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Existence and Uniqueness of a Solution to a Semilinear Partial Delay Differential Equation with an Integral Condition

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Abstract: In this work we consider a semilinear delay partial differential equation with an integral condition. We apply the method of semi-discretization in time, also known as the method of lines, to establish the existence and uniqueness of solutions. We also study the the continuation of the solution to the maximal interval of existence. Finally we give examples to demonstrate the applications of our results.

Keywords: *Diffusion equation; retarded argument; weak solution; method of lines.*

Mathematics Subject Classification (2000): 34K30, 34G20, 47H06.

1 Introduction

In this paper we are concerned with the following semilinear partial delay differential equation with an integral condition,

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t,u(x,t),u(x,t-\tau)), \quad (x,t) \in (0,1) \times [0,T], \tag{1}$$

$$u(x,t) = \Phi(x,t), \quad t \in [-\tau, 0], \quad x \in (0,1), \tag{2}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad t \in [0,T], \tag{3}$$

$$\int_0^1 u(x,t)dx = 0, \quad t \in [0,T], \tag{4}$$

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where $\tau > 0$, $0 < T < \infty$, the map f is defined from $(0,1) \times [0,T] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ and the history function Φ is defined from $(0,1) \times [-\tau, 0]$ into $\mathbb{R}$. Our aim is to apply the method of semi-discretization in time, also known as the method of lines or Rothe's method, to establish the existence, uniqueness of a solution and the unique continuation of a solution to the maximal interval of existence. We note that there is no loss of generality in considering the homogeneous conditions in (3) and (4) as the more general problem (1)–(4) with u, f and Φ replaced by v, g, Ψ and conditions (3) and (4) replaced by

$$\frac{\partial v(0,t)}{\partial x} = U_0(t), \quad t \in [0,T], \tag{5}$$

$$\int_0^1 v(x,t)dx = U_1(t), \quad t \in [0,T], \tag{6}$$

respectively, may be reduced to (1)–(4) using the transformations

$$u(x,t) = v(x,t) - U_0(t)\left(x - \frac{1}{2}\right) - U_1(t)$$

and

$$\begin{aligned} f(x,t,r,s) = g\bigg(x,\, t,\, r + U_0(t)\left(x - \frac{1}{2}\right) - U_1(t),\, s + U_0(t-\tau)\left(x - \frac{1}{2}\right) - U_1(t-\tau)\bigg) \\ - \left(x - \frac{1}{2}\right)\frac{dU_0(t)}{dt} - \frac{dU_1(t)}{dt}, \\ \Phi(x,t) = \Psi(x,t) - U_0(t)\left(x - \frac{1}{2}\right) - U_1(t), \end{aligned}$$

with $U_0(t-\tau) = U_0(0)$ and $U_1(t-\tau) = U_1(0)$ for $t \leq \tau$.

The initial work on heat equations with integral conditions has been carried out by Cannon [7]. Subsequently, similar studies have been done by Kamynin [11], Ionkin [8]. Beilin [5] has considered the wave equation with an integral condition using the method of separation of variables and Fourier series.

Pulkina [14] has dealt with a hyperbolic problem with two integral conditions and has established the existence and uniqueness of generalized solutions using the fixed point arguments.

Our analysis is motivated by the works of Bouziani and Merazga [12, 6] and Bahuguna and Shukla [4]. In [12, 6] the authors have used the method of semi-discretization to (1)-(4) without delays. In [4] the method of semigroups of bounded linear operators in a Banach space is used to study a partial differential equation involving delays arising in the population dynamics. We use the method of semi-discretizaion in time first to establish the local existence of a unique solution of (1)–(4) on a subinterval $[-\tau, T_0]$, $0 < T_0 \leq T$ and then extend it either to the whole interval $[-\tau, T]$ or to the maximal subinterval $[-\tau, T_{\max}) \subset [-\tau, T]$ of existence with $\lim_{t \to T_{\max}-} \|u(t)\| = +\infty$.

2 Preliminaries

The problem (1)–(4) may be treated as an abstract equation in the real Hilbert space $\mathbf{H} = L^2(0,1)$ of square-integrable functions defined from $(0,1)$ into $\mathbb{R}$ with the inner product

$$(u, v) = \int_0^1 u(x)v(x)\,dx, \quad u, v \in \mathbf{H},$$

and the corresponding norm

$$\|u\|^2 = \int_0^1 |u(x)|^2 dx.$$

For $k \in \mathbb{N}$, the Sobolev space $\mathbf{H}^k$ is the Hilbert space of all functions $u \in \mathbf{H}$ such that the distributional derivative $u^{(j)} \in \mathbf{H}$ with the inner product

$$(u, v)_k = \sum_{j=0}^{k} (u^{(j)}, v^{(j)}), \quad u, v \in \mathbf{H}^k,$$

and the corresponding norm

$$\|u\|_k^2 = \sum_{j=0}^{k} \|u^{(j)}\|^2.$$

We shall incorporate the integral condition (4) with the space itself under consideration by taking $\mathbf{V} \subset \mathbf{H}$ defined by

$$\mathbf{V} = \left\{ u \in \mathbf{H} : \int_0^1 u(x)\,dx = 0 \right\}.$$

$\mathbf{V}$ is a closed subspace of $\mathbf{H}$ and hence is a Hilbert space itself with the inner product $(\cdot, \cdot)$, and the corresponding norm $\|\cdot\|$.

