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The Connection Between Boundedness and Periodicity in Nonlinear Functional Neutral Dynamic Equations on a Time Scale

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Abstract: Let $\mathbb{T}$ be a time scale that is unbounded above. We use a direct mapping and then utilize a Krasnosel'skiĭ fixed point theorem to show the existence of solutions of the nonlinear functional neutral dynamic system with delay

$$x^{\Delta}(t) = f(t, x(t), x^{\Delta}(t - h(t))) + g(t, x(t), x(t - h(t))),\ t, t - h(t) \in \mathbb{T}.$$

Then, we consider a special form of the above mentioned system and use the contraction mapping principle and show the existence of a uniform bound on all solutions and then conclude the existence of a unique periodic solution. Finally, the connection between the boundedness of solutions and the existence of periodic solutions leads us to the extension of Massera's theorem to functional differential equations on general periodic time scales.

Keywords: *connection between boundedness and periodic solutions; existence; functional; neutral; time scale.*

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1 Introduction

We assume the reader is familiar with the notation and basic results for dynamic equations on time scales. For a review of this topic we direct the reader to the monographs [5], [6] and [10].

Let $\mathbb{T}$ be a time scale that is unbounded above. By the notation $[a, b]$ we mean

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$$

unless otherwise specified. The intervals $[a, b), (a, b]$, and (a, b) are defined similarly. In this note we examine the existence of solutions of the nonlinear functional neutral dynamical equation

$$x^{\Delta}(t) = f\big(t, x(t), x^{\Delta}(t - h(t))\big) + g\big(t, x(t), x(t - h(t))\big);\ t, t - h(t) \in \mathbb{T}, \qquad (1)$$

where f, g and h are continuous, $h : \mathbb{T} \to [0, h_0]$ for some positive constant $h_0 \in \mathbb{T}$, and $f, g : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

The first and third authors have considered a variation of (1); namely, in [11] they studied the existence of periodic solutions of the neutral dynamical system

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + c(t)x^{\Delta}(t - h(t)) + g\big(x(t), x(t - h(t))\big),\ t, t - h(t) \in \mathbb{T}, \qquad (2)$$

where $\mathbb{T}$ is a periodic time scale and a, b and h are periodic. In [12], the first and third authors showed

$$x^{\Delta}(t) = -a(t)x^{\sigma}(t) + \big(Q(t, x(t), x(t - g(t)))\big)^{\Delta} + G\big(t, x(t), x(t - g(t))\big), t \in \mathbb{T}, \qquad (3)$$

has a periodic solution. In both papers, the authors obtained the existence of a periodic solution using a Krasnosel'skiĭ fixed point theorem. Moreover, under a slightly more stringent inequality they showed that the periodic solution is unique using the contraction mapping principle. The authors also showed that the zero solution was asymptotically stable using the contraction mapping principle provided that $Q(t, 0, 0) = G(t, 0, 0) = 0$.

In obtaining the existence of a periodic solution of (2) and (3) and the stability of the zero solution of (3), the authors inverted (2) and (3) and generated a variation of parameters-like formula. This formula was the sum of two mappings; one mapping was shown to be compact and the other was shown to be a contraction. We remark that the inversion of either (2) or (3) was made possible by the linear term $-a(t)x^{\sigma}(t)$, a luxury that (1) does not enjoy.

A neutral differential equation is an equation where the immediate growth rate is affected by the past growth rate. This can be observed in the behavior of a stock price or the growth of a child. Also, in the case $\mathbb{T} = \mathbb{R}$, neutral equations arise in circuit theory (see [3]) and in the study of drug administration and populations, (see [4], [13], [14]). This paper extends the results of [7] to time scales. Also, it is worth mentioning that the book [8] contains a wealth of information regarding stability and periodicity using fixed point theory.

Now we state Krasnosel'skiĭ's fixed point theorem which enables us to prove the existence of a solution. For its proof we refer the reader to [16].

Theorem 1.1 (Krasnosel'skiĭ) *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $\big(\mathbb{B}, \|\cdot\|\big)$. Suppose that A and B map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{M}$, implies $Ax + By \in \mathbb{M}$,

(ii) A is compact and continuous,

(iii) B is a contraction mapping.

Then there exists $z \in \mathbb{M}$ with $z = Az + Bz$.

