



Topological Pressure of Nonautonomous Dynamical Systems^{*}

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Abstract: We define and study topological pressure for the non-autonomous discrete dynamical systems given by a sequence $\{f_i\}_{i=1}^{\infty}$ of continuous self-maps of a compact metric space. In this paper, we obtain the basic properties and the invariant with respect to equiconjugacy of topological pressure for the non-autonomous discrete dynamical systems.

Keywords: *Topological pressure; sequence of continuous self-maps; non-autonomous system.*

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1 Introduction

Entropies are fundamental to our current understanding of dynamical systems. The notion of topological entropy was introduced by Adler, Konheim and McAndrew as an invariant of topological conjugacy. Topological entropy provides a numerical measure for the complexity of an endomorphism of a compact topological space [1]. Later Bowen and Dinaburg gave a new, but equivalent, definition in the case when the space under consideration is metrizable [2]. S. Kolyada and L. Snoha studied topological entropy for the non-autonomous discrete dynamical systems given by a sequence $\{f_i\}_{i=1}^{\infty}$ of continuous

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self-maps of a compact topological space [3]. Topological pressure is a generalization to topological entropy for a dynamical system [4].

Our purpose is to introduce and study the notion of topological pressure for the non-autonomous discrete dynamical systems given by a sequence $\{f_i\}_{i=1}^{\infty}$ of continuous self-maps of a compact topological space.

First, some notation and definitions are established.

Throughout the paper, (X, d) will be a compact metric space and $C(X, X)$ be the set of continuous maps from (X, d) into itself, $C(X, R)$ be the functional space containing all continuous, real-valued functions on X .

Let $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ be a sequence of continuous maps from X to X . The identity map on X will be denoted by id_X or shortly by id . Let N, R, Z be the set of all positive integers, real and integers, respectively. For any $i \in N$ let $f_i^0 = f_i^{-0} = id_X$ and for any $i, n \in N$ set $f_i^n = f_{i+(n-1)} \circ \cdots \circ f_{i+1} \circ f_i$ (first apply f_i) and $f_i^{-n} = f_i^{-1} \circ f_{i+1}^{-1} \circ \cdots \circ f_{i+(n-1)}^{-1}$ (the last notations will be applied to sets, we do not assume that the maps f_i are invertible). Finally, denote by $f_{1,\infty}^n$ the sequence of maps $\{f_{in+1}^n\}_{i=0}^{\infty}$ and by $f_{1,\infty}^{-1}$ the sequence $\{f_i^{-1}\}_{i=0}^{\infty}$.

Now we are going to describe the main results of the paper and how it is organized. For the precise statements of the results and for the definitions used see corresponding sections.

Let $f_{1,\infty} \in C(X, X)$ and $\varphi \in C(X, R)$. In this paper we will define and study the topological pressure $P(f_{1,\infty}, \varphi)$ of non-autonomous discrete dynamical systems given by a sequence $\{f_i\}_{i=1}^{\infty}$ with respect to φ .

In Section 1, we give the basic definition of topological pressure for the non-autonomous discrete dynamical systems given by a sequence $\{f_i\}_{i=1}^{\infty}$ of continuous self-maps of a compact metric space. In Section 2, we study the basic properties of topological pressure for the non-autonomous discrete dynamical systems.

2 Topological Pressure of a Sequence of Maps on a Compact Metric Space

We are going to define the topological pressure of a non-autonomous dynamical system $(X; \{f_i\}_{i=1}^{\infty})$ analogously to the topological pressure of a autonomous dynamical system $(X; f)$ ([4]). Of course, for $f_1 = f_2 = \cdots = f$ we get the classical definition.

