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Nontrivial Solutions of Boundary Value Problems of Second-Order Dynamic Equations on an Isolated Time Scale

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Abstract: We will use Clark's theorem to show the existence of multiple solutions to the self–adjoint dynamic boundary value problem

$$\begin{aligned} &\left(p(t)u^{\Delta}(t)\right)^{\nabla} + q(t)u(t) + \lambda h(t, u(t)) = 0, \quad t \in [a, b]_{\mathbb{T}}, \\ &u(\rho(a)) = u(\sigma(b)) = 0, \end{aligned}$$

where λ is a sufficiently large positive parameter and $\mathbb{T}$ is an isolated time scale. Examples of our results will be given.

Keywords: *Clark's theorem; isolated time scales; critical point theory.*

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1 Introduction

A great deal of work has been done concerning the existence of solutions to discrete boundary value problems. Recently, techniques from critical point theory have been employed to show the existence of nontrivial solutions to discrete boundary value problems [4], [11], [13],[7]. These techniques are complementary to the fixed point theory that has also been utilized to study this area.

Throughout this paper, we assume the time scale $\mathbb{T}$ is isolated. Let $m = \min \mathbb{T}$ and $M = \max \mathbb{T}$. Then $\mathbb{T}$ is isolated if $\rho(t) < t < \sigma(t)$ $\forall t \in \mathbb{T}, t \neq m, M$ and

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$\rho(M) < M = \sigma(M)$, $\rho(m) = m < \sigma(m)$. Consider $[a,b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$, $a < b$. By the interval $[a,b]_{\mathbb{T}}$ we mean the set $[a,b] \cap \mathbb{T}$. To avoid a trivialized problem, we assume throughout that there is at least one point in the time scale between the endpoints a and b. We will be concerned with the boundary value problem:

$$\left(p(t)u^{\Delta}(t)\right)^{\nabla} + q(t)u(t) + \lambda h(t, u(t)) = 0, \quad t \in [a,b]_{\mathbb{T}}, \tag{1}$$

$$u(\rho(a)) = u(\sigma(b)) = 0, \tag{2}$$

where λ is a positive parameter. The $\left(pu^{\Delta}\right)^{\nabla}$ term generalizes the central difference. The second–order mixed derivative problem was originally introduced in [1]. By examining this boundary value problem, we are extending the work done in [4]. Anderson considered the existence of solutions to a related second–order mixed derivative problem in [2]. We define the linear operator $\mathcal{L}$ on $\{u : [\rho(a), \sigma(b)]_{\mathbb{T}} \to \mathbb{R}\}$ by

$$\mathcal{L}u(t) = \left(p(t)u^{\Delta}(t)\right)^{\nabla} + q(t)u(t), \; t \in [a,b]_{\mathbb{T}}.$$

Then the formally self-adjoint nonlinear equation (1) can be written as

$$\mathcal{L}u = -\lambda h(t, u).$$

We assume:

$$p, \; q : [a,b]_{\mathbb{T}} \to \mathbb{R} \quad \text{and} \quad p > 0, \quad q < 0 \quad \text{on} \quad [a,b]_{\mathbb{T}}, \tag{3}$$

$$h : [a,b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R} \quad \text{is continuous with respect to the second variable}, \tag{4}$$

$$\exists \, \alpha > 0 \text{ such that } h(t, \alpha) = 0 \text{ and } h(t, u) > 0 \text{ for } u \in (0, \alpha), \; t \in [a,b]_{\mathbb{T}}, \tag{5}$$

$$h(t, u) \quad \text{is odd in } u. \tag{6}$$

This boundary value problem generalizes the important Sturm-Liouville problem. The time scale calculus was developed by Stefan Hilger [10] in 1988. The references [5], [6] provide excellent introductions to the theory of time scales. The following theorem provides a useful relationship between nabla and delta derivatives.

Theorem 1.1 [6] *If $\mathbb{T}$ is isolated and $f : \mathbb{T} \to \mathbb{R}$, then*

$$f^{\nabla}(t) = f^{\Delta}(\rho(t)),$$
$$f^{\Delta}(t) = f^{\nabla}(\sigma(t)), \; \forall t \in \mathbb{T}.$$

Before proceeding, we need a few useful definitions and theorems pertaining to critical point theory.

Definition 1.1 Let E be a real Banach space and let $\varphi : E \to \mathbb{R}$ be a mapping. We say φ is *Fréchet differentiable* at $u \in E$ if there exists a continuous linear map $L = L(u) : E \to \mathbb{R}$ satisfying

$$\lim_{x \to u} \frac{\varphi(x) - \varphi(u) - L(x - u)}{\|x - u\|_E} = 0.$$

The mapping L will be denoted by $\varphi'(u)$. A *critical point* u of φ is a point at which $\varphi'(u) = 0$, i.e., $\varphi'(u)v = 0 \;\; \forall \, v \in E$. We write $\varphi \in C^1(E, \mathbb{R})$ provided $\varphi'(u)$ is continuous $\forall \, u \in E$.