2 Existence

For emphasis, we consider

$$x^{\Delta}(t) = f\big(t, x(t), x^{\Delta}(t-h(t))\big) + g\big(t, x(t), x(t-h(t))\big);\ t, t-h(t) \in \mathbb{T}, \tag{4}$$

where f, g and h are continuous, $h : \mathbb{T} \to [0, h_0]$ for some positive constant $h_0 \in \mathbb{T}$, $f, g : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Since our equation has a delay and the derivative enters nonlinearly on the right side, we must ask for an initial function $\eta \in C^1([-h_0, 0), \mathbb{R})$ whose Δ-derivative at 0 satisfies the expression

$$\eta^{\Delta}(0) = f\big(0, \eta(0), \eta^{\Delta}(-h(0))\big) + g\big(0, \eta(0), \eta(-h(0))\big). \tag{5}$$

In the next definition, we state what we mean by a solution for (4) in terms of a given initial function.

Definition 2.1 Let $\eta \in C^1([-h_0, 0), \mathbb{R})$ be a given bounded initial function that satisfies (5). We say $x(t, \eta)$ is a solution of (4) on an interval $[-h_0, r), r > 0, r \in \mathbb{T}$, if $x(t, \eta) = \eta(t)$ on $[-h_0, 0]$ and satisfies (4) on $[0, r)$.

The above definition allows us to continue the solution on $[r, r_1)$, for some $r_1 > r$ under the requirement of certain conditions.

Let $C_{rd} = C_{rd}([-h_0, r], \mathbb{R})$ be the space of all rd-continuous functions and define the set S by

$$S = \left\{\varphi \in C_{rd} :\ \varphi(t) = \eta^{\Delta}(t) \text{ on } [-h_0, 0]\right\}.$$

Then $\big(S, \nu\big)$ is a complete metric space, where $\nu(\varphi, \psi) = \|\varphi - \psi\| = \sup_{t\in[0,r]} \{|\varphi(t) - \psi(t)|\}$.

If $\varphi \in S$, we define

$$\Phi(t) = \begin{cases} \eta(t), & t \in [-h_0, 0], \\ \eta(0) + \int_0^t \varphi(s)\Delta s, & t \in [0, r]. \end{cases}$$

It is clear that $\Phi^{\Delta}(t) = \varphi(t)$ on $[0, r]$ and $\Phi \in C_{rd}$. Next, we suppose there is an $a > 0$ and define a subset S^* of S by

$$S^* = \{\varphi \in S : |\varphi(t) - \eta^{\Delta}(0)| \le a\},$$

such that there are an $\alpha > 0$ and $\beta < 1/2$ so that for $\varphi, \psi \in S^*$ we have

$$\begin{aligned} &\Big| f\big(t, \Phi(t), \varphi(t-h(t))\big) - f\big(t, \Psi(t), \psi(t-h(t))\big)\Big| \\ &\qquad\qquad \le \alpha|\Phi(t) - \Psi(t)| + \beta\big|\varphi(t-h(t)) - \psi(t-h(t))\big|. \end{aligned} \tag{6}$$

To be able to use Krasnosel'skiĭ's fixed point theorem, we define the two required mappings as follow. For $\varphi, \psi \in S^*$:

$$(A\varphi)(t) = g\big(t, \Phi(t), \varphi(t-h(t))\big), \quad (B\varphi)(t) = f\big(t, \Phi(t), \varphi(t-h(t))\big).$$

Then (6) implies that

$$\big|(B\varphi)(t) - (B\psi)(t)\big| \le \alpha|\Phi(t) - \Psi(t)| + \beta|\varphi(t-h(t)) - \psi(t-h(t))|. \tag{7}$$

It is obvious from the constructions of sets S and S^*, that fixed points of S^* are solutions of (4).

Theorem 2.1 *If η satisfies* (5) *and* (7)*, then there is an $r > 0$ such that the solution $x(t, \eta)$ of* (4) *exists on* $[0, r)$.

