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A Note on a Generalization of Sturm's Comparison Theorem

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Abstract: In this note, an attempt is made to generalize the Sturm's comparison theorem. Let t_1 and t_2 be two consecutive zeros of a solution y of an implicit equation

$$g_1(y''(t)) + r(t)g_2(y(t)) = 0$$

and x be a solution of

$$f_1(x''(t)) + q(t)f_2(x(t)) = 0.$$

Under certain conditions stated on the given functions f_1, f_2, g_1, g_2, q and r, we show that x has a zero between t_1 and t_2. Sturm's comparison theorem turns out to be a consequence of the established result.

Keywords: *Implicit differential equations; comparison theory.*

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1 Introduction

Sturm's comparison theorem plays an important role in the theory of oscillations. In this note an attempt is made to generalize the Sturm's comparison theorem. Let f_1, f_2, g_1 and $g_2 \in C(\Re, \Re)$ and $q, r \in C(\Re^+, \Re)$ be given functions. Let x and y be solutions of the implicit second order equations

$$f_1(x''(t)) + q(t)f_2(x(t)) = 0, \tag{1.1}$$

$$g_1(y''(t)) + r(t)g_2(y(t)) = 0. \tag{1.2}$$

In this note, we assume the existence of solutions x and y of (1.1) and (1.2) on $J = [0, \alpha]$, $\alpha > 0$. Under certain conditions on f_1, f_2, g_1, g_2, q and r, we establish that between any two consecutive zeros t_1, t_2 of y, there is a zero of x, where $[t_1, t_2] \subseteq J$. Hypotheses along with the main result are stated in Section 2. Section 3 is devoted to a few consequences and examples of this result.

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2 Main Result

This section deals with the main result. Let f_1, f_2, g_1 and $g_2 \in C(\Re, \Re)$ and $q, r \in C(\Re^+, \Re)$. We need the following hypotheses for further study

$$\mathbf{H_1}: \qquad \min_{0\neq u\in\Re} \frac{g_1(u)}{u} \geq \max_{0\neq u\in\Re} \frac{f_1(u)}{u},$$

$$\mathbf{H_2}: \qquad \min_{0\neq u\in\Re} \frac{f_2(u)}{u} \geq \max_{0\neq u\in\Re} \frac{g_2(u)}{u},$$

$$\mathbf{H_3}: \quad uf_1(u) > 0, \;\; ug_2(u) > 0, \;\; \forall\, 0 \neq u \in \Re.$$

Theorem 2.1 *Let x and y be nontrivial solutions of (2.1) and (2.2) respectively.*

$$f_1(x''(t)) + q(t)f_2(x(t)) = 0. \tag{2.1}$$

$$g_1(y''(t)) + r(t)g_2(y(t)) = 0. \tag{2.2}$$

Assume that the hypotheses $H_1 - H_3$ hold. Let t_1, t_2 be consecutive zeros of y and $q(t) > r(t) > 0$ for $t \in I = (t_1, t_2)$. Then, x vanishes at least once on I.

Proof By hypothesis, $y(t_1) = 0 = y(t_2)$ and $y(t) \neq 0$, $\forall\, t \in I$. Suppose $x(t) \neq 0$, $\forall\, t \in I$. Suppose if, $y''(t) = 0$ for some $t \in I$, then from H_1, H_3 and (2.2), we have $g_2(y(t)) = 0$ as well, but this contradicts the non vanishing of $y(t)$ in I. So $y''(t) \neq 0$, $\forall\, t \in I$. Similarly $x''(t) \neq 0$, $\forall\, t \in I$. From H_1–H_3, we have

$$\begin{aligned} g_1(y''(t))/y''(t) \geq f_1(x''(t))/x''(t) > 0, \quad \forall\, t \in I \\ \text{and} \quad f_2(x(t))/x(t) \geq g_2(y(t))/y(t) > 0, \quad \forall\, t \in I. \end{aligned}$$

From the above two inequalities and since $1/r(t) > 1/q(t) > 0$, for all $t \in I$, we have

$$\frac{y(t)g_1(y''(t))}{g_2(y(t))r(t)y''(t)} > \frac{x(t)f_1(x''(t))}{q(t)f_2(x(t))x''(t)}. \tag{2.3}$$

Define $m(t) = x(t)y'(t) - x'(t)y(t),\ t \in I$. Then, $m'(t) = x(t)y''(t) - x''(t)y(t)$.

