Nonlinear Dynamics and Systems Theory, **9** (1) (2009) 51–68

Oscillation Criteria for Half-Linear Delay Dynamic Equations on Time Scales

Lynn Erbe[1], Taher S. Hassan[1,2], Allan Peterson[1*] and Samir H. Saker[2]

[1] *Department of Mathematics, University of Nebraska–Lincoln Lincoln, NE 68588-0130, USA*
[2] *Department of Mathematics, Faculty of Science, Mansoura University Mansoura, 35516, Egypt*

Received: December 6, 2007; Revised: December 19, 2008

Abstract: This paper is concerned with oscillation of the second-order half-linear delay dynamic equation

$$(r(t)(x^{\Delta})^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0,$$

on a time scale $\mathbb{T}$, where $\gamma \geq 1$ is the quotient of odd positive integers, $p(t)$, and $\tau : \mathbb{T} \to \mathbb{T}$ are positive rd-continuous functions on $\mathbb{T}$, $r(t)$ is positive and (delta) differentiable, $\tau(t) \leq t$, and $\lim_{t\to\infty} \tau(t) = \infty$. We establish some new sufficient conditions which ensure that every solution oscillates or converges to zero. Our results in the special cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ involve and improve some oscillation results for second-order differential and difference equations; and when $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = q^{\mathbb{N}_0}$ and $\mathbb{T} = \mathbb{N}^2$ our oscillation results are essentially new. Some examples illustrating the importance of our results are also included.

Keywords: *oscillation; delay half-linear dynamic equations; time scales.*

Mathematics Subject Classification (2000): 34K11, 39A10, 39A99.

* Corresponding author: apeterson1@math.unl.edu

1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph. D. Thesis in 1988 in order to unify continuous and discrete analysis, see [10]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [3]). This new theory of these so-called "dynamic equations" not only unifies the corresponding theories for the differential equations and difference equations cases, but it also extends these classical cases to cases "in between". That is, we are able to treat the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0 \text{ for } q > 1\}$ (which has important applications in quantum theory (see [12])) and can be applied to different types of time scales like $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$ the set of the harmonic numbers. The books on the subject of time scales by Bohner and Peterson [3], [4] summarize and organize much of time scale calculus. In the last few years, there has been increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations on time scales, and we refer the reader to the papers [1], [5], [7], [8], [9], [15] and the references cited therein. In this paper, we are concerned with oscillation behavior of the second-order half-linear delay dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $\gamma \geq 1$ is a quotient of odd positive integers, p is a positive $rd-$continuous function on $\mathbb{T}$, $r(t)$ is positive and (delta) differentiable and the so-called delay function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) \leq t$ for $t \in \mathbb{T}$ and $\lim_{t\to\infty} \tau(t) = \infty$. Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. By a solution of (1.1) we mean a nontrivial real–valued function $x \in C^1_{rd}[T_x, \infty)$, $T_x \geq t_0$ which has the property that $r(t)(x^{\Delta}(t))^{\gamma} \in C^1_{rd}[T_x, \infty)$ and satisfies equation (1.1) on $[T_x, \infty)$, where C_{rd} is the space of $rd-$continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Note that if $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$, $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, and (1.1) becomes the half-linear delay differential equation

$$\left(r(t)\left(x'(t)\right)^{\gamma}\right)' + p(t)x^{\gamma}(\tau(t)) = 0. \tag{1.2}$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1$, $\mu(t) = 1$, $f^{\Delta}(t) = \Delta f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, and (1.1) becomes the half-linear delay difference equation

$$\Delta(r(t)\left(\Delta x(t)\right)^{\gamma}) + p(t)x^{\gamma}(\tau(t)) = 0. \tag{1.3}$$

If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\sigma(t) = t+h$, $\mu(t) = h$, $y^{\Delta}(t) = \Delta_h y(t) := \frac{y(t+h)-y(t)}{h}$, $\int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a+kh)h$, and (1.1) becomes the second-order half-linear delay difference equation

$$\Delta_h(r(t)\left(\Delta_h x(t)\right)^{\gamma}) + p(t)x^{\gamma}(\tau(t)) = 0. \tag{1.4}$$

If $\mathbb{T} = \{t : t = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$, then $\sigma(t) = qt$, $\mu(t) = (q-1)t$, $x^{\Delta}(t) = \Delta_q x(t) = (x(q\,t) - x(t))/(q-1)\,t$, $\int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=n_0}^{\infty} f(q^k)\mu(q^k)$, where $t_0 = q^{n_0}$, and (1.1) becomes the second-order $q-$half-linear delay difference equation

$$\Delta_q(r(t)\,(\Delta_q x(t))^{\gamma}) + p(t)x^{\gamma}(\tau(t)) = 0. \tag{1.5}$$

If $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$, then $\sigma(t) = (\sqrt{t}+1)^2$, $\mu(t) = 1 + 2\sqrt{t}$, $\Delta_N y(t) = \frac{y((\sqrt{t}+1)^2) - y(t)}{1+2\sqrt{t}}$, and (1.1) becomes the second-order half-linear delay difference equation

$$\Delta_N(r(t)\,(\Delta_N x(t))^{\gamma}) + p(t)x^{\gamma}(\tau(t)) = 0. \tag{1.6}$$

If $\mathbb{T} = \{H_n : n \in \mathbb{N}\}$ where H_n is the so-called n-th harmonic number defined by $H_0 = 0$, $H_n = \sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}_0$, then $\sigma(H_n) = H_{n+1}$, $\mu(H_n) = \frac{1}{n+1}$, $y^{\Delta}(t) = \Delta_{H_n} y(H_n) = (n+1)\Delta y(H_n)$ and (1.1) becomes the second-order half-linear delay difference equation

$$\Delta_{H_n}(r(H_n)\,(\Delta_{H_n} x(H_n))^{\gamma}) + p(H_n)x^{\gamma}(\tau(H_n)) = 0. \tag{1.7}$$

Recall that for a discrete time scale

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)_{\mathbb{T}}} f(t)\mu(t).$$

In the following, we state some oscillation results for differential and difference equations that will be related to our oscillation results for (1.1) on time scales and explain the important contributions of this paper. We will see that our results not only unify some of the known oscillation results for differential and difference equations but also give new oscillation criteria which include the equations (1.3)–(1.7), where in many cases the oscillatory behavior of their solutions was not known. In 1948 Hille [11] considered the linear differential equation

$$x''(t) + p(t)x(t) = 0, \tag{1.8}$$

where $p(t)$ is a positive function, and proved that if

$$\liminf_{t\to\infty} t\int_t^{\infty} p(s)ds > \frac{1}{4}, \tag{1.9}$$

then every solution of (1.8) oscillates. In 1957 Nehari [13] proved that if

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} s^2 p(s)ds > \frac{1}{4}, \tag{1.10}$$

then every solution of (1.8) oscillates. We note that the inequalities (1.9) and (1.10) are exact and can not be weakened. Indeed, let $p(t) = 1/4t^2$ for $t \geq 1$. Then we have