Proof Let a be given as in the set S^*. Since the functions f and g are continuous in their respective arguments, then for $\varphi \in S^*$, we can find a positive constant $L(a)$ depending on a, such that

$$\big|(B\varphi)(t)\big| + \big|(A\varphi)(t)\big| \le L(a),\ t \in \mathbb{T}.$$

Moreover, AS^* is equicontinuous. Now, for a fixed $a > 0$, we claim that there is an $r > 0$ such that and for all $t \in [0, r]_{\mathbb{T}}$,

$$\Big|f\big(t, \Phi(t), \varphi(t-h(t))\big) - f\big(0, \eta(0), \eta^{\Delta}(-h(0))\big)\Big| \le a/2. \tag{8}$$

To see this

$$\begin{aligned}
&\Big|f\big(t, \Phi(t), \varphi(t-h(t))\big) - f\big(0, \eta(0), \eta^{\Delta}(-h(0))\big)\Big| \\
&\qquad \le \alpha|\Phi(t) - \eta(0)| + \beta\big|\varphi(t-h(t)) - \eta^{\Delta}(-h(0))\big| \\
&\qquad \le \alpha \sup_{t\in[0,r]_{\mathbb{T}}} \left|\eta(0) + \int_0^t \varphi(s)\,\Delta s - \eta(0)\right| + \beta\big|\varphi(t-h(t)) - \eta^{\Delta}(-h(0))\big| \\
&\qquad \le \alpha\, t|\varphi(\xi)| + \beta\big|\varphi(t-h(t)) - \eta^{\Delta}(-h(0))\big|,\ \xi \in (0, t)_{\mathbb{T}},
\end{aligned}$$

where $|\varphi(\xi)| = \sup\limits_{\xi\in(0,t)} |\varphi(\xi)|$.

On the one hand, suppose $h(0) = 0$. Then $0 \le t - h(t) \le t$. As a consequence, for $\varphi \in S^*$, we have

$$\beta\big|\varphi(t-h(t)) - \eta^{\Delta}(-h(0))\big| = \beta\big|\varphi(t-h(t)) - \eta^{\Delta}(0)\big| \le \beta a < a/2.$$

On the other hand, if $h(0) > 0$, then there is an $r_1 > 0$ such that $t - h(t) \le 0$ for $t \in [0, r_1]$. Hence, $\varphi(t-h(t)) = \eta^{\Delta}(t-h(t))$ for $t \in [0, r_1]$. Since η^{Δ} is continuous, there is an $r_2 > 0$ so that $|\varphi(t-h(t)) - \eta^{\Delta}(0)| < a,$ for $t \in [0, r_2]$. Thus, if we choose $r^* \in (0, \min\{r_1, r_2\})$, then we can set $r = r^*$, so that for $t \in [0, r^*]$, we have

$$\big|\varphi(t-h(t)) - \eta^{\Delta}(-h(0))\big| \le a. \tag{9}$$

Due to inequality (9) and since $\beta < 1/2$, we can find a positive number $q < a/2$ so that

$$\beta\big|\varphi(t-h(t)) - \eta^{\Delta}(-h(0))\big| \le q.$$

Moreover, since $\varphi \in S^*$, we can choose an r, $r \in (0, r^*)$ so that for $t \in [0, r]$, we have

$$\alpha\, t|\varphi(\xi)| + \beta\big|\varphi(t-h(t)) - \eta^{\Delta}(-h(0))\big| \le \alpha\, t|\varphi(\xi)| + q \le a/2,$$

which proves (8).

This shows that A is compact.

Finally, we claim that we can make r small enough so that for $\varphi \in S^*$, $t \in [0, r]$, we have

$$\big|g(t, \Phi(t), \Phi(t-h(t))) - g(0, \eta(0), \eta(-h(0)))\big| \le a/2. \tag{10}$$

The proof of the claim follows from the fact g is uniformly continuous on any bounded set. For $0 \le t - h(t)$, we have

$$
\begin{aligned}
&\Big| g\big(t, \Phi(t), \Phi(t-h(t))\big) - g\big(0, \eta(0), \eta(-h(0))\big) \Big| \\
&\quad \le |t-0| + |\Phi(t) - \eta(0)| + \big|\Phi(t-h(t)) - \eta(-h(0))\big| \\
&\quad \le t + t|\varphi(\xi)| + \left| \eta(0) + \int_0^{t-h(t)} \varphi(s)\,\Delta s - \eta(-h(0)) \right| \\
&\quad \le t + t|\varphi(\xi)| + (t - h(t)|\varphi(\xi)| \\
&\quad \le r[1 + 2|\varphi(\xi)|],
\end{aligned}
$$

where $|\varphi(\xi)| = \sup\limits_{\xi \in (0,r)} |\varphi(\xi)|$, which can be made arbitrary small. Due to the continuity of η, the case $t - h(t) < 0$ readily follows. This completes the proof of (10).