For any Banach space X, with the norm $\|\cdot\|_X$ and an interval $I = [a, b],\ -\infty < a < b < \infty$, we shall denote by $C(I; X)$ the space of all continuous functions u from $[a, b]$ into X with the norm

$$\|u\|_{C(I;X)} = \max_{a \le t \le b} \|u(t)\|_X.$$

The space $L^2(I; X)$ consists of all square-Bochner integrable functions (equivalent classes) u for which the norm

$$\|u\|_{L^2(I;X)}^2 = \int_a^b \|u(t)\|_X^2\,dt.$$

Similarly $L^\infty(I; X)$ is the Banach space of all essentially bounded functions from I into X with the norm

$$\|u\|_{L^\infty(I;X)} = \operatorname*{ess\,sup}_{t \in I} \|u(t)\|_X,$$

and the Banach space $C^{0,1}(I; X)$ is the space of all Lipschitz continuous functions from I into X with the norm

$$\|u\|_{C^{0,1}(I;X)} = \|u\|_{C(I;X)} + \sup_{t,s \in I;\ t \ne s} \frac{\|u(t) - u(s)\|}{|t - s|}.$$

In addition to the spaces mentioned above, we need the space $B_2^1(0, 1)$ introduced by Merazga and A. Bouziani [12] being the completion of the space $C_0(0, 1)$ of all real continuous functions having compact supports in (0,1) with the inner product

$$(u, v)_{B_2^1} = \int_0^1 \Im_x u . \Im_x v\,dx,$$

where $\Im_x v = \int_0^x v(\xi)\, d\xi$ for every fixed $x \in (0,1)$ and the corresponding norm

$$\|u\|^2_{B_2^1} = \int_0^1 (\Im_x u)^2 dx.$$

It follows that the following inequality

$$\|v\|^2_{B_2^1} \le \frac{1}{2}\|v\|^2$$

holds for every $v \in L^2(0,1)$, and the embedding $L^2(0,1) \to B_2^1(0,1)$ is continuous.

Given a function $h:\ (0,1)\times[a,b] \to \mathbb{R}$ such that for each $t \in [a,b]$, $h(\cdot,t):\ [a,b] \to \mathbf{H}$, we may identify it with the function $h : [a,b] \to \mathbf{H}$ given by $h(t)(x) = h(x,t)$. For a given Lipschitz continuous function $g:\ (0,1)\times[a,b]\times\mathbb{R} \to \mathbb{R}$ and h as above, we may identify it with a function $g:\ [a,b]\times\mathbf{H} \to \mathbf{H}$ by $g(t,h(t))(x) = g(x,t,h(x,t))$.

We assume the following conditions.

(A1) $f(t,u,v) \in \mathbf{H}$ for $(t,u,v) \in [0,T]\times\mathbf{H}\times\mathbf{H}$ and the Lipschitz condition

$$\|f(t_1,u_1,v_1) - f(t_2,u_2,v_2)\|_{B_2^1} \le l_f\left[|t_1-t_2| + \|u_1-u_2\|_{B_2^1} + \|v_1-v_2\|_{B_2^1}\right]$$

for all $t_i \in [0,T]$, $u_i, v_i \in \mathbf{V}$, $i = 1,2$, holds for some positive constant l_f.

(A2) For each $x \in (0,1)$, $\Phi(x,\cdot):\ [-\tau,0] \to \mathbf{H}^2 \cap C^{0,1}([-\tau,0];\mathbf{H})$ with the uniform Lipchitz constant l_Φ.

(A3) $\dfrac{d\Phi(0,x)}{dx} = 0$ and $\int_0^1 \Phi(0,x)\, dx = 0$.

Definition 2.1 By a weak solution of (1)–(4) we mean a function $u\colon [-\tau,T] \to \mathbf{H}$

(i) $u = \Phi$ on $[-\tau,0]$;

(ii) $u \in L^\infty([0,T];\mathbf{V}) \cap C^{0,1}([0,T];B_2^1(0,1))$;

(iii) u has ($a.e$ in $[0,T]$) a strong derivative $\dfrac{du}{dt} \in L^\infty([0,T];B_2^1(0,1))$;

(iv) for all $\phi \in \mathbf{V}$ and a.e. $t \in [0,T]$, the identity

$$\left(\frac{du(t)}{dt}, \phi\right)_{B_2^1} + (u(t),\phi) = (f(t,u(t),u(t-\tau)),\ \phi)_{B_2^1}, \tag{7}$$

is satisfied.