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} s^2 p(s)ds = \liminf_{t\to\infty} t\int_t^{\infty} p(s)ds = \frac{1}{4},$$

and the second-order Euler–Cauchy differential equation

$$x''(t) + \frac{1}{4t^2}x(t) = 0, \quad t \geq 1, \tag{1.11}$$

has a nonoscillatory solution $x(t) = \sqrt{t}$. In other words the constant $1/4$ is the lower bound for oscillation for all solutions of (1.11). In 1971 Wong [17] generalized the Hille-type condition (1.9) for the delay equation

$$x''(t) + p(t)x(\tau(t)) = 0, \tag{1.12}$$

where $\tau(t) \geq \alpha t$ with $0 < \alpha < 1$, and proved that if

$$\liminf_{t\to\infty} t \int_t^\infty p(s)ds > \frac{1}{4\alpha}, \tag{1.13}$$

then every solution of (1.12) is oscillatory. In 1973 Erbe [6] improved the condition (1.13) and proved that if

$$\liminf_{t\to\infty} t \int_t^\infty p(s)\frac{\tau(s)}{s}ds > \frac{1}{4}, \tag{1.14}$$

then every solution of (1.12) oscillates where $\tau(t) \leq t$. In 1984 Ohriska [14] proved that, if

$$\limsup_{t\to\infty} t \int_t^\infty p(s)\left(\frac{\tau(s)}{s}\right)ds > 1, \tag{1.15}$$

then every solution of (1.12) oscillates. Note that when $p(t) = \frac{\lambda}{t\tau(t)}$, with $\tau(t) \leq t$, (1.12) reduces to the second-order delay differential equation

$$x''(t) + \frac{\lambda}{t\tau(t)}x(\tau(t)) = 0, \quad t \geq t_0. \tag{1.16}$$

From (1.14) we see that every solution of (1.16) is oscillatory if $\lambda > \frac{1}{4}$ and nonoscillatory if $\lambda \leq \frac{1}{4}$, with oscillation constant 1/4 (see [1]). For oscillation of half-linear differential equations, Agarwal et al [2] considered the equation

$$\left((x'(t))^\gamma\right)' + p(t)x^\gamma(\tau(t)) = 0, \tag{1.17}$$

and extended the condition (1.15) of Ohriska and proved that if

$$\limsup_{t\to\infty} t^\gamma \int_t^\infty p(s)\left(\frac{\tau(s)}{s}\right)^\gamma ds > 1, \tag{1.18}$$

then every solution of (1.17) oscillates. It is clear that the condition (1.18) reduces to (1.15) when $\gamma = 1$. From which, we can easily see that the oscillation condition (1.18) that has been established by Agarwal et al [2] for (1.17) is not a sharp sufficient condition for oscillation of (1.17), since the condition (1.15) that has been established by Ohriska [14] is not sharp. For oscillation of half-linear difference equations, Thandapani et al [16] considered the difference equation

$$\Delta((\Delta x(n)))^\gamma) + p(n)x^\gamma(n) = 0, \quad n \geq n_0, \tag{1.19}$$

where $\gamma > 0$, $p(n)$ is a positive sequence, and proved that every solution is oscillatory, if

$$\sum_{n=n_0}^\infty p(n) = \infty. \tag{1.20}$$

We note that the condition (1.20) can not be applied to the difference equation

$$\Delta((\Delta x(n)))^\gamma) + \frac{\beta}{n^\gamma}x^\gamma(n) = 0, \quad \text{for } \gamma > 1. \tag{1.21}$$

In view of the above comments, we shall establish oscillation criteria for the dynamic equation (1.1) on a time scale $\mathbb{T}$ which as a special case when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$:

(i) involve the oscillation conditions (1.9) and (1.10) that have been given by Hille [11] and Nehari [13] for equation (1.8)

(ii) involve the oscillation condition (1.14) that was given by Erbe [6] for delay equation (1.12),

(iii) improve the oscillation condition (1.18) that was given by Agarwal et al [2] for half-linear differential equation (1.17),

(iv) improve the oscillation condition (1.20) that was established by Thandapani et al [16] for half-linear difference equation (1.19).

This paper is organized as follows: In Section 2, we establish some sufficient conditions for oscillation of (1.1) when $r(t) = 1$, which partially anwers the above question. Also, by using the Riccati transformation technique we will establish some new oscillation criteria for (1.1) in its general form when $r(t) \not\equiv 1$, which can be considered as a generalization of the results that have been established by Saker [15] and as a special case involve some results established by Agarwal et al [2] for half-linear differential equations. In Section 3, we give several examples which illustrate the importance of our main results. Note that the results are essentially new for equations (1.2)–(1.7). To the best of our knowledge nothing is known regarding the oscillatory behavior of half-linear delay dynamic equations on time scales until now so this paper initiates this study.

2 Main Results

Thoughout the paper we assume that $r^{\Delta}(t) \geq 0$ and

$$\int_{t_0}^{\infty} \tau^{\gamma}(t)p(t)\Delta t = \infty \tag{2.1}$$

is satisfied. Before stating our main results, we begin with the following lemma which will play an important role in the proof of our main results.

Lemma 2.1 *Assume that*

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \infty \tag{2.2}$$

holds and (1.1) *has a positive solution* x *on* $[t_0, \infty)_{\mathbb{T}}$. *Then there exists a* $T \in [t_0, \infty)_{\mathbb{T}}$, *sufficiently large, so that*

$$x^{\Delta}(t) > 0, \quad x^{\Delta\Delta}(t) < 0, \quad x(t) > tx^{\Delta}(t), \quad \left(\frac{x(t)}{t}\right)^{\Delta} < 0 \quad \text{on } [T, \infty)_{\mathbb{T}}.$$