We now go back to the proof of the theorem. It readily follows form (7) that for $\varphi, \psi \in S^*$, there is a $\lambda < 1$ so that the

$$
\|B\varphi - B\psi\| \le \lambda \|\varphi - \psi\|. \tag{11}
$$

Next we show that if $\varphi, \psi \in S^*$, then $A\psi + B\varphi \in S^*$. We remark that $(A\psi)(0) + (B\psi)(0) = \eta^{\Delta}(0)$, where $\eta^{\Delta}(0)$ is given by (5). As a consequence, we have by (8) and (10)

$$
\begin{aligned}
|(A\psi)(t) \;&+\; (B\varphi)(t) - \eta^{\Delta}(0)| \\
&= |(A\psi)(t) - (A\psi)(0) + (B\varphi)(t) - (B\psi)(0)| \\
&= \big|(A\psi)(t) - g\big(0, \eta(0), \eta(-h(0))\big) + (B\varphi)(t) - f\big(0, \eta(0), \eta^{\Delta}(-h(0))\big)\big| \\
&\le \big|(A\psi)(t) - g\big(0, \eta(0), \eta(-h(0))\big)\big| + \big|(B\varphi)(t) - f\big(0, \eta(0), \eta^{\Delta}(-h(0))\big)\big| \\
&\le a/2 + a/2 = a.
\end{aligned}
$$

This completes the proof of $A\psi + B\varphi \in S^*$.

Also, (11) shows that B is a contraction. Hence all the conditions of Theorem 1.1 are satisfied, which imply that there is $\varphi \in S^*$, such that $\varphi = A\varphi + B\varphi$. □

3 Connection Between Boundedness and Periodicity

Intuitively, in the study of stability or periodic solutions in dynamical systems one will have to ask for the existence of solutions in the sense that solutions exist for all time or remain bounded. Thus, we may look at boundedness of solutions as a necessary condition before studying stability or attempt to search for a periodic solution. In this section, we examine the relationship between the boundedness of solutions and the existence of a periodic solution of the nonlinear non-autonomous delay dynamical system of the form

$$
x^{\Delta}(t) = -a(t)x^{\sigma}(t) + b(t)g\big(x(t - r(t))\big) + q(t), \tag{12}
$$

where $\mathbb{T}$ is unbounded above and below.

We assume that $a, b : \mathbb{T} \to \mathbb{R}$ are continuous and $q : [0, \infty) \to \mathbb{R}$ is continuous. In order for the function $x(t - r(t))$ to be well-defined over $\mathbb{T}$, we assume that $r : \mathbb{T} \to \mathbb{R}$ and that $id - r : \mathbb{T} \to \mathbb{T}$ is strictly increasing.

The proof of Lemma 3.2 below can be easily deduced from [11], and hence we omit the proof. But first, we state some facts about the exponential function. A function

$p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \operatorname{Log}(1 + \mu(z)p(z))\,\Delta z\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma, [5, Theorem 2.36].

Lemma 3.1 *Let $p, q \in \mathcal{R}$. Then*
(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$;
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ *where,* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$.

Lemma 3.2 *x is a solution of equation* (12) *if and only if*

$$x(t) = x(0)e_{\ominus a}(t,0) + \int_0^t b(s)g(x(s - r(s))e_{\ominus a}(t,s)\,\Delta s + \int_0^t q(s)e_{\ominus a}(t,s)\,\Delta s.$$

Let $\psi : (-\infty, 0] \to \mathbb{R}$ be a given bounded Δ-differentiable initial function. We say $x := x(\cdot, 0, \psi)$ is a solution of (12) if $x(t) = \psi(t)$ for $t \le 0$ and satisfies (12) for $t \ge 0$. If $\psi : (-\infty, 0] \to \mathbb{R}$, then we set

$$\|\psi\| = \sup_{s \in (-\infty,0]} |\psi(s)|.$$

Definition 3.1 Let ψ be as defined above. We say solutions of (12) are uniformly bounded if for each $B_1 > 0$, there exists $B_2 > 0$ such that

$$[t_0 \ge 0,\ \|\psi\| \le B_1,\ t \le t_0] \Rightarrow |x(\cdot, 0, \psi)| < B_2.$$