Theorem 2.1 *Suppose that the conditions (A1)–(A3) are satisfied. Then problem (1)–(4) has a unique weak solution on* $[-\tau,T_0]$*, for some* $0 < T_0 \le T$*. Furthermore,* u *can be continued uniquely either on the whole interval* $[-\tau,T]$ *or there exists a maximal interval* $[0,T_{\max})$*,* $0 < T_{\max} < T$*, of existence with* $\lim_{t\to T_{max}-} \|u(t)\| = +\infty$.

3 Discretization Scheme and A Priori Estimates

In this section we establish existence and uniqueness of a weak solution to (1)–(4). For the the application of the method of line we proceed as follows. We choose $T_0,\ 0 < T_0 = \min\{\tau, T\}$, for $n \in \mathbb{N}$. Let $h_n = \frac{T_0}{n}$. We set $u_0^n = \Phi(0)$ for all $n \in \mathbb{N}$ and define each of $\{u_j^n\}_{j=1}^n$ as the unique solution of the time-discretized problems

$$\frac{u_j^n - u_{j-1}^n}{h_n} - \frac{d^2 u_j^n}{dx^2} = f_j^n, \quad x \in (0,1), \tag{8}$$

$$\frac{du_j^n}{dx}(0) = 0, \tag{9}$$

$$\int_0^1 u_j^n(x)\, dx = 0, \tag{10}$$

where $f_j^n = f(t_j^n, u_{j-1}^n, \Phi(t_{j-1}^n - \tau))$. The existence of unique $u_j^n \in \mathbf{H^2}$ satisfying (8),(9) is ensured as established in [13] Lemma 3.1. We first prove the estimates for u_j^n and difference quotients $\{(u_j^n - u_{j-1}^n)/h_n\}$ using (A1)–(A3). We introduce sequences $\{U^n\}$ of polygonal functions from $U^n \colon [-\tau, T_0] \to H^2(0,1) \cap V$ defined by

$$U^n(t) = \begin{cases} \Phi(t), & t \in [-\tau, 0], \\ u_{j-1}^n + \dfrac{t - t_{j-1}^n}{h_n}(u_j^n - u_{j-1}^n), & t \in [t_{j-1}^n, t_j^n], \end{cases} \tag{11}$$

and prove the convergence of $\{U^n\}$ to a unique solution u of (1)–(4) in $C([-\tau, T_0], B_2^1(0,1))$ as $n \to \infty$. For the notational convenience, we some time suppress the superscript n, throughout, C will represent a generic constant independent of j, h_n and n.

Lemma 3.1 *Assume that the hypotheses (A1)–(A3) are satisfied. Then there exists a positive constant C, independent of j, h and n such that.*

$$\|u_j\| \le C, \tag{12}$$

$$\|\delta u_j\|_{B_2^1} \le C, \tag{13}$$

$n \ge 1$ and $j = 1, \dots n$.

Proof Taking the inner product in $B_2^1(0,1)$ of (8) with any $\phi \in \mathbf{V}$,

$$(\delta u_j, \phi)_{B_2^1} - \left(\frac{d^2 u_j}{dx^2}, \phi\right)_{B_2^1} = (f_j, \phi)_{B_2^1}. \tag{14}$$

Using (9) and integration by parts

$$\left(\frac{d^2 u_j}{dx^2}, \phi\right)_{B_2^1} = \int_0^1 \frac{du_j(x)}{dx} \Im_x \phi\, dx = u_j(x) \Im_x \phi|_{x=0}^{x=1} - \int_0^1 u_j \phi\, dx.$$

Since

$$\left(\frac{d^2 u_j}{dx^2}, \phi\right)_{B_2^1} = -(u_j, \phi),$$

(14) becomes

$$(\delta u_j, \phi)_{B_2^1} + (u_j, \phi) = (f_j, \phi)_{B_2^1}. \tag{15}$$

Taking $j = 1$ in (15), and $\phi = u_1$ we have

$$\begin{aligned} \frac{1}{h_n}(u_1, u_1)_{B_2^1} + (u_1, u_1) &= \left(f_1 + \frac{1}{h_n}\Phi(0), u_1\right)_{B_2^1}, \\ \|u_1\|_{B_2^1} \le h_n \max_{t\in[0,T_0]} \|f(t_1, \Phi(0), \Phi(-\tau))\|_{B_2^1} &+ \|\Phi(0)\|_{B_2^1} = C. \end{aligned} \tag{16}$$