Case 1. $x(t) > 0,\ y(t) > 0$ on I.

In this case, $m(t_1) > 0,\ m(t_2) < 0$. This implies that

$$m(t_2) - m(t_1) < 0. \tag{2.4}$$

From (2.1) we have,

$$f_1(x''(t)) < 0, \quad t \in I.$$

The sector condition H_3 on f_1 now implies $x''(t) < 0$ for all $t \in I$. Similarly $y''(t) < 0$ for all $t \in I$. We notice that

$$m'(t) = -\frac{x(t)f_1(x''(t))y''(t)}{q(t)f_2(x(t))} + \frac{y(t)g_1(y''(t))x''(t)}{g_2(y(t))r(t)} \quad \forall\, t \in I. \tag{2.5}$$

By (2.3), we have $m'(t) > 0,\ t \in I$,

$$m(t_2) - m(t_1) > 0. \tag{2.6}$$

Inequalities (2.4) and (2.6) lead to a contradiction. Thus, x vanishes at least once between t_1 and t_2. The cases when $x(t) > 0,\ y(t) < 0;\ x(t) < 0,\ y(t) > 0;\ x(t) < 0,\ y(t) < 0$ are similarly dealt. These proofs are omitted for brevity. □

Remark 2.1 When f_1, f_2, g_1 and g_2 are identity functions, the celebrated Sturm's comparison theorem is a particular case of Theorem 2.1, see [3, 4]. Theorem 2.1 can also be viewed as a nonlinear version of the Sturm's comparison theorem.

3 A Few Consequences

This section primarily deals with consequences concerning Theorem 2.1. We observe that the hypotheses H_1 and H_2 can be relaxed. Let $S_1 = \{x(t): t \in I\}$, $S_2 = \{y(t): t \in I\}$ and $S \supseteq S_1 \cup S_2$. Then the condition in H_1 and H_2, we can replace $u \in \Re$ by $u \in S$. In practice it is hard to figure out what S is? For nonlinear equations these conditions could be impracticable unless we have a prior bounds on the solutions. Secondly, none of the condition have monotonicity criteria, see [1, 2] but sector condition is a part and parcel of it. Thirdly, we can derive results by comparing the implicit equations with an explicit equation for nonoscillation also, as shown below,

Proposition 3.1 *Let y be any nontrivial solution of*

$$y''(t) + f(y''(t)) + \frac{1}{5t^2}y(t) = 0, \quad t > 0, \tag{3.1}$$

where, $f : \Re \to \Re$ be any continuous function satisfying $uf(u) > 0, \forall\, 0 \neq u \in \Re$. Then (3.1) is non oscillatory.

Proof Consider the differential equation

$$x''(t) + \frac{1}{4t^2}x(t) = 0. \tag{3.2}$$

(3.1) and (3.2) can be identified as (2.2) and (2.1) respectively, with $f_1(u) = u = f_2(u) = u$, $q(t) = \frac{1}{4t^2}$, $g_1(u) = u + f(u)$, $uf(u) > 0$, $\forall\, 0 \neq u \in \Re$, $g_2(u) = u$, $r(t) = \frac{1}{5t^2}$. Equation (3.2) is nonoscillatory, as $y(t) = t^{\frac{1}{2}}$ is a solution of (3.2). It is easy to see that f_1, f_2, g_1 and g_2 satisfy the hypotheses $H_1 - H_3$. So, Theorem 2.1 implies that (3.1) is nonoscillatory.

