Nonlinear Dynamics and Systems Theory, **8** (2) (2008) 109–122

Conditions of Ultimate Boundedness of Solutions for a Class of Nonlinear Systems

A.Yu. Aleksandrov * and A.V. Platonov

Saint Petersburg State University,
198504, Universitetskij ave., 35, Petrodvorets, St. Petersburg, Russia

Received: May 12, 2007; Revised: March 13, 2008

Abstract: A class of nonlinear differential equations systems is considered. An approach for the construction of Lyapunov's functions for these systems is suggested. By the use of functions constructed the conditions of ultimate boundedness of solutions for systems investigated are obtained.

Keywords: *Nonlinear systems; Lyapunov's functions; ultimate boundedness; large scale systems.*

Mathematics Subject Classification (2000): 34D20, 93D20, 93D30.

1 Introduction

In a variety of control systems design problems it is often required not only to stabilize given programmed motions but to ensure also boundedness for every solution of system investigated. Of great practical interest is the case when all the solutions enter a neighborhood of the origin and remain within it thereafter. Generally the time period needed for the solution to enter this neighborhood depends on the initial values of the solution. In this case solutions are called ultimately bounded [14].

The main approach for finding the conditions of boundedness of solutions for nonlinear systems is the Lyapunov direct method. Using this method, numerous results on various types of boundedness are obtained [6, 11, 12, 14, 16]. However, there are still no general constructive approaches for the construction of Lyapunov's functions.

In the present paper, a certain class of differential equations systems is considered. An approach for the construction of Lyapunov's functions for these systems is suggested. The conditions for the existence of Lyapunov's functions in the given form, satisfying the assumptions of the Yoshizawa ultimate boundedness theorem [14] are investigated. The results obtained are used for the analysis of the asymptotic behavior of solutions of essentially nonlinear complex systems.

* Corresponding author: alex@vrm.apmath.spbu.ru

2 Statement of the Problem

Consider the system of differential equations

$$\dot{x}_s = a_s f_s(x_s) + \sum_{j=1}^{k_s} b_{sj} f_1^{\alpha_{s1}^{(j)}}(x_1) \dots f_n^{\alpha_{sn}^{(j)}}(x_n), \quad s = 1, \dots, n. \tag{2.1}$$

Here a_s and b_{sj} are constant coefficients, $\alpha_{si}^{(j)}$ are nonnegative rationals with odd denominators, functions $f_s(x_s)$ are defined and continuous for $x_s \in (-\infty, +\infty)$ and possess the following properties: $x_s f_s(x_s) > 0$ for $x_s \neq 0$, $f_s(x_s) \to -\infty$ as $x_s \to -\infty$, $f_s(x_s) \to +\infty$ as $x_s \to +\infty$.

System (2.1) is a generalization of this one

$$\dot{x}_s = \sum_{j=1}^{n} b_{sj} f_j(x_j), \quad s = 1, \dots, n,$$

which is widely used in automatic control systems design [3, 7, 9].

In this paper we shall assume that coefficients a_s and b_{sj} in (2.1) satisfy the conditions

$$a_s < 0, \quad b_{sj} > 0, \quad j = 1, \dots, k_s, \quad s = 1, \dots, n. \tag{2.2}$$

For instance, inequalities (2.2) are valid if (2.1) is obtained as comparison system for a complex system [4, 13].

Consider, at first, the case where

$$\sum_{i=1}^{n} \alpha_{si}^{(j)} > 0, \quad j = 1, \dots, k_s, \quad s = 1, \dots, n. \tag{2.3}$$

Then system (2.1) has the zero solution.

Definition 2.1 [2] System (2.1) is called absolutely stable if the zero solution of this system is asymptotically stable for any admissible functions $f_1(x_1), \dots, f_n(x_n)$.

The criterion of absolute stability for (2.1) was established in [2].

Definition 2.2 [2] System (2.1) satisfies the Martynyuk–Obolenskij condition [8] (MO-condition) if for any $\delta > 0$ there exists solution $\tilde{\theta}_1, \dots, \tilde{\theta}_n$ of the system

$$a_s \theta_s + \sum_{j=1}^{k_s} b_{sj} \theta_1^{\alpha_{s1}^{(j)}} \dots \theta_n^{\alpha_{sn}^{(j)}} < 0, \quad s = 1, \dots, n, \tag{2.4}$$

such that $0 < \tilde{\theta}_s < \delta$, $s = 1, \dots, n$.

It was proved [2] that (2.1) is absolutely stable if and only if it satisfies the MO-condition.

The proof of necessity of this criterion is based on the fact, that for the special choice of admissible functions $f_1(x_1), \dots, f_n(x_n)$ system (2.1) is Wazewskij's one, and for it the general criterion of asymptotic stability of autonomous Wazewskij's systems [8] is applicable.

To prove the sufficiency, it was suggested [2] to construct Lyapunov's function for (2.1) in the form

$$V(x) = \sum_{s=1}^{n} \lambda_s \int_0^{x_s} f_s^{\mu_s}(\tau) d\tau, \tag{2.5}$$

where $x = (x_1, \dots, x_n)^*$, $\lambda_s > 0$ are constant coefficients, $\mu_s > 0$ are rationals with odd numerators and denominators. It was shown that (2.1) is absolutely stable if and only if for this system there exists Lyapunov's function in the form (2.5), satisfying the assumptions of the Lyapunov asymptotic stability theorem.

Definition 2.3 We call (2.1) absolutely ultimately bounded if solutions of this system are ultimately bounded for any admissible functions $f_1(x_1), \dots, f_n(x_n)$.

The main goal of the present paper is to obtain the criterion of absolute ultimate boundedness for (2.1). To solve this problem, let us determine the conditions under which for system investigated there exists Lyapunov's function in the form (2.5), satisfying the assumptions of the Yoshizawa ultimate boundedness theorem [14].

Remark 2.1 In what follows, we do not assume the fulfilment of inequalities (2.3).