The following remark will be useful.

Remark 1.1 If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is defined near $u^0 = (u_1^0, \cdots, u_n^0) \in \mathbb{R}^n$, and differentiable at u^0, then each of the partial derivatives $\frac{\partial \varphi}{\partial u_k}$ exists at u^0 and the Fréchet derivative of φ at u^0 is represented by the gradient:

$$\varphi'(u^0) = \nabla_u \varphi(u^0),$$

where

$$\nabla_u \varphi = \left(\frac{\partial \varphi}{\partial u_1}, \cdots, \frac{\partial \varphi}{\partial u_n} \right)$$

is the gradient of φ with respect to u.

Definition 1.2 [Palais–Smale condition] Let E be a real Banach space. A function $\varphi \in C^1(E, \mathbb{R})$ satisfies the *Palais–Smale condition* if every sequence $\{u_j\}$ in E such that $\{\varphi(u_j)\}$ is bounded and $\varphi'(u_j) \to 0$ as $j \to \infty$ contains a convergent subsequence.

We state Clark's theorem, which is crucial to proving the main results of the paper. Clark's Theorem was originally stated in [8]. The version we cite here comes from Rabinowitz [12] and Bai [4]. Let E be a real Banach space, with zero vector denoted by 0. Let Σ denote the family of sets $A \subset E\backslash\{0\}$ such that A is closed in E and symmetric to 0, i.e., $u \in A$ implies $-u \in A$. Suppose $u \in E$ satisfies Definition 1.1. In the case when $I : E \to \mathbb{R}$ is an even mapping, we say that $(u, -u)$ is a pair of critical points for I.

Theorem 1.2 (Clark's theorem) *Let E be a real Banach space, $I \in C^1(E, \mathbb{R})$ with I even, bounded from below, and satisfying the Palais–Smale condition. Suppose $I(0) = 0$, there is a set $K \in \Sigma$ such that K is homeomorphic to S^{j-1} (j-1 dimensional unit sphere in $\mathbb{R}^j$) by an odd map, and* $\sup_K I < 0$. *Then I has at least j distinct pairs of critical points.*

2 Preliminary Results

Definition 2.1 Real-valued functions α, β on $[\rho(a), \sigma(b)]_{\mathbb{T}}$ are called *lower and upper solutions,* respectively, for the BVP (1), (2) if

$$\begin{cases} (p\alpha^{\Delta})^{\nabla}(t) + q(t)\alpha(t) \geq -\lambda h(t, \alpha(t)), & \forall t \in [a, b]_{\mathbb{T}}, \\ \alpha(\rho(a)) \leq 0, \ \alpha(\sigma(b)) \leq 0 \end{cases}$$

and

$$\begin{cases} (p\beta^{\Delta})^{\nabla}(t) + q(t)\beta(t) \leq -\lambda h(t, \beta(t)), & \forall t \in [a, b]_{\mathbb{T}}, \\ \beta(\rho(a)) \geq 0, \ \beta(\sigma(b)) \geq 0. \end{cases}$$

Define $h_1 : [a, b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ by

$$h_1(t, s) = \begin{cases} 0, & s > \alpha, \\ h(t, s), & |s| \leq \alpha, \\ 0, & s < -\alpha, \end{cases}$$

where α is as assumed in (5).

Lemma 2.1 *Let α be as in* (5)*. If u satisfies the BVP*

$$\mathcal{L}u = -\lambda h_1(t,u), \quad t \in [a,b]_{\mathbb{T}}, \tag{7}$$

$$u(\rho(a)) = u(\sigma(b)) = 0, \tag{8}$$

then

$$\|u\|_1 := \sum_{i=1}^{T} u_i \le \alpha$$

and consequently u is a solution of the BVP (1)*,* (2)*.*

Proof We first observe that by definition, $-\alpha$, α are lower and upper solutions, respectively, of the BVP (1), (2). We claim that $u(t) \le \alpha$ on $[\rho(a), \sigma(b)]_{\mathbb{T}}$. Suppose not, then $w(t) := u(t) - \alpha > 0$ for at least one point in $[a,b]_{\mathbb{T}}$. Since $w(\rho(a)) \le 0$ and $w(\sigma(b)) \le 0$, we get that w has a positive maximum at some point $t_0 \in [a,b]_{\mathbb{T}}$. Furthermore, we may assume that t_0 is the last such maximum in $[a,b]_{\mathbb{T}}$, i.e., $w(t) < w(t_0)$ for $t \in (t_0, b]_{\mathbb{T}}$. Hence, by [6, Lemma 6.17],

$$w(t_0) > 0,\ w^{\Delta}(t_0) \le 0,\ (pw^{\Delta})^{\nabla}(t_0) \le 0.$$

This implies that

$$u(t_0) > \alpha,\ u^{\Delta}(t_0) \le 0,\ (pu^{\Delta})^{\nabla}(t_0) \le 0.$$

So

$$(pu^{\Delta})^{\nabla}(t_0) + q(t_0)u(t_0) - \alpha q(t_0) < 0.$$

But

$$(pu^{\Delta})^{\nabla}(t_0) + q(t_0)u(t_0) - \alpha q(t_0) \ge -\lambda h_1(t_0, u(t_0)) = 0.$$