Again for $j = 1$ in (15) and $(\Phi(0), \phi) = -\left(\frac{d^2\Phi(0)}{dx^2}, \phi\right)_{B_2^1}$, we get

$$(\delta u_1, \phi)_{B_2^1} + h_n(\delta u_1, \phi) = \left(f_1 + \frac{d^2\Phi(0)}{dx^2}, \phi\right)_{B_2^1}$$

Testing this equality with $\phi = \delta u_1 = \frac{u_1 - \Phi(0)}{h_n} \in V$,

$$\|\delta u_1\|^2_{B_2^1} + h_n\|\delta u_1\|^2 \le \left[\|f_1\|_{B_2^1} + \left\|\frac{d^2\Phi(0)}{dx^2}\right\|_{B_2^1}\right]\|\delta u_1\|_{B_2^1},$$

consequently we get

$$\|\delta u_1\|_{B_2^1} \le \max_{t\in[0,T_0]} \|f(t_1, \Phi(0), \Phi(-\tau))\|_{B_2^1} + \left\|\frac{d^2\Phi(0)}{dx^2}\right\|_{B_2^1} = C. \tag{17}$$

Let $2 \le j \le n$. Subtracting (15) for $j-1$ from (15) for j and putting $\phi = \delta u_j$, we get

$$(\delta u_j - \delta u_{j-1}, \delta u_j)_{B_2^1} + (u_j - u_{j-1}, \delta u_j) = (f_j - f_{j-1}, \delta u_j)_{B_2^1},$$

or

$$\|\delta u_j\|^2_{B_2^1} + \frac{1}{h_n}\|u_j - u_{j-1}\|^2 \le (\|f_j - f_{j-1}\|_{B_2^1} + \|\delta u_{j-1}\|_{B_2^1})\|\delta u_j\|_{B_2^1},$$

which finally gives

$$\|\delta u_j\|_{B_2^1} \le \|f_j - f_{j-1}\|_{B_2^1} + \|\delta u_{j-1}\|_{B_2^1}.$$

By assumption $(A1)$ we have for $j \ge 2$,

$$\begin{aligned} \|f_j - f_{j-1}\|_{B_2^1} &= \|f(t_j, u_{j-1}, \Phi(t_{j-1} - \tau)) - f(t_{j-1}, u_{j-2}, \Phi(t_{j-2} - \tau))\|_{B_2^1} \\ &\le l_f[|t_j - t_{j-1}| + h_n\|\delta u_{j-1}\|_{B_2^1} + l_\Phi|t_{j-1} - t_{j-2}|] \\ &\le Ch_n[1 + \|\delta u_{j-1}\|_{B_2^1}]. \end{aligned}$$

Hence above equation becomes

$$\begin{aligned} \|\delta u_j\|_{B_2^1} &\le (1 + Ch_n)\|\delta u_{j-1}\|_{B_2^1} + Ch_n \\ &\le (1 + Ch_n)^2\|\delta u_{j-2}\|_{B_2^1} + Ch_n[1 + (1 + Ch_n)]. \end{aligned}$$

By iterative process we obtain

$$\|\delta u_j\|_{B_2^1} \le (1 + Ch_n)^{j-1}\left[\|\delta u_1\|_{B_2^1} + Ch_n(j-1)\right] \tag{18}$$

Replacing $(1+Ch_n)^{j-1} \le e^{CT}$ and $Ch_n(j-1) \le CT$ we get the second required estimate. Now for the first estimate, we take $\phi = u_j$ in (15), $j = 1, 2, \ldots, n$, to get

$$\frac{1}{h_n}\|u_j\|^2_{B^1_2} + \|u_j\|^2 \le \left(\|f_j\|_{B^1_2} + \frac{1}{h_n}\|u_{j-1}\|_{B^1_2}\right)\|u_j\|_{B^1_2},$$

which implies

$$\|u_j\|_{B^1_2} \le h_n\|f_j\|_{B^1_2} + \|u_{j-1}\|_{B^1_2}. \tag{19}$$

By assumption $(A1)$, we have for all $j \ge 1$,

$$\begin{aligned}\|f_j\|_{B^1_2} &\le \|f(t_j, u_{j-1}, \Phi(t_{j-1} - \tau)) - f(t_j, 0, 0)\|_{B^1_2} + \|f(t_j, 0, 0)\|_{B^1_2} \\ &\le C(1 + \|u_{j-1}\|_{B^1_2}).\end{aligned}$$