Proof Let x be as in the statement of this theorem. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $x(\tau(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Since x is a positive solution of (1.1), we have $(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} = -p(t)x^{\gamma}(\tau(t)) < 0$, for $t \in [t_1, \infty)_{\mathbb{T}}$. Then $r(t)\left(x^{\Delta}(t)\right)^{\gamma}$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. We claim that $r(t)\left(x^{\Delta}(t)\right)^{\gamma} > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume not, then there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $r(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma} =: c < 0$. Therefore, $r(t)\left(x^{\Delta}(t)\right)^{\gamma} \leq r(t_2)\left(x^{\Delta}(t_2)\right)^{\gamma} = c$, for $t \in [t_2, \infty)_{\mathbb{T}}$, and therefore $x^{\Delta}(t) \leq \frac{a}{r^{\frac{1}{\gamma}}(t)}$, for $t \in [t_2, \infty)_{\mathbb{T}}$ where $a := c^{\frac{1}{\gamma}} < 0$. Integrating, we find that

$$x(t) = x(t_2) + \int_{t_2}^{t} x^{\Delta}(s)\Delta s \leq x(t_2) + a\int_{t_2}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} \to -\infty \quad \text{as } t \to \infty,$$

which implies that $x(t)$ is eventually negative. This is a contradiction. Hence $r(t)\left(x^{\Delta}(t)\right)^{\gamma} > 0$ on $[t_1,\infty)_{\mathbb{T}}$ and so $x^{\Delta}(t) > 0$ on $[t_1,\infty)_{\mathbb{T}}$. We now show that $x^{\Delta\Delta}(t) < 0$. Since $(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} < 0$ on $[t_1,\infty)_{\mathbb{T}}$, we have after differentiation that

$$r^{\Delta}(t)\left(x^{\Delta}(t)\right)^{\gamma} + r^{\sigma}(t)\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0. \tag{2.3}$$

Using the Pötzsche chain rule ([3, Theorem 1.90])

$$(f\circ g)^{\Delta}(t) = \int_0^1 f'(g(t)+h\mu(t)g^{\Delta}(t))dh\; g^{\Delta}(t), \tag{2.4}$$

we have

$$\begin{aligned}\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} &= \gamma\int_0^1 \left[x^{\Delta}(t)+h\mu(t)x^{\Delta\Delta}(t)\right]^{\gamma-1}\; dh\; x^{\Delta\Delta}(t)\\ &= \gamma x^{\Delta\Delta}(t)\int_0^1 \left[x^{\Delta}(t)+h[x^{\Delta\sigma}(t)-x^{\Delta}(t)\right]^{\gamma-1} dh\\ &= \gamma x^{\Delta\Delta}(t)\int_0^1 \left[hx^{\Delta\sigma}(t)+(1-h)x^{\Delta}(t)\right]^{\gamma-1} dh. \end{aligned}\tag{2.5}$$

From (2.3) we have that

$$r^{\sigma}(t)\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < -r^{\Delta}(t)\left(x^{\Delta}(t)\right)^{\gamma} \leq 0,$$

since $r^{\Delta}(t) \geq 0$ and $x^{\Delta}(t) > 0$ and so it follows that

$$r^{\sigma}(t)\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0.$$

This shows by (2.5) that $x^{\Delta\Delta}(t) < 0$, since the integral in (2.5) is positive. Next, we show that $\left(\frac{x(t)}{t}\right)^{\Delta} < 0$. To see this, let $U(t) := x(t) - tx^{\Delta}(t)$, then $U^{\Delta}(t) = -\sigma(t)x^{\Delta\Delta}(t) > 0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. This implies that $U(t)$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$. We claim there is a $t_2\in[t_1,\infty)_{\mathbb{T}}$ such that $U(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Assume not, then $U(t) < 0$ on $[t_1,\infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t)-x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} > 0, \quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{2.6}$$

Pick $t_3\in[t_1,\infty)_{\mathbb{T}}$ so that $\tau(t)\geq\tau(t_1)$, for $t\geq t_3$. Then

$$x(\tau(t))/\tau(t) \geq x(\tau(t_1))/\tau(t_1) =: d > 0,$$

so that $x(\tau(t)) \geq d\tau(t)$ for $t\geq t_3$. Now by integrating both sides of the dynamic equation (1.1) from t_3 to t we have

$$r(t)\left(x^{\Delta}(t)\right)^{\gamma} - r(t_3)\left(x^{\Delta}(t_3)\right)^{\gamma} + \int_{t_3}^{t} p(s)x^{\gamma}(\tau(s))\Delta s = 0.$$

This implies that

$$r(t_3)\left(x^{\Delta}(t_3)\right)^{\gamma} \geq \int_{t_3}^{t} p(s)x^{\gamma}(\tau(s))\Delta s \geq d^{\gamma}\int_{t_3}^{t} p(s)\tau^{\gamma}(s)\Delta s. \tag{2.7}$$

Letting $t \to \infty$ we obtain a contradiction to assumption (2.1). Hence there is a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $U(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Consequently,

$$\left(\frac{x(t)}{t}\right)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} < 0, \quad t \in [t_2, \infty)_{\mathbb{T}}, \tag{2.8}$$

and we have that $\left(\frac{x(t)}{t}\right)^{\Delta} < 0$ on $[t_2, \infty)_{\mathbb{T}}$. This completes the proof of Lemma 2.1.

In the following, we consider the equation (1.1) in the special case $r(t) \equiv 1$, namely,

$$((x^{\Delta}(t))^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \tag{2.9}$$

where $\gamma \geq 1$ is the quotient of odd positive integers and $p(t)$ is an $rd-$continuous and positive function and $\tau(t) \leq t$. We introduce the following notation.

$$p_* := \liminf_{t\to\infty} t^{\gamma} \int_{\sigma(t)}^{\infty} P(s)\Delta s, \quad q_* := \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} s^{\gamma+1} P(s)\Delta s, \tag{2.10}$$

where $P(t) := p(t)\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma}$ and assume that $l := \liminf_{t\to\infty} \frac{t}{\sigma(t)}$. Note that $0 \leq l \leq 1$. In order for the definition of p_* to make sense we assume that

$$\int_{t_0}^{\infty} P(t)\Delta t < \infty. \tag{2.11}$$

Theorem 2.1 *Assume that $l > 0$ and (2.11) holds. Let $x(t)$ be an eventually positive solution of (2.9) such that $x(t)$ and $x(\tau(t)) > 0$ for $t \geq t_1 > t_0$. Let $w(t) = \left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}$ and*

$$r := \liminf_{t\to\infty} t^{\gamma} w^{\sigma}(t), \quad \textit{and} \quad R := \limsup_{t\to\infty} t^{\gamma} w^{\sigma}(t), \tag{2.12}$$

then

$$p_* \leq r - l^{\gamma} r^{1+\frac{1}{\gamma}} \quad \textit{and} \quad p_* + q_* \leq \frac{1}{l^{\gamma(\gamma+1)}}. \tag{2.13}$$

Proof From Lemma 2.1 we get there is a $T \in [t_1, \infty)_{\mathbb{T}}$, sufficiently large, so that $x(t)$ satisfies the conclusions of Lemma 2.1. This implies that $w(t)$ is positive. Using the quotient rule and equation (2.9), we get

$$w^{\Delta}(t) = -\left(\frac{x(\tau(t))}{x^{\sigma}(t)}\right)^{\gamma} p(t) - \frac{(x^{\Delta}(t))^{\gamma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}.$$