This is a contradiction. Hence $u(t) \le \alpha$ for $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$. A similar argument shows that $-\alpha \le u(t)\ \forall t \in [a,b]_{\mathbb{T}}$. It follows that $u(t)$ is a solution of the BVP (1), (2). Thus, the lemma is proved. $\Box$

Let

$$E = \{u : [\rho(a), \sigma(b)]_{\mathbb{T}} \to \mathbb{R} : u(\rho(a)) = u(\sigma(b)) = 0\}.$$

Let $|S|$ denote the cardinality of the set S. Note that E can be identified with $\mathbb{R}^T$, where $T := |[a,b]_{\mathbb{T}}|$, by the correspondence

$$(0, u(a), u(\sigma(a)), \cdots, u(b), 0) \leftrightarrow (x_1, \cdots, x_T),$$

where $x_i = u^{\sigma^{i-1}}(a), 1 \le i \le T$.
Define an inner product on E by

$$< u,v >_E = \sum_{t \in [a, \sigma(b)]_{\mathbb{T}}} \nu(t) \left[p^{\rho}(t) u^{\nabla}(t) v^{\nabla}(t) - q(t)u(t)v(t)\right]$$

with corresponding norm

$$\|u\|_E^2 =< u,u >_E = \sum_{t \in [a, \sigma(b)]_{\mathbb{T}}} \nu(t) \left[p^{\rho}(t) \left(u^{\nabla}(t)\right)^2 - q(t)u^2(t)\right].$$

We note that since E is finite dimensional, E equipped with this inner product is a Hilbert space. In this definition of E, it is important that $\mathbb{T}$ is isolated to guarantee that E equipped with this inner product is indeed a Hilbert space. Work similar to that done in [3] would be invaluable to extend this work to more general time scales.

Definition 2.2 We define the nonlinear functional $I : E \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\|u\|_E^2 - \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)H(t,u(t)), \quad \forall\, u \in E,$$

where $H(t,z) := \int_0^z h_1(t,s)ds$.

For our application, we will be interested in computing the Fréchet derivative of I. Here is a remark to aid in this calculation:

Remark 2.1 Let $\mathcal{H}$ be a real Hilbert space, let $f : \mathcal{H} \to \mathbb{R}$ be the function defined by $f(x) = \|x\|^2$ and let $u \in \mathcal{H}$. Then the Fréchet derivative of f at u is the linear functional on $\mathcal{H}$ given by $f'(u)x := 2 < x, u >$.

One could use Remark 1.1 to prove Remark 2.1. It is also an easy exercise to prove Remark 2.1 using the definition of the Fréchet derivative.

With the aid of Remark 2.1, we calculate the Fréchet derivative of our functional I:

Theorem 2.1 *For $u, v \in E$,*

$$\begin{aligned} I'(u)v &= < u,v >_E -\lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)h_1(t,u(t))v(t) \\ &= -\sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\mathcal{L}u(t)v(t) - \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)h_1(t,u(t))v(t). \end{aligned}$$

Proof Let

$$I_1(u) = \frac{1}{2}\|u\|_E^2 \ \text{ and } \ I_2(u) = \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)H(t,u(t)).$$

Then $I = I_1 - I_2$. By Remark 2.1,

$$I_1'(u)v =< u,v >_E = \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\left[p^\rho(t)u^\nabla(t)v^\nabla(t) - q(t)u(t)v(t)\right].$$

Using integration by parts, properties of the integral discussed in [5] and Theorem 1.1, we see

$$\begin{aligned} < u,v >_E &= p(t)u^\Delta(t)v(t)\Big|_{\rho(a)}^{\sigma(b)} \\ &\qquad - \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\left[\left[p(t)u^\Delta(t)\right]^\nabla + q(t)u(t)\right]v(t) \\ &= -\sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\left[\left[p(t)u^\Delta(t)\right]^\nabla + q(t)u(t)\right]v(t) \end{aligned}$$

by the boundary conditions on u.

