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Novel Qualitative Methods of Nonlinear Mechanics and their Application to the Analysis of Multifrequency Oscillations, Stability, and Control Problems ⋄

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Abstract: The method of oriented manifolds is developed to study geometric properties of the sets of trajectories of nonlinear differential systems with control. This method is conceptually connected with the classical methods of Lyapunov, Poincaré, and Levi–Civita and is a natural extension and development of results of the Donetsk school of mechanics. In terms of the method of oriented manifolds, sufficient conditions for stabilizability of nonlinear control systems are established.

A new method for stability investigation of nonlinear differential systems of perturbed motions is created on the basis of the concept of matrix-valued Lyapunov functions. This method is generalized for the systems with impulse action and aftereffect, differential equations with explosive right-hand sides and hybrid systems.

New conditions of practical stability of motion for nonlinear systems with impulse action are established on the basis of two auxiliary Lyapunov functions and the condition of exponential stability for linear impulse systems in a Hilbert space.

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General theory of the Fredholm boundary-value problems is constructed for systems of functional-differential equations, a classification of resonance boundary-value problems is elaborated, efficient coefficient criteria of existence of solutions are obtained and bifurcation and branching conditions for solutions to such problems are established.

New matrix methods are developed for the analysis of stability, localization of spectrum and representation of solutions of arbitrary order linear differential and difference systems. The methods of comparison and robust stability analysis are worked out for nonlinear dynamic systems in partially ordered space.

The averaging technique and the method of integral manifolds are developed for nonlinear resonance oscillating systems with slowly varying frequencies. New exact error estimations are established for the averaging technique in the initial and boundary-value problems for multifrequency systems and systems with impulse action.

New statements on stability and instability of linear approach to solutions of evolutionary equations in a Banach space are made. Absolute stability conditions are established for systems with aftereffect. In particular, a process of aircraft undercarriage galloping is studied at landing on the ground airfield with constant velocity. Also, stability conditions are established for the metal cutting process at turning behind a track with constant angular velocity of spindle rotation.

Keywords: *stability; robust stability; practical stability of motion; initial and boundary-value problems; differential and difference systems; systems with impulse action and aftereffect; comparison principle; Lyapunov functions; matrix equation; generalized Lyapunov equation; cone inequality.*

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1 Introduction

This paper presents a survey of main results of a series of investigations competing for the State Prize of Ukraine in the Field of Science and Technology in 2008.

First, it should be noted that the fruitful ideas by Lyapunov have enabled his successors to develop constructive approaches for the analysis of dynamical behaviour of nonlinear systems.

Remarkable results of N.M. Krylov and N.N. Bogoliubov, which became a groundwork for a new direction in the field of mathematical physics, called "nonlinear mechanics", have become a source of many investigations of systems with small parameter, both of theoretical and practical importance.

The discovery of the principle of maximum in the mathematical theory of optimal control made by L.S. Pontryagin proved to be a profound synthesis of the theory of differential equations and the variational calculus whose development is associated with the name of outstanding mathematician of the 18-th century L. Euler.

A range of problems whose solutions are proposed in the monographs [1–12] and papers [13–43] was formed according to the needs of new fields of science and technology such as exploration of the near-Earth and outer space, automatic control of production processes by computers, mathematical biology, etc. A key role in the solution of these problems is played by the ideas and methods set out in the remarkable works by Lyapunov–Bogoliubov–Pontryagin.

Several hundred references in the publications [1–43] give an idea about the directions of the investigations mentioned in the title of this series of works and bring the reader to the boundary beyond which new areas for further investigations are opened up in these challenging scientific directions which constitute the basis for the technological advance in the beginning of the third millenium.

2 Qualitative Theory of Nonlinear Control Systems

In the papers [1, 2, 13, 14, 16], [19]–[24], qualitative properties of the family of trajectories of nonlinear systems of differential equations of the type

$$\dot{x} = f(x, u), \quad x \in D \subset \mathbb{R}^n, \; u \in U \subset \mathbb{R}^m, \tag{2.1}$$

are studied, where x is the state vector and u is the control. The function $f(x, u)$ is assumed to be continuously differentiable sufficient number of times in $D \times \overline{U}$. In papers by A.M. Kovalev [13, 14], the notion of a set oriented with respect to control system was introduced and the method of oriented manifolds was proposed. This method is conceptually connected with the method of Lyapunov functions and the Poincaré–Levi–Civita method of invariant relations.

Definition 2.1 A manifold $K \subset D$ is called oriented with respect to system (2.1) in the domain D if it coincides with its positive ($K = Or^+K$) or negative ($K = Or^-K$) orbit. Positive orbit Or^+K of the set K is a set of points attainable from the set K along the trajectories of system (2.1) and negative orbit Or^-K is a set of points from which the set K can be attained.