For each $n \geq 1$ there is a positive integer. Define the metric in X by $d_n(x, y) = \max_{0 \leq j \leq n-1} d(f_1^j(x), f_1^j(y))$. A subset E of the space X is called (n, ε) -separated if for any two distinct points $x, y \in E$, $d_n(x, y) > \varepsilon$. Let $C(X, R)$ be the space of real-valued continuous functions of X . For $\varphi \in C(X, R)$ and $n \in N$ we denote $\sum_{i=0}^{n-1} \varphi(f_1^i(x))$ by $(S_n \varphi)(x)$. For $\varepsilon > 0$, $x \in X$, we put

$$P_n(f_{1,\infty}, \varphi, \varepsilon) := \sup \left\{ \sum_{x \in E} e^{(S_n \varphi)(x)} \mid E \text{ is a } (n, \varepsilon) \text{ separated set for } X \right\}.$$

Then we put

$$P(f_{1,\infty}, \varphi, \varepsilon) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_n(f_{1,\infty}, \varphi, \varepsilon)$$

and we define the topological pressure of $f_{1,\infty}$ with respect to φ as

$$P(f_{1,\infty}, \varphi) = \lim_{\varepsilon \rightarrow 0} P(f_{1,\infty}, \varphi, \varepsilon).$$

It is clear that $P(f_{1,\infty}, 0) = h(f_{1,\infty})$.

A set $F \subset X$ (n, ε) – spans another set $K \subset X$ provided that for each $x \in K$ there is $y \in F$ for which $d_n(x, y) \leq \varepsilon$. For $\varepsilon > 0, x \in X$, we put

$$Q_n(f_{1,\infty}, \varphi, \varepsilon) := \inf \left\{ \sum_{x \in E} e^{(S_n \varphi)(x)} \mid E \text{ is a } (n, \varepsilon) \text{ spanning set for } X \right\}.$$

Remark 2.1 $Q_n(f_{1,\infty}, \varphi, \varepsilon) \leq P_n(f_{1,\infty}, \varphi, \varepsilon)$.

Proof It follows from the fact that $e^{(S_n \varphi)(x)} > 0$ and a (n, ε) separated set which cannot be enlarged to a (n, ε) separated set must be a (n, ε) spanning set of X . \square

Remark 2.2 If $\delta > 0$ is such that $d(x, y) < \frac{\varepsilon}{2}$ implies that $|\varphi(x) - \varphi(y)| < \delta$ then $P_n(f_{1,\infty}, \varphi, \varepsilon) \leq e^{n\delta} Q_n(f_{1,\infty}, \varphi, \varepsilon)$.

Proof Let E be a (n, ε) separated set and F is a $(n, \frac{\varepsilon}{2})$ spanning set. Define $\phi : E \rightarrow F$ by choosing, for each $x \in E$, some point $\phi(x) \in F$ with $d_n(x, \phi(x)) \leq \frac{\varepsilon}{2}$. Then ϕ is injective and

$$\begin{aligned} \sum_{y \in F} e^{(S_n \varphi)(y)} &\geq \sum_{y \in \phi(E)} e^{(S_n \varphi)(y)} \geq \left(\min_{x \in E} e^{(S_n \varphi)(\phi(x)) - (S_n \varphi)(x)} \right) \sum_{x \in E} e^{(S_n \varphi)(x)} \\ &\geq e^{-n\delta} \sum_{x \in E} e^{(S_n \varphi)(x)}. \end{aligned}$$

Therefore $Q_n(f_{1,\infty}, \varphi, \varepsilon) \leq e^{-n\delta} P_n(f_{1,\infty}, \varphi, \varepsilon)$. \square

Remark 2.3 By (1) and (2), if we put

$$P(f_{1,\infty}, \varphi, \varepsilon) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log Q_n(f_{1,\infty}, \varphi, \varepsilon)$$

we will have

$$P(f_{1,\infty}, \varphi) = \lim_{\varepsilon \rightarrow 0} Q(f_{1,\infty}, \varphi, \varepsilon).$$

Let α be an open cover of X . For $x \in X$, we put

$$q_n(f_{1,\infty}, \varphi, \alpha) := \inf \left\{ \sum_{B \in \beta} \inf_{x \in B} e^{(S_n \varphi)(x)} \mid \beta \text{ is a finite subcover of } \bigvee_{i=0}^{n-1} f_1^{-i} \alpha \right\}$$

and put

$$p_n(f_{1,\infty}, \varphi, \alpha) := \inf \left\{ \sum_{B \in \beta} \sup_{x \in B} e^{(S_n \varphi)(x)} \mid \beta \text{ is a finite subcover of } \bigvee_{i=0}^{n-1} f_1^{-i} \alpha \right\}.$$

Clearly $q_n(f_{1,\infty}, \varphi, \alpha) \leq p_n(f_{1,\infty}, \varphi, \alpha)$. In addition similar to the case of the autonomous systems we have the following Proposition.