Since

$$\frac{x(\tau(t))}{\tau(t)} \geq \frac{x(t)}{t} \geq \frac{x^{\sigma}(t)}{\sigma(t)} \quad \text{and} \quad x^{\Delta}(t) \geq x^{\Delta\sigma}(t),$$

we get the inequality

$$w^{\Delta}(t) \leq -\left(\frac{\tau(t)}{\sigma(t)}\right)^{\gamma} p(t) - \frac{(x^{\Delta\sigma}(t))^{\gamma}(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}, \tag{2.14}$$

since $x^{\Delta\Delta}(t) < 0$. By the Pötzsche chain rule, and the fact that $x^{\Delta}(t) > 0$, we obtain

$$\begin{aligned}(x^{\gamma}(t))^{\Delta} &= \gamma \int_0^1 \left[x(t) + h\mu(t)x^{\Delta}(t)\right]^{\gamma-1} dh\, x^{\Delta}(t) \\ &\geq \gamma \int_0^1 (x(t))^{\gamma-1}\, dh\, x^{\Delta}(t) \\ &= \gamma (x(t))^{\gamma-1} x^{\Delta}(t). \end{aligned} \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$\begin{aligned} w^{\Delta}(t) &\leq -P(t) - \frac{\left(x^{\Delta\sigma}(t)\right)^{\gamma} \gamma (x(t))^{\gamma-1} x^{\Delta}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}} \\ &= -P(t) - \gamma w^{\sigma}(t) w^{\frac{1}{\gamma}}(t). \end{aligned}$$

Then $w(t)$ satisfies the dynamic Riccati inequality

$$w^{\Delta}(t) + P(t) + \gamma w^{\sigma}(t) w^{\frac{1}{\gamma}}(t) \leq 0, \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.16}$$

Since $P(t) > 0$ and $w(t) > 0$ for $t \geq t_1$, it follows from (2.16) that $w^{\Delta}(t) < 0$ and hence $w(t)$ is strictly decreasing for $t \in [T, \infty)_{\mathbb{T}}$. It follows from Lemma 2.1 that

$$w(t) = \left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma} < \frac{1}{t^{\gamma}}, \quad t \in [T, \infty)_{\mathbb{T}}, \tag{2.17}$$

which implies that $\lim_{t\to\infty} w(t) = 0$ and that

$$0 \leq r \leq R \leq 1. \tag{2.18}$$

Now, we prove that the first inequality in (2.13) holds. Let $\epsilon > 0$, then by the definition of p_* and r we can pick $t_2 \in [T, \infty)_{\mathbb{T}}$, sufficiently large, so that

$$t^{\gamma} \int_{\sigma(t)}^{\infty} P(s)\Delta s \geq p_* - \epsilon, \quad \text{and} \quad t^{\gamma} w^{\sigma}(t) \geq r - \epsilon,$$

for $t \in [t_2, \infty)_{\mathbb{T}}$. Integrating (2.16) from $\sigma(t)$ to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w^{\sigma}(t) \geq \int_{\sigma(t)}^{\infty} P(s)\Delta s + \gamma \int_{\sigma(t)}^{\infty} w^{\frac{1}{\gamma}}(s) w^{\sigma}(s)\Delta s, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.19}$$

It follows from (2.19) that

$$\begin{aligned} t^{\gamma} w^{\sigma}(t) &\geq t^{\gamma} \int_{\sigma(t)}^{\infty} P(s)\Delta s + \gamma t^{\gamma} \int_{\sigma(t)}^{\infty} w^{\frac{1}{\gamma}}(s) w^{\sigma}(s)\Delta s \\ &\geq (p_* - \epsilon) + \gamma t^{\gamma} \int_{\sigma(t)}^{\infty} \frac{s\left(w^{\sigma}(s)\right)^{\frac{1}{\gamma}} s^{\gamma} w^{\sigma}(s)}{s^{\gamma+1}} \Delta s \\ &\geq (p_* - \epsilon) + (r - \epsilon)^{1+\frac{1}{\gamma}}\, t^{\gamma} \int_{\sigma(t)}^{\infty} \frac{\gamma}{s^{\gamma+1}} \Delta s. \end{aligned} \tag{2.20}$$

Using the Pötzsche chain rule we get

$$\begin{aligned}\left(\frac{-1}{s^\gamma}\right)^\Delta &= \gamma\int_0^1 \frac{1}{[s+h\mu(s)]^{\gamma+1}}dh \\ &\leq \int_0^1 \left(\frac{\gamma}{s^{\gamma+1}}\right)dh = \frac{\gamma}{s^{\gamma+1}}. \end{aligned}\tag{2.21}$$

Then from (2.20) and (2.21), we have

$$t^\gamma w^\sigma(t) \geq (p_* - \epsilon) + (r-\epsilon)^{1+\frac{1}{\gamma}}\left(\frac{t}{\sigma(t)}\right)^\gamma.$$

Taking the lim inf of both sides as $t\to\infty$ we get that

$$r \geq p_* - \epsilon + (r-\epsilon)^{1+\frac{1}{\gamma}}\, l^\gamma.$$

Since $\epsilon > 0$ is arbitrary, we get the desired result

$$r \geq p_* + (r)^{1+\frac{1}{\gamma}}\, l^\gamma.$$

To complete the proof it remains to prove the second inequality in (2.13). Since $w^\Delta(t) \leq 0$, we have $w(t) \geq w^\sigma(t)$, and (2.16) becomes

$$w^\Delta(t) + P(t) + \gamma\,(w^\sigma)^\lambda \leq 0, \quad t\in[T,\infty)_{\mathbb{T}}, \tag{2.22}$$

where $\lambda = 1+\frac{1}{\gamma}$. Multiplying both sides (2.22) by $t^{\gamma+1}$, and integrating from T to t ($t \geq T$) we get

$$\int_T^t s^{\gamma+1}w^\Delta(s)\Delta s \leq -\int_T^t s^{\gamma+1}P(s)\Delta s - \gamma\int_T^t s^{\gamma+1}\left(w^\sigma(s)\right)^\lambda \Delta s.$$

Using integration by parts, we obtain

$$\begin{aligned} t^{\gamma+1}w(t) &\leq T^{\gamma+1}w(T) - \int_T^t s^{\gamma+1}P(s)\Delta s - \gamma\int_T^t s^{\gamma+1}\left(w^\sigma(s)\right)^\lambda\Delta s \\ &+ \int_T^t \left(s^{\gamma+1}\right)^{\Delta_s} w^\sigma(s)\Delta s. \end{aligned}$$