By means of the method of oriented manifolds, a general controllability criterion for nonlinear systems is proved.

Theorem 2.1 [13] *System* (2.1) *is controllable iff there are no manifolds K with smooth boundary oriented with respect to this system such that $K \neq \emptyset, D$.*

As compared with known results in the control theory, Theorem 2.1 does not assume infinite differentiability (or analiticity) of the vector fields of a control system. The equations of oriented manifolds obtained in [13] are of independent interest. Their relationship with the Levi–Civita equations of invariant manifolds and Lyapunov equations for functions ensuring motion instability is established. This relationship was used in the investigation of the problem on sufficient conditions for stabilizability of nonlinear controlled systems and the synthesis of a feedback law with respect to all and a part of variables [22]. To formulate the main result of the paper, we designate the ε-neighborhood of the point $x = 0$ by $B(0, \varepsilon)$.

Theorem 2.2 [22] *Let $0 \in \operatorname{int} D$, $0 \in U$, $f(0, 0) = 0$, U be a compact and, for some $\varepsilon > 0$, each point of the set $B(0, \varepsilon) \setminus \{0\}$ is a point of local controllability of system (2.1). Then there exists a feedback control $u : B(0, \varepsilon) \to U, u(0) = 0$ (generally speaking, discontinuous) which ensures non-asymptotic stability of the solution $x = 0$ of the closed-loop system*

$$\dot{x} = f(x, u(x)). \tag{2.2}$$

Besides, the solutions of system (2.2) are defined in the sense of A.F. Filippov.

Examples are constructed which demonstrate that this result can not be refined (i.e. it is final). For a control affine system, it is proved that the set of discontinuity points of the feedback is contained in some set whose dimensions are smaller than the dimension of the state space.

In order to generalize controllability conditions for the case of manifolds with smooth boundary, properties of attainability domains of linear systems in the presence of joint restrictions on the control and the initial state were studied in [2, Ch. 1]. A formula for the gage function of attainability set was obtained which simplifies the further analysis and allows one to construct the external and internal estimates of the attainability set. In monographs [1, 2], problems on motion control for a rigid body and systems of bodies were considered with the application of estimates of attainability sets. New estimates of attainability sets of a system of differential equations modelling the rotational motion of a rigid body under the action of a control torque were proposed. A problem in restricted statement and a case of translational and rotational motion were studied. In particular, in [13] equations of rigid body motion with respect to a center of masses under the action of jet force were considered without taking into account mass changes

$$\begin{aligned} A_1\dot{\omega}_1 &= (A_2 - A_3)\omega_2\omega_3 + e_1 u, \\ A_2\dot{\omega}_2 &= (A_3 - A_1)\omega_1\omega_3 + e_2 u, \\ A_3\dot{\omega}_3 &= (A_1 - A_2)\omega_1\omega_2 + e_3 u, \end{aligned} \tag{2.3}$$

where A_1, A_2, A_3 are the principal central moments of inertia of the body; ω_1, ω_2, ω_3 are the projections of the angular velocity vector ω on the main central axes; $e = (e_1, e_2, e_3)$ is a unit vector of direction of the jet force moment; u is a control characterizing the magnitude of the jet moment. It is established that system (2.3) is uncontrollable under any of the conditions

$$A_1(A_2 - A_3)e_3^2 = A_3(A_1 - A_2)e_1^2, \tag{2.4}$$

$$A_2(A_3 - A_1)e_1^2 = A_1(A_2 - A_3)e_2^2, \tag{2.5}$$

$$A_3(A_1 - A_2)e_2^2 = A_2(A_3 - A_1)e_3^2. \tag{2.6}$$

In paper [13], it is shown that if the parameters of system (2.3) do not satisfy conditions (2.4)–(2.6) then system (2.3) is controllable according to Theorem 2.1. As compared with the previous papers, the application of the method of oriented manifolds enabled a unified description of controllability conditions for system (2.3) to be obtained in all cases of dynamically symmetric and asymmetric rigid body.

