 65080, VAN – TURKEY

Received: October 12, 2006; Revised: August 22, 2007

Abstract: We present some new non-oscillation criteria for a class of non-linear differential equations of third order. Depending on these criteria, our results include and improve some well-known results in the literature.

Keywords: *Differential equation, third order, non-oscillation.*

Mathematics Subject Classification (2000): 34C11.

1 Introduction

We are concerned with non-oscillation of solutions of third-order nonlinear differential equations of the form

$$(r(t)y''(t))' + q(t)y'(t) + p(t)y^{\alpha}(g(t)) = f(t), \quad t \geq t_0 \tag{1}$$

and

$$(r(t)y''(t))' + q(t)\left(y'\left(g_1(t)\right)\right)^{\beta} + p(t)y^{\alpha}(g(t)) = f(t), \quad t \geq t_0, \tag{2}$$

where $t_0 \geq 0$ is a fixed real number, $f,\ p,\ q,\ r,\ g$ and $g_1 \in C\left([0,\infty), \Re\right)$ such that $r(t) > 0$ and $f(t) \geq 0$ for all $t \in [0,\infty)$. Throughout the paper, it is assumed, for all $g(t)$, $g_1(t)$, α and β appeared in (1) or (2), that $g(t) \leq t$ and $g_1(t) \leq t$ for all $t \geq t_0$; $\lim\limits_{t\to\infty} g(t) = \infty$ and $\lim\limits_{t\to\infty} g_1(t) = \infty$; both $\alpha > 0$ and $\beta > 0$ are quotients of odd integers.

In the relevant literature, till now, oscillation and non-oscillation behaviors of solutions of linear and non-linear second order, third order etc. differential equations have been the subject of intensive investigations for many authors. For instance, one can refer to [1–36] as some related papers or books on the subject. Now, to the best of our

* Corresponding author: cemtunc@yahoo.com.

knowledge, some results obtained in the literature on the topic of this paper can be summarized, briefly, as follows: First, in 1974, Kusano and Onose [12] studied the oscillatory and asymptotic behavior of solutions of the differential equation

$$x^{(n)}(t) + p(t)f\left(x(g(t))\right) = g(t),$$

and they established two theorems on the topic. In the same year, Kartsatos and Manougian [10] provided some sets of criteria sufficient for oscillation of either all solutions of equation

$$x^{(n)}(t) + P(t)f\left(x(g(t))\right) = Q(t), \quad n \geq 2,$$

or all bounded solutions of the same equation. Later, in 1979, Singh [30] discussed the asymptotic oscillatory behavior of the solutions of the differential equations

$$\left(r(t)y'(t)\right)^{(n-1)} + F\left(h\left(y(g(t))\right), t\right) = 0, \quad n \geq 2,$$
$$\left(r(t)y'(t)\right)' + a(t)h\left(y(g(t))\right) = f(t)$$

and

$$\left(r(t)y'(t)\right)' + p(t)y(t) + a(t)h\left(y(g(t))\right) = f(t).$$

Afterward, in 1985, Grace and Lalli [7] established some oscillation and non-oscillation criteria for the n–order nonlinear differential equation

$$x^{(n)}(t) + f\left(t,\, x(t),\, x[g(t)]\right) = h(t).$$

In 1981, N. Parhi [16] and in 1983, 1985, 1986 and 1987, N. Parhi and S. Parhi [26–29] discussed the qualitative behavior, oscillation and non-oscillation of solutions of a third order differential equation of the form

$$\left(r(t)y''\right)' + q(t)\left(y'\right)^{\beta} + p(t)y^{\alpha} = f(t).$$

Similarly, in 1993, Parhi [18] established some sufficient conditions for oscillation of all solutions of the second order forced differential equation of the form

$$\left(r(t)y'(t)\right)' + p(t)y^{\alpha}(g(t)) = f(t)$$

and non-oscillation of all bounded solutions of the equations

$$\left(r(t)y'(t)\right)' + q(t)\left(y'(t)\right)^{\beta} + p(t)y^{\alpha}(g(t)) = f(t)$$

and

$$\left(r(t)y'(t)\right)' + q(t)\left(y'\left(g_1(t)\right)\right)^{\beta} + p(t)y^{\alpha}(g(t)) = f(t),$$

where the real-valued functions $f,\ p,\ q,\ r,\ g$ and g_1 are continuous on $[0,\infty)$ with $r(t) > 0$ and $f(t) \geq 0$; $g(t) \leq t,\ g_1(t) \leq t$ for $t \geq t_0$; $\lim\limits_{t\to\infty} g(t) = \infty,\ \lim\limits_{t\to\infty} g_1(t) = \infty$, and both $\alpha > 0$ and $\beta > 0$ are quotients of odd integers. In addition, in 1994, Parhi and Das [22] considered nonlinear third-order differential equations of the form

$$y'''(t) + a(t)y''(t) + b(t)y'(t) + c(t)F\left(y(g(t))\right) = 0,$$

and they study the oscillatory and asymptotic behavior of solutions of this equation. In the same year, the same authors, Parhi and Das [22], also established some results for non-oscillation of solutions of equation

$$\left(r(t)y''(t)\right)' + q(t)y' + p(t)y = f(t)$$

and its associated homogeneous equation

$$(r(t)y''(t))' + q(t)y' + p(t)y = 0.$$

In 1996, Nayak and Choudhury [13] considered the differential equation

$$(r(t)y''(t))' - q(t)\,(y'(t))^{\beta} - p(t)y^{\alpha}(g(t)) = f(t)$$

and they gave certain sufficient conditions on the functions involved for all bounded solutions of the above equation to be non-oscillatory.

Later, in 2001, Adamets and Lomtatidze [1] investigated oscillatory properties of solutions of the third order linear differential equation

$$u''' + p(t)u = 0,$$

where p is a locally integrable function on $[0, \infty)$ which is eventually of one sign.

In the same year, Parhi and Padhi [24] also gave sufficient conditions ensuring that all nontrivial solutions of the third-order linear differential equation

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0$$

are oscillatory.

After that, in 2002, the same authors, Parhi and Padhi [24] proved several theorems provided sufficient conditions for the above equation to have oscillatory solutions, and they also studied the nature of nonoscillatory solutions of the same equation. Namely, sufficient conditions were given for the set of nonoscillatory solutions of the above equation to form a one-dimensional subspace of the solution space.

In 2003, Candan and Dahiya [5] investigated oscillatory and asymptotic properties of solutions of the third order forced differential equation

$$\left(\left(b(t)\left(a(t)x'\right)^{\alpha}\right)'\right)' + q(t)f(x(g(t))) = r(t).$$

Finally, more recently, in 2005, Agarwal et al. [3] established some new criteria for the bounded oscillation of a fourth order functional differential equation. Besides, in the same year, Zhong et al. [35] considered a third order linear neutral delay difference equation with positive and negative coefficients. By using the Banach contraction principle the authors established some sufficient conditions which ensure that the equation considered has a nonoscillatory solution.

In this paper, we restrict our considerations to the real solutions of equations (1) and (2) which exist on the half-line $[T, \infty)$, where T (≥ 0) depends on the particular solution, are non-trivial in any neighborhood of infinity. It is well-known that a solution $y(t)$ of (1) or (2) is said to be non-oscillatory on $[T, \infty)$ if there exists a $t_1 \geq T$ such that $y(t) \neq 0$ for $t \geq t_1$; it is said to be oscillatory if for any $t_1 \geq T$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$ such that $y(t_2) > 0$ and $y(t_3) < 0$; $y(t)$ is said to be a Z–type solution if it has arbitrarily large zeros but is ultimately non-negative or non-positive.

Now, it is reasonable to ask why the equations (1) and (2) have been investigated here. When one considers the papers and equations mentioned above, we think the importance of the investigation of behaviors of equations (1) and (2) may be acceptable.

2 Non-Oscillation Behaviors of Solutions of (1)

In this section, some sufficient conditions have been established for non-oscillation of all bounded solutions of (1). In order to reach our main results, first, we dispose of the following lemma.

Lemma 2.1 *Consider second order linear differential equation*

$$(r(t)z')' + q(t)z = 0, \tag{3}$$

where r and q are the same as in (1). If $z(t)$ is a non-oscillatory solution of equation (3) such that $z(t) > 0$ or $z(t) < 0$ for $t \in [a, \infty)$, $a > 0$, and if u is once continuously differentiable function on $[a, \infty)$, such that $u(b) = u(c) = 0$, $a < b < c$ and $u(t) \neq 0$ on $[b, c]$, then

$$\int_b^c \left[r(t)\ (u'(t))^2 - q(t)\,(u(t))^2 \right] dt > 0.$$

Proof See [28]. □

Next, in this section, we give the following four theorems.