But, by the Pötzsche chain rule,

$$\begin{aligned}\left(s^{\gamma+1}\right)^\Delta &= (\gamma+1)\int_0^1 [s+h\mu(s)]^\gamma dh \\ &\leq (\gamma+1)\int_0^1 [\sigma(s)]^\gamma dh \\ &= (\gamma+1)[\sigma(s)]^\gamma. \end{aligned}\tag{2.23}$$

Hence

$$\begin{aligned} t^{\gamma+1}w(t) &\leq T^{\gamma+1}w(T) - \int_T^t s^{\gamma+1}P(s)\Delta s + \int_T^t (\gamma+1)(\sigma(s))^\gamma w^\sigma(s)\Delta s \\ &- \gamma\int_T^t s^{\gamma+1}[w^\sigma(s)]^\lambda\Delta s. \end{aligned}$$

Let $0 < \epsilon < l$ be given, then using the definition of l, we can assume, without loss of generality, that T is sufficiently large so that

$$\frac{s}{\sigma(s)} > l - \epsilon, \quad s \geq T.$$

It follows that

$$\sigma(s) \leq Ks, \qquad s \geq T \quad \text{where} \quad K := \frac{1}{l - \epsilon}.$$

We then get that

$$\begin{aligned} t^{\gamma+1} w(t) &\leq T^{\gamma+1} w(T) - \int_T^t s^{\gamma+1} P(s) \Delta s \\ &+ \int_T^t [(\gamma+1) K^\gamma s^\gamma w^\sigma(s) - \gamma s^{\gamma+1} [w^\sigma(s)]^\lambda] \Delta s. \end{aligned}$$

Let

$$u(s) := s^\gamma w^\sigma(s),$$

then

$$(u(s))^{\frac{\gamma+1}{\gamma}} = s^{\gamma+1} [w^\sigma(s)]^\lambda.$$

It follows that

$$\begin{aligned} t^{\gamma+1} w(t) &\leq T^{\gamma+1} w(T) - \int_T^t s^{\gamma+1} P(s) \Delta s \\ &+ \int_T^t [(\gamma+1) K^\gamma u(s) - \gamma [u(s)]^\lambda] \Delta s. \end{aligned}$$

Using the inequality

$$Bu - Au^\lambda \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}, \tag{2.24}$$

where A, B are constants, we get

$$\begin{aligned} t^{\gamma+1} w(t) &\leq T^{\gamma+1} w(T) - \int_T^t s^{\gamma+1} P(s) \Delta s \\ &+ \int_T^t \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{[(\gamma+1) K^\gamma]^{\gamma+1}}{\gamma^\gamma} \Delta s \\ &\leq T^{\gamma+1} w(T) - \int_T^t s^{\gamma+1} P(s) \Delta s + K^{\gamma(\gamma+1)} (t - T). \end{aligned}$$

It follows from this that

$$t^\gamma w(t) \leq \frac{T^{\gamma+1} w(T)}{t} - \frac{1}{t} \int_T^t s^{\gamma+1} P(s) \Delta s + K^{\gamma(\gamma+1)} \frac{(t-T)}{t}.$$

Since $w^\sigma(t) \leq w(t)$, we get

$$t^\gamma w^\sigma(t) \leq \frac{T^{\gamma+1} w(T)}{t} - \frac{1}{t} \int_T^t s^{\gamma+1} P(s) \Delta s + K^{\gamma(\gamma+1)} \frac{(t-T)}{t}.$$

Taking the lim sup of both sides as $t \to \infty$ we obtain

$$R \leq -q_* + K^{\gamma(\gamma+1)} = -q_* + \frac{1}{(l-\epsilon)^{\gamma(\gamma+1)}}.$$

Since $\epsilon > 0$ is arbitrary, we get that

$$R \leq -q_* + \frac{1}{l^{\gamma(\gamma+1)}}.$$

Using this and the first inequality in (2.13) we get

$$p_* \leq r - l^{\gamma} r^{1+\frac{1}{\gamma}} \leq r \leq R \leq -q_* + \frac{1}{l^{\gamma(\gamma+1)}},$$

which gives us the desired second inequality in (2.13).

Using Theorem 2.1 we can now easily prove the following oscillation result.

Theorem 2.2 *If*

$$p_* = \liminf_{t\to\infty} t^{\gamma} \int_{\sigma(t)}^{\infty} p(s) \left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma} \Delta s > \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}}, \tag{2.25}$$

then (2.9) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof Assume (2.9) is nonoscillatory on $[t_0, \infty)_{\mathbb{T}}$, then the hypotheses of Theorem 2.1 hold. From the first inequality in (2.13) we have that

$$p_* \leq r - l^{\gamma} r^{\frac{\gamma+1}{\gamma}}.$$

Using the inequality (2.24), with $B = 1$ and $A = l^{\gamma}$ we get that

$$p_* \leq \frac{\gamma^{\gamma}}{l^{\gamma^2}(\gamma+1)^{\gamma+1}},$$

which contradicts (2.25).

Using the second inequality in Theorem 2.1 we easily get the following result

Theorem 2.3 *If*

$$p_* + q_* > \frac{1}{l^{\gamma(\gamma+1)}}, \tag{2.26}$$

then (2.9) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Remark 2.1 Note that when $\mathbb{T} = \mathbb{R}$, $\sigma(t) = t$ and the condition (2.25) becomes

$$\liminf_{t\to\infty} t^{\gamma} \int_{t}^{\infty} p(s) \left(\frac{\tau(s)}{s}\right)^{\gamma} ds > \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}, \tag{2.27}$$

which is a sufficient condition for oscillation of (1.17). We note that the condition (2.27) generalizes the condition (1.14) that has been established by Erbe [6]. Also when $\mathbb{T} = \mathbb{N}$, $\sigma(t) = t + 1$ and condition (2.25) becomes

$$\liminf_{t\to\infty} t^{\gamma} \sum_{s=t+1}^{\infty} p(s) \left(\frac{\tau(s)}{s+1}\right)^{\gamma} > \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}, \tag{2.28}$$

which is a sufficient condition for oscillation of (1.19). We note that the condition (2.28) may be viewed as an extension of the oscillation condition (1.20) that has been established by Thandapani et al [16]. As a special case when $\tau(t) = t$, the condition (2.27) becomes the Hille condition (1.9).

Remark 2.2 We give an example which shows that the inequality (2.27) and hence the inequality (2.19) can not be weakened. To see this let $\mathbb{T} = [1, \infty)$, and

$$p(t) := \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \frac{1}{t^{\gamma+1}}, \quad t \geq 1,$$

we have that

$$p_* = \liminf_{t\to\infty} t^\gamma \int_t^\infty p(s)ds = \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}},$$

and the second-order half-linear differential equation

$$\left((x'(t))^\gamma\right)' + p(t)x^\gamma(t) = 0,$$

has a nonoscillatory solution $x(t) = t^{\frac{\gamma}{\gamma+1}}$. This shows that the constant $\frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}$ is sharp for the oscillation for all solutions of this equation. Note in the case when $\gamma = 1$ this constant is $\frac{1}{4}$.