For the next theorem we assume the following. There is a positive constant Q so that

$$\int_0^t |q(s)|e_{\ominus a}(t,s)\,\Delta s \le Q, \tag{13}$$

$$\int_0^t a(s)\,\Delta s \to \infty, \tag{14}$$

there is an $\alpha < 1$ so that

$$\int_0^t |b(s)|e_{\ominus a}(t,s)\,\Delta s < \alpha, \tag{15}$$

$$0 \le r(t),\ t - r(t) \to \infty \text{ as } t \to \infty, \tag{16}$$

and if $x, y \in \mathbb{R}$, then

$$g(0) = 0 \text{ and } \ |g(x) - g(y)| < |x - y|. \tag{17}$$

Theorem 3.1 *If* (13)–(17) *hold, then solutions of* (12) *are uniformly bounded at* $t_0 = 0$.

Proof First by (14), there is a constant $k > 1$ so that $e_{\ominus a}(t, 0) \leq k$. Let B_1 be given so that if $\psi : (-\infty, 0] \to \mathbb{R}$ be a given bounded initial function, $\|\psi\| \leq B_1$. Define the constant B_2 by $B_2 = \frac{kB_1+Q}{1-\alpha}$. Let

$$S = \{\varphi \in C_{rd} : \varphi(t) = \psi(t) \text{ if } t \in (-\infty, 0],\ \|\varphi\| \leq B_2\}.$$

Then $\big(S, \|\cdot\|\big)$ is a complete metric space where $\|\cdot\|$ is the supremum norm.

For $\varphi \in S$, define the mapping P

$$\big(P\varphi\big)(t) = \psi(t), t \leq 0$$

and

$$\begin{aligned}\big(P\varphi\big)(t) &= \psi(0)e_{\ominus a}(t, 0) + \int_0^t b(s)g(\varphi(s - r(s))e_{\ominus a}(t, s)\,\Delta s \\ &\quad + \int_0^t q(s)e_{\ominus a}(t, s)\,\Delta s,\ t \geq 0.\end{aligned}$$

It follows from (17) that

$$|g(x)| = |g(x) - g(0) + g(0)| \leq |g(x) - g(0)| + |g(0)| \leq |x|.$$

This implies that

$$|(P\varphi)(t)| \leq kB_1 + \alpha B_2 + Q = B_2.$$

Thus, $P : S \to S$. It is easy to show, using (17), that P is a contraction with contraction constant α. Hence there is a unique fixed point in S, which solves (12). □

We end this paper by examining the existence of a periodic solution of (12). We must first define what we mean by a periodic time scale.

Definition 3.2 We say that a time scale $\mathbb{T}$ is *periodic* if there exists a $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the *period* of the time scale.

The above definition is due to Kaufmann and Raffoul [12]. Other definitions of periodic time scales are due to Atici *et. al* [2], C. D. Ahlbrandt and J. Ridenhour [1], and J. J. DaCunha and J. M. Davis [9].

Example 3.1 The following time scales are periodic.

1. $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [(2i - 1)h, 2ih]$, $h > 0$ has period $p = 2h$.
2. $\mathbb{T} = h\mathbb{Z}$ has period $p = h$.
3. $\mathbb{T} = \mathbb{R}$.
4. $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z}, m \in \mathbb{N}_0\}$ where, $0 < q < 1$ has period $p = 1$.

Remark 3.1 Using the above definition, all periodic time scales are unbounded above and below.

Definition 3.3 Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period p. We say that the function $f\colon \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 3.2 If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$. Consequently, the graininess function μ satisfies $\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period p.

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. Define $P_T = \{\varphi \in C(\mathbb{T}, R) : \varphi(t+T) = \varphi(t)\}$, where $C(\mathbb{T}, R)$ is the space of all real valued continuous functions on $\mathbb{T}$. Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \sup_{t \in [0,T]} |x(t)|.$$

Here we let the function $q : (-\infty, \infty) \to \mathbb{R}$. Since we are searching for a periodic solution, we must ask that

$$a(t+T) = a(t),\ b(t+T) = b(t),\ r(t+T) = r(t), \text{ and } q(t+T) = q(t). \tag{18}$$

Lemma 3.3 *Suppose* (14)–(18) *hold. If* $x(t) \in P_T$, *then* $x(t)$ *is a solution of equation* (12) *if and only if*

$$x(t) = \int_{-\infty}^{t} b(s)g(x(s - r(s)))e_{\ominus a}(t,s)\,\Delta s + \int_{-\infty}^{t} q(s)e_{\ominus a}(t,s)\,\Delta s. \tag{19}$$