Proposition 2.1 Let $f_{1,\infty} \in C(X, X)$ and $\varphi \in C(X, R)$.

(1) If α is an open cover of X with Lebesgue δ then $q_n(f_{1,\infty}, \varphi, \alpha) \leq Q_n(f_{1,\infty}, \varphi, \varepsilon)$.

(2) If $\varepsilon > 0$ and γ is an open cover with $\text{diam}(\gamma) \leq \varepsilon$ then $P_n(f_{1,\infty}, \varphi, \varepsilon) \leq p_n(f_{1,\infty}, \varphi, \gamma)$.

(3) If α is an open cover of X , then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log p_n(f_{1,\infty}, \varphi, \alpha)$$

exists and equals to $\inf_n \frac{1}{n} \log p_n(f_{1,\infty}, \varphi, \alpha)$.

(4) If α, γ are open covers of X and $\alpha \prec \gamma$ (i.e. for each $C \in \gamma$, there is an $A \in \alpha$ such that $C \subset A$), then $q_n(f_{1,\infty}, \varphi, \alpha) \leq q_n(f_{1,\infty}, \varphi, \gamma)$.

(5) If $d(x, y) < \text{diam}(\alpha)$ implies $|f(x) - f(y)| \leq \delta$ then $p_n(f_{1,\infty}, \varphi, \alpha) \leq \varepsilon n \delta q_n(f_{1,\infty}, \varphi, \gamma)$.

(6) $P(f_{1,\infty}, \varphi) = \lim_{k \rightarrow \infty} [\lim_{n \rightarrow \infty} \frac{1}{n} \log p_n(f_{1,\infty}, \varphi, \alpha_k)] = \lim_{k \rightarrow \infty} [\limsup_{n \rightarrow \infty} \frac{1}{n} \log q_n(f_{1,\infty}, \varphi, \alpha_k)]$ if α_k is a sequence of open covers with $\text{diam}(\alpha_k) \rightarrow 0$.

(7) $P(f_{1,\infty}, \varphi) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log P_n(f_{1,\infty}, \varphi, \varepsilon)$.

(8) $P(f_{1,\infty}, \varphi) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log Q_n(f_{1,\infty}, \varphi, \varepsilon)$.

The proof of Proposition 2.1 is similar to the case of the autonomous systems (for detailed proof see [4]), we omitted it.

3 Properties of Pressure of a Sequence of Maps on a Compact Metric Space

We now study the properties of $P(f_{1,\infty}, \cdot) : C(X, X) \rightarrow R \cup \infty$. In particular we see that either $P(f_{1,\infty}, \cdot)$ never takes the value ∞ or is identical to ∞ .

Theorem 3.1 Let $f_{1,\infty} : X \rightarrow X$ be a continuous maps of a compact metric space X and $\varphi \in C(X, R), \varepsilon > 0$. Then $P(f_{1,\infty}^k, S_k \varphi) \leq kP(f_{1,\infty}, \varphi)$ (here $(S_k \varphi)(x) = \sum_{i=0}^{k-1} \varphi(f_{1,\infty}^i(x))$) for any $k \geq 1$.

Proof If F is (nk, ε) spanning for $f_{1,\infty}$ then F is (n, ε) spanning for $f_{1,\infty}^k$. Here $Q_n(f_{1,\infty}^k, S_k \varphi, \varepsilon) \leq Q_{nk}(f_{1,\infty}, \varphi, \varepsilon)$ so that $P(f_{1,\infty}^k, S_k \varphi) \leq kP(f_{1,\infty}, \varphi)$. \square

Remark 3.1 In general we cannot claim that $P(f_{1,\infty}^k, S_k \varphi) = kP(f_{1,\infty}, \varphi)$ for any $k \geq 1$.