Theorem 2.4 *Assume that* (2.2) *holds and that*

$$\limsup_{t\to\infty} \frac{t^\gamma}{r(t)} \int_t^\infty p(s) \left(\frac{\tau(s)}{s}\right)^\gamma \Delta s > 1. \tag{2.29}$$

Then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof Assume x is an eventually positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Using Lemma 2.1 there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x^\Delta(t) > 0, \quad x^{\Delta\Delta}(t) < 0, \quad \frac{x(t)}{t} > x^\Delta(t),$$

on $[t_1, \infty)_{\mathbb{T}}$ and $\frac{x(t)}{t}$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Then integrating both sides of the dynamic equation (1.1) from t to T, $T \geq t \geq t_1$ we obtain

$$\int_t^T p(s)x^\gamma(\tau(s))\Delta s = r(t)(x^\Delta(t))^\gamma - r(T)(x^\Delta(T))^\gamma.$$

Since $x^\Delta(t) > 0$, we get that

$$\frac{1}{r(t)} \int_t^T p(s)x^\gamma(\tau(s))\Delta s \leq (x^\Delta(t))^\gamma.$$

Since $\frac{x(t)}{t}$ is strictly decreasing and using $x^\Delta(t) < \frac{x(t)}{t}$, we obtain

$$\frac{1}{r(t)} \int_t^T p(s) \left(\frac{\tau(s)}{s}\right)^\gamma x^\gamma(s)\Delta s \leq \frac{x^\gamma(t)}{t^\gamma}.$$

Since $x(t)$ is increasing, we get

$$\frac{t^\gamma}{r(t)}\int_t^T p(s)\left(\frac{\tau(s)}{s}\right)^\gamma \Delta s \le 1,$$

which implies that

$$\frac{t^\gamma}{r(t)}\int_t^\infty p(s)\left(\frac{\tau(s)}{s}\right)^\gamma \Delta s \le 1,$$

which gives us the contradiction

$$\limsup_{t\to\infty}\frac{t^\gamma}{r(t)}\int_t^\infty p(s)\left(\frac{\tau(s)}{s}\right)^\gamma \Delta s \le 1.$$

Remark 2.3 When $\mathbb{T}=\mathbb{R}$, Theorem 2.4 improves the results established by Ohriska [14] and Agarwal et al [2] for differential equations. In the case when $\mathbb{T}=\mathbb{N}$ and $r(t)=1$ the condition (2.29) becomes

$$\limsup_{t\to\infty} t^\gamma \sum_{s=t}^{\infty} p(s)\left(\frac{\tau(s)}{s}\right)^\gamma > 1, \tag{2.30}$$

which is a new oscillation condition for (1.19).

Motivated by Theorem 3.1 in [15], we can prove the following result which is a new oscillation result for equation (1.1).

Theorem 2.5 *Assume that* (2.2) *holds. Furthermore, assume that there exists a positive* $\Delta-$*differentiable function* $\delta(t)$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^t \left[\delta(s)p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^\gamma - \frac{r(s)((\delta^\Delta(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)}\right]\Delta s = \infty, \tag{2.31}$$

where $d_+(t) := \max\{d(t),0\}$ *is the positive part of any function* $d(t)$*. Then every solution of equation* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof Assume (1.1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusions of Lemma 2.1 on $[t_1,\infty)_{\mathbb{T}}$ with $x(\tau(t)) > 0$ on $[t_1,\infty)_{\mathbb{T}}$. Let $\delta(t)$ be a positive Δ differentiable function and consider the generalized Riccati substitution

$$w(t) = \delta(t)r(t)\left(\frac{x^\Delta(t)}{x(t)}\right)^\gamma.$$

Then by Lemma 2.1, we see that the function $w(t)$ is positive on $[t_1,\infty)_{\mathbb{T}}$. By the product rule and then the quotient rule (suppressing arguments)

$$\begin{aligned}
w^\Delta &= \delta^\Delta\left(\frac{r(x^\Delta)^\gamma}{x^\gamma}\right)^\sigma + \delta\left(\frac{r(x^\Delta)^\gamma}{x^\gamma}\right)^\Delta \\
&= \frac{\delta^\Delta}{\delta^\sigma}w^\sigma + \delta\frac{x^\gamma(r(x^\Delta)^\gamma)^\Delta - r(x^\Delta)^\gamma(x^\gamma)^\Delta}{x^\gamma x^{\gamma\sigma}} \\
&= \frac{\delta^\Delta}{\delta^\sigma}w^\sigma + \frac{\delta x^\gamma(-px^{\tau\gamma})}{x^\gamma(x^\sigma)^\gamma} - \frac{\delta r(x^\Delta)^\gamma(x^\gamma)^\Delta}{x^\gamma(x^\sigma)^\gamma} \\
&= \frac{\delta^\Delta}{\delta^\sigma}w^\sigma - p\delta\left(\frac{x^\tau}{x^\sigma}\right)^\gamma - \delta\frac{r(x^\Delta)^\gamma(x^\gamma)^\Delta}{x^\gamma(x^\sigma)^\gamma}.
\end{aligned}$$

Using the fact that $\frac{x(t)}{t}$ and $r(t)(x^{\Delta}(t))^{\gamma}$ are strictly decreasing (from Lemma 2.1) we get

$$\frac{x^{\tau}(t)}{x^{\sigma}(t)} \geq \frac{\tau(t)}{\sigma(t)} \quad \text{and} \quad r(t)(x^{\Delta}(t))^{\gamma} \geq r^{\sigma}(t)(x^{\Delta}(t))^{\gamma\sigma}.$$

From these last two inequalites we obtain

$$w^{\Delta} \leq \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \delta \frac{r^{\sigma}(x^{\Delta\sigma})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}.$$

Using (2.15) and the definition of w we have that

$$\begin{aligned} w^{\Delta} &\leq \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{\delta^{\sigma}} \frac{x^{\Delta}}{x} w^{\sigma} \\ &= \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{\delta^{\sigma}} \frac{r^{\frac{1}{\gamma}} x^{\Delta}}{r^{\frac{1}{\gamma}} x} w^{\sigma}. \end{aligned}$$