Proposition 3.2 *Let x be any nontrivial solution of*

$$x''(t) + 2(k^2x(t) + f(x(t))) = 0, \tag{3.3}$$

where, $f:\ \Re \to \Re$ be any continuous function satisfying $uf(u) > 0$ for all $0 \neq u \in \Re$, $k > 0$. Then (3.3) is oscillatory.

Proof Consider the differential equation

$$y''(t) + \frac{k^2}{4}y(t) = 0. \tag{3.4}$$

With $f_1(u) = u$, $f_2(u) = k^2u + f(u)$, $uf(u) > 0, \forall\, 0 \neq u \in \Re$, $g_1(u) = u$, $g_2(u) = \frac{k^2}{4}u$, $q(t) = 2$, $r(t) = 1$ (3.3) and (3.4) can be identified as (2.1) and (2.2). It is easy to see that f_1, f_2, g_1 and g_2 satisfy the hypotheses H_1–H_3. So, Theorem 2.1 implies that x vanishes between any two consecutive zeros of $y(t) = \sin\frac{k}{2}t$. Since (3.4) is oscillatory. So, by Theorem 2.1, (3.3) is oscillatory. □

Remark 3.1 In proving (3.3) is oscillatory, we are using conditions different from what has been used in [1, Remark 1] and [2].

Example 3.1 Consider the differential equations

$$f_1(x''(t)) + q(t)f_2(x(t)) = 0, \tag{3.5}$$
$$g_1(y''(t)) + r(t)g_2(y(t)) = 0, \tag{3.6}$$

where, $f_1(u) = ue^{-|u|}$, $f_2(u) = u$, $q(t) = e^{-|\sin t|}$, $g_1(u) = u = g_2(u)$, $r(t) = e^{-1}$. Then, $x(t) = \sin t$ satisfies (3.5). Compare this with the equation

$$y''(t) + r(t)y(t) = 0,$$

with the solution $y(t) = \sin(t/\sqrt{e})$ on $[0, \pi\sqrt{e}]$. It is trivial to check that f_1, f_2, g_1 and g_2 satisfy the hypotheses $H_1 - H_3$. Also $q(t) > r(t) > 0$. Theorem 2.1 implies that not only must $\sin t$ vanish in $[0, \pi\sqrt{e}]$, which is clear, but also so must every other nontrivial solution of (3.5).

Example 3.2 Consider the differential equations

$$f_1(x''(t)) + q(t)f_2(x(t)) = 0, \tag{3.7}$$
$$g_1(y''(t)) + r(t)g_2(y(t)) = 0, \tag{3.8}$$

where $f_1(u) = u$, $f_2(u) = ue^{|u|}$, $q(t) = 2$, $g_1(u) = u = g_2(u)$, $r(t) = 1$. Here f_1, f_2, g_1 and g_2 satisfy the hypotheses H_1–H_3. Also $q(t) > r(t) > 0$. Let x be a nontrivial solution of (3.7). Then, Theorem 2.1 implies that x vanishes at least once between any two consecutive zeros of $y(t) = \sin t$. Since (3.8) is oscillatory, so (3.7) is oscillatory.

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References

[1] Man Kam Kwong and Wong, J.S.W. An application of integral inequality to second order nonlinear oscillation. *J. Diff. Eqns* **46** (1982) 63–77.

[2] Wong, P.J.Y. and Agarwal, R.P. Oscillatory behavior of solution of certain second order nonlinear differential equations. *J. Math. Anal. Appl.* **198** (1996) 337–354.

[3] Lakshmikantham, V., Deo, S.G. and Raghavendra, V. *Ordinary Differential Equations.* TMH, India, 1997.

[4] Simmons, G.F. *Differential Equations with Applications and Historical Notes.* TMH, 1974.