By Remark 1.1,

$$I_2'(u)v = \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)h_1(t,u(t))v(t).$$

Thus,

$$I'(u)v = -\sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t)\left[\left[p(t)u^{\Delta}(t)\right]^{\nabla} + q(t)u(t)\right]v(t) - \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)h_1(t,u(t))v(t),$$

as desired. □

Corollary 2.1 *Let $u \in E$. The following are equivalent:*
1. u is a critical point of I,
2. u is a solution of (1), (2).
Furthermore, $I \in C^1(E,\mathbb{R})$.

Proof Let $u \in E$. Then

$$u \text{ is a critical point of } I$$

if and only if

$$I'(u)v = 0 \quad \forall\, v \in E,$$

if and only if

$$\sum_{t\in[\rho(a),\sigma(b)]_{\mathbb{T}}} \nu(t)\left[\left[p(t)u^{\Delta}(t)\right]^{\nabla} + q(t)u(t)\right]v(t) \\ +\lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)h_1(t,u(t))v(t) = 0 \quad \forall\, v \in E,$$

if and only if

$$u \text{ is a solution of (1), (2).}$$

To see that the last statement holds, for any $m \in [a,b]_{\mathbb{T}}$, let

$$v_m(t) = \begin{cases} 1, & \text{if } t = m, \\ 0, & \text{if } t \neq m. \end{cases}$$

Then $v_m \in E$ and $I'(u)v_m = 0 \ \ \forall\, m \in [a,b]_{\mathbb{T}}$. But this implies: $\nu(t)\left[p(t)u^{\Delta}(t)\right]^{\nabla} + q(t)u(t) - \lambda\nu(t)h_1(t,u(t)) = 0, \quad \forall\, t \in [a,b]_{\mathbb{T}}$. As $\nu(t) > 0$ on $\mathbb{T}$, these critical points correspond to solutions of (7), (8). By Lemma 2.1, we equivalently have solutions to (1), (2).

As E and $\mathbb{R}$ are Euclidean spaces, the continuity of h guarantees that $I \in C^1(E,\mathbb{R})$. □

3 Main Result and Proof

We note that if u is a solution of (1), (2) then $-u$ also solves (1), (2) and we say that $(u,-u)$ is a pair of solutions to (1), (2). The main result of this paper is:

Theorem 3.1 *Let* (3)–(6) *be satisfied. Then there exists a* $\lambda^* > 0$ *such that if* $\lambda > \lambda^*$, (1), (2) *has at least* $T := |[a,b]_{\mathbb{T}}|$ *distinct pairs of nontrivial solutions. Furthermore, each nontrivial solution* u *satisfies* $|u(t)| \leq \alpha$, $t \in [a,b]_{\mathbb{T}}$ *and* α *as in* (5).

Proof We will use Theorem 1.2 and Lemma 2.1 to prove this result. As h_1 is odd in its second variable, we know that I is an even functional. Indeed,

$$\begin{aligned}
I(-u) &= \frac{1}{2}\|-u\|_E^2 - \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t) H(t,-u(t)) \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} \int_0^{-u(t)} h_1(t,s)\, ds \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} \left(- \int_{-u(t)}^0 h_1(t,s)\, ds \right) \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} \int_{-u(t)}^0 h_1(t,-s)\, ds \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} \int_{u(t)}^0 -h_1(t,\tau)\, d\tau \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} \int_0^{u(t)} h_1(t,\tau)\, d\tau \\
&= \frac{1}{2}\|u\|_E^2 - \sum_{t\in[a,b]_{\mathbb{T}}} H(t,u(t)) \\
&= I(u).
\end{aligned}$$

By construction, $I(0) = 0$. As $h_1(t,s) = 0$ for $|s| \geq \alpha$,

$$\begin{aligned}
\sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t) H(t,u(t)) &= \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t) \int_0^{u(t)} h_1(t,s)\, ds \\
&\leq \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t) \int_{-\alpha}^{\alpha} |h_1(t,s)|\, ds =: C \quad \forall\, u \in E.
\end{aligned}$$

This implies:

$$I(u) \geq \frac{1}{2}\|u\|_E^2 - \lambda C \geq -\lambda C, \quad \forall u \in E.$$

Hence, I is bounded from below.

Now we verify the Palais-Smale condition. Let $\{u_m\} \subset E$ be any sequence such that $\{I(u_m)\}$ is bounded and $I'(u_m) \to 0$ as $m \to \infty$. Then there exist c_1, c_2 such that $c_1 \leq I(u_m) \leq c_2$, $m \in \mathbb{N}$. Then

$$\begin{aligned}
I(u_m) &= \frac{1}{2}\|u_m\|_E^2 - \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t) H(t,u_m(t)) \\
&\geq \frac{1}{2}\|u_m\|_E^2 - \lambda C
\end{aligned}$$

which implies that

$$\begin{aligned}\|u_m\|_E^2 &\leq 2I(u_m) + 2\lambda C \\ &\leq 2c_2 + 2\lambda C, \quad \forall\, m \in \mathbb{N}.\end{aligned}$$

Therefore, $\{u_m\}$ is a bounded sequence in a finite-dimensional space E and so has a convergent subsequence in E. Thus, the Palais-Smale condition is satisfied.