Theorem 2.1 *Let us consider the equation (1), and let $f(t) - |p(t)| > 0$. If equation (3) is non-oscillatory, then all solution of equation (1), which are bounded above by 1, are non-oscillatory.*

Proof Let $y(t)$ be a bounded solution of (1) on $[T_y, \infty)$, $T_y > 0$, such that $|y(t)| \leq 1$. Since $\lim\limits_{t\to\infty} g(t) = \infty$, then there exists a $t_1 > t_0$ such that $g(t) \geq T_y$ for $t \geq t_1$. Now, if possible, let $y(t)$ be of non-negative Z–type solution with consecutive double zeros at a and b $(T_y \leq a < b)$ such that $y(t) > 0$ for $t \in (a, b)$. So, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Multiplying equation (1) through by $y'(t)$, we obtain

$$(r(t)y'(t)y''(t))' = r(t)\,(y''(t))^2 - q(t)\,(y'(t))^2 - p(t)y^{\alpha}(g(t))y'(t) + f(t)y'(t). \tag{4}$$

Integrating (4) from a to c, we get

$$\begin{aligned} 0 &= \int_a^c \left[r(t)\,(y''(t))^2 - q(t)\,(y'(t))^2 \right] dt + \int_a^c \left[f(t) - p(t)y^{\alpha}(g(t)) \right] y'(t)\,dt \\ &\geq \int_a^c \left[f(t) - |p(t)|\ |y^{\alpha}(g(t))| \right] y'(t)\,dt \geq \int_a^c \left[f(t) - |p(t)| \right] y'(t)\,dt > 0, \end{aligned}$$

a contradiction.

Next, let $y(t)$ be of non-positive Z–type solution with consecutive double zeros at a and b $(T_y \leq a < b)$. Then, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$. Integrating (4) from c to b, we have

$$\begin{aligned} 0 &= \int_c^b \left[r(t)\,(y''(t))^2 - q(t)\,(y'(t))^2 \right] dt + \int_c^b \left[f(t) - p(t)y^{\alpha}(g(t)) \right] y'(t)\,dt \\ &\geq \int_c^b \left[f(t) - |p(t)|\ |y^{\alpha}(g(t))| \right] y'(t)\,dt \geq \int_c^b \left[f(t) - |p(t)| \right] y'(t)\,dt > 0, \end{aligned}$$

a contradiction.

Now, if possible let $y(t)$ be oscillatory with consecutive double zeros at a, b and a' ($T_y < a < b < a'$) such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$, $y(t) < 0$ for $t \in (a,b)$ and $y(t) > 0$ for $t \in (b,a')$. Therefore, there exist $c \in (a,b)$ and $c' \in (b,a')$ such that $y'(c) = y'(c') = 0$ and $y'(t) > 0$ for $t \in (c,b)$ and $t \in (b,c')$. Integrating (3) from c to c' , we obtain

$$\begin{aligned} 0 &= \int_c^{c'} \left[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(t)\right)^2 \right] dt + \int_c^b \left[f(t) - p(t) y^\alpha(g(t)) \right] y'(t)\, dt \\ &\quad + \int_b^{c'} \left[f(t) - p(t) y^\alpha(g(t)) \right] y'(t)\, dt \\ &\geq \int_c^b \left[f(t) - |p(t)|\, |y^\alpha(g(t))| \right] y'(t)\, dt + \int_b^{c'} \left[f(t) - |p(t)|\, |y^\alpha(g(t))| \right] y'(t)\, dt \\ &\geq \int_c^b \left[f(t) - |p(t)| \right] y'(t)\, dt + \int_b^{c'} \left[f(t) - |p(t)| \right] y'(t)\, dt > 0, \end{aligned}$$

a contradiction. This completes the proof of Theorem 2.1. □

Remark 2.1 It should be noted that there is no sign restriction on $p(t)$ and $q(t)$, which appear in equation (1), in Theorem 2.1. Our result, Theorem 2.1, improves the results established in N. Parhi [17; Theorem 2.4, Theorem 2.5] and N. Parhi and S. Parhi [28; Theorem 1.1].

Theorem 2.2 *If equation (3) admits a non-oscillatory solution and* $\lim\limits_{t\to\infty} \frac{f(t)}{|p(t)|} = \infty$, *then all bounded solution of (1) are non-oscillatory.*

Proof Because of the fact that $\lim\limits_{t\to\infty} \frac{f(t)}{|p(t)|} = \infty$, there exists a $t_2 \geq t_1$ such that $f(t) \geq M^\alpha |p(t)|$ for all $t \geq t_2$, where M is a positive constant and α is defined as in (1). The remaining of the proof of Theorem 2.2 follows a similar way as shown in proof of Theorem 2.1, except some minor modifications; hence we omit the detailed proof. □

Remark 2.2 It is interesting to note that there is no sign restriction on $p(t)$ and $q(t)$, which appear in equation (1), in Theorem 2.2. The author in [33], Tunç, proved a different result, when $g(t) = t$ in (1), under the conditions whenever equation (3) is non-oscillatory, $p(t) \leq 0$ and $\lim\limits_{t\to\infty} \frac{f(t)}{-p(t)} = \infty$.

Theorem 2.3 *Consider the equation (1). If equation (3) admits a non-oscillatory solution and* $f(t) \geq K^\alpha |p(t)|$ *for large* t, *where* K *is a positive constant, then all* $y(t)$ *solutions of (1), which satisfy the inequality* $y(g(t)) \leq K$ *in any interval where* $y(t) > 0$, *are non-oscillatory.*

Proof The proof of this theorem, Theorem 2.3, is similar to the proof of Theorem 2.1 and hence is omitted. □

Remark 2.3 The motivation for Theorem 2.3 has been inspired basically by N. Parhi and S. Parhi [26; Theorem 2.4], in which $g(t) = t$, $p(t) \geq 0$ and $q(t) \geq 0$. Next, there is no sign restriction on $p(t)$ and $q(t)$ in Theorem 2.3 proved here, and the inequality $f(t) \geq K^\alpha |p(t)|$ does not implies $\frac{f(t)}{|p(t)|} \to \infty$ as $t \to \infty$. Therefore, our conditions far less restrictive than those established in N. Parhi and S. Parhi [26; Theorem 2.4].

Theorem 2.4 *Consider the equation (1) with $\alpha \geq 1$. Suppose that $p(t) \geq 0$ and $q(t) \leq 0$, $q(t)$ once continuously differentiable such that $q'(t) \geq 0$. If $\lim\limits_{t\to\infty} \frac{q'(t)}{p(t)} = \infty$, then all bounded solutions of (1) are non-oscillatory.*

Proof Let $y(t)$ be a bounded solution of equation (1) on $[T_y, \infty)$, $T_y \geq 0$, such that $|y(t)| \leq M$ for all $t \geq T_y$, where M is a positive constant. Since $\lim\limits_{t\to\infty} g(t) = \infty$, then there exists a $t_1 > t_0$ such that $g(t) \geq T_y$ for $t \geq t_1$. In view of the assumption $\lim\limits_{t\to\infty} \frac{q'(t)}{p(t)} = \infty$, it follows that there exists a $t_2 \geq t_1$ such that $q'(t) \geq M^{\alpha-1}p(t)$ for $t \geq t_2$. Now, if possible, let $y(t)$ be of non-negative Z–type solution with consecutive double zeros at a and b ($t_2 < a < b$) such that $y(t) > 0$ for $t \in (a, b)$. Thus, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Clearly, $y''(a) \geq 0$ and $y''(c) \leq 0$. Thereby, integrating (1) from a to c, we obtain

$$
\begin{aligned}
0 &> -\int_a^c q(t)y'(t)\,dt - \int_a^c p(t)y^{\alpha}(g(t))\,dt \\
&\geq -q(c)y(c) + \int_a^c q'(t)y(t)\,dt - \int_a^c p(t)y^{\alpha}(g(t))\,dt \\
&\geq \int_a^c q'(t)y(t)\,dt - \int_a^c M^{\alpha-1}p(t)y(g(t))\,dt \\
&\geq \int_a^c \left[q'(t) - M^{\alpha-1}p(t)\right] y(g(t))\,dt > 0
\end{aligned}
$$

or

$$
\geq \int_a^c \left[q'(t) - M^{\alpha-1}p(t)\right] y(t)\,dt > 0
$$

a contradiction.

Similarly, it can be shown that $y(t)$ can not be of non-positive Z–type solution and oscillatory. Hence this completes the proof of the theorem. □

Remark 2.4 N. Parhi and S. Parhi [28; Theorem 6] proved a result, when $g(t) = t$ in (1), under the conditions $p(t) \geq 0$, $q(t) \leq 0$, $q'(t) \geq 0$ and $\lim\limits_{t\to\infty} \frac{q'(t)}{p(t)} = \infty$. Our conditions and equation (1) are different from the equation considered and the conditions established by N. Parhi and S. Parhi [28; Theorem 6].

3 Non-Oscillation Behaviors of Solutions of (2)

In this section, some results have been proved for non-oscillation of all bounded solutions of (2). The first one is the following.