Putting it into (19) we have

$$\|u_j\|_{B^1_2} \le (1 + Ch_n)\|u_{j-1}\|_{B^1_2} + Ch_n.$$

Repeating the last inequality we estimate

$$\|u_j\|_{B^1_2} \le (1 + Ch_n)^{j-1}[\|u_1\|_{B^1_2} + Ch_n(j-1)]. \tag{20}$$

Again replacing $(1 + Ch_n)^{j-1} \le e^{CT}$ and $Ch_n(j-1) \le CT$. We get

$$\|u_j\|_{B^1_2} \le C. \tag{21}$$

Now taking $\phi = u_j - u_{j-1}$ in (15) and using the identity,

$$(u_j,\, u_j - u_{j-1}) = \frac{1}{2}\big(\|u_j - u_{j-1}\|^2 + \|u_j\|^2 - \|u_{j-1}\|^2\big),$$

we get

$$h_n\|\delta u_j\|^2_{B^1_2} + \frac{1}{2}\|u_j - u_{j-1}\|^2 + \frac{1}{2}\|u_j\|^2 = (f_j,\, u_j - u_{j-1})_{B^1_2} + \frac{1}{2}\|u_{j-1}\|^2.$$

Ignoring the first two terms in the left hand side, we have

$$\begin{aligned}\|u_j\|^2 &\le 2h_n\|f_j\|_{B^1_2}\|\delta u_j\|_{B^1_2} + \|u_{j-1}\|^2 \\ &\le Ch_n(1 + \|u_{j-1}\|_{B^1_2})\|u_{j-1}\|_{B^1_2} + \|u_{j-1}\|^2 \\ &\le Ch_n + \|u_{j-1}\|^2.\end{aligned}$$

Repeating the above inequality we get the required estimate. This completes the proof of the lemma. □

Definition 3.1 We define Rothe's sequence $\{U^n\}$ by (11). Furthermore, we define another sequences $\{X^n\}$ of step functions from $[-h_n, T_0]$ into $\mathbf{H}^2 \cap \mathbf{V}$ given by

$$X^n(t) = \begin{cases}\Phi(0), & t \in [-h_n, 0], \\ u_j, & t \in (t_{j-1}, t_j].\end{cases}$$

Remark 3.1 From Lemma 3.1 it follows that the function U^n is Lipschitz continuous on $[0,T_0]$. The sequences $\{U^n\}$ and $\{X^n\}$ are bounded in $C([0,T_0];B^1_2(0,1))$ uniformly in $n \in \mathbb{N}$ and $t \in [0,T_0]$

$$\|U^n\| \leq C, \quad \|X^n\| \leq C, \quad \left\|\frac{dU^n(t)}{dt}\right\|_{B^1_2} \leq C, \quad \|U^n(t) - U^n(s)\| \leq C|t-s|,$$
$$\|X^n(t) - U^n(t)\|_{B^1_2} \leq \frac{C}{n}, \quad \text{and} \quad \|X^n(t) - X^n(t-h_n)\|_{B^1_2} \leq \frac{C}{n}.$$

For notational convenience, let

$$f^n(t) = f(t_j, X^n(t-h_n), \Phi(t_{j-1} - \tau)), \quad t \in (t_{j-1}, t_j], \quad 1 \leq j \leq n.$$

Then (15) may be rewritten as

$$\left(\frac{dU^n(t)}{dt}, \phi\right)_{B^1_2} + (X^n(t), \phi) = (f^n(t), \phi)_{B^1_2}, \tag{22}$$

for all $\phi \in \mathbf{V}$ and a.e. $t \in (0,T_0]$.

Lemma 3.2 *There exists $u \in C([0,T_0];B^1_2(0,1))$ such that $U^n(t) \to u(t)$ uniformly on I. Moreover $u(t)$ is Lipschitz continuous on $[0,T_0]$.*

Proof From (22) for a.e. $t \in (0,T_0]$, we have

$$\left(\frac{d}{dt}(U^n(t) - U^k(t)), U^n(t) - U^k(t)\right)_{B^1_2} + (X^n(t) - X^k(t), U^n(t) - U^k(t))$$
$$= (f^n(t) - f^k(t), U^n(t) - U^k(t))_{B^1_2}.$$

From the above equality, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|U^n(t) - U^k(t)\|^2_{B^1_2} &+ \|X^n(t) - X^k(t)\|^2 \\ &= (X^n(t) - X^k(t), X^n(t) - X^k(t) - U^n(t) + U^k(t)) \\ &+ (f^n(t) - f^k(t), U^n(t) - U^k(t))_{B^1_2}. \end{aligned} \tag{23}$$

From (22), $\|f^n(t)\|_{B^1_2} \leq C$, and thus the identity

$$(X^n(t), \phi) = \left(f^n(t, X^n(t-h_n), \Phi(t_j - \tau)) - \frac{dU^n}{dt}, \phi\right)_{B^1_2}$$

gives

$$|(X^n(t), \phi)| \leq \left[\|f^n\|_{B^1_2} + \left\|\frac{dU^n}{dt}\right\|_{B^1_2}\right]\|\phi\|_{B^1_2} \leq C\|\phi\|_{B^1_2}. \tag{24}$$