Proof Due to condition (14) and the fact that $p(t)$ is periodic, condition (13) is satisfied. Thus, by Theorem 3.1, solutions of (12) are bounded for all $t \in (-\infty, \infty)$. As a consequence, if we multiply both sides of (12) by $e_a(s, 0)$, and then integrate from $-\infty$ to t we obtain (19). By taking the Δ-derivative on both sides of (19) we obtain (12). $\square$

Theorem 3.2 *Assume the hypothesis of Lemma 3.3. Then* (12) *has a unique T-periodic solution.*

Proof For $\phi \in P_T$, define a mapping $H : P_T \to P_T$ by

$$(H\phi)(t) = \int_{-\infty}^{t} b(s)g(\phi(s - r(s)))e_{\ominus a}(t,s)\,\Delta s + \int_{-\infty}^{t} p(s)e_{\ominus a}(t,s)\,\Delta s.$$

It is easy to verify that H is periodic and defines a contraction on P_T. Thus, H has a unique fixed point in P_T by the contraction mapping principle, which solves (12) by Lemma 3.8. $\square$

We remark that Lemma 3.3 and Theorem 3.2 show a clear connection between boundedness and the existence of a periodic solution. In the case $\mathbb{T} = \mathbb{R}$, this result is known as Massera's theorem, see [15]. Below we state and prove Massera's theorem for the general case of $\mathbb{T}$ being a periodic time scale. We begin with a lemma that will be needed in the proof.

Lemma 3.4 *Let $\mathbb{T}$ be a periodic time scale with period $T > 0$. Let $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $F(t + T, x) = F(t, x)$ and that $F(t, x)$ satisfy a local Lipschitz condition with respect to x.*

1. If $x(t)$ is a solution of $x^\Delta = F(t, x)$, then $x(t+T)$ is also a solution of $x^\Delta = F(t, x)$.

2. The equation $x^\Delta = F(t, x)$ has a T-periodic solution if and only if there is a (t_0, x_0) with $x(t_0+T; t_0, x_0) = x_0$ where $x(t; t_0, x_0)$ is the unique solution of $x^\Delta = F(t, x), x(t_0) = x_0$.

Proof For part (1), let $q(t) = x(t + T)$. Then,

$$q^\Delta(t) = x^\Delta(t + T) = F(t + T, x(t + T)) = F(t, q(t)),$$

and the proof of part (1) is complete.

For part (2), first suppose that $x(t; t_0, x_0)$ is T-periodic. Then, $x(t_0 + T; t_0, x_0) = x(t_0; t_0, x_0) = x_0$.

Now suppose that there exists (t_0, x_0) such that $x(t_0 + T; t_0, x_0) = x_0$. From part (1), $q(t) \equiv x(t + T; t_0, x_0)$ is also a solution of $x^\Delta = F(t, x)$. Since $q(t_0) = x(t_0 + T; t_0, x_0) = x_0$, then by the uniqueness of solutions of initial value problems, $x(t+T; t_0, x_0) = q(t) = x(t; t_0, x_0)$. This completes the proof of part (2). □

Theorem 3.3 *Let $\mathbb{T}$ be a periodic time scale with period $T > 0$. Let $F : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $F(t+T, x) = F(t, x)$ and that $F(t, x)$ satisfies a local Lipschitz condition with respect to x. If the equation*

$$x^\Delta = F(t, x) \tag{20}$$

has a solution bounded in the future, then it has a T-periodic solution.

Proof Let $x(t)$ be the solution of (20) such that $|x(t)| \leq M$ for all $t \in \mathbb{T}, t \geq 0$. Define the sequence $\{x_n(t)\}$ by $x_n(t) = x(t + nT), n = 0, 1, 2, \ldots$. By Lemma 3.4, $x_n(t)$ is a solution of (20) for each n and furthermore, $|x_n(t)| \leq M$ for $t \geq 0$. There are two cases to consider.

Case 1: Suppose that for some n, $x_n(0) = x_{n+1}(0)$. By uniqueness of solutions for initial value problems we have $x(t + nT) = x(t + (n + 1)T)$ for all $t \in \mathbb{T}$. Thus, $x(t)$ is a T-periodic solution of (20).