Recall that $T = |[a,b]_{\mathbb{T}}|$. We take $\{y_i\}_{i=1}^T$ as an orthonormal basis of E. Define

$$K(r) = \left\{ \sum_{i=1}^T \beta_i y_i : \sum_{i=1}^T \beta_i^2 = r^2 \right\}, \quad r > 0.$$

Then $0 \notin K(r)$ and $K(r)$ is symmetric with respect to 0.

It is immediate to see that $K(r)$ is compact. Indeed, fix $r > 0$. Define a map $f : K(r) \to S^{T-1}$ by

$$f(u) = f(\beta_1 y_1 + \cdots + \beta_T y_T) = \frac{<\beta_1, \cdots, \beta_T>}{r}.$$

Then f is an isomorphism that preserves inner products. As the inner product determines the topology of our spaces, it follows that f is a homeomorphism. Moreover, f is an odd map, so we have verified that $K(r)$ is homeomorphic to S^{T-1} by an odd map for any $r > 0$.

Now, let $u \in K(r)$. Then with the aid of Hölder's inequality,

$$\begin{aligned}\|u\|_E^2 &= \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t) \left[p^\rho(t) \left(\sum_{i=1}^T \beta_i y_i^\nabla(t) \right)^2 - q(t) \left(\sum_{i=1}^T \beta_i y_i(t) \right)^2 \right] \\ &\leq \sum_{t\in[a,\sigma(b)]_{\mathbb{T}}} \nu(t) \left[p^\rho(t) \sum_{i=1}^T \beta_i^2 \sum_{i=1}^T (y_i^\nabla(t))^2 - q(t) \sum_{i=1}^T \beta_i^2 \sum_{i=1}^T y_i^2(t) \right] \\ &= r^2 \sum_{i=1}^T \sum_{t\in[a,\sigma(b)]} \nu(t) \left[p^\rho(t)(y_i^\nabla(t))^2 - q(t) y_i^2(t) \right] \\ &= r^2 \|y_i\|_E^2 T \\ &= r^2 T, \quad \text{since } \{y_i\}_{i=1}^T \text{ is an orthonormal basis of E.} \end{aligned} \tag{9}$$

As $\dim E < \infty$, there exists a $c_0 > 0$ such that $\|u\|_1 \leq c_0 \|u\|_E$ for all $u \in E$. Fix r such that $0 < r \leq \dfrac{\alpha}{c_0\sqrt{T}}$. Using (9), we see that for $u \in K(r)$,

$$\|u\|_1 \leq c_0 \|u\|_E \leq c_0 r \sqrt{T} \leq \alpha.$$

Hence, $h(t, u(t)) = h_1(t, u(t))$ for all $u \in K(r)$. From assumption (5), we see that for $u \in K(r)$,

$$H(t, u(t)) = \int_0^{u(t)} h(t, s)ds > 0$$

if $u(t) \neq 0,\ t \in [a,b]_{\mathbb{T}}$. Since we know that $0 \notin K(r)$, we have

$$\sum_{t\in[a,b]_{\mathbb{T}}} \nu(t) H(t, u(t)) = \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t) \int_0^{u(t)} h(t,s)ds > 0.$$

Let $\tau = \inf_{u\in K(r)} \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)H(t,u(t))$. If $\tau = 0$, then by the compactness of $K(r)$, $0 \in K(r)$, which is a contradiction. Hence $\tau > 0$. Define $\lambda^* := \dfrac{\alpha^2}{2\tau c_0^2}$. For $u \in K(r)$, if $\lambda > \lambda^*$,

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_E^2 - \lambda \sum_{t\in[a,b]_{\mathbb{T}}} \nu(t)H(t,u(t)) \\ &\le \frac{1}{2}\|u\|_E^2 - \lambda^*\tau \\ &\le \frac{r^2}{2}T - \lambda^*\tau \\ &< \frac{\alpha^2}{2c_0^2} - \lambda^*\tau \\ &= 0. \end{aligned}$$

Thus, all the conditions of Clark's theorem (Theorem 1.2) are satisfied. Hence I has at least T distinct pairs of nonzero critical points. By construction and Lemma 2.1, the BVP (1), (2) has at least T distinct pairs of nontrivial solutions. □

We now examine two basic examples in which we find approximate upper bounds for λ^* explicitly, as predicted by Theorem 3.1. Since the 1970s, the theory of nonlinear difference equations has been widely studied due to its numerous applications in areas such as computer science, economics, and ecology, to name a few [9]. Analysis using time scales calculus could be used to extend and generalize these applications. These examples show a generalization of the important Sturm-Liouville problems to time scales, where $u^{\Delta\nabla}$ generalizes the central difference. Here the time scales are chosen to show the effect of the graininess ν on the value for λ^*.