Now using (24), we have the estimate

$$\begin{aligned} (X^n(t) - X^k(t),\ & X^n(t) - X^k(t) - U^n(t) + U^k(t)) \\ &\leq 2C\Big(\|X^n(t) - U^n(t)\|_{B^1_2} + \|X^k(t) - U^k(t)\|_{B^1_2}\Big) \\ &\leq 4C\left(\frac{1}{n} + \frac{1}{k}\right). \end{aligned} \tag{25}$$

By inequality $\alpha\beta \le \dfrac{\alpha^2+\beta^2}{2}$, $\alpha, \beta \in \mathbb{R}$, we may write

$$\begin{aligned}(f^n(t) - f^k(t), U^n(t) - U^k(t))_{B_2^1} &\le \|f^n(t) - f^k(t)\|_{B_2^1} \|U^n(t) - U^k(t)\|_{B_2^1} \\ &\le \frac{1}{2}\Big[\|f^n(t) - f^k(t)\|^2_{B_2^1} + \|U^n(t) - U^k(t)\|^2_{B_2^1}\Big].\end{aligned} \tag{26}$$

Using assumption (A1), we have

$$\begin{aligned}\|f^n(t) - f^k(t)\|_{B_2^1} &= \|f(t_j, X^n(t-h_n), \Phi(t_{j-1} - \tau)) - f(t_l, X^k(t-h_k), \Phi(t_{l-1} - \tau))\|_{B_2^1} \\ &\le \delta_{nk}(t) + l_f \|X^n(t) - X^k(t)\|_{B_2^1},\end{aligned}$$

where

$$\begin{aligned}\delta_{nk}(t) = l_f[|t_j - t_l| &+ \|X^n(t-h_n) - X^n\|_{B_2^1} + \|X^k(t-h_k) - X^k(t)\|_{B_2^1} \\ &+ \|\Phi(t_{j-1} - \tau) - \Phi(t_{l-1} - \tau)\|_{B_2^1}],\end{aligned}$$

for $t \in (t_{j-1}, t_j]$ and $t \in (t_{l-1}, t_l]$, $1 \le j \le n$, $1 \le l \le k$. Clearly $\delta_{nk}(t) \to 0$ uniformly on $[0, T_0]$ as $n, k \to \infty$. Also

$$\|f^n(t) - f^k(t)\|^2_{B_2^1} \le \delta^1_{nk}(t) + l_f^2 \|X^n(t) - X^k(t)\|^2_{B_2^1}.$$

Hence (26) becomes

$$\begin{aligned}(f^n(t) - f^k(t), U^n(t) - U^k(t))_{B_2^1} &\le \frac{1}{2}\delta^1_{nk} + \frac{1}{2} l_f^2 \|X^n(t) - X^k(t)\|^2_{B_2^1} \\ &+ \frac{1}{2}\|U^n(t) - U^k(t)\|^2_{B_2^1}, \quad \forall\, t \in (0, T_0].\end{aligned} \tag{27}$$

Now combining (25), (26) and (27) then (23) becomes

$$\begin{aligned}\frac{d}{dt}\|U^n(t) - U^k(t)\|^2_{B_2^1} + 2\|X^n(t) - X^k(t)\|^2 &\le 2C\left(\frac{1}{n} + \frac{1}{k}\right) + l_f^2\|X^n(t) - X^k(t)\|^2 \\ &+ \delta^1_{nk} + \|U^n(t) - U^k(t)\|^2_{B_2^1}, \quad \forall\, t \in (0, T_0],\end{aligned}$$

or

$$\begin{aligned}\frac{d}{dt}\|U^n(t) - U^k(t)\|^2_{B_2^1} + (2 - l_f^2)\|X^n(t) - X^k(t)\|^2 &\le 2C\left(\frac{1}{n} + \frac{1}{k}\right) + \delta^1_{nk} \\ &+ \|U^n(t) - U^k(t)\|^2_{B_2^1}, \quad \forall\, t \in (0, T_0],\end{aligned}$$

where δ^1_{nk} is a sequence of numbers converging to zero as $n, k \to \infty$. Integrating over $(0, s)$, $0 < s \le t \le T_0$, taking the supremum over $(0, t)$ and using the fact that $U^n = \Phi$ on $[-\tau, 0]$ for all n we get

$$\|U^n - U^k\|^2_{B_2^1} \le 2CT\left(\frac{1}{n} + \frac{1}{k}\right) + CT\delta^1_{nk} + C\int_0^t \|U^n - U^k\|^2_{B_2^1}\, ds,$$

where C is a positive constant independent of j, h and n. Applying Gronwall's inequality, we conclude that there exists a function $u \in C([0, T_0]; B_2^1(0,1))$ such that $U^n \to u$ in this space and by Remark 3.1 it follows that u is Lipschitz continuous on $[-\tau, T_0]$. This completes the proof of the lemma. $\square$

In consequence of Remark 3.1 and Lemma 3.2 we have the following remark on the weak convergence (denoted by $\rightharpoonup$) U^n and its strong derivative to the function u and its strong derivative, respectively.