Case 2: Suppose that $x_n(0) \neq x_{n+1}(0)$ for all n. We may assume, without loss of generality, that $x(0) < x_1(0)$. By uniqueness, we have $x(t) < x_1(t)$ for all $t \in \mathbb{T}, t \geq 0$. In particular, $x_n(0) = x(0 + nT) < x_1(0 + nT) = x_{n+1}(0)$. Hence, $x_n(t) < x_{n+1}(t)$ for all $t \in \mathbb{T}, t \geq 0$. Thus, $\{x_n(t)\}$ is an increasing sequence bounded above by M. Thus $x_n(t) \to x^*(t)$ for each $t \in \mathbb{T}, t \geq 0$ as $n \to \infty$. Since $|F(t, x)| \leq J$ for $t \in \mathbb{T}$ and $|x| \leq M$, then $|x^\Delta(t)| \leq J, t \in \mathbb{T}$.

¿From the Mean Value Theorem (see [5, Corollary 1.68]) we have $|x_n(t) - x_n(s)| \leq \sup_{r\in[s,t]^\kappa} |F^\Delta(r, x_n)|\,|t-s| \leq J|t-s|$ for $t, s \in \mathbb{T}$ with $0 \leq s \leq t, n \geq 0$. Using the Arzela-Ascoli Theorem, we get that on any compact subinterval of $\mathbb{T}$ there exists a subsequence of $\{x_n(t)\}$ that converges uniformly. We know that the original sequence is monotone, and so, the original sequence is convergent on any compact interval. Since for each n,

$$x_n(t) = x_n(0) + \int_0^t F(s, x_n(s))\,\Delta s,$$

then the limiting function $x^*(t)$ is a solution of (20). Finally, since $x^*(T) = \lim x_n(T) = \lim x_{n+1}(0) = x^*(0)$, then by Lemma 3.4, the limiting function is a T-periodic solution of (20) and the proof is complete. □

References

[1] Ahlbrandt, C. D. and Ridenhour, J. Floquet theory for time scales and Putzer representations of matrix logarithms, *J. Difference Equ. Appl.* **9**(1) (2003) 77–92.

[2] Atici, F. M. , Guseinov, G. Sh. and Kaymakcalan, B. Stability criteria for dynamic equations on time scales with periodic coefficients, *Proceedings of the International Confernce on Dynamic Systems and Applications, III* **3** (1999) 43–48.

[3] Belden, A., Guglielmi, N., and Ruehli, A. Methods for linear systems of circuit delay differential equations of neutral type, *IEEE Transactions on Circuits and Systems* **46**(1) (1999) 212–216.

[4] Beretta, E. , Solimano, F. and Takeuchi, Y. A mathematical model for drug administration by using the phagocytosis of red blood cells, *J. Math. Biol.* **35**(1) (1996) 1–19.

[5] Bohner, M. and Peterson, A. *Dynamic Equations on Time Scales, An introduction with Applications*, Birkhäuser, Boston, 2001.

[6] Bohner, M. and Peterson, A. *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[7] Burton, T. A. An existence theorem for a neutral equation, *Nonlinear Studies* **5** (1998) 1-6.

[8] Burton, T. A. Stability by Fixed Point Theory for Functional Differential Equations, *Dover Publications*, New York, 2006.

[9] DaCunha, J. J. and Davis, J. M. A unified Floquet theory for hybrid periodic linear systems. (Submitted).

[10] Hilger, S. Ein Masskettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten. PhD thesis, Universität Würzburg, 1988.

[11] Kaufmann, E. R. and Raffoul, Y. N. Periodic solutions for a neutral nonlinear dynamical equation on a time scale, *J. Math. Anal. Appl.* **319**(1) (2006) 315–325.

[12] Kaufmann, E. R. and Raffoul, Y. N. Periodicity and stability in neutral nonlinear dynamic equations with functional delay on a time scale, *Electron. J. Diff. Eqns.* **2007**(27) (2007) 1–12.

[13] Kuang, Y. *Delay Differential Equations with Applications to Populations Dynamics*, Academic Press, Boston, 1993.

[14] Kuang, Y. Global stability in one or two species neutral delay population models, *Canadian Appl. Math. Quart.* **1** (1993) 23–45.

[15] Massera, J. L. The existence of periodic solutions of systems of differential equations, *Duke Math. J.* **17** (1950) 457–475.

[16] Smart, D. R. *Fixed points theorems*, Cambridge Univ. Press, Cambridge, 1980.