Theorem 3.1 *Let $q(t) \leq 0$. If $\lim\limits_{t\to\infty} \frac{f(t)}{|p(t)|} = \infty$, the all bounded solutions of (2) are non-oscillatory.*

Proof Let $y(t)$ be a bounded solution of equation (2) on $[T_y, \infty)$, $T_y > 0$, such that $|y(t)| \leq M$. Because of $\lim\limits_{t\to\infty} g(t) = \infty$, there exists a $t_1 > t_0$ such that $g(t) \geq T_y$ for $t \geq t_1$. Next, since $\lim\limits_{t\to\infty} \frac{f(t)}{|p(t)|} = \infty$, then it follows that there exists a $t_2 \geq t_1$ such that $f(t) > M^{\alpha}|p(t)|$ for $t \geq t_2$, where M is a positive constant and α is defined as the same in

(2). Now, if possible let $y(t)$ be of non-negative Z -type solution with consecutive double zeros at a and b $(t_2 < a < b)$ such that $y(t) > 0$ for $t \in (a,b)$. So, there exists $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a,c)$. Multiplying equation (2) through by $y'(t)$, we get

$$(r(t)y'(t)y''(t))' = r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)^\beta y'(t) - p(t)y^\alpha(g(t))y'(t) + f(t)y'(t). \quad (5)$$

Integrating (5) from a to c , we get

$$\begin{aligned} 0 &= \int_a^c \left[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)\beta y'(t)\right] dt + \int_a^c \left[f(t) - p(t)y^\alpha(g(t))\right] y'(t)\, dt \\ &\geq \int_a^c \left[f(t) - p(t)y^\alpha(g(t))\right] y'(t)\, dt \\ &\geq \int_a^c \left[f(t) - M^\alpha|p(t)|\,\right] y'(t)\, dt > 0, \end{aligned}$$

a contradiction.

Let $y(t)$ be of non-positive Z–type solution with consecutive double zeros at a and b $(t_2 < a < b)$. Then, there exists $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c,b)$.

Integrating (5) from c to b, we have

$$\begin{aligned} 0 &= \int_c^b \left[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)^\beta y'(t)\right] dt + \int_c^b \left[f(t) - p(t)y^\alpha(g(t))\right] y'(t)\, dt \\ &\geq \int_c^b \left[f(t) - |p(t)|\; |y^\alpha(g(t))|\,\right] y'(t)\, dt \geq \int_c^b \left[f(t) - M^\alpha|p(t)|\,\right] y'(t)\, dt > 0, \end{aligned}$$

a contradiction.

Now, if possible let $y(t)$ be oscillatory with consecutive double zeros at a, b and a' $(t_2 < a < b < a')$ such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$, $y(t) < 0$ for $t \in (a,b)$ and $y(t) > 0$ for $t \in (b,a')$. Therefore, there exist $c \in (a,b)$ and $c' \in (b,a')$ such that $y'(c) = y'(c') = 0$ and $y'(t) > 0$ for $t \in (c,b)$ and $t \in (b,c')$. Now, integrating (5) from c to c', we obtain

$$\begin{aligned} 0 &= \int_c^{c'} \left[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)^\beta y'(t)\right] dt + \int_c^{c'} \left[f(t) - p(t)y^\alpha(g(t))\right]\, y'(t)\, dt \\ &\geq \int_c^b \left[f(t) - p(t)y^\alpha(g(t))\right]\, y'(t)\, dt + \int_b^{c'} \left[f(t) - p(t)y^\alpha(g(t))\right]\, y'(t)\, dt \\ &\geq \int_c^b \left[f(t) - |p(t)|\; |y^\alpha(g(t))|\,\right]\, y'(t)\, dt + \int_b^{c'} \left[f(t) - |p(t)|\; |y^\alpha(g(t))|\,\right]\, y'(t)\, dt \\ &\geq \int_c^b \left[f(t) - M^\alpha|p(t)|\,\right]\, y'(t)\, dt + \int_b^{c'} \left[f(t) - M^\alpha|p(t)|\,\right]\, y'(t)\, dt > 0, \end{aligned}$$

a contradiction. Hence $y(t)$ is non-oscillatory. □

Remark 3.1 For the special case $g(t) = g_1(t) = 0$ in (2), under the acceptations $p(t) \geq 0$, $q(t) \leq 0$ and $\lim_{t\to\infty} \frac{f(t)}{p(t)} = \infty$, Theorem 3.1 has been proved by N. Parhi and S. Parhi [26]. Our result improves the result established in N. Parhi and S. Parhi [26].

Theorem 3.2 *If* $\lim_{t\to\infty} \dfrac{f(t)}{|p(t)|+|q(t)|} = \infty$*, the all solutions of equation (2), which are bounded together with their first derivatives, are non-oscillatory.*

Proof Let $y(t)$ be a solution of (2) on $[T_y, \infty)$, $T_y > 0$, such that $y(t)$ and $y'(t)$ are bounded. Hence, there exists positive constants M_1 and M_2 such that $|y(t)| \leq M_1$ and $|y'(t)| \leq M_2$ for all $t \geq T_y$. Further, in view of $\lim_{t\to\infty} g(t) = \infty$ and $\lim_{t\to\infty} g_1(t) = \infty$, it follows that there exists a $t_0 > 0$ such that $g(t) \geq T_y$ and $g_1(t) \geq T_y$ for $t \geq t_0$. Next, owing to the fact $\lim_{t\to\infty} \dfrac{f(t)}{|p(t)|+|q(t)|} = \infty$, clearly, there exists a $t_1 > t_0$ such that $f(t) > L(|p(t)| + |q(t)|)$ for all $t \geq t_1$. Now, if possible let $y(t)$ be of non-negative Z–type solution with consecutive double zeros at a and b $(t_1 < a < b)$ such that $y(t) > 0$ for $t \in (a, b)$. Thus, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a, c)$. Integrating (5) from a to c, we obtain

$$\begin{aligned}
0 &= \int_a^c r(t)\,(y''(t))^2\,dt + \int_a^c \Big[f(t) - \Big\{q(t)\,(y'(g_1(t)))^{\beta} + p(t)y^{\alpha}(g(t))\Big\}\Big]\,y'(t)\,dt \\
&\geq \int_a^c \Big[f(t) - \Big\{M_2^{\beta}|q(t)| + M_1^{\alpha}|p(t)|\Big\}\Big]\,y'(t)\,dt \\
&\geq \int_a^c \left[f(t) - L\left\{|q(t)| + |p(t)|\right\}\right]\,y'(t)\,dt > 0,
\end{aligned}$$

a contradiction, where $L = \max\Big\{M_1^{\alpha},\, M_2^{\beta}\Big\}$.

Now, if possible, let $y(t)$ be of non-positive Z–type solution with consecutive double zeros at a and b $(t_1 < a < b)$. Then, there exists $c \in (a, b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, b)$.

Integrating (5) from c to b, we have

$$\begin{aligned}
0 &= \int_c^b \Big[r(t)\,(y''(t))^2 - q(t)\,(y'(g_1(t)))^{\beta}\,y'(t) - p(t)y^{\alpha}(g(t))y'(t) + f(t)y'(t)\Big]\,dt \\
&\geq \int_c^b \Big[f(t) - \Big\{M_2^{\beta}|q(t)| + M_1^{\alpha}|p(t)|\Big\}\Big]\,y'(t)\,dt \\
&\geq \int_a^c \left[f(t) - L\left\{|q(t)| + |p(t)|\right\}\right]\,y'(t)\,dt > 0,
\end{aligned}$$

a contradiction, where $L = \max\Big\{M_1^{\alpha},\, M_2^{\beta}\Big\}$.

Finally, if possible, let $y(t)$ be oscillatory with consecutive double zeros at a, b and a' $(t_1 < a < b < a')$ such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$, $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. So, there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = y'(c') = 0$ and $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. Integrating (5) from c to c', we have

$$\begin{aligned}
0 &= \int_c^{c'} r(t)\,(y''(t))^2\,dt + \int_c^{c'} \Big[f(t) - q(t)\,(y'(g_1(t)))^{\beta} - p(t)y^{\alpha}(g(t))\Big]\,y'(t)\,dt \\
&\geq \int_c^{c'} \Big[f(t) - \Big\{M_2^{\beta}|q(t)| + M_1^{\alpha}|p(t)|\Big\}\Big]\,y'(t)\,dt \\
&\geq \int_c^{c'} \left[f(t) - L\left\{|q(t)| + |p(t)|\right\}\right]\,y'(t)\,dt > 0,
\end{aligned}$$

a contradiction, where $L = \max\left\{M_1^{\alpha},\, M_2^{\beta}\right\}$. Therefore, we conclude that $y(t)$ is non-oscillatory. Thus the theorem is proved. □

Remark 3.2 For the special case $g(t) = g_1(t) = t$ and $\alpha = \beta$ in (2), subject to the assumptions $p(t) \geq 0$, $q(t) \geq 0$ and $\lim\limits_{t\to\infty} \dfrac{f(t)}{p(t)+q(t)} = \infty$, Theorem 3.2 has been proved by Tunç [34]. Our result improves the result established in Tunç [34].