3 Conditions of Ultimate Boundedness for Wazewskij's Systems

Let us note, just as in [2], that in the case where $f_1(x_1), \dots, f_n(x_n)$ are nondecreasing functions, system (2.1) is Wazewskij's one. Therefore, we shall investigate, first, conditions of ultimate boundedness of solutions for autonomous Wazewskij's systems of the general form.

Consider the system

$$\dot{x} = g(x), \tag{3.1}$$

where $x = (x_1, \dots, x_n)^*$ and vector function $g(x)$ is defined and continuous for all $x \in R^n$.

Assume that system (3.1) possesses the following properties:

(a) for any $t_0 \in (-\infty, +\infty)$ and any $x_0 \in R^n$ the initial value problem for (3.1) has unique solution $x(t, x_0, t_0)$;

(b) system (2.1) is Wazewskij's one;

(c) there exists a number $D > 0$ such that in the region $\|x\| \geq D$ there is no equilibrium position of (3.1).

Here $\|\cdot\|$ is the Euclidean norm of the vector.

Remark 3.1 Condition (c) is a necessary one for the solutions of (3.1) to be ultimately bounded.

Furthermore, we shall assume that nonnegative cone $K^+ = \{x \in R^n : \, x_s \geq 0, \, s = 1, \dots, n\}$ is an invariant set for (3.1).

Definition 3.1 The solutions of (3.1) are ultimately bounded in K^+ if there exists a $H > 0$ and if, corresponding to every $Q > 0$, one can choose a $T > 0$ such that for any $t_0 \in (-\infty, +\infty)$ and for any $x_0 \in K^+$, $\|x_0\| < Q$, the inequality $\|x(t, x_0, t_0)\| < H$ holds for all $t \geq t_0 + T$.

Definition 3.2 We shall say that (3.1) satisfies the $\widetilde{MO}$-condition if for any $\Delta > 0$ there exists vector $\theta = (\theta_1, \dots, \theta_n)^*$, $\|\theta\| > \Delta$, such that $\theta > 0$ and $g(\theta) < 0$ (the inequalities are componentwise).

By the use of Lemmas 3.1 and 3.2 from [8], we get the validity of the following

Theorem 3.1 (necessary condition of ultimate boundedness) *If the solutions of (3.1) are ultimately bounded in K^+, then this system satisfies the $\widetilde{MO}$-condition.*

Remark 3.2 The $\widetilde{MO}$-condition is a necessary one for the ultimate boundedness of solutions. However, it is not, generally, the sufficient condition.

Example 3.1 Let system (3.1) be of the form

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2, \\ \dot{x}_2 &= -x_2. \end{aligned} \tag{3.2}$$

This system possesses properties (a) – (c) and K^+ is an invariant set for it.

Consider the inequalities

$$-\theta_1 + \theta_1^2\theta_2 < 0, \qquad -\theta_2 < 0. \tag{3.3}$$

For given $\Delta > 0$ there exists positive vector $\tilde{\theta} = (1/(2\Delta), \Delta)^*$, satisfying (3.3), such that $\|\tilde{\theta}\| > \Delta$. Thus, for system (3.2) the $\widetilde{MO}$-condition is fulfilled.

At the same time, (3.2) has the solution $x(t) = (2e^t, e^{-t})^*$. Hence, solutions of this system are not ultimately bounded in K^+.

Definition 3.3 We shall say that (3.1) satisfies the $\overline{MO}$-condition if for any $\Delta > 0$ there exists vector $\theta = (\theta_1, \ldots, \theta_n)^*$ such that $\theta_s > \Delta$, $s = 1, \ldots, n$, and $g(\theta) < 0$.

Using Lemma 3.3 from [8], it is easy to show the validity of the following

Theorem 3.2 (sufficient condition of ultimate boundedness) *If system (3.1) satisfies the $\overline{MO}$-condition, then its solutions are ultimately bounded in K^+.*

Remark 3.3 The $\overline{MO}$-condition is a sufficient one for the ultimate boundedness of solutions. However, it is not, generally, the necessary condition.

Example 3.2 Let the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2, \\ \dot{x}_2 &= -x_2 \end{aligned} \tag{3.4}$$

be given. This system satisfies all the above assumptions (properties (a)–(c) and invariance of K^+).

By the direct integration, one can verify that solutions of (3.4) are ultimately bounded.

On the other hand, if for a positive vector $\theta = (\theta_1, \theta_2)^*$ the inequalities

$$-\theta_1 + \theta_1\theta_2 < 0, \qquad -\theta_2 < 0,$$

are valid, then $\theta_2 < 1$. Hence, for (3.4) the $\overline{MO}$-condition is not fulfilled.

4 Construction of Lyapunov's Functions

Now, let us investigate the problem of absolute ultimate boundedness for system (2.1).

Construct Lyapunov's function for this system in the form (2.5), where, as before, $\lambda_s > 0$ are constant coefficients and $\mu_s > 0$ are rationals with odd numerators and denominators.

Function $V(x)$ is positive for all $x \neq 0$, and $V(x) \to +\infty$ as $\|x\| \to \infty$. On differentiating this function with respect to (2.1), one arrives at

$$\left.\frac{dV}{dt}\right|_{(2.1)} = \sum_{s=1}^{n} \lambda_s a_s f_s^{\mu_s+1}(x_s) + \sum_{s=1}^{n} \lambda_s f_s^{\mu_s}(x_s) \sum_{j=1}^{k_s} b_{sj} f_1^{\alpha_{s1}^{(j)}}(x_1) \dots f_n^{\alpha_{sn}^{(j)}}(x_n).$$

Hence, $V(x)$ satisfies the assumptions of the Yoshizawa ultimate boundedness theorem for any admissible functions $f_1(x_1), \dots, f_n(x_n)$, if coefficients λ_s and exponents μ_s, $s = 1, \dots, n$, are chosen for the function

$$W(y) = \sum_{s=1}^{n} \lambda_s a_s y_s^{\mu_s+1} + \sum_{s=1}^{n} \lambda_s y_s^{\mu_s} \sum_{j=1}^{k_s} b_{sj} y_1^{\alpha_{s1}^{(j)}} \dots y_n^{\alpha_{sn}^{(j)}} \tag{4.1}$$

to be negative in a region $\|y\| > R$. Here $y = (y_1, \dots, y_n)^*$, while $R > 0$ is a constant.