Example 3.1 Consider the following difference equation boundary value problem:

$$\nabla\Delta u(t) - u(t) + \lambda\sin(\pi u(t)) = 0,\ t \in \{1,2\}, \tag{10}$$

$$u(0) = 0 = u(3). \tag{11}$$

Then conditions (3)–(6) are satisfied, where $p(t) \equiv 1,\ q(t) \equiv -1$ for $t \in \{1,2\}$, $h(t,s) = \sin(\pi s)$, and $\alpha = 1$. Using the Gram–Schmidt procedure, we can find an orthonormal basis for E. One such orthonormal basis is

$$y_1 = \left\langle \frac{1}{\sqrt{3}}, 0 \right\rangle \quad \text{and} \quad y_2 = \left\langle \frac{1}{6}\sqrt{\frac{3}{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} \right\rangle.$$

We also need to find a constant $c_0 > 0$ such that $\|u\|_1 \le c_0\|u\|_E$. Then we know:

$$\begin{aligned} \|u\|_E^2 &= (\nabla u(1))^2 + u^2(1) + (\nabla u(2))^2 + u^2(2) + (\nabla u(3))^2 + u^2(3) \\ &= 3u^2(1) + 3u^2(2) - 2u(2)u(1). \end{aligned}$$

Due to symmetry, we may without loss of generality assume $u(1) \le u(2)$. Hence:

$$\begin{aligned} \|u\|_E^2 &\ge 3u^2(1) + 3u^2(2) - 2u^2(2) \\ &\ge u^2(1) + u^2(2). \end{aligned}$$

So

$$\|u\|_E \geq \|u\|_2 \geq \frac{1}{\sqrt{2}}\|u\|_1.$$

So we can take $c_0 = \sqrt{2}$.

According to the proof of Theorem 3.1, we fix $0 < r \leq \frac{\alpha}{c_0\sqrt{T}}$. We take r as large as possible, so here $r = \frac{1}{2}$. Then $u \in K(\frac{1}{2})$ if and only if

$$u = \beta_1 \left\langle \frac{1}{\sqrt{3}}, 0 \right\rangle + \beta_2 \left\langle \frac{1}{6}\sqrt{\frac{3}{2}}, \frac{1}{2}\sqrt{\frac{3}{2}} \right\rangle,$$

where $\beta_1^2 + \beta_2^2 = \frac{1}{4}$.

Finally, we compute

$$\tau = \inf_{u \in K(\frac{1}{2})} \sum_{t \in \{1,2\}} \int_0^{u(t)} h(t,s)ds.$$

Note:

$$\begin{aligned}\sum_{t \in \{1,2\}} \int_0^{u(t)} h(t,s)\, ds &= \int_0^{\frac{\beta_1}{\sqrt{3}} + \frac{\beta_2}{6}\sqrt{\frac{3}{2}}} \sin \pi s\, ds + \int_0^{\frac{\beta_2}{2}\sqrt{\frac{3}{2}}} \sin \pi s\, ds \\ &= \frac{1}{\pi}\left[2 - \cos\left[\pi\left(\frac{\beta_1}{\sqrt{3}} + \frac{\beta_2}{6}\sqrt{\frac{3}{2}}\right)\right] - \cos\left(\pi\frac{\beta_2}{2}\sqrt{\frac{3}{2}}\right)\right].\end{aligned}$$

Thus, to find τ, we minimize

$$f(x,y) = \frac{1}{\pi}\left[2 - \cos\left[\pi\left(\frac{x}{\sqrt{3}} + \frac{y}{6}\sqrt{\frac{3}{2}}\right)\right] - \cos\left(\pi\frac{y}{2}\sqrt{\frac{3}{2}}\right)\right]$$

subject to the constraint $x^2 + y^2 = \frac{1}{4}$. Solving for x:

$$x = \pm\sqrt{\frac{1}{4} - y^2}, \quad -\frac{1}{2} \leq y \leq \frac{1}{2}.$$

So we minimize

$$f_-(y) = \frac{1}{\pi}\left[2 - \cos\left[\pi\left(-\sqrt{\frac{\frac{1}{4} - y^2}{3}} + \frac{y}{6}\sqrt{\frac{3}{2}}\right)\right] - \cos\left(\pi\frac{y}{2}\sqrt{\frac{3}{2}}\right)\right]$$

and

$$f_+(y) = \frac{1}{\pi}\left[2 - \cos\left[\pi\left(\sqrt{\frac{\frac{1}{4} - y^2}{3}} + \frac{y}{6}\sqrt{\frac{3}{2}}\right)\right] - \cos\left(\pi\frac{y}{2}\sqrt{\frac{3}{2}}\right)\right],$$

$-\frac{1}{2} \leq y \leq \frac{1}{2}$. Running a script in Matlab, we find that we can take $\tau = 0.0957$. A graph of f_- and f_+ is shown above. Thus, by Theorem 3.1, if $\lambda > 2.61 \geq \lambda^*$, the boundary value problem (10), (11) has two distinct pairs of nontrivial solutions. Furthermore, each solution u satisfies $|u(t)| \leq 1,\ t \in \{1,2\}$.