Theorem 3.3 *Consider equation (2) with $\alpha = \beta$ and $g(t) = g_1(t)$. Let $p(t) \geq 0$. If $\lim\limits_{t\to\infty} \dfrac{f(t)}{|q(t)| + p(t)(g(t))^{\alpha}} = \infty$, then all bounded solutions of equation (2), for which their first derivatives, $y'(t)$, are also bounded for large t, are non-oscillatory.*

Proof Let $y(t)$ be a solution of equation (2) on $[T_y, \infty)$, $T_y > 0$, such that $y(t)$ and $y'(t)$ are bounded. So, there exists positive constants M_1 and M_2 such that $|y(t)| \leq M_1$ and $|y'(t)| \leq M_2$ for all $t \geq T_y$. Next, from the fact $\lim\limits_{t\to\infty} g(t) = \infty$, it follows that there exists a $t_0 > 0$ such that $g(t) \geq T_y$ for $t \geq t_0$. By virtue of $\lim\limits_{t\to\infty} \dfrac{f(t)}{|q(t)| + p(t)(g(t))^{\alpha}} = \infty$, evidently, there exists a $t_1 > t_0$ such that $f(t) > L[|q(t)| + p(t)(g(t))^{\alpha}]$ for all $t \geq t_1$, where $L = \max\{M_1^{\alpha},\, M_2^{\alpha}\}$. Now, if possible, let $y(t)$ be of non-negative Z–type solution with consecutive double zeros at a and b $(t_1 < a < b)$ such that $y(t) > 0$ for $t \in (a,b)$. So, there exists $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (a,c)$. Consequently, there exists $d \in (a,c)$ such that $y''(d) = 0$ and $y''(t) > 0$ for $t \in (a,d)$. Next, clearly, $y'(t) \geq \frac{y(t)}{t}$ for large t_1 and $t \in [a,d]$. Hence, $y'(g(t)) \geq \dfrac{y(g(t))}{g(t)}$ for large t_1 and $t \in [a,d]$. Now, integrating equality (5) from a to d, we obtain

$$\begin{aligned}
0 &> \int_a^d r(t)\left(y''(t)\right)^2 dt + \int_a^d \left[f(t) - \left\{q(t)\left(y'(g_1(t))\right)^{\alpha} + p(t)y^{\alpha}(g(t))\right\}\right] y'(t)\,dt \\
&> \int_a^d \left[f(t) - \{|q(t)| + p(t)(g(t))^{\alpha}\}\left(y'(g(t))\right)^{\alpha}\right] y'(t)\,dt \\
&> \int_a^d \left[f(t) - M_2^{\alpha}\{|q(t)| + p(t)(g(t))^{\alpha}\}\right] y'(t)\,dt > 0,
\end{aligned}$$

a contradiction.

If possible, let $y(t)$ be of non-positive Z–type solution with consecutive double zeros at a and b $(t_1 < a < b)$. Then, there exists $c \in (a,b)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c,b)$.

Integrating (5) from c to b yields

$$\begin{aligned}
0 &= \int_c^b \Big[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)^{\alpha} y'(t) - p(t)y^{\alpha}(g(t))y'(t) + f(t)y'(t)\Big]\,dt \\
&\geq \int_c^b \left[f(t) - q(t)\left(y'(g_1(t))\right)^{\alpha}\right] y'(t)\,dt \geq \int_c^b \left[f(t) - |q(t)|\left(y'(g_1(t))\right)^{\alpha}\right] y'(t)\,dt \\
&\geq \int_c^b \left[f(t) - M_2^{\alpha}|q(t)|\right] y'(t)\,dt > 0,
\end{aligned}$$

a contradiction.

Now, if possible let $y(t)$ be oscillatory with consecutive double zeros at a, b and a' $(t_1 < a < b < a')$ such that $y'(a) \leq 0$, $y'(b) \geq 0$, $y'(a') \leq 0$, $y(t) < 0$ for $t \in (a, b)$ and $y(t) > 0$ for $t \in (b, a')$. Thus, there exist $c \in (a, b)$ and $c' \in (b, a')$ such that $y'(c) = y'(c') = 0$ and $y'(t) > 0$ for $t \in (c, b)$ and $t \in (b, c')$. Integrating (5) from c to b, we have

$$\begin{aligned}
0 &\geq r(b)y'(b)y''(b) \\
&= \int_c^b \left[r(t)\left(y''(t)\right)^2 - q(t)\left(y'(g_1(t))\right)^{\alpha} y'(t) - p(t)y^{\alpha}(g(t))y'(t) + f(t)y'(t) \right] dt \\
&\geq \int_c^b \left[f(t) - q(t)\left(y'(g(t))\right)^{\alpha} \right] y'(t)\, dt \geq \int_c^b \left[f(t) - |q(t)|\, \left(y'(g(t))\right)^{\alpha} \right] y'(t)\, dt \\
&\geq \int_c^b \left[f(t) - M_2^{\alpha} |q(t)| \right] y'(t)\, dt > 0,
\end{aligned}$$

a contradiction. In conclusion, $y''(b) > 0$. Besides, since $y''(c') < 0$, there exists $d \in (b, c')$ such that $y''(d) = 0$ and $y''(t) > 0$ for $t \in [b, d)$. For that reason, $y'(t) \geq \frac{y(t)}{t}$ for $t \in [b, d]$ and for sufficiently large t_1. Again, integrating equality in (5) from b to d, we get

$$\begin{aligned}
0 &\geq -r(b)y'(b)y''(b) \\
&= \int_b^d r(t)\left(y''(t)\right)^2 dt + \int_b^d \left[f(t) - q(t)\left(y'(g_1(t))\right)^{\alpha} - p(t)y^{\alpha}(g(t)) \right] y'(t)dt \\
&> \int_b^d \left[f(t) - \{ |q(t)| + p(t)(g(t))^{\alpha} \} \left(y'(g(t))\right)^{\alpha} \right] y'(t)\, dt \\
&> \int_b^d \left[f(t) - M_2^{\alpha} \{ |q(t)| + p(t)(g(t))^{\alpha} \} \right] y'(t)\, dt,
\end{aligned}$$

a contradiction. This completes the proof of the theorem. □

Remark 3.3 Theorem 3.3 includes, respectively, the results obtained by Tunç [33, Theorem 7] and improves the result established by of S. Parhi and N. Parhi [27, Theorem 2.5, Theorem 2.6].

Theorem 3.4 *Let $q(t) \leq 0$. If $f(t) \geq K^{\alpha}|p(t)|$ for large t, where K is a positive constant and α is defined as the same in equation (2). Then all solutions $y(t)$ of (2), which satisfy the inequality $y(g(t)) \leq K$ in any interval where $y(t) > 0$, are non-oscillatory.*

Proof The proof of the theorem is straightforward and hence is omitted. □

Remark 3.4 It should be noted that Theorem 3.4 is different than Theorem 2.3 just proved above because of $\beta \neq 1$ and $g_1(t) \neq t$.

References

[1] Adamets, L. and Lomtatidze, A. Oscillation conditions for a third-order linear equation. *Differ. Uravn.* **37**(6) (2001) 723–729, 861. [Russian]. Translation in *Differ. Equ.* **37**(6) (2001) 755–762.

[2] Agarwal, R.P., Grace, S.R. and O'Regan, D. Non-oscillatory solutions for higher order dynamic equations. *J. London Math. Soc.* (2) **67** (2003)(1) 165–179.

[3] Agarwal, R.P., Grace, S.R. and Wong, P.J.Y. On the bounded oscillation of certain fourth order functional differential equations. *Nonlinear Dyn. Syst. Theory* **5** (2005) 215–227.

[4] Agarwal, R.P., Grace, S.R. and Smith, T. Oscillation of certain third order functional differential equations. *Adv. Math. Sci. Appl.* **16**(1) (2006) 69–94.

[5] Candan, T. and Dahiya, R.S. Oscillation of third order nonlinear differential equations with and without delay. *Int. J. Differ. Equ. Appl.* **4**(2) (2002) 125–139.

[6] Dub, S.G. and Mingarelli, A.B. A non-oscillation theorem for differential matrix systems. *J. Math. Anal. Appl.* **306**(1) (2005) 349–363.

[7] Grace, S.R. and Lalli, B.S. On oscillation and nonoscillation of general functional-differential equations. *J. Math. Anal. Appl.* **109**(2) (1985) 522–533.

[8] Graef, J.R. and Gregus, M. Oscillatory properties of solutions of certain nonlinear third order differential equations. *Nonlinear Stud.* **7**(1) (2000) 43–50.

[9] Hartman, P. *Ordinary Differential Equations.* (Corrected reprint of the second (1982) edition), Classics in Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.

[10] Kartsatos, A.G. and Manougian, M.N. Perturbations causing oscillations of functional-differential equations. *Proc. Amer. Math. Soc.* **43** (1974) 111–117.

[11] Kusano, T. and Onose, H. Nonlinear oscillation of a sublinear delay equation of arbitrary order. *Proc. Amer. Math. Soc.* **40** (1973) 219–224.

[12] Kusano, T. and Onose, H. Oscillations of functional differential equations with retarded argument. *J. Differential Equations* **15** (1974) 269–277.

[13] Nayak, P.C. and Choudhury, R. Oscillation and nonoscillation theorems for third order functional-differential equation. *J. Indian Math. Soc. (N. S.)* **62**(1–4) (1996) 89–96.

[14] Padhi, S. On oscillatory solutions of third order differential equations. *Mem. Differential Equations Math. Phys.* **31** (2004) 109–111.