Let us denote $h_s = 1/(\mu_s + 1)$, $s = 1, \dots, n$. By the use of generally-homogeneous functions properties [15], we get that $W(y)$ might be negative for all $\|y\| > R$ only in the case, where following inequalities are valid:

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j)} h_i \leq 0, \quad j = 1, \dots, k_s, \quad s = 1, \dots, n, \tag{4.2}$$

Remark 4.1 If there exist positive rationals $h_1, \dots, h_n$ for which all the inequalities in (4.2) are strict, i.e.

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j)} h_i < 0, \quad j = 1, \dots, k_s, \quad s = 1, \dots, n, \tag{4.3}$$

then for corresponding values of μ_s and for any admissible values of a_s, b_{sj} and λ_s, $j = 1, \dots, k_s$, $s = 1, \dots, n$, one can choose a constant $R > 0$ such that $W(y) < 0$ for $\|y\| > R$.

5 Auxiliary Results

Let us determine the conditions of the existence of positive solutions for systems (4.2) and (4.3).

Remark 5.1 It is known [4, 13], that in the case where $k_s = 1$, $s = 1, \dots, n$, for the existence of a positive solution for (4.3) it is necessary and sufficient for the matrix

$$A = \begin{pmatrix} \alpha_{11}^{(1)} - 1 & \alpha_{12}^{(1)} & \dots & \alpha_{1n}^{(1)} \\ \alpha_{21}^{(1)} & \alpha_{22}^{(1)} - 1 & \dots & \alpha_{2n}^{(1)} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1}^{(1)} & \alpha_{n2}^{(1)} & \dots & \alpha_{nn}^{(1)} - 1 \end{pmatrix}$$

to satisfy the Sevast'yanov–Kotelyanskij conditions.

Lemma 5.1 *If there exists a positive solution for (4.3), then for system (2.1) the* $\overline{MO}$*-condition is fulfilled.*

Proof Let inequalities (4.3) be valid for positive constants $\tilde{h}_1, \dots, \tilde{h}_n$. Then for given number $\Delta > 0$, one can choose $\tau > 0$ so large that the constants $\tilde{\theta}_s = \tau^{\tilde{h}_s}$, $s = 1, \dots, n$, satisfy inequalities (2.4), and $\tilde{\theta}_s > \Delta$ for $s = 1, \dots, n$. $\square$

Along with (4.2), consider the system

$$-h_s + \sum_{i=1}^{n} \alpha_{si}^{(j)} h_i = c_s^{(j)}, \quad j = 1, \dots, k_s, \quad s = 1, \dots, n,$$

where $c_s^{(j)}$ are nonpositive constants. This system can be splitted into n subsystems

$$A_s h = c_s, \quad s = 1, \dots, n. \tag{5.1}$$

Here $h = (h_1, \dots, h_n)^*$, $c_s = (c_s^{(1)}, \dots, c_s^{(k_s)})^*$,

$$A_1 = \begin{pmatrix} \alpha_{11}^{(1)} - 1 & \alpha_{12}^{(1)} & \dots & \alpha_{1n}^{(1)} \\ \dots & \dots & \dots & \dots \\ \alpha_{11}^{(k_1)} - 1 & \alpha_{12}^{(k_1)} & \dots & \alpha_{1n}^{(k_1)} \end{pmatrix}, \quad \dots \quad, \quad A_n = \begin{pmatrix} \alpha_{n1}^{(1)} & \alpha_{n2}^{(1)} & \dots & \alpha_{nn}^{(1)} - 1 \\ \dots & \dots & \dots & \dots \\ \alpha_{n1}^{(k_n)} & \alpha_{n2}^{(k_n)} & \dots & \alpha_{nn}^{(k_n)} - 1 \end{pmatrix}.$$

Let us apply to (5.1) the modified Gaussian elimination procedure. On the s-th step of this procedure each of the equations with negative coefficient of h_s is used for the elimination of h_s from the $(s+1)$-th, etc., and n-th subsystems. This results in a new set of subsystems with (generally) the other number of equations than in the initial system.

Lemma 5.2 *System (4.2) possesses a positive solution if and only if the above modified Gaussian elimination procedure reduces system (5.1) to the form*

$$B_s h = \tilde{c}_s, \quad s = 1, \dots, n,$$

where

$$B_1 = \begin{pmatrix} \beta_{11}^{(1)} & \dots & \beta_{1n}^{(1)} \\ \dots & \dots & \dots \\ \beta_{11}^{(q_1)} & \dots & \beta_{1n}^{(q_1)} \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & \beta_{22}^{(1)} & \dots & \beta_{2n}^{(1)} \\ \dots & \dots & \dots & \dots \\ 0 & \beta_{22}^{(q_2)} & \dots & \beta_{2n}^{(q_2)} \end{pmatrix}, \; \dots, B_n = \begin{pmatrix} 0 & \dots & 0 & \beta_{nn}^{(1)} \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \beta_{nn}^{(q_n)} \end{pmatrix},$$

$$\tilde{c}_1 = \left(\tilde{c}_1^{(1)}, \dots, \tilde{c}_1^{(q_1)}\right)^*, \quad \dots, \quad \tilde{c}_n = \left(\tilde{c}_n^{(1)}, \dots, \tilde{c}_n^{(q_n)}\right)^*,$$

$\tilde{c}_s^{(j)} \leq 0$, $\beta_{ss}^{(j)} \leq 0$, $\beta_{si}^{(j)} \geq 0$ *for* $j = 1, \dots, q_s$, $i = s+1, \dots, n$, $s = 1, \dots, n$, *and if* $\beta_{ss}^{(j)} = 0$ *for some values of indices* s *and* j, *then* $\beta_{si}^{(j)} = 0$, $i = s+1, \dots, n$, *for all such* s *and* j.