Since

$$r(t)(x^{\Delta}(t))^{\gamma} \geq r^{\sigma}(t)(x^{\Delta}(t))^{\gamma\sigma}, \quad \text{and} \quad x^{\sigma}(t) \geq x(t),$$

we get that

$$w^{\Delta} \leq \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{\delta^{\sigma}} \frac{(r^{\frac{1}{\gamma}} x^{\Delta})^{\sigma}}{r^{\frac{1}{\gamma}} x^{\sigma}} w^{\sigma}.$$

Using the definition of w we finally obtain

$$w^{\Delta} \leq \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda}, \tag{2.32}$$

where $\lambda := \frac{\gamma+1}{\gamma}$. It follows from (2.32) that

$$w^{\Delta} \leq \frac{(\delta^{\Delta})_{+}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda}. \tag{2.33}$$

Define $A \geq 0$ and $B \geq 0$ by

$$A^{\lambda} := \frac{\gamma\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda}, \quad B^{\lambda-1} := \frac{r^{\frac{1}{\gamma+1}}}{\lambda(\gamma\delta)^{\frac{1}{\lambda}}} (\delta^{\Delta})_{+}.$$

Then, using the inequality ($\lambda \geq 1$)

$$\lambda A B^{\lambda-1} - A^{\lambda} \leq (\lambda - 1) B^{\lambda},$$

we get that

$$\begin{aligned} \frac{(\delta^{\Delta})_{+}}{\delta^{\sigma}} w^{\sigma} - \gamma \frac{\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda} &= \lambda A B^{\lambda-1} - A^{\lambda} \\ &\leq (\lambda - 1) B^{\lambda} \\ &\leq \frac{r \left(\delta^{\Delta}\right)_{+}^{\gamma+1}}{(\gamma+1)^{\gamma+1} \delta^{\gamma}}. \end{aligned}$$

From this last inequality and (2.33) we get

$$w^{\Delta} \leq \frac{r((\delta^{\Delta})_{+})^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}.$$

Integrating both sides from t_1 to t we get

$$-w(t_1) \leq w(t) - w(t_1) \leq \int_{t_1}^{t} \left[\frac{r((\delta^{\Delta})_{+})^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}\right] \Delta s,$$

which leads to a contradiction, since the right hand side tends to $-\infty$ by (2.31).

By Theorem 2.5, by choosing $\delta(t) = 1$, $t \geq t_0$ we have the following oscillation result which as a special case gives the oscillation theorem established by Agarwal et al [2, Theorem 2.8].

Corollary 2.6 *Assume that* (2.2) *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma} p(s)\Delta s = \infty, \tag{2.34}$$

hold. Then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Similarly letting $\delta(t) = t$ in Theorem 2.5 we get the following result.

Corollary 2.7 *Assume that* (2.2) *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[sp(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma} - \frac{r(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right] \Delta s = \infty, \tag{2.35}$$

hold. Then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Note that again when $\mathbb{T} = \mathbb{N}$, Theorem 2.5 and Corollaries 2.6 and 2.7 improve the oscillation results that have been established by Thandapani et al [16]. In the following, we assume that

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} < \infty, \tag{2.36}$$

holds and establish some sufficient conditions which ensure that every solution $x(t)$ of (1.1) oscillates or converges to zero. The proof is similar to the proof of Theorem 3.3 in [15] and hence is omitted.

Theorem 2.8 *Assume that* (2.36) *and*

$$\int_{t_0}^{\infty} \left[\frac{1}{r(t)} \int_{t_0}^{t} p(s)\Delta s\right]^{\frac{1}{\gamma}} \Delta t = \infty, \tag{2.37}$$

hold. If one of the conditions (2.31), (2.34), *and* (2.35) *holds, then every solution of* (1.1) *oscillates or converges to zero.*

3 Examples

In this section we give some examples to illustrate our main results.

Example 3.1 Consider the half-linear delay dynamic equation

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+p(t)x^{\gamma}(\tau(t))=0, \tag{3.1}$$

where $p(t):=\frac{\beta}{t^{\gamma+1}}\left(\frac{\sigma(t)}{\tau(t)}\right)^{\gamma}$, where β is a positive constant and $\gamma\geq 1$ is the quotient of odd positive integers. It is clear that

$$\int_{t_0}^{\infty}\tau^{\gamma}(t)p(t)\Delta t=\beta\int_{t_0}^{\infty}\left(\frac{\sigma(t)}{t}\right)^{\gamma}\frac{1}{t}\Delta t\geq\beta\int_{t_0}^{\infty}\frac{\Delta t}{t}=\infty,$$

(i.e., (2.1) holds). For equation (3.1), we have

$$\begin{aligned} p_* &= \liminf_{t\to\infty} t^{\gamma}\int_{\sigma(t)}^{\infty}p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}\Delta s \\ &= \beta\liminf_{t\to\infty} t^{\gamma}\int_{\sigma(t)}^{\infty}\frac{\Delta s}{s^{\gamma+1}}. \end{aligned}$$

But, by the Pötzsche chain rule

$$\left(-\frac{1}{t^{\gamma}}\right)^{\Delta}=\gamma\int_0^1\frac{dh}{(t+h\mu(t))^{\gamma+1}}\leq\gamma\int_0^1\frac{dh}{t^{\gamma+1}}=\frac{\gamma}{t^{\gamma+1}},$$

so we get that

$$p_*\geq\frac{\beta}{\gamma}\liminf_{t\to\infty}\left(\frac{t}{\sigma(t)}\right)^{\gamma}=\frac{\beta}{\gamma}l^{\gamma}.$$

So if

$$\beta>\frac{\gamma^{\gamma+1}}{l^{\gamma(\gamma+1)}(\gamma+1)^{\gamma+1}},$$

then (2.25) holds and we have by Theorem 2.2 that (3.1) is oscillatory.

Note that in the case $\mathbb{T}=\mathbb{R}$, $\tau(t)=t$, $\gamma=1$, we get that $l=1$ and we see that $\beta>\frac{1}{4}$ which is the sharp condition for the Euler–Cauchy differential equation to be oscillatory (see [1] for related results for the delay case). Also, note that the results by Agarwal et al [2] and Thandapani et al [16] can not be applied to equation (3.1) in the cases of differential and difference equations.