[15] Padhi, S. On oscillatory linear third order forced differential equations. *Differential Equations Dynam. Systems* **13**(3–4) (2005) 343–358.

[16] Parhi, N. Nonoscillatory behaviour of solutions of nonhomogeneous third order differential equations. *Applicable Anal.* **12**(4) (1981) 273–285.

[17] Parhi, N. Nonoscillation of solutions of a class of third order differential equations. *Acta Math. Hungar.* **54**(1–2) (1989) 79–88.

[18] Parhi, N. Sufficient conditions for oscillation and nonoscillation of solutions of a class of second order functional-differential equations. *Analysis* **13**(1–2) (1993) 19–28.

[19] Parhi, N. On non-homogeneous canonical third-order linear differential equations. *J. Austral. Math. Soc. Ser. A* **57**(2) (1994) 138–148.

[20] Parhi, N. and Das, P. Oscillation criteria for a class of nonlinear differential equations of third order. *Ann. Polon. Math.* **57**(3) (1992) 219–229.

[21] Parhi, N. and Das, P. On asymptotic property of solutions of linear homogeneous third order differential equations. *Boll. Un. Mat. Ital. B* (7) **7**(4) (1993) 775–786.

[22] Parhi, N. and Das, P. Oscillatory and asymptotic behaviour of a class of nonlinear functional-differential equations of third order. *Bull. Calcutta Math. Soc.* **86**(3) (1994) 253–266.

[23] Parhi, N. and Das, P. On nonoscillation of third order differential equations. *Bull. Inst. Math. Acad. Sinica* **22**(3) (1994) 267–274.

[24] Parhi, N. and Padhi, S. On oscillatory linear differential equations of third order. *Arch. Math.* (Brno) **37**(1) (2001) 33–38.

[25] Parhi, N. and Padhi, S. On oscillatory linear third order differential equations. *J. Indian Math. Soc.* (N.S.) **69**(1–4) (2002) 113–128.

[26] Parhi, N. and Parhi, S. Oscillation and nonoscillation theorems for nonhomogeneous third order differential equations. *Bull. Inst. Math. Acad. Sinica* **11**(2) (1983) 125–139.

[27] Parhi, N. and Parhi, S. Nonoscillation and asymptotic behaviour for forced nonlinear third order differential equations. *Bull. Inst. Math. Acad. Sinica* **13**(4) (1985) 367–384.

[28] Parhi, N. and Parhi, S. On the behaviour of solutions of the differential equations $(r(t)y'')'+q(t)\,(y')\,\beta+p(t)y^{\alpha}=f(t)$. *Ann. Polon. Math.* **47**(2) (1986) 137–148.

[29] Parhi, N. and Parhi, S. Qualitative behaviour of solutions of forced nonlinear third order differential equations. *Riv. Mat. Univ. Parma* **13**(4) (1987) 201–210.

[30] Singh, B. Necessary and sufficient condition for maintaining oscillations and nonoscillations in general functional equations and their asymptotic properties. *SIAM J. Math. Anal.* **10**(1) (1979) 18–31.

[31] Swanson, C.A. *Comparison and Oscillation Theory of Linear Differential Equations.* Mathematics in Science and Engineering, Vol. 48. Academic Press, New York-London, 1968.

[32] Tiryaki, A. and Pasinlioglu, S. Oscillatory behaviour of a class of nonlinear differential equations of third order. *Acta Math. Sci. Ser. B* (Engl. Ed.) **21**(2) (2001) 182–188.

[33] Tunç, C. On the nonoscillation of solutions of nonhomogeneous third order differential equations. *Soochow J. Math.* **23**(1) (1997) 1–7.

[34] Tunç, C. Non-oscillation criteria for a class of nonlinear differential equations of third order. *Bull. Greek Math. Soc.* **39** (1997) 131–137.

[35] Zhong, Xiao-Zhu; Xing, Hai-Long; Shi, Yan; Liang, Jing-Cui and Wang, Dong-Hua. Existence of nonoscillatory solution of third order linear neutral delay difference equation with positive and negative coefficients. *Nonlinear Dyn. Syst. Theory* **5**(2) (2005) 201–214.

[36] Zhong, Xiaozhu; Liang, Jingcui; Shi, Yan; Wang, Donghua and Ge, Lixia. Existence of nonoscillatory solution of high-order nonlinear difference equation. *Nonlinear Dyn. Syst. Theory* **6**(2) (2006) 205–210.

Nonlinear Dynamics and Systems Theory, **7** (4) (2007) 431–438

Generalized Monotone Iterative Technique for Functional Differential Equations with Retardation and Anticipation

A.S. Vatsala *

University of Louisiana at Lafayette, Department of Mathematics, Lafayette, Louisiana 70504 USA

Received: July 12, 2006; Revised: June 14, 2007

Abstract: The method of monotone iterative technique together with coupled lower and upper solutions is employed to prove the existence of coupled extremal solutions when the forcing function is the sum of an increasing and decreasing functions. This is referred to as generalized monotone method. This will include the usual monotone method results as special cases. Further using uniqueness condition uniqueness results for functional differential equations involving retardation and anticipation are also established.

Keywords: *Generalized monotone method, equations with retardation and anticipation.*

Mathematics Subject Classification (2000): 34C12, 34K05, 34K99.

1 Introduction

Qualitative and quantitative study of the functional differential equations with retardation and anticipation has very useful applications. Such dynamic systems occur in chaotic epidemic model and financial models, specifically stock exchange models. A typical model that arises is of the form

$$\begin{aligned} x'(t) &= F(t, x(t), y(t+\tau)) - ax(t), \\ y'(t) &= G(x(t-\tau) - by(t). \end{aligned} \tag{1}$$

* Corresponding author: aghalaya@gmail.com

See [3, 4, 5] for more details. Some numerical,computational and simulation methods are suggested for such equations in [3, 4, 5]. One formulation of such models [6] can be considered as

$$\begin{aligned} &x'(t) = f(t, x(t), x_t, x^t), \quad t \in I = [t_0, T], \\ &x_{t_0} = \phi_0, \quad x^T = \psi_0, \quad t_0 \geq 0, \quad t_0 < T. \end{aligned} \tag{2}$$

See [6] for other possible formulation. In [6] the authors developed existence theory for the general functional differential equations which involved with both retardation and anticipation, indicating other possible formulations. They achieved this by suitably applying results of [1]. Recently, in [7] the authors developed usual monotone iterative method by assuming the forcing function to be nondecreasing in the unknown function and its retardation term and non-increasing nature in the anticipation term. In this paper we develop the generalized monotone method as in [9, 11] for the functional differential equation with retardation and anticipation when the forcing function is the sum of a nondecreasing and non-increasing function in all its components. This yields the results of the usual monotone method [8, 10] as special cases. Using the method of coupled upper and lower solutions we develop sequences which converge to coupled minimal and maximal solutions. Further, using uniqueness condition we can prove that the nonlinear functional differential equations with retardation and anticipation problem has a unique solution. For more details on the monotone method and delay differential equations equations see [2] and the references therein.

2 Main Results

The usual monotone method developed in literature proves the existence of extremal solutions of

$$\begin{aligned} &x'(t) = f(t, x(t), x_t, x^t), \quad t \in I = [t_0, T], \\ &x_{t_0} = \phi_0, \quad x^T = \psi_0, \quad t_0 \geq 0, \quad t_0 < T, \end{aligned} \tag{3}$$

when $f(t, x, \phi, \psi)$ is either nondecreasing in x, ϕ, ψ or could be made nondecreasing by adding appropriate linear terms. This is precisely the onesided Lipschitz condition in x, ϕ, ψ. In this paper we develop monotone method for the following functional differential equation with retardation and anticipation, given by

$$\begin{aligned} x'(t) = f(t, x(t), x_t, x^t) + g(t, x(t), x_t, x^t), \quad t \in I = [t_0, T], \\ x_{t_0} = \phi_0, \quad x^T = \psi_0, \quad t_0 \geq 0, \quad t_0 < T, \end{aligned} \tag{4}$$

where $\mathcal{C}_1 = C([-h_1, 0], R)$, $\mathcal{C}_2 = C([0, h_2], R)$, $\phi_0 \in \mathcal{C}_1$, $\psi_0 \in \mathcal{C}_2$ and $f, g \in C(I \times R \times \mathcal{C}_1 \times \mathcal{C}_2, R)$, $h_1, h_2 > 0$. Here and in what follows, the symbols $x_t = x_t(s) = x(t+s)$, $-h_1 \leq s \leq 0$, $x^t = x^t(\sigma) = x(t+\sigma)$, $0 \leq \sigma \leq h_2$, representing retardation and anticipation, respectively. We plan to employ the generalized monotone iterative technique for proving the existence of unique solution for (4) utilizing coupled lower and upper solutions for (4) if of two different types. Through this paper we assume that f is nondecreasing in all its components or could be made nondecreasing by adding appropriate linear functions whereas g is non-increasing in all its components. Before we proceed further, we need to list the following known results relative to linear functional differential inequalities in a suitable form [10].