Remark 5.2 In the case where $\beta_{ss}^{(j)} < 0$, $j = 1, \dots, q_s$, $s = 1, \dots, n$, there exist positive numbers $\tilde{h}_1, \dots, \tilde{h}_n$ satisfying strict inequalities (4.3).

Lemma 5.3 *If for (2.1) the* $\overline{MO}$*-condition is fulfilled, then there exists a positive solution for system (4.2).*

The proofs of Lemmas 5.2 and 5.3 are similar to those ones of Lemmas 4.2 and 4.3 from [2].

Remark 5.3 The proof of Lemma 5.2 contains a constructive algorithm for finding a positive solution $\tilde{h}_1, \dots, \tilde{h}_n$ for (4.2). Moreover, let us note that using this algorithm one may choose $\tilde{h}_1, \dots, \tilde{h}_n$ for the numbers $\mu_s = 1/\tilde{h}_s - 1$, $s = 1, \dots, n$, to be positive rationals with odd numerators and denominators.

6 Conditions of Absolute Ultimate Boundedness

Consider the necessary conditions of absolute ultimate boundedness for system (2.1).

Theorem 6.1 *If system (2.1) is absolutely ultimately bounded, then it satisfies the $\widetilde{MO}$-condition.*

Proof Suppose that (2.1) is absolutely ultimately bounded. Then its solutions are ultimately bounded for any admissible functions $f_1(x_1), \dots, f_n(x_n)$.

Let $f_s(x_s) = x_s^{m_s}$, $s = 1, \dots, n$, where m_s are odd positive integers such that $\alpha_{si}^{(j)} m_i \geq 1$ for $j = 1, \dots, k_s$, $i, s = 1, \dots, n$. For chosen admissible functions, system (2.1) possesses all the properties from Section 3 (properties (a)–(c) and invariance of K^+). Using Theorem 3.1, we get that (2.1) satisfies the $\widetilde{MO}$-condition. □

Theorem 6.2 *Let there exist positive constants $\tilde{h}_1, \dots, \tilde{h}_n$, and for every $s = 1, \dots, n$ there exists at least one value of $j \in \{1, \dots, k_s\}$ such that*

$$-\tilde{h}_s + \sum_{i=1}^{n} \alpha_{si}^{(j)} \tilde{h}_i > 0.$$

Then system (2.1) is not absolutely ultimately bounded.

Proof Choose, again, functions $f_1(x_1), \dots, f_n(x_n)$ for the obtained system to satisfy all the assumptions from Section 3.

Consider the numbers $\tilde{\theta}_s = \tau^{\tilde{h}_s}$, $\tau > 0$, $s = 1, \dots, n$. For all sufficiently large values of τ the inequalities

$$a_s \tilde{\theta}_s + \sum_{j=1}^{k_s} b_{sj} \tilde{\theta}_1^{\alpha_{s1}^{(j)}} \dots \tilde{\theta}_n^{\alpha_{sn}^{(j)}} > 0, \quad s = 1, \dots, n,$$

are fulfilled. Applying Lemma 3.4 from [8], we get that for chosen admissible functions, the solutions of system (2.1) are not ultimately bounded. □

Consider, next, the sufficient conditions of absolute ultimate boundedness.

Theorem 6.3 *If for system (2.1) the $\overline{MO}$-condition is fulfilled, then it is absolutely ultimately bounded.*

Proof Let us show that for system investigated there exists Lyapunov's function in the form (2.5), satisfying the assumptions of the Yoshizawa ultimate boundedness theorem.

According to Lemma 5.3, we obtain that if for (2.1) the $\overline{MO}$-condition is fulfilled, then one can choose positive rationals $\mu_1, \dots, \mu_n$ with odd numerators and denominators such that for the numbers $\tilde{h}_s = 1/(\mu_s + 1)$, $s = 1, \dots, n$, inequalities (4.2) are valid. We shall take these values of $\mu_1, \dots, \mu_n$ as exponents in Lyapunov's function (2.5).

Let us show the existence of positive coefficients $\lambda_1, \dots, \lambda_n$ under which function (4.1) is negative in a region $\|y\| > R$, where $R > 0$ is a constant.

Consider inequalities (4.2) for $h_s = \tilde{h}_s$, $s = 1, \dots, n$. It should be noted that in the case where for some values of indices j and s the corresponding inequalities are strict, one can construct, instead of (4.1), new function $\widehat{W}(y)$ by setting $b_{sj} = 0$ for all such j and s. If there exist positive coefficients $\lambda_1, \dots, \lambda_n$ for which $\widehat{W}(y)$ is negative definite, then for these values of $\lambda_1, \dots, \lambda_n$ and for some number $R > 0$, the inequality $W(y) < 0$ holds in the region $\|y\| > R$. Therefore, we may assume, without loss of generality, that for the numbers $\tilde{h}_1, \dots, \tilde{h}_n$ all the inequalities in (4.2) turn to equalities.

The rest part of the proof is similar to that one of Theorem 5.1 from [2]. $\square$

Remark 6.1 Theorem 6.3 looks similar to Theorem 3.2. However, in comparison with the conditions of ultimate boundedness of solutions for autonomous Wazewskij's systems obtained in Section 3, Theorem 6.3 states that the only $\overline{MO}$-condition is a sufficient one for the absolute ultimate boundedness of (2.1), i.e. the other assumptions from Section 3 (properties (a)–(c) and invariance of K^+) are redundant.

Corollary 6.1 *Let system (4.2) has a positive solution. Then (2.1) is absolutely ultimately bounded if and only if there exists at least one set of positive constants $\tilde{\theta}_1, \dots, \tilde{\theta}_n$, satisfying inequalities (2.4).*

Proof The necessity follows from Theorem 6.1.