Remark 3.2 (i) $u \in L^\infty([0,T_0];\mathbf{V}) \cap C^{0,1}([0,T_0];B_2^1(0,1))$;

(ii) u is strongly differentiable a.e. in $[0,T_0]$ and $\dfrac{du}{dt} \in L^\infty([0,T_0];B_2^1(0,1))$;

(iii) $U^n(t)$ and $X^n(t) \rightharpoonup u(t)$ in V for all $t \in I$;

(iv) $\dfrac{dU^n(t)}{dt} \rightharpoonup \dfrac{du}{dt}$ in $L^\infty([0,T_0];B_2^1(0,1))$.

Proof of Theorem 2.1

First we prove the existence on $[-\tau,T_0]$. Integrating the (22) over $(0,t) \subset [0,T_0]$ and invoking the fact that $U^n(0) = \Phi(0)$, we have

$$(U^n(t) - \Phi(0), \phi)_{B_2^1} + \int_0^t (X^n(s),\phi)\,ds = \int_0^t (f^n,\phi)_{B_2^1}\,ds. \tag{28}$$

Since $U^n(t) \rightharpoonup u(t)$ in $\mathbf{V}$ for all $t \in [0,T_0]$ and $\forall\,\phi \in \mathbf{V}$ and the linear functional $v \to (v,\phi)_{B_2^1}$ is bounded on $\mathbf{V}$, we have

$$(U^n(t),\phi)_{B_2^1} \to (u(t),\phi)_{B_2^1}, \quad \forall\, t \in [0,T_0]. \tag{29}$$

Now by the Lipschitz continuity of f and Remark 3.1 we get

$$f^n(s, X^n(s-h_n), \Phi(s-\tau)) \to f(s,u(s),\Phi(s-\tau)) \quad \text{in} \quad B_2^1(0,1) \tag{30}$$

as $n \to \infty$. Now from (22) and (24) the functions $|(f^n,\phi)_{B_0^1}|$ and $|(X^n,\phi)|$ are uniformly bounded. Now by bounded convergence theorem and (22), we obtain, as $n \to \infty$,

$$(u(t) - \Phi(0),\, \phi)_{B_2^1} + \int_0^t (u(s),\phi)\,ds = \int_0^t (f(s,u(s),\Phi(s-\tau)),\phi)_{B_2^1}\,ds$$

for all $\phi \in \mathbf{V}$ and $t \in [0,T_0]$. Differentiating the identity we get the required relation. Now we prove the uniqueness. Let u_1 and u_2 be two such solutions of (1)–(4). Then we have

$$\left(\frac{dU(t)}{dt}, U(t)\right)_{B_2^1} + \|U(t)\|^2 = (f(t,u_1(t),\Phi(t-\tau)) - f(t,u_2(t),\Phi(t-\tau)), U(t))_{B_2^1},$$

where $U(t) = u_1(t) - u_2(t)$. Integrating over $(0,s)$ for $0, s \le t \le T_0$ and using the fact that $U(0) = 0$, we get

$$\begin{aligned}\|U(t)\|_{B_2^1}^2 + 2\int_0^t \|U(t)\|^2 ds = 2\int_0^t &(f(s,u_1(s),\Phi(s-\tau)) \\ &- f(s,u_2(s),\Phi(s-\tau)), U(s))_{B_2^1}\,ds \le 2l_f \int_0^t \|U(s)\|_{B_2^1}^2\,ds.\end{aligned}$$

Application of Gronwall's inequality implies that $U \equiv 0$ on $[-\tau,T_0]$.

Now, we prove the unique continuation of the solution u to either on whole interval $[-\tau,T]$ or to the maximal interval $[-\tau,T_{\max})$ of existence where $0 < T_{\max} < T$ and if

$T_{\max} < T$ then $\lim_{t\to T_{\max}-} \|u(t)\| = \infty$. Suppose $T_0 < T$ and $\|u(T_0)\| < \infty$. Consider the problem

$$\begin{aligned}\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} &= \tilde{f}(x,t,w(t),w(t-\tau)), \quad x \in (0,1), \quad 0 < t \le T - T_0,\\ w(x,t) &= \tilde{\Phi}(x,t), \quad x \in (0,1), \quad t \in [-\tau - T_0, 0],\\ \frac{\partial w(0,t)}{\partial x} &= 0, \quad t \in [0, T-T_0],\\ \int_0^1 w(x,t)\,dx &= 0, \quad x \in (0,1), \quad t \in [0, T-T_0],\end{aligned} \tag{31}$$

where $\tilde{f}(x,t,w(t),w(t-\tau)) = f(x,t+T_0,w(t),w(t-\tau)),\ x \in (0,1),\ 0 < t \le T - T_0,$

$$\tilde{\Phi}(t) = \begin{cases}\Phi(t+T_0), & t \in [-\tau - T_0, -T_0],\\ u(t+T_0), & t \in [-T_0, 0].\end{cases}$$