Lemma 2.1 *Assume that*

(i) $p \in C([t_0 - h_1, T + h_2], R)$, *p is continuously differentiable on* $I = [t_0, T]$ *and*

$$p'(t) \leq -Mp(t) - N \int_{-h_1}^{0} p_t(s)\, ds, \quad t \in I;$$

(ii) $p_{t_0}(s) \leq 0, \ -h_1 \leq s \leq 0, \ p \in C^1([t_0 - h_1, t_0], R), \ p'(s) \leq \dfrac{\lambda}{T + h_1}$ *where* $\min_{[t_0 - h_1, t_0]} p(s) = -\lambda, \ \lambda \geq 0$ *and* $[M + Nh_1](T + h_1) \leq 1$.

Then $p(t) \leq 0$ *on* $t_0 \leq t \leq T$.

This lemma is the suitable part of Lemma 2.1 in [10].

Lemma 2.2 *Suppose that* $p \in C([t_0 - h_1, T + h_2], R)$, $p'(s)$ *exists and is continuous on* I *and*

$$p'(t) \leq -Lp(t) + N_1 \int_{-h_1}^{0} p_t(s)\, ds + N_2 \int_{0}^{h_2} p^t(\sigma)\, d\sigma, \quad t \in I,$$

where $L, N_1, N_2 > 0$ *satisfying* $N_1 h_1 + N_2 h_2 < L$. *Then* $p_{t_0} \leq 0$, $p^T \leq 0$ *implies* $p(t) \leq 0$ *on* I.

Proof If the conclusion is false, there exists a $t_1 \in (t_0, T)$ and an $\epsilon > 0$ such that $p(t_1) = \epsilon, \ p(t) \leq \epsilon$ on I. It then follows that

$$0 = p'(t_0) \leq -L\epsilon + N_1 \epsilon h_1 + N_2 \epsilon h_2 < 0,$$

by assumptions proving $p(t) \leq 0$ on I. $\square$

Let us list the following assumptions relative to (4) for convenience.

We call α_0, β_0 as type I or of type II coupled lower and upper solutions of (4) respectively if (i) or (ii) below are satisfied.

(i) $\alpha_0, \beta_0 \in C^1(I, R)$ satisfies

$$\begin{aligned} \alpha_0'(t) &\leq f(t, \alpha_0(t), \alpha_{0t}, \alpha_0^t) + g(t, \beta_0(t), \beta_{0t}, \beta_0^t), \quad \alpha_{0t_0} = \phi_1, \quad \alpha_0^T = \psi_1, \\ \beta_0'(t) &\geq f(t, \beta_0(t), \beta_{0t}, \beta_0^t) + g(t, \alpha_0(t), \alpha_{0t}, \alpha_0^t), \quad \beta_{0t_0} = \phi_2, \quad \beta_0^T = \psi_2, \end{aligned}$$

such that $\phi_1 \leq \phi_0 \leq \phi_2, \ \psi_1 \leq \psi_0 \leq \psi_2, \ \alpha_0(t) \leq \beta_0(t)$ on I and $\phi_1, \phi_2 \in \mathcal{C}_1$, $\psi_1, \psi_2 \in \mathcal{C}_2$.

(ii) $\alpha_0, \beta_0 \in C^1(I, R)$ satisfies

$$\begin{aligned} \alpha_0'(t) &\leq f(t, \beta_0(t), \beta_{0t}, \beta_0^t) + g(t, \alpha_0(t), \alpha_{0t}, \alpha_0^t), \quad \alpha_{0t_0} = \phi_1, \quad \alpha_0^T = \psi_1, \\ \beta_0'(t) &\geq f(t, \alpha_0(t), \alpha_{0t}, \alpha_0^t) + g(t, \beta_0(t), \beta_{0t}, \beta_0^t), \quad \beta_{0t_0} = \phi_2, \quad \beta_0^T = \psi_2, \end{aligned}$$

such that $\phi_1 \leq \phi_0 \leq \phi_2, \ \psi_1 \leq \psi_0 \leq \psi_2, \ \alpha_0(t) \leq \beta_0(t)$ on I and $\phi_1, \phi_2 \in \mathcal{C}_1$, $\psi_1, \psi_2 \in \mathcal{C}_2$.

(iii)

$$f(t,x,\phi,\psi) = F(t,x,\phi,\psi) - M_1 x - N_1 \int_{-h_1}^{0} \phi(s)\,ds,$$

where $f(t,x,\phi,\psi)$ is nondecreasing in (x,ϕ,ψ) for each t,

$$g(t,x,\phi,\psi) = G(t,x,\phi,\psi) - M_2 x - N_2 \int_{-h_1}^{0} \phi(s)\,ds,$$

where $G(t,x,\phi,\psi)$ is non-increasing in (x,ϕ,ψ) for each t, whenever $\alpha_0(t) \le x \le \beta_0(t)$, $\alpha_{0t} \le \phi \le \beta_{0t}$, $\xi \in \mathcal{C}_2$ such that $M_1, N_1, M_2, N_2 \ge 0$. Also $M_1 + M_2 > 0$ and $N_1 + N_2 > 0$.

(iv) $\alpha_{0t_0} - \phi_0$, $\phi_0 - \beta_{0t_0}$ satisfying the assumptions (ii) of Lemma 2.1.

The type I and II of coupled lower and upper solutions assumed in (i)and (ii) in the assumption are utilized in [8, 9, 11] fruitfully. We are now in a position to state and prove our main result relative to coupled lower and upper solutions of type I.

Theorem 2.1 *Suppose that assumptions (i) to (iv) except (ii) are satisfied. Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ such that $\alpha_n(t) \to \rho(t)$, $\beta_n(t) \to r(t)$ uniformly as $n \to \infty$ on $[t_0 - h_1, T + h_2]$ and that (ρ, r) are coupled minimal and maximal solutions of (4). That is $\rho(t)$, $r(t)$ satisfies*

$$\rho' = f(t,\rho,\rho_t,\rho^t) + G(t,r,r_t,r^t) - M_2\rho(t) - N_2 \int_{-h_1}^{0} \rho_t(s)\,ds, \tag{5}$$

$$r' = f(t,r,r_t,r^t) + G(t,\rho,\rho_t,\rho^t) - M_2 r(t) - N_2 \int_{-h_1}^{0} r_t(s)\,ds. \tag{6}$$

If, in addition,

(v) $f(t,x,\phi_1,\psi_1) - f(t,y,\phi_2,\psi_2)$

$$\le -L_1(x-y) + N_{11}\int_{-h_1}^{0} (\phi_1-\phi_2)(s)\,ds + N_{12}\int_{0}^{h_2} (\psi_1-\psi_2)(\sigma)\,d\sigma$$

and

$G(t,y,\phi_2,\psi_2) - G(t,x,\phi_1,\psi_1)$

$$\le -L_2(x-y) + N_{21}\int_{-h_1}^{0} (\phi_1-\phi_2)(s)ds + N_{22}\int_{0}^{h_2} (\psi_1-\psi_2)(\sigma)\,d\sigma$$

where $L_1, L_2, N_{11}, N_{12}, N_{21}, N_{22} \ge 0$, such that $(L_1 + L_2) > 0$, $(N_{11} + N_{21}) > 0$ $(N_{12} + N_{22}) > 0$ for $\alpha_0(t) \le y \le x \le \beta_0(t)$, $\alpha_{0t} \le \phi_2 \le \phi_1 \le \beta_{0t}$, $\alpha_0^T \le \psi_1 \le \psi_2 \le \beta_0^T$ and $(N_{11} + N_{21} - N_2)h_1 + (N_{21} + N_{22})h_2 < L_1 + L_2 + M_2$, holds, then $\rho(t) = r(t) = x(t)$ is the unique solution of (4) on I.

Proof Consider the following linear problem for each $n = 1, 2, 3, \ldots$

$$\begin{aligned} \alpha'_{n+1} = {} & F(t,\alpha_n,\alpha_{nt},\alpha_n^t) - M_1\alpha_{n+1}(t) - N_1\int_{-h_1}^{0} (\alpha_{(n+1),t}(s)\,ds \\ & + G(t,\beta_n,\beta_{nt},\beta_n^t) - M_2\alpha_{n+1}(t) - N_2\int_{-h_1}^{0} \alpha_{(n+1),t}(s)\,ds \end{aligned} \tag{7}$$

$$\begin{aligned}\beta'_{n+1} = \; & F(t,\beta_n,\beta_{nt},\beta_n^t) - M_1\beta_{n+1}(t) - N_1\int_{-h_1}^{0}\beta_{(n+1),t}(s)\,ds \\ & + G(t,\alpha_n,\alpha_{nt},\alpha_n^t) - M_2\beta_{n+1}(t) - N_2\int_{-h_1}^{0}\beta_{(n+1),t}(s)\,ds\end{aligned} \tag{8}$$

with $\alpha_{(n+1)t_0} = \phi_0,\ \beta_{(n+1)t_0} = \phi_0$ and $\alpha^T_{n+1},\ \beta^T_{n+1}$ are chosen such that

$$\alpha_0^T \le \alpha_n^T \le \alpha_{n+1}^T \le \psi_0 \le \beta_{n+1}^T \le \beta_n^T \le \beta_0^T \tag{9}$$

and α_n^T, β_n^T converge uniformly to ψ_0 on $[0,h_2]$ (see Remark 2.1).