Example 3.2 Consider the half-linear delay dynamic equation

$$\left(t^{\gamma-1}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\frac{\alpha}{t^2}\left(\frac{\sigma(t)}{\tau(t)}\right)^{\gamma}x^{\gamma}(\tau(t))=0, \tag{3.2}$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, where α is a positive constant and $\gamma\geq 1$ is the quotient of odd positive integers. Here $p(t)=\frac{\alpha}{t^2}\left(\frac{\sigma(t)}{\tau(t)}\right)^{\gamma}$ and $r(t)=t^{\gamma-1}$. It is clear that condition (2.1) holds and condition (2.2) is satisfied, since

$$\int_{t_0}^{\infty}\frac{\Delta t}{t^{\frac{\gamma-1}{\gamma}}}=\infty,\quad\text{for}\quad\gamma\geq 1,$$

by Example 5.60 in [4]. To apply Corollary 2.7, it remains to prove that condition (2.35) holds. To see this note that

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_0}^{t}\left[sp(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}-\frac{r(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s\\ =\left(\alpha-\frac{1}{(\gamma+1)^{\gamma+1}}\right)\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\Delta s}{s}=\infty,\end{aligned}$$

if $\alpha > \frac{1}{(\gamma+1)^{\gamma+1}}$. We conclude, by Corollary 2.7, that if

$$\alpha > \frac{1}{(\gamma+1)^{\gamma+1}},$$

then every solution of (3.2) is oscillatory.

Example 3.3 Consider the half-linear delay dynamic equation

$$\left(t^{\gamma+1}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\beta\left(\frac{\sigma(t)}{\tau(t)}\right)^{\gamma}x^{\gamma}(\tau(t))=0, \tag{3.3}$$

for $t\in[t_0,\infty)_{\mathbb{T}}$, where β is a positive constant and $\gamma\geq 1$ is the quotient of odd positive integers. In this case $p(t)=\beta\left(\frac{\sigma(t)}{\tau(t)}\right)^{\gamma}$ and $r(t)=t^{\gamma+1}$. It is clear that (2.1) holds and $r(t)$ satisfies condition (2.36) since

$$\int_{t_0}^{\infty}\frac{\Delta t}{t^{\frac{\gamma+1}{\gamma}}}<\infty,\quad \gamma\geq 1,$$

for those time scales $[t_0,\infty)_{\mathbb{T}}$, where $\int_{t_0}^{\infty}\frac{1}{t^p}\Delta t<\infty$ when $p>1$. This holds for many time scales (see Theorems 5.64 and 5.65 in [4] and see Example 5.63 where this result does not hold). To see that (2.37) holds note that

$$\begin{aligned}\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t}p(s)\Delta s\right]^{\frac{1}{\gamma}}\Delta t &= \int_{t_0}^{\infty}\left[\frac{1}{t^{\gamma+1}}\int_{t_0}^{t}\beta\left(\frac{\sigma(s)}{\tau(s)}\right)^{\gamma}\Delta s\right]^{\frac{1}{\gamma}}\Delta t\\ &\geq \int_{t_0}^{\infty}\left[\frac{1}{t^{\gamma+1}}\int_{t_0}^{t}\beta\Delta s\right]^{\frac{1}{\gamma}}\Delta t\\ &= \beta^{\frac{1}{\gamma}}\int_{t_0}^{\infty}\left(\frac{t-t_0}{t}\right)^{\frac{1}{\gamma}}\frac{\Delta t}{t}=\infty.\end{aligned}$$

To apply Theorem 2.8, it remains to prove that the condition (2.31) holds.To see this note that if $\delta(t)=1$, then

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_0}^{t}\left[\delta(s)p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}-\frac{r(s)((\delta^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\Delta s\\ =\beta\int_{t_0}^{\infty}\Delta t=\infty.\end{aligned}$$

We conclude that $[t_0,\infty)_{\mathbb{T}}$ is a time scale where $\int_{t_0}^{\infty}\frac{1}{t^p}\Delta t<\infty$ when $p>1$, then, by Theorem 2.8, every solution of (3.3) is oscillatory or converges to zero.

Example 3.4 One can use Theorem 2.4 to show that if $\beta > 1$, then the equation

$$\left(t^{\gamma-1}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \frac{\beta t^{\gamma-1}}{\tau^{\gamma}(t)\sigma(t)} x^{\gamma}(\tau(t)) = 0,$$

is oscillatory for any time scale where $\int_{t_0}^{\infty} \frac{t^{\gamma-1}}{\sigma(t)} \Delta t = \infty$. We leave the details to the reader.

Acknowledgement

The second author would like to thank the Egyptian Government for the support he received while visiting the University of Nebraska-Lincoln.

References

[1] Agarwal, R. P., Bohner, M. and Saker, S. H. Oscillation of second order delay dynamic equation. *Canadian Appl. Math. Quart.* **13** (2005) 1–17.

[2] Agarwal, R. P., Shien, S. L. and Yeh, C. C. Oscillation criteria for second-order retarded differential equations. *Math. Comp. Modelling* **26** (1997) 1–11.

[3] Bohner, M. and Peterson, A. Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston, 2001.

[4] Bohner, M. and Peterson, A. Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston, 2003.

[5] Bohner, M. and Saker, S. H. Oscillation of second order nonlinear dynamic equations on time scales. *Rocky Mountain J. Math.* **34**(4) (2004) 1239–1254.

[6] Erbe, L. Oscillation criteria for second order nonlinear delay equations. *Canad. Math. Bull.* **16** (1973) 49–56.

[7] Erbe, L. Peterson, A. and Saker, S. H. Oscillation criteria for second-order nonlinear dynamic equations on time scales. *J. London Math. Soc.* **76** (2003) 701–714.

[8] Erbe, L. Peterson, A. and Saker, S. H. Kamenev-type oscillation criteria for second-order linear delay dynamic equations. *Dynam. Syst. Appl.* **15** (2006) 65–78.

[9] Hassan, T. S. Oscillation criteria for half-linear dynamic equations on time scales. *J. Math. Anal. Appl.* **345** (2008) 176–185.

[10] Hilger, S. Analysis on measure chains — a unified approach to continuous and discrete calculus. *Results Math.* **18** (1990) 18–56.

[11] Hille, E. Non-oscillation theorems. *Trans. Amer. Math. Soc.* **64** (1948) 234–252.

[12] Kac, V. and Chueng, P. Quantum Calculus. Universitext, New York, NY: Springer. IX, 2002, 112 pp.

[13] Nehari, Z. Oscillation criteria for second-order linear differential equations. *Trans. Amer. Math. Soc.* **85** (1957) 428–445.

[14] Ohriska, J. Oscillation of second order delay and ordinary differential equations. *Czech. Math. J.* **34** (1984) 107–112.

[15] Saker, S. H. Oscillation criteria of second-order half-linear dynamic equations on time scales. *J. Comp. Appl. Math.* **177** (2005) 375–387.

[16] Thandapani, E., Ravi, K. and Graef, J. Oscillation and comparison theorems for half-linear second order difference equations. *Comp. Math. Appl.* **42** (2001) 953–960.

[17] Wong, J. S. W. Second order oscillation with retarded arguments. In: Ordinary differential equations (Proc. Conf., Math. Res. Center, Naval Res. Lab., Washington, D.C., 1971), 581–596, Academic press, New York and London, 1972.