To prove the sufficiency, suppose that the positive vectors $\tilde{h} = (\tilde{h}_1, \dots, \tilde{h}_n)^*$ and $\tilde{\theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_n)^*$ are solutions of systems (4.2) and (2.4) correspondingly. Then the numbers $\hat{\theta}_s = \tau^{\tilde{h}_s}\tilde{\theta}_s$, $\tau > 0$, $s = 1, \dots, n$, satisfy inequalities (2.4) for all sufficiently large values of τ. Hence, the $\overline{MO}$-condition is fulfilled for (2.1). $\square$

Corollary 6.2 *For system (2.1) there exists Lyapunov's function in the form (2.5), satisfying the assumptions of the Yoshizawa ultimate boundedness theorem, if and only if the $\overline{MO}$-condition is fulfilled for this system.*

Proof The sufficiency follows from Theorem 6.3. Let us prove the necessity.

In Section 4 it was noted that if function (2.5) satisfies the assumptions of the Yoshizawa ultimate boundedness theorem, then system (4.2) possesses a positive solution. On the other hand, the existence of such Lyapunov's function provides the absolute ultimate boundedness for (2.1). Then, according to Theorem 6.1, there exists a positive solution for system (2.4). By analogy with the proof of Corollary 6.1, we get that for (2.1) the $\overline{MO}$-condition is fulfilled. $\square$

Consider now, along with (2.1), the perturbed system

$$\dot{x}_s = a_s f_s(x_s) + \sum_{j=1}^{k_s} b_{sj} f_1^{\alpha_{s1}^{(j)}}(x_1) \dots f_n^{\alpha_{sn}^{(j)}}(x_n) + \psi_s(t, x), \quad s = 1, \dots, n. \tag{6.1}$$

Here functions $\psi_s(t, x)$ are continuous for all $t \in (-\infty, +\infty)$, $x \in R^n$, and satisfy the inequalities $|\psi_s(t, x)| \le \gamma_s + \varepsilon_s |f_s(x_s)|$, where γ_s and ε_s are positive constants, $s = 1, \dots, n$.

Corollary 6.3 *Let for (2.1) the* $\overline{MO}$*-condition be fulfilled. Then solutions of (6.1) are uniformly ultimately bounded for sufficiently small values of* $\varepsilon_1, \dots, \varepsilon_n$*.*

Proof Construct for system (2.1) Lyapunov's function in the form (2.5), satisfying the assumptions of the Yoshizawa ultimate boundedness theorem. It is easily shown that for sufficiently large number $R > 0$ and for sufficiently small values of $\varepsilon_1, \dots, \varepsilon_n$ the derivative of Lyapunov's function constructed with respect to (6.1) is negative in the region $\|x\| > R$. □

Remark 6.2 In a similar way, the conditions of absolute ultimate boundedness can be obtained for the case when the inequalities $b_{sj} > 0$ in (2.2) are replaced by those connecting coefficients b_{sj} and a basis $\omega_1, \dots, \omega_n$: $b_{sj}\omega_s\omega_1^{\alpha_{s1}^{(j)}} \dots \omega_n^{\alpha_{sn}^{(j)}} > 0$ for $j = 1, \dots, k_s$, $s = 1, \dots, n$ [9]. Here every constant $\omega_1, \dots, \omega_n$ takes either of the values $+1$ or -1.

7 Systems with the Special Structure of Connections

In the previous section it was proved that for (2.1) to be absolutely ultimately bounded it is sufficient the fulfilment of the $\overline{MO}$-condition. Consider now some types of systems of the form (2.1) with the special structure of connections for which the $\overline{MO}$-condition is not only sufficient one but also a necessary one for absolute ultimate boundedness.

Example 7.1 Consider system (2.1) with $k_1 = 1, \dots, k_n = 1$:

$$\dot{x}_s = a_s f_s(x_s) + b_s f_1^{\alpha_{s1}}(x_1) \dots f_n^{\alpha_{sn}}(x_n), \quad s = 1, \dots, n. \tag{7.1}$$

Here $a_s < 0$ and $b_s > 0$ are constant coefficients, α_{si} are nonnegative rationals with odd denominators. For (7.1) the corresponding system of inequalities (4.2) is of the form

$$-h_s + \sum_{i=1}^{n} \alpha_{si} h_i \le 0, \quad s = 1, \dots, n. \tag{7.2}$$

According to Corollary 6.1, we get that if (7.2) possesses a positive solution, then for (7.1) to be absolutely ultimately bounded it is necessary and sufficient for this system to satisfy the $\overline{MO}$-condition.

On the other hand, if (7.2) has no positive solutions, and $\alpha_{si} > 0$ for $s \ne i$, then there exist positive constants $\tilde{h}_1, \dots, \tilde{h}_n$ such that

$$-\tilde{h}_s + \sum_{i=1}^{n} \alpha_{si} \tilde{h}_i > 0, \quad s = 1, \dots, n.$$

Hence (v. Theorem 6.2), system (7.1) is not absolutely ultimately bounded.

Thus, in the case where $\alpha_{si} > 0$ for $s \ne i$, $i, s = 1, \dots, n$, system (7.1) is absolutely ultimately bounded if and only if the $\overline{MO}$-condition is fulfilled for this system.

Example 7.2 Let system (2.1) be of the form

$$\begin{aligned} \dot{x}_1 &= a_1 f_1(x_1) + b_1 f_n^{\alpha_1}(x_n), \\ \dot{x}_i &= a_i f_i(x_i) + b_i f_{i-1}^{\alpha_i}(x_{i-1}), \quad i = 2, \dots, n-1, \\ \dot{x}_n &= a_n f_n(x_n) + b_n f_1^{\beta_1}(x_1) \dots f_{n-1}^{\beta_{n-1}}(x_{n-1}), \end{aligned} \tag{7.3}$$

where $a_s < 0$ and $b_s > 0$ are constant coefficients, $\alpha_j > 0$ and $\beta_j \geq 0$ are rationals with odd denominators, $j = 1, \dots, n-1$, $s = 1, \dots, n$.