Clearly each linear problem has a unique solution on $[t_0-h_1, T+h_2]$. We wish to show that

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \ldots \le \alpha_n \le \beta_n \le \ldots \le \beta_2 \le \beta_1 \le \beta_0 \quad \text{on} \quad I. \tag{10}$$

We claim first that $\alpha_0 \le \alpha_1$ on I. Since $\alpha_0(t) \le \beta_0(t)$ on I and using (iv) we get

$$\begin{aligned}\alpha_0'(t) &\le f(t,\alpha_0(t),\alpha_{0t},\alpha_0^t) + g(t,\beta_0(t),\beta_{0t},\beta_0^t) \\ &\le F(t,\alpha_0(t),\alpha_{0t},\alpha_0^t) - M_1\alpha_0(t) - N_1\int_{-h_1}^{0}\alpha_{0t}(s)\,ds \\ &\quad + G(t,\beta_0(t),\beta_{0t},\beta_0^t) - M_2\alpha_0(t) - N_2\int_{-h_1}^{0}\alpha_{0t}(s)\,ds.\end{aligned}$$

Now set $p = \alpha_0 - \alpha_1$ so that it follows from (7), (9) and condition (i),

$$\begin{aligned}p' = \alpha_0' - \alpha_1' &\le F(t,\alpha_0,\alpha_{0t},\alpha_0^t) - F(t,\alpha_0,\alpha_{0t},\alpha_0^t) + G(t,\beta_0,\beta_{0t},\beta_0^t) - G(t,\beta_0,\beta_{0t},\beta_0^t) \\ &\quad + (M_1+M_2)(\alpha_1-\alpha_0) + (N_1+N_2)\int_{-h_1}^{0}(\alpha_{1t}-\alpha_{0t})(s)\,ds \\ &\le -(M_1+M_2)p - (N_1+N_2)\int_{-h_1}^{0}p_t(s)\,ds, \quad t\in I\end{aligned}$$

and

$$p_{t_0} = \alpha_{0t_0} - \alpha_{1t_0} \le 0.$$

By Lemma 2.1, in view of assumption (iii), this implies $\alpha_0 \le \alpha_1$ on I. Similarly, we can show that $\beta_1 \le \beta_0$ on I.

Next we prove that $\alpha_1 \le \beta_1$ on I. Setting $p = \alpha_1 - \beta_1$, we obtain in view of (7), (8), for $t \in I$,

$$\begin{aligned}p' = \alpha_1' - \beta_1' &= F(t,\alpha_0,\alpha_{0t},\alpha_0^t) - F(t,\beta_0,\beta_{0t},\beta_0^t) + G(t,\beta_0,\beta_{0t},\beta_0^t) - G(t,\alpha_0,\alpha_{0t},\alpha_0^t) \\ &\quad - (M_1+M_2)(\alpha_1-\beta_1) - (N_1+N_2)\int_{-h_1}^{0}(\alpha_{1t}-\beta_{1t})(s)\,ds.\end{aligned}$$

Since $F(t,x,\phi,\psi)$ and $G(t,x,\phi,\psi)$ is nondecreasing and non-increasing in (x,ϕ,ψ) respectively for each t, we get

$$p' = \alpha_1' - \beta_1' \le -(M_1+M_2)(p(t)) - (N_1+N_2)\int_{-h_1}^{0}p_t(s)\,ds$$

and from Lemma 2.1 it follows that $p(t) \leq 0$ which proves that $\alpha_1(t) \leq \beta_1(t)$ on I. As a result, it follows that

$$\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0 \quad \text{on} \quad I. \tag{11}$$

Now suppose that for some $k > 1$, we have

$$\alpha_{k-1} \leq \alpha_k \leq \beta_k \leq \beta_{k-1} \quad \text{on} \quad I. \tag{12}$$

We shall show that

$$\alpha_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta_k \quad \text{on} \quad I. \tag{13}$$

To do this, let $p = \alpha_k - \alpha_{k+1}$ so that $p_{t_0} = 0$ and

$$\begin{aligned}
p' = \alpha_k' - \alpha_{k+1}' &= F(t, \alpha_{(k-1)}, \alpha_{(k-1)t}, \alpha_{(k-1)}^t) - M_1(\alpha_k) - N_1 \int_{-h_1}^0 \alpha_{kt}(s)\,ds \\
&\quad + G(t, \beta_(k-1), \beta_{(k-1)t}, \beta_(^t k-1)) - M_2\alpha_k(t) - N_2 \int_{-h_1}^0 \alpha_{(k),t}(s)\,ds \\
&\quad - (F(t, \alpha_k, \alpha_{kt}, \alpha_k^t) - M_1\alpha_{k+1}(t) - N_1 \int_{-h_1}^0 \alpha_{(k+1),t}(s)ds) \\
&\quad - (G(t, \beta_n, \beta_{nt}, \beta_k^t) - M_2\alpha_{k+1}(t) - N_2 \int_{-h_1}^0 \alpha_{(k+1),t}(s)ds).
\end{aligned}$$

Substituting for F in terms of f and using the monotone nature of f, G and simplifying we get

$$p' \leq (M_1 + M_2)(\alpha_k - \alpha_{(k+1)} + (N_1 + N_2) \int_{-h}^0 (\alpha_{kt} - \alpha_{k+1t})(s)\,ds \quad \text{and} \quad p_{t_0} = 0.$$

This implies by Lemma 2.1 that $\alpha_k \leq \alpha_{k+1}$ on I. Similarly, we can show that $\beta_{k+1} \leq \beta_k$ on I. To prove $\alpha_{k+1} \leq \beta_{k+1}$ on I, consider $p = \alpha_{k+1} - \beta_{k+1}$ so that $p_{t_0} = 0$ and arguing as before, one can show that

$$p' \leq -(M_1 + M_2)p - (N_1 + N_2) \int_{-h_1}^0 p_t(s)\,ds,$$

and $p_{t_0} = 0$, which yields $\alpha_{k+1} \leq \beta_{k+1}$, on I. Thus we have (13) and therefore by induction, we see that (10)is valid on I. This together with (9) follows that (10) is also true on $[t_0, T + h_2]$.

Since the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are bounded by (10), employing the standard arguments [8, 9] namely Ascoli-Arzela and Dini theorems, one can conclude that $\{\alpha_n\}$, $\{\beta_n\}$ converge uniformly on $[t_0, T]$, that is, $\alpha_n \to \rho$, $\beta_n \to r$ uniformly on $[t_0, T]$.

Also, it is easy to show that (ρ, r) satisfy (5) and (5) respectively with $\rho \leq r$ on I and $\rho^T = r^T$. To show that (ρ, r) are coupled minimal and maximal solutions of (4), let $x(t)$ be any solution of (4) with $x_{t_0} = \phi_0$, $x^T = \psi_0$ such that $\alpha_0 \leq x \leq \beta_0$ on I. Then it is enough to show that $\rho \leq x \leq r$ since by definition of (ρ, r) we already have $\rho^T = x^T = r^T$. Setting $p = \alpha_1 - x$ so that $p_{t_0} = 0$ and

$$\begin{aligned}
p' = \alpha_1' - x' &= F(t, \alpha_0, \alpha_{0t}, \alpha_0^t) - M_1\alpha_1 - N_1 \int_{-h_1}^0 \alpha_{1t}(s)\,ds - f(t, x, x_t, x^t) \\
&\quad + G(t, \beta_0, \beta_{0t}, \beta_0^t) - M_2\alpha_1 - N_2 \int_{-h_1}^0 \alpha_{1t}(s)\,ds - g(t, x, x_t, x^t).
\end{aligned}$$

Since $\alpha_0 \le x(t) \le \beta_0$, substituting for (F, g) in terms of (f, G) from (iii) and using the nondecreasing and non-increasing nature of f, G the above equation simplifies to the

$$p' \le -((M_1 + M_2)(\alpha_1 - x) - (N_1 + N_2)\int_{-h_1}^{0} (\alpha_{1t} - x_t)(s)\, ds.$$

Thus we get from Lemma 2.1, $\alpha_1 \le x$ on I. Similarly, $x \le \beta_1$ on I. By proceeding similarly and by induction, it is easy to show that $\alpha_{n+1} \le x \le \beta_{n+1}$ on I for all n. Hence (ρ, r) are coupled minimal and maximal solutions of (4).