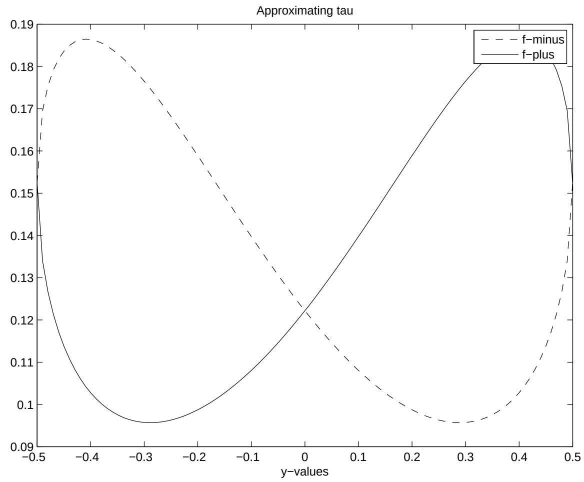

Figure 3.1: Approximating tau.

Example 3.2 Consider the following dynamic equation boundary value problem:

$$u^{\Delta\nabla}(t) - u(t) + \lambda \sin(\pi u(t)) = 0,\ t \in \left\{\frac{1}{4}, 2\right\}, \tag{12}$$

$$u(0) = 0 = u(3). \tag{13}$$

Then conditions (3)–(6) are satisfied, where $p(t) \equiv 1,\ q(t) \equiv -1$ for $t \in \left\{\frac{1}{4}, 2\right\}$, $h(t,s) = \sin(\pi s)$, and $\alpha = 1$. Using the Gram–Schmidt procedure, we can find an orthonormal basis for E. One such orthonormal basis is

$$y_1 = \left\langle \frac{2}{3}\sqrt{\frac{7}{15}}, 0 \right\rangle \quad \text{and} \quad y_2 = \left\langle \frac{32}{45}\sqrt{\frac{15}{1757}}, 6\sqrt{\frac{15}{1757}} \right\rangle.$$

We also need to find a constant $c_0 > 0$ such that $\|u\|_1 \le c_0 \|u\|_E$. There are two cases to consider:
Case 1: $|u\left(\frac{1}{4}\right)| \le |u(2)|$. Then we know:

$$\begin{aligned}
\|u\|_E^2 &= \frac{1}{4}\left[\left(u^{\nabla}\left(\frac{1}{4}\right)\right)^2 + u^2\left(\frac{1}{4}\right)\right] + \frac{7}{4}\left[(u^{\nabla}(2))^2 + u^2(2)\right] + (u^{\nabla}(3))^2 + u^2(3) \\
&\ge \frac{135}{28}u^2\left(\frac{1}{4}\right) + \frac{93}{28}u^2(2) - \frac{8}{7}|u(2)|\left|u\left(\frac{1}{4}\right)\right| \\
&\ge \frac{135}{28}u^2\left(\frac{1}{4}\right) + \frac{61}{28}u^2(2) \\
&\ge \frac{61}{28}\left(u^2\left(\frac{1}{4}\right) + u^2(2)\right).
\end{aligned}$$

Case 2: $|u\left(\frac{1}{4}\right)| \geq |u(2)|$. Similarly,

$$\begin{aligned}
\|u\|_E^2 &\geq \frac{135}{28}u^2\left(\frac{1}{4}\right) + \frac{93}{28}u^2(2) - \frac{8}{7}|u(2)|\left|u\left(\frac{1}{4}\right)\right| \\
&\geq \frac{103}{28}u^2\left(\frac{1}{4}\right) + \frac{93}{28}u^2(2) \\
&\geq \frac{103}{28}\left(u^2\left(\frac{1}{4}\right) + u^2(2)\right).
\end{aligned}$$

Hence, for all $u \in E$,

$$\|u\|_E \geq \frac{1}{2}\sqrt{\frac{61}{28}}\;\|u\|_2 \geq \frac{1}{2}\sqrt{\frac{61}{14}}\;\|u\|_1.$$

So we can take $c_0 = 2\sqrt{\frac{14}{61}}$.