Consider inequalities (2.4), corresponding to system (7.3). We get

$$\theta_1 > -\frac{b_1}{a_1}\,\theta_n^{\alpha_1}, \qquad \theta_i > -\frac{b_i}{a_i}\,\theta_{i-1}^{\alpha_i}, \;\; i = 2, \dots, n-1, \qquad \theta_n > -\frac{b_n}{a_n}\,\theta_1^{\beta_1} \dots \theta_{n-1}^{\beta_{n-1}}.$$

It is easily shown that (7.3) is absolutely ultimately bounded if and only if the inequality

$$\alpha_1\beta_1 + \alpha_1\alpha_2\beta_2 + \dots + \alpha_1 \dots \alpha_{n-1}\,\beta_{n-1} \leq 1 \tag{7.4}$$

holds, and in the case where (7.4) turns to equality, the condition

$$\left(-\frac{b_1}{a_1}\right)^{\xi_1}\left(-\frac{b_2}{a_2}\right)^{\xi_2} \dots \left(-\frac{b_{n-1}}{a_{n-1}}\right)^{\xi_{n-1}}\left(-\frac{b_n}{a_n}\right) < 1$$

is fulfilled. Here $\xi_i = \beta_i + \alpha_{i+1}\xi_{i+1}$, $i = 1, \dots, n-2$, $\xi_{n-1} = \beta_{n-1}$. Hence, the absolute ultimate boundedness of system (7.3) implies that this system satisfies the $\overline{MO}$-condition.

Thus, for (7.3) the sufficient condition of absolute ultimate boundedness is also a necessary one.

Example 7.3 Consider the system

$$\begin{aligned} \dot{x}_i &= a_i f_i(x_i) + b_i f_n^{\alpha_i}(x_n), \quad i = 1, \dots, n-1, \\ \dot{x}_n &= a_n f_n(x_n) + b_n f_1^{\beta_1}(x_1) \dots f_{n-1}^{\beta_{n-1}}(x_{n-1}). \end{aligned} \tag{7.5}$$

Here parameters a_s, b_s, α_j and β_j, $j = 1, \dots, n-1$, $s = 1, \dots, n$, possess the same properties as in the previous example. We get, again, that if the system investigated is absolutely ultimately bounded, then it satisfies the $\overline{MO}$-condition.

It can be easily shown that for the absolute ultimate boundedness of (7.5) it is necessary and sufficient the validity of the inequality $\alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_{n-1}\beta_{n-1} \leq 1$. If this inequality turns to equality, then for coefficients a_s and b_s the condition

$$\left(-\frac{b_1}{a_1}\right)^{\beta_1}\left(-\frac{b_2}{a_2}\right)^{\beta_2} \dots \left(-\frac{b_{n-1}}{a_{n-1}}\right)^{\beta_{n-1}}\left(-\frac{b_n}{a_n}\right) < 1$$

should be fulfilled.

8 Systems with Additive Connections

Let system (2.1) be of the form

$$\dot{x}_s = \sum_{j=1}^{n} p_{sj} f_j^{\alpha_{sj}}(x_j), \quad s = 1, \dots, n, \tag{8.1}$$

where $\alpha_{sj} > 0$ are rationals with odd denominators, $\alpha_{ss} = 1$, and p_{sj} are constant coefficients, $p_{ss} < 0$, $p_{sj} \geq 0$ for $j \neq s$, $j, s = 1, \dots, n$. Thus, connections in the equations considered are additive.

Theorem 8.1 *System (8.1) is absolutely ultimately bounded if and only if it satisfies the $\overline{MO}$-condition.*

Proof The sufficiency follows from Theorem 6.3. Let us prove the necessity.

Suppose that (8.1) is absolutely ultimately bounded. Then (v. Theorem 6.1) the $\widetilde{MO}$-condition is fulfilled for this system. Therefore, there exists a sequence of positive vectors $\tilde{\theta}^{(m)} = (\tilde{\theta}_{1m}, \dots, \tilde{\theta}_{nm})^*$ such that $\|\tilde{\theta}^{(m)}\| \to \infty$ as $m \to \infty$, and for every $m = 1, 2, \dots$ vector $\tilde{\theta}^{(m)}$ is a solution of the system

$$\sum_{j=1}^{n} p_{sj}\theta_j^{\alpha_{sj}} < 0, \quad s = 1, \dots, n. \tag{8.2}$$

One may assume, without loss of generality, that $\tilde{\theta}_{sm} \to +\infty$ as $m \to \infty$ for $s = 1, \dots, k$, where $1 \le k \le n$, and for $s > k$ the sequences $\{\tilde{\theta}_{sm}\}$ are bounded.

If $k = n$, then for system (8.1) the $\overline{MO}$-condition is fulfilled. Consider, further, the case where $k < n$. The inequalities

$$\sum_{j=1}^{n} p_{sj}\tilde{\theta}_{jm}^{\alpha_{sj}} < 0, \quad s = k+1, \dots, n,$$

are valid for $m = 1, 2, \dots$. Hence, (8.1) can be splitted into the following two subsystems:

$$\dot{x}_s = \sum_{j=1}^{k} p_{sj} f_j^{\alpha_{sj}}(x_j) + \sum_{j=k+1}^{n} p_{sj} f_j^{\alpha_{sj}}(x_j), \quad s = 1, \dots, k, \tag{8.3}$$

$$\dot{x}_s = \sum_{j=k+1}^{n} p_{sj} f_j^{\alpha_{sj}}(x_j), \quad s = k+1, \dots, n. \tag{8.4}$$

Since (8.1) is absolutely ultimately bounded, then subsystem (8.4) possesses the same property. For this subsystem, one can to repeat the above arguments. Continuing this process, we get that (8.1) can be splitted into a ordered set of subsystems such that every subsystem does not influence the subsequent ones.

The last subsystem satisfies the $\overline{MO}$-condition. Let us show that for the set of the last two, three, etc. subsystems the $\overline{MO}$-condition is also fulfilled.

We shall assume, without loss of generality, that (8.1) is splitted only into the two subsystems: (8.3) and (8.4).