If, in addition, condition (v) holds, since $\rho \le r$, we let $p = r - \rho$ and find using $\rho^t \le r^t$ and (v),

$$\begin{aligned} p' = r' - \rho' = f(t, r, r_t, r^t) - f(t, \rho, \rho_t, \rho^t) + G(t, \rho, \rho_t, \rho^t) - M_2 r - N_2 \int_{h_1}^{0} r_t(s)\, ds \\ - \left\{ G(t, r, r_t, r^t) - M_2 \rho - N_2 \int_{h_1}^{0} \rho_t(s) ds \right\}. \end{aligned}$$

Using (v) this simplifies to

$$\begin{aligned} p' &\le -L_1(r - \rho) + N_{11} \int_{-h_1}^{0} (r_t - \rho_t)(s) ds + N_{12} \int_{0}^{h_2} (r^t - \rho^t(\sigma) d\sigma \\ &\quad + (-L_2 - M_2)(r - \rho) + (N_{21} - N_2) \int_{-h_1}^{0} (r_t - \rho_t)(s) ds + N_{22} \int_{0}^{h_2} (r^t - \rho^t(\sigma)\, d\sigma \\ &\le -(L_1 + L_2 + M_2)(p) + (N_{11} + N_{21} - N_2) \int_{-h_1}^{0} (p_t)(s) ds + (N_{12} + N_{22}) \int_{0}^{h_2} p^t(\sigma)\, d\sigma \end{aligned}$$

and

$$p_{t_0} = 0, \quad p^T = 0.$$

This implies by Lemma 2.2, $p(t) \le 0$ on I, which means $\rho = r$ on I. This proves that $x = \rho = r$ is the unique solution of (4) with $x_{t_0} = \phi_0$, and $x^T = \psi_0$. The proof is therefore complete. □

Here we recall the remark of [7] for completion of our Theorem 2.1.

Remark 2.1 A simple choice of (9) would be to take for α_n^T, β_n^T, suitable translates of ψ_0 such that $\alpha_n^T = \psi_0 - \epsilon_n$, $\beta_n^T = \psi_0 + \eta_n$, with $\alpha_n(T) = \psi_0(T) - \epsilon_n$, $\beta_n(T) = \psi_0(T) + \eta_n$, for each n, where ϵ_n, $\eta_n > 0$ are monotone sequences tending to zero as $n \to \infty$. To make life simpler still, one can assume that $\alpha_0^T = \beta_0^T = \psi_0$. Note also that given any ϕ_0 with $\alpha_{0t_0} \le \phi_0 \le \beta_{0t_0}$, ψ_0 need to satisfy the inequality $\alpha_1(T) \le \psi_0(T) \le \beta_1(T)$ so that the choice (9) is possible.

Remark 2.2 If $g \equiv 0$ then the results of [7] related to (3) can be obtained as a special case of Theorem 2.1.

Recall that the method of lower and upper solutions provides existence results in the closed set generated by lower and upper solutions. In general coupled upper and lower solutions of type II can be easily constructed. See [9, 11] for details. Generalized monotone method using coupled upper and lower solutions of type II will be discussed elsewhere.

References

[1] Bernfeld, S.R., Lakshmikantham, V. and Reddy, Y.M. Fixed point theorems of operators with PPF dependence in Banach spaces. *Applicable Analysis* **6** (1977) 271–280.

[2] Dshalalow, J.H. and Martynyuk, A.A. Personage in Science (Professor V. Lakshmikantham). *Nonlinear Dynamics and Systems Theory* **5**(2) (2005) 107–111.

[3] Dubois, D.M. Theory of incursive synchronization and application to the anticipation of a chaotic epidemic. *Int. J. Computing Anticipatory Systems* **10** (2001) 3–18.

[4] Dubois, D.M. Extension of the Kalder-Kaleeki model of business cycle with computational anticipated capital stock. *Organisational and Social Change* **1** (2004) 63–80.

[5] Dubois, D.M. Theory of computing anticipatory systems based n differential delayed-advanced difference equations. *Computing Anticipatory Systems: CASYS 2001- Fifth International Conference, American Institute of Physics*, AIP CP **627** (2002) 3–16.

[6] Gnana Bhaskar, T. and Lakshmikantham, V. Functional differential systems with retardation and anticipation. *Nonlinear Analysis* (to appear).

[7] Gnana Bhaskar, T., Lakshmikantham, V. and Vasundhara Devi, J. Monotone iterative method for functional differential systems with retardation and anticipation. *Nonlinear Analysis* (to appear).

[8] Ladde, G.S., Lakshmikantham, V. and Vatsala, A.S. *Monotone Iterative Techniques for Nonlinear Differential Equations*. Pitman Publishing, Inc., Boston, 1985.

[9] Lakshmikantham, V. and Köksal, S. *Monotone Flows and Rapid Convergence for Nonlinear PDEs*. Taylor and Francis, London 2003.

[10] Lakshmikantham, V. and Zhang, B.G. Monotone iterative technique for delay differential equations. *Applicable Analysis* **22** (1986) 227–233.

[11] West, Ianna H. and Vatsala, A.S. Generalized monotone iterative method integro differential equations with periodic bondary conditions. *Mathematical Inequalities and Applications* (to appear).

Nonlinear Dynamics and Systems Theory, **7** (4) (2007) 439–440

Contents of Volume 7, 2007

Volume 7 Number 1 March 2007

Preface v

Prior-free Inference for Objective Bayesian Analysis and Model Selection 1
Koki Kyo

Inverse Determination of Model Parameters of Nonlinear Heat Conduction Problem Using Hybrid Genetic Algorithm 23
Shouju Li and Yingxi Liu

Continuous-Time Optimal Portfolio Selection Using Mean-CaR Models 35
Zhong-Fei Li, Kai W. Ng and Xiao-Tie Deng

A Simple Nonlinear Adaptive-Fuzzy Passivity-Based Control of Power Systems 51
H.E. Psillakis and A.T. Alexandridis

Performance Analysis of Communication Networks Based on Conditional Value-at-Risk 69
Jun Wu, Wuyi Yue and Shouyang Wang

Analysis of Network Revenue Management under Uncertainty 85
Yifan Xu, Shu-Cherng Fang and Youyi Feng

An Online Learning Algorithm with Adaptive Forgetting Factors for Feedforward Neural Networks in Financial Time Series Forecasting 97
Lean Yu, Shouyang Wang and Kin Keung Lai

Volume 7 Number 2 June 2007

PERSONAGE IN SCIENCE
Alexander Mikhaylovich Liapunov 113
Yu.A. Mitropolskii, P. Borne and A.A. Martynyuk

Observation for the descriptor systems with disturbances 121
Xinkai Chen and Guisheng Zhai

Near-Time-Optimal Path Planning of a Rigid Machine with Multiple Axes 141
J.-Y. Dieulot and A. Chatti

$\mathcal{H}_\infty$ Filtering for Uncertain Bilinear Stochastic Systems 151
Huijun Gao, James Lam, Xuerong Mao and Peng Shi

Estimations of Solutions Convergence of Hybrid Systems Consisting of Linear Equations with Delay 169
D. Khusainov, R. Langerak and O. Kuzmych

Robustly Global Exponential Stability of Time-varying Linear Impulsive Systems with Uncertainty 187
Bin Liu and Fangyi Liu

Transformation Synthesis for Euler-Lagrange Systems 197
M. Mabrouk, S. Ammar and J.-C. Vivalda

Stability Results for Large-Scale Difference Systems via Matrix-Valued Liapunov Functions 217
A.A. Martynyuk and V.I. Slyn'ko

Volume 7 Number 3 September 2007

Elements of Stability Theory of A.M. Liapunov for Dynamic Equations on Time Scales ... 225
M. Bohner and A.A. Martynyuk

Lyapunov Based on Cascaded Non-linear Control of Induction Machine 253
H. Chekireb, M. Tadjine and M. Djemaï

Comparison of Transfer Orbits in the Restricted Three and Four-Body Problems 267
A.A. Corrêa, A.F.B.A. Prado, T.J. Stuchi and C. Beaugé

Rendezvous Maneuvers under Thrust Deviations and Mass Variation 279
A.D.C. Jesus and T.N. Teles

A Successive Approximation Algorithm to Optimal Feedback Control of Time-varying LPV State-delayed Systems 289
Hamid Reza Karimi

The Sufficient Conditions of Local Controllability for Linear Systems with Random Parameters 303
Yu.V. Masterkov and L.I. Rodina

Uniform Convergence to Global Attractors for Discrete Disperse Dynamical Systems 315
A.J. Zaslavski

A Study on Stabilization of Nonholonomic Systems Via a Hybrid Control Method 327
Guisheng Zhai, Isatada Matsune, Tomoaki Kobayashi and Joe Imae

Volume 7 Number 4 December 2007

Positive Invariance and Differential Inclusions with Periodic Right-Hand Side 339
I. Benedetti and E.A. Panasenko

An LMI Approach to H_∞ Filtering for Linear Parameter-Varying Systems with Delayed States and Outputs 351
H.R. Karimi, B. Lohmann and C. Buskens

Robust Stability for Nonlinear Uncertain Neural Networks with Delay 369
X.Y. Lou and B.T. Cui

Compact Explicit MPC Law with Guarantees of Feasibility for Reference Tracking 379
Sorin Olaru and Didier Dumur

Observer Design for a Class of Nonlinear Systems with Non-Full Relative Degree 399
K. Röbenack

Third-Body Perturbation Using Single Averaged Model: Application to Lunisolar Perturbations 409
C.R.H. Solórzano and A.F.B.A. Prado

On the Non-Oscillation of Solutions of Some Nonlinear Differential Equations of Third Order 419
Cemil Tunç

Generalized Monotone Iterative Technique for Functional Differential Equations with Retardation and Anticipation 431
A.S. Vatsala

Contents of Volume 7, 2007 439