Example 3.1 Indeed, on $X = I = [0, 1]$ take the standard tent map $g(x) = 1 - |2x - 1|$, $\varphi = 0$ and

$$f_{1,\infty} = \left\{ g, \frac{1}{2} id_{S^1}, g^2, \frac{1}{4} id_{S^1}, \dots, g^n, \frac{1}{2^n} id_{S^1}, \dots \right\}.$$

Since $f_{1,\infty}^{2n-1} = g^n$ for every n , we have $s(f_{1,\infty}, 2n, \varepsilon) = s(g, n, \varepsilon)$ and therefore $P(f_{1,\infty}, \varphi) = h(f_{1,\infty}) \geq \frac{1}{2} h(g) = \frac{1}{2} \log 2$. On the other hand, $f_{1,\infty}^2 = \{f_{1,\infty}^2, f_{1,\infty}^3, \dots, f_{1,\infty}^{2n-1}, \dots\}$, where for any $n \in N$ and for any $x \in I$, $f_{1,\infty}^{2n-1}(x) \leq \frac{1}{2^n}$. Therefore $\limsup_{n \rightarrow \infty} \frac{1}{n} \log s(f_{1,\infty}^2, n, \varepsilon) = 0$ for every $\varepsilon > 0$ and so $P(f_{1,\infty}^2, S_2 \varphi) = h(f_{1,\infty}^2) = 0$.

Thus $h(f_{1,\infty}^2) < 2h(f_{1,\infty})$, i.e. $P(f_{1,\infty}^2, S_2 \varphi) < 2P(f_{1,\infty}, \varphi)$.

So if we wish to have the equality instead of the inequality in Theorem 3.1, we need additional assumptions. We present here one result of this kind, we restrict ourselves to compact metric spaces and sequences of equicontinuous maps.

Theorem 3.2 *Let $f_{1,\infty} : X \rightarrow X$ be a sequence of equicontinuous self-maps of the compact metric space X . $P(f_{1,\infty}^k, S_k\varphi) = kP(f_{1,\infty}, \varphi)$ (here $(S_k\varphi)(x) = \sum_{i=0}^{k-1} \varphi(f_1^i(x))$) for any $k \geq 1$.*

Proof For $k = 1$ this is trivial. Take any $k \geq 2$. In view of Theorem 3.1 it suffices to prove that $P(f_{1,\infty}^k, S_k\varphi) \geq kP(f_{1,\infty}, \varphi)$. To this end, for every $\varepsilon > 0$ take $\delta(\varepsilon) \geq \varepsilon$ such that $\delta(\varepsilon) \rightarrow 0$ if $\varepsilon \rightarrow 0$ and $d(f_i^m(x), f_i^m(y)) \leq \delta(\varepsilon)$ whenever $i \geq 1, m \in \{1, 2, \dots, k-1\}$ and $d(x, y) \leq \varepsilon$. Take any positive integer n , then any $(nk, \delta(\varepsilon))$ -separated set for $f_{1,\infty}$ is (n, ε) -separated set for $f_{1,\infty}^k$ and so $P_n(f_{1,\infty}^k, S_k\varphi, \delta(\varepsilon)) \geq P_{nk}(f_{1,\infty}, \varphi, \varepsilon)$. Therefore $P(f_{1,\infty}^k, S_k\varphi) \geq kP(f_{1,\infty}, \varphi)$. \square

In the sequel, let us consider the following situation: (X, d) and (Y, ρ) are compact metric spaces, $f_{1,\infty}$ is a sequence of continuous maps from X into itself and $g_{1,\infty}$ is a sequence of continuous maps from Y into itself.

Kolyada and Snoha proved the topological entropy of sequence of continuous maps is invariant with equiconjugacy [3]. Now we mainly show the topological pressure of sequence of continuous maps is invariant with equiconjugacy.

Suppose that $\pi_{1,\infty}$ is a sequence of continuous maps from X into Y such that $\pi_{i+1} \circ f_i = g_i \circ \pi_i$ for every $i \geq 1$. There are two special cases when we can compare the pressure of $f_{1,\infty}$ and $g_{1,\infty}$. They are the following.