System (8.4) and the system

$$\dot{x}_s = \sum_{j=1}^{k} p_{sj} f_j^{\alpha_{sj}}(x_j), \quad s = 1, \dots, k,$$

satisfy the $\overline{MO}$-condition. Hence, for every $\Delta > 0$ there exist numbers $\hat{\theta}_1, \dots, \hat{\theta}_n$ such that $\hat{\theta}_s > \Delta$, $s = 1, \dots, n$, and

$$\sum_{j=1}^{k} p_{sj}\hat{\theta}_j^{\alpha_{sj}} < 0, \ \ s = 1, \dots, k; \qquad \sum_{j=k+1}^{n} p_{sj}\hat{\theta}_j^{\alpha_{sj}} < 0, \ \ s = k+1, \dots, n.$$

By the use of Lemma 5.3, we get that the system

$$-h_s + \alpha_{sj} h_j \le 0, \quad j \ne s, \quad j, s = 1, \dots, k,$$

has a positive solution $\tilde{h}_1, \dots, \tilde{h}_k$. Then for the numbers $\bar{\theta}_j = \tau^{\tilde{h}_j} \hat{\theta}_j$, $j = 1, \dots, k$, $\tau > 1$, following inequalities are valid:

$$\sum_{j=1}^{k} p_{sj} \bar{\theta}_j^{\alpha_{sj}} < 0, \quad s = 1, \dots, k,$$

Thus, for sufficiently large values of τ the vector $(\bar{\theta}_1, \dots, \bar{\theta}_k, \hat{\theta}_{k+1}, \dots, \hat{\theta}_n)^*$ is a solution of system (8.2), and all the entries of this vector are greater than Δ. Hence, (8.1) satisfies the $\overline{MO}$-condition. $\square$

Remark 8.1 Theorem 8.1 states that for systems with additive connections the $\overline{MO}$-condition is not only sufficient one but also a necessary condition for the absolute ultimate boundedness. On the other hand, the $\widetilde{MO}$-condition is a necessary one for (8.1) to be absolutely ultimately bounded. However, this condition is not a sufficient one.

Example 8.1 Let system (8.1) be of the form

$$\begin{aligned} \dot{x}_1 &= -f_1(x_1), \\ \dot{x}_2 &= -f_2(x_2) + f_3^2(x_3), \\ \dot{x}_3 &= -f_3(x_3) + f_2^2(x_2). \end{aligned} \tag{8.5}$$

It is easily verified that the $\widetilde{MO}$-condition is fulfilled for (8.5). At the same time, if $f_s(x_s) = x_s$, $s = 1, 2, 3$, then solutions of this system are not ultimately bounded.

9 Conditions of Ultimate Boundedness for Large Scale Systems

Let us show now that the results obtained in the present paper can be used for the determination of conditions of ultimate boundedness of solutions for essentially nonlinear complex systems.

Consider the system

$$\dot{x}_s = F_s(x_s) + \sum_{j=1}^{k_s} Q_{sj}(t, x), \quad s = 1, \dots, n, \tag{9.1}$$

where $x_s \in R^{m_s}$, $x = (x_1^*, \dots, x_n^*)^*$; the elements of the vectors $F_s(x_s)$ are continuous homogeneous functions of the orders $\sigma_s > 0$; the vector functions $Q_{sj}(t, x)$ are continuous for $t \geq 0$, $x \in R^m$ $(m = m_1 + \dots + m_n)$. We will assume that in the region $t \geq 0$, $\|x\| \geq H$ ($H > 0$ is a constant) functions $Q_{sj}(t, x)$ satisfy the inequalities

$$\|Q_{sj}(t, x)\| \leq c_{sj} \|x_1\|^{\beta_{s1}^{(j)}} \dots \|x_n\|^{\beta_{sn}^{(j)}}, \quad c_{sj} > 0, \quad \beta_{si}^{(j)} \geq 0.$$

System (9.1) describes the dynamics of complex system composed of n interconnected subsystems [4, 13]. Here x_s are state vectors, the functions $F_s(x_s)$ define the interior connections of subsystems while the functions $Q_{sj}(t, x)$ characterize the interaction between the subsystems.

Consider the isolated subsystems

$$\dot{x}_s = F_s(x_s), \quad s = 1, \dots, n. \tag{9.2}$$

Let the zero solutions of subsystems (9.2) be asymptotically stable. In [1, 2, 5, 10] the conditions are obtained under which asymptotic stability of the zero solutions of (9.2) implies that the zero solution of (9.1) is also asymptotically stable. In the present section, we will look for the conditions of ultimate boundedness of solutions for system (9.1).

Assume that for isolated subsystems there exist Lyapunov's functions $V_s(x_s)$, $s = 1,\dots,n$, with the following properties:

(a) $V_s(x_s)$ are positive definite;

(b) $V_s(x_s)$ are continuously differentiable for all $x_s \in R^{m_s}$;

(c) $V_s(x_s)$ are positive homogeneous functions of orders $\gamma_s - \sigma_s + 1$;

(d) the derivatives of $V_s(x_s)$ with respect to (9.2) are negative functions.

Remark 9.1 In the case where $F_1(x_1),\dots,F_n(x_n)$ are continuously differentiable functions, the existence of such Lyapunov's functions it was proved in [17].

Remark 9.2 In the capacity of $\gamma_1,\dots,\gamma_n$ one may choose arbitrary numbers such that $\gamma_s > \sigma_s$, $s = 1,\dots,n$.

By the use of the homogeneous functions properties [17], we get that functions $V_1(x_1),\dots,V_n(x_n)$ satisfy the inequalities $a_{1s}\|x_s\|^{\gamma_s-\sigma_s+1} \le V_s(x_s) \le a_{2s}\|x_s\|^{\gamma_s-\sigma_s+1}$,

$$\left\|\frac{\partial V_s}{\partial x_s}\right\| \le a_{3s}\|x_s\|^{\gamma_s-\sigma_s}, \qquad \left(\frac{\partial V_s}{\partial x_s}\right)^* F_s \le -a_{4s}\|x_s\|^{\gamma_s}$$

for all $x_s \in R^{m_s}$, where a_{1s}, a_{2s}, a_{3s}, a_{4s} are positive constants, $s = 1,\dots,n$.