According to the proof of Theorem 3.1, we fix $0 < r \leq \frac{\alpha}{c_0\sqrt{T}}$. We take r as large as possible, so here $r = \frac{1}{4}\sqrt{\frac{61}{7}}$. Then $u \in K\left(\frac{1}{4}\sqrt{\frac{61}{7}}\right)$ if and only if

$$u = \beta_1\left\langle \frac{2}{3}\sqrt{\frac{7}{15}}, 0\right\rangle + \beta_2\left\langle \frac{32}{45}\sqrt{\frac{15}{1757}}, 6\sqrt{\frac{15}{1757}}\right\rangle$$

where $\beta_1^2 + \beta_2^2 = \frac{61}{112}$.

Finally, we compute

$$\tau = \inf_{u \in K\left(\frac{1}{4}\sqrt{\frac{61}{7}}\right)} \sum_{t \in \{\frac{1}{4}, 2\}} \int_0^{u(t)} h(t,s)ds.$$

Note: $\sum_{t \in \{\frac{1}{4},2\}} \int_0^{u(t)} h(t,s)\; ds$

$$\begin{aligned}
&= \frac{1}{4}\int_0^{\frac{2\beta_1}{3}\sqrt{\frac{15}{1757}} + \frac{32\beta_2}{45}\sqrt{\frac{15}{1757}}} \sin \pi s\; ds + \frac{7}{4}\int_0^{6\beta_2\sqrt{\frac{15}{1757}}} \sin \pi s\; ds \\
&= \frac{1}{\pi}\left[2 - \frac{1}{4}\cos\left[\pi\left(\frac{2\beta_1}{3}\sqrt{\frac{7}{15}} + \frac{32\beta_2}{45}\sqrt{\frac{15}{1757}}\right)\right] - \cos\left(6\pi\beta_2\sqrt{\frac{15}{1757}}\right)\right].
\end{aligned}$$

Thus, to find τ, we minimize

$$g(x,y) = \frac{1}{\pi}\left[2 - \frac{1}{4}\cos\left[\pi\left(\frac{2x}{3}\sqrt{\frac{7}{15}} + \frac{32y}{45}\sqrt{\frac{15}{1757}}\right)\right] - \cos\left(6\pi y\sqrt{\frac{15}{1757}}\right)\right]$$

subject to the constraint $x^2 + y^2 = \frac{61}{112}$. Solving for x:

$$x = \pm\sqrt{\frac{61}{112} - y^2}, \quad -\frac{1}{4}\sqrt{\frac{61}{7}} \leq y \leq \frac{1}{4}\sqrt{\frac{61}{7}}.$$

So we minimize

$$g_-(y) = \frac{1}{\pi}\left[2 - \frac{1}{4}\cos\left[\pi\left(-\frac{2}{3}\sqrt{\frac{7}{15}}\sqrt{\frac{61}{112} - y^2} + \frac{32y}{45}\sqrt{\frac{15}{1757}}\right)\right] - \cos\left(6\pi y\sqrt{\frac{15}{1757}}\right)\right]$$

and

$$g_+(y) = \frac{1}{\pi}\left[2 - \frac{1}{4}\cos\left[\pi\left(\frac{2}{3}\sqrt{\frac{7}{15}}\sqrt{\frac{61}{112} - y^2} + \frac{32y}{45}\sqrt{\frac{15}{1757}}\right)\right] - \cos\left(6\pi y\sqrt{\frac{15}{1757}}\right)\right],$$

$-\frac{1}{4}\sqrt{\frac{61}{7}} \le y \le \frac{1}{4}\sqrt{\frac{61}{7}}$. Running a script in Matlab, we find that we can take $\tau = 0.0403$. A graph of g_- and g_+ is shown below.

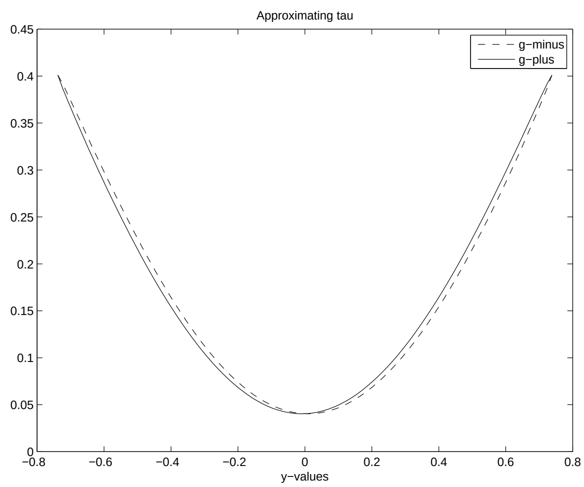

Figure 3.2: Approximating tau.

Thus, by Theorem 3.1, if $\lambda > 13.52 \ge \lambda^*$, the boundary value problem (12), (13) has two distinct pairs of nontrivial solutions. Furthermore, each solution u satisfies $|u(t)| \le 1,\ t \in \{\frac{1}{4}, 2\}$.

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