Since $\|\tilde{\Phi}(0)\| = \|u(T_0)\| < \infty$ and $\tilde{f}$ satisfies $(A1)$ on $[0, T-T_0]$, we may proceed as before and prove the existence of a unique $w(t) \in C([-\tau - T_0, T_1]; B_2^1(0,1)),\ 0 < T_1 \le T - T_0$, such that w is Lipschitz continuous on $[0, T_1]$ and w satisfies

$$\begin{aligned}\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} &= \tilde{f}(x,t,w(t),w(t-\tau)), \quad x \in (0,1), \quad 0 < t \le T_1,\\ w(x,t) &= \tilde{\Phi}(x,t), \quad x \in (0,1), \quad t \in [-\tau - T_0, 0],\\ \frac{\partial w(0,t)}{\partial x} &= 0, \quad t \in [0, T-T_0],\\ \int_0^1 w(x,t)\,dx &= 0, \quad t \in [0, T-T_0].\end{aligned} \tag{32}$$

Then the function

$$\overline{u}(t) = \begin{cases}u(t), & t \in [-\tau, T_0],\\ w(t-T_0), & t \in [T_0, T_0+T_1],\end{cases}$$

is Lipschitz continuous on $[0, T_0+T_1]$, $\overline{u}(t) \in C([0, T_0+T_1], B_2^1(0,1))$ for $t \in [0, T_0+T_1]$ and satisfies (1) on $[0, T_0+T_1]$. Continuing this way we may prove the existence on the whole interval $[-\tau, T]$ or there is the maximal interval $[-\tau, T_{\max}),\ 0 < T_{\max} \le T$, such that u is the weak solution of (1)–(4) on every subinterval $[-\tau, \tilde{T}], 0 < \tilde{T} < T_{\max}$. In the later case, if $\lim_{t\to t_{max}} \|u(t)\| < \infty$ then we may continue the solution beyond $T_{\max}$ but this will contradict the definition of maximal interval of existence. This completes the proof of Theorem 2.1.

4 Applications

In this section we consider problems arising in the population dynamics (cf. Engel and Nagel [9]).

Example 4.1 Consider the following partial differential equation with delay,

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = -d(t)u(x,t) + b(t)u(x,t-\tau), \quad (x,t) \in (0,1) \times [0,T], \tag{33}$$

$$u(x,t) = \Phi(x,t), \quad t \in [-\tau, 0], \quad x \in (0,1), \tag{34}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad t \in [0,T], \tag{35}$$

$$\int_0^1 u(x,t)\,dx = \psi(t), \quad t \in [0,T]. \tag{36}$$

Here $u(x,t)$ denotes the size of a population at time t and at the point $x \in [0,1]$. The term $\dfrac{\partial^2 u}{\partial x^2}$ represents the internal migration. The continuous functions b and d on $[0,T]$ represent the birth and death rates and τ is the delay due to pregnancy. The function $\psi(t)$ may be viewed as a control on the average population size at time t. Thus, we have no-flux condition at the left end and the right end is free so there may be a flux at this end but the average population size is being controlled by the integral condition. Here we take $f\colon\ [0,T] \times \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ given by $f(t,\chi,\psi) = -d(t)\chi + b(t)\psi$, $t \in [0,T]$, $\chi, \psi \in \mathbf{H}$. Our analysis of the earlier sections may be applied to this problem to ensure the well-posedness of the model.

Example 4.2 In this example we consider the following problem,

$$\frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial x^2} = r(t)u(x,t-\tau))(1-u(x,t)), \quad (x,t) \in (0,\pi) \times [0,T], \tag{37}$$

$$u(x,t) = \Phi(x,t), \quad t \in [-\tau, 0], \quad x \in (0,\pi), \tag{38}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad t \in [0,T], \tag{39}$$

$$\int_0^\pi u(x,t)\,dx = \psi(t), \quad t \in [0,T]. \tag{40}$$

The equation (37) arises in the study of a population density with a time delay and self-regulation (cf. Turyn [16]). In this problem we take $T = \tau$ and assume that Φ is bounded on $(0,\pi) \times [-\tau, 0]$. Also, we take $f\colon\ [0,\tau] \times \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ given by $f(t,\chi,\psi) \equiv f(t,\chi)(x) = r(t)\Phi(x,t-\tau)(1-\chi(x))$, $t \in [0,\tau]$, $\chi \in \mathbf{H}$. Here again we have considered no-flux condition on the left end and the average population size is being controlled by the function ψ in place of the Dirichlet boundary condition on u as taken in [16]. The results of the earlier sections may be used to ensure the well-posedness of this model. We shall be dealing with the problem involving the Dirichlet condition together with an integral condition in our subsequent study.

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