On differentiating $V_s(x_s)$ with respect to (9.1), one can deduce that the estimations

$$\frac{dV_s}{dt}\Big|_{(9.1)} \le -a_{4s}\|x_s\|^{\gamma_s} + a_{3s}\|x_s\|^{\gamma_s-\sigma_s}\sum_{j=1}^{k_s} c_{sj}\|x_1\|^{\beta_{s1}^{(j)}}\dots\|x_n\|^{\beta_{sn}^{(j)}}$$

are valid for $t \ge 0,\ \|x\| \ge H,\ s = 1,\dots,n$.

Consider the function

$$V(x) = \sum_{s=1}^{n}\lambda_s V_s(x_s),$$

where $\lambda_1,\dots,\lambda_n$ are positive coefficients. For all $t \ge 0$ and $\|x\| \ge H$ we obtain

$$\frac{dV}{dt}\Big|_{(9.1)} \le -\sum_{s=1}^{n}\lambda_s a_{4s}\|x_s\|^{\gamma_s} + \sum_{s=1}^{n}\lambda_s a_{3s}\|x_s\|^{\gamma_s-\sigma_s}\sum_{j=1}^{k_s} c_{sj}\|x_1\|^{\beta_{s1}^{(j)}}\dots\|x_n\|^{\beta_{sn}^{(j)}}.$$

Hence, to prove the ultimate boundedness of solutions for (9.1) it is sufficient to show that one can choose coefficients $\lambda_1,\dots,\lambda_n$ for the function

$$W(y) = -\sum_{s=1}^{n}\lambda_s a_{4s} y_s^{\gamma_s} + \sum_{s=1}^{n}\lambda_s a_{3s} y_s^{\gamma_s-\sigma_s}\sum_{j=1}^{k_s} c_{sj}\, y_1^{\beta_{s1}^{(j)}}\dots y_n^{\beta_{sn}^{(j)}}$$

to be negative in a region $\|y\| > R$. Here $R > 0$ is a constant.

Suppose that parameters $\gamma_1,\dots,\gamma_n$ satisfy the inequalities

$$-\frac{\sigma_s}{\gamma_s} + \sum_{i=1}^{n}\frac{\beta_{si}^{(j)}}{\gamma_i} \le 0, \quad j = 1,\dots,k_s, \quad s = 1,\dots,n. \tag{9.3}$$

In this case, by analogy with the proof of Theorem 6.3, we get the validity of the following

Theorem 9.1 *If there exist positive numbers $\tilde{\theta}_1, \dots, \tilde{\theta}_n$ such that*

$$-a_{4s}\tilde{\theta}_s^{\sigma_s} + a_{3s}\sum_{j=1}^{k_s} c_{sj}\tilde{\theta}_1^{\beta_{s1}^{(j)}} \dots \tilde{\theta}_n^{\beta_{sn}^{(j)}} < 0, \quad s = 1, \dots, n, \tag{9.4}$$

then solutions of (9.1) are uniformly ultimately bounded.

Remark 9.3 Coefficients a_{3s}, a_{4s} in (9.4) depend, in general, on the chosen values of $\gamma_1, \dots, \gamma_n$.

Remark 9.4 If for chosen values of $\gamma_1, \dots, \gamma_n$ all the inequalities in (9.3) are strict, then solutions of (9.1) are uniformly ultimately bounded (the verification of the existence of the positive numbers $\tilde{\theta}_1, \dots, \tilde{\theta}_n$, satisfying the inequalities (9.4), in this case is redundant).

References

[1] Aleksandrov, A.Yu. On the stability of complex systems in critical cases. *Autom. i Telemekh* (9) (2001) 3–13. [Russian]

[2] Aleksandrov, A.Yu. and Platonov, A.V. Construction of Lyapunov's functions for a class of nonlinear systems. *Nonlinear Dynamics and Systems Theory* **6**(1) (2006) 17–29.

[3] Barbashin, Ye.A. *The Lyapunov Functions.* Nauka, Moscow, 1967. [Russian]

[4] Grujic, Lj.T., Martynyuk, A.A. and Ribbens-Pavella, M. *Stability of Large Scale Systems under Structural and Singular Perturbations.* Naukova Dumka, Kiev, 1984. [Russian]

[5] Kosov, A.A. On the stability of complex systems by the nonlinear approximation. *Differ. Uravn.* **33**(10) (1997) 1432–1434. [Russian]

[6] LaSalle, J.P. and Lefschetz, S. *Stability by Liapunov's Direct Method with Applications.* Academic Press, New York, London, 1961.

[7] Lur'e, A.I. *Certain Nonlinear Problems of Automatic Control Theory.* Gostekhizdat, Moscow, 1951. [Russian]

[8] Martynyuk, A.A. and Obolenskij, A.Yu. On the stability of solutions for the Wazewskij autonomous systems. *Differ. Uravn.* **16**(8) (1980) 1392–1407. [Russian]

[9] Persidsky, S.K. On the problem of absolute stability. *Autom. i Telemekh.* (12) (1969) 5–11. [Russian]

[10] Platonov, A.V. On stability of complex nonlinear systems. *Izv. RAN: Teoriya i Sistemy Upravleniya* (4) (2004) 41–46. [Russian]

[11] Pliss, V.A. *Nonlocal Problems of Oscillations Theory.* Nauka, Moscow, Leningrad, 1964. [Russian]

[12] Rouche, N., Habets, P. and Laloy, M. *Stability Theory by Liapunov's Direct Method.* Springer, New York etc., 1977.

[13] Siljak, D.D. *Decentralized Control of Complex Systems.* Academic Press, Cambridge, MA, 1991.

[14] Yoshizawa, T. *Stability Theory by Liapunov's Second Method.* The Math. Soc. of Japan, Tokyo, 1966.

[15] Zubov, V.I. *Mathematical Methods for the Study of Automatical Control Systems.* Sudostroenie, Leningrad, 1959. [Russian]

[16] Zubov, V.I. *Oscillations in Nonlinear and Controled Systems.* Sudostroenie, Leningrad, 1962. [Russian]

[17] Zubov, V.I. *Stability of Motion.* Vysshaya Shkola, Moscow, 1973. [Russian]