(i) When $\pi_{1,\infty}$ is a sequence of equicontinuous surjective (i.e., onto) maps from X onto Y . In this case we say that $\pi_{1,\infty}$ topologically equisemiconjugates $f_{1,\infty}$ with $g_{1,\infty}$, $\pi_{1,\infty}$ is a topological equisemiconjugacy between $f_{1,\infty}$ and $g_{1,\infty}$ and the dynamical systems $(X, f_{1,\infty})$ is topologically equisemiconjugate with $(Y, g_{1,\infty})$. The system $(Y, g_{1,\infty})$ is an equifactor of $(X, f_{1,\infty})$.

(ii) When $\pi_{1,\infty}$ is an equicontinuous sequence of homeomorphisms such that the sequence $\pi_{1,\infty}^{-1} = \{\pi_i^{-1}\}_{i=1}^\infty$ of inverse homeomorphisms is also equicontinuous. In this case we say that $\pi_{1,\infty}$ topologically equiconjugates $f_{1,\infty}$ with $g_{1,\infty}$, $\pi_{1,\infty}$ is a topological equiconjugacy between $f_{1,\infty}$ and $g_{1,\infty}$ and the dynamical systems $(X, f_{1,\infty})$ is topologically equiconjugate with $(Y, g_{1,\infty})$.

Theorem 3.3 *Let (X, d) and (Y, ρ) be compact metric spaces, $f_{1,\infty}$ be a sequence of continuous maps from X into itself and $g_{1,\infty}$ be a sequence of continuous maps from Y into itself. If the system $(X, f_{1,\infty})$ is topologically equisemiconjugate with $(Y, g_{1,\infty})$ (denote the equisemiconjugacy by $\pi_{1,\infty}$) then*

$$P(g_{1,\infty}, \varphi) \leq P(f_{1,\infty}, \varphi \circ \pi_{1,\infty}),$$

for any $\varphi \in C(Y, R)$.

Proof Since $\pi_{1,\infty}$ is a sequence of equicontinuous maps from X to Y , given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $\rho(\pi_i(x), \pi_i(y)) > \varepsilon$ for some $i \geq 1$, then $d(x, y) > \delta(\varepsilon)$. Let $F \subset Y$ be a $(n, \varepsilon, g_{1,\infty}, \rho)$ -separated set, then $\pi_1^{-1}(F)$ is an $(n, \delta(\varepsilon), f_{1,\infty}, d)$ -separated set. Thus

$$\sum_{x \in F} e^{\varphi(x) + \varphi(g_1(x)) + \dots + \varphi(g_1^{n-1}(x))} = \sum_{y \in \pi_1^{-1}(F)} e^{\varphi(\pi_1(y)) + \varphi(\pi_1 f_1(y)) + \dots + \varphi(\pi_1 f_1^{n-1}(y))}.$$

Therefore $P(g_{1,\infty}, \varphi, \varepsilon) \leq P(f_{1,\infty}, \varphi \circ \pi_{1,\infty}, \delta(\varepsilon))$. It follows that

$$P(g_{1,\infty}, \varphi) \leq P(f_{1,\infty}, \varphi \circ \pi_{1,\infty}).$$

□

Corollary 3.1 *Let (X, d) and (Y, ρ) be compact metric spaces, $f_{1,\infty}$ be a sequence of continuous maps from X into itself and $g_{1,\infty}$ is a sequence of continuous maps from Y into itself. If the system $(X, f_{1,\infty})$ is topologically equiconjugate with $(Y, g_{1,\infty})$ then*

$$P(g_{1,\infty}, \varphi) = P(f_{1,\infty}, \varphi \circ \pi_{1,\infty}).$$

Proof Denote the conjugacy by $\pi_{1,\infty}$. We have $P(g_{1,\infty}, \varphi) \leq P(f_{1,\infty}, \varphi \circ \pi_{1,\infty})$ since $\pi_{1,\infty}$ is a semiequiconjugacy between $f_{1,\infty}$ and $g_{1,\infty}$ and $P(g_{1,\infty}, \varphi) \geq P(f_{1,\infty}, \varphi \circ \pi_{1,\infty})$ since $\pi_{1,\infty}^{-1}$ is a semiequiconjugacy between $g_{1,\infty}$ and $f_{1,\infty}$. □

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