The evolution of geometric methods of nonlinear control theory led to the necessity of constructive description of the class of flat-systems, i.e. the systems which admit exact linearization by means of an endogenous feedback. The theory of flat-systems, appeared in the works by M. Fliess, J. Lévine, P. Rouchon, Ph. Martin, is being developed in the papers [2, 14, 16]. In these works, the method of invariant relations is applied for solving inverse control problems, observation, identification, convertibility, and functional controllability problems. General theorem on identifiability of nonlinear systems was proved. It states the identifiability of any system with respect to the measurements of its phase vector under a condition of its nonrepresentability by means of a smaller number of parameters. Conditions of observability and identifiability of nonlinear systems with respect to a part of variables are established [2, Ch. 5]. For general type systems, a functional controllability criterion is proposed, a property of invertibility is studied, a notion of inverse system is introduced, and an algorithm of its construction is presented

[14]. A generalized flat-algorithm proposed in [16] allows a considerable extension of the class of nonlinear control systems which admit explicit solution of motion planning problems. The concept of a generalized flat-system on the trajectory set is applied to study the problems of observation and identification of phase coordinates and parameters of motion equation of a rigid body in the force field. Observability conditions are used to substantiate the choice of output functions which are measured at probe navigation. In this direction, a class of problems on the determination of the mass center motion and rigid body orientation is solved [2, Ch. 6]. The results obtained in the field of identification of nonlinear systems are used to investigate problems of determining the moments of inertia and aerodynamical characteristics of a rigid body by the available information about motion.

A method of transforming the dynamical system with impulse control to the system with jumps realized on some surfaces in a phase space is proposed in [19], and new notions of impulses of high degrees and orders are introduced which are necessary for the investigation of systems nonlinear with respect to control. By employing impulse effects, a series of control and stabilization problems are solved and numerical methods are justified which can be used for an approximate construction of solutions to impulse systems. The results are applied for the problems on controlled stabilization of mechanical systems. In particular, a solution for the problem on stabilization of the Brockett integrator is obtained. An algorithm is proposed for constructing control system for nonholonomic models with independent quasivelocities as a control.

The notion of a control Lyapunov function with respect to a part of variables is introduced in [20]. These functions are employed in the proof of the theorem on partial stabilizability of nonlinear nonautonomous system. For control affine systems, an efficient method of constructing a stabilizing feedback is proposed. This result extends a theorem of Z. Artstein for the case of partial stabilization. The apparatus of control Lyapunov functions allowed one to solve a series of model problems on partial stabilization of a rigid body orientation. In particular, a model problem is considered for the motion of a satellite as an absolutely rigid body around its center of mass in the restricted statement under the action of jet control moments. Also, the case is studied when the control moments are implemented by means of a pair of flywheels [21]. In [23], control and stabilization algorithms are developed for motion of a satellite with elastic antennas and rods. The proposed control technique incorporates the mathematical model of a hybrid mechanical system in the form of differential Euler–Lagrange equations with infinite number of degrees of freedom. For a preassigned arbitrary number of elastic modes, an approximated finite-dimensional nonlinear system is constructed for which a stabilizing control with feedback is found. The above-mentioned control ensures asymptotic stability of the equilibrium state with respect to the combinations of elastic coordinates and body orientation. Besides, stability in the sense of Lyapunov is reached with respect to all phase coordinates. Observability of a model of hybrid system is proved with respect to the measurements of sensors of elastic element deformations. This allows one to substantiate the possibility of technical implementation of the proposed control laws.

A new approach is proposed in [24] for the investigation of stabilizability conditions for nonlinear controlled system by means of critical Hamiltonians. New stabilizability conditions are obtained for nonlinear affine control system defined by two homogeneous vector fields.

3 Rigid Body Dynamics and Motion Stability of Mechanical Systems

In [18] the author stated and solved a problem on the inclusion of given invariant manifold into the family of integral manifolds for a system of ordinary differential equations of the type

$$\dot{x} = \varphi(x), \quad x \in D \subset \mathbb{R}^n. \tag{3.1}$$

Assume that for every $x_0 \in D$ system (3.1) has a unique solution $x(t;x_0)$, $t \geq 0$ satisfying the initial condition $x(0;x_0) = x_0$. We shall introduce necessary definition.

Definition 3.1 A manifold $M \subset D$ is called invariant for system (3.1) if $x(t;x_0) \in M$ for all $t \geq 0$ and $x_0 \in M$. If $F_i(x)$, $i = 1,2,...,k$ are independent integrals of system (3.1), the set $N = \{x : F_i(x) = c_i,\ i = 1,2,...,k\}$ is called the integral manifold of system (3.1).

Now we shall formulate the main result on the inclusion of an invariant manifold into the family of integral manifolds.

Theorem 3.1 [18] *Any integral manifold M of dimension $n-k$ in a neighborhood of a nonsingular point of system (3.1) is contained in some k-parametric set of integral manifolds.*

It is proved that such an inclusion is locally possible only if the invariant manifold under consideration is not a $(n-1)$-dimensional manifold consisting of singular points.

By means of the Levi–Civita equations of integral manifolds assertions describing the structure of the including family are proved. The results obtained are applied in the investigation of motion equations of the Hess gyroscope in special coordinate axes [18]

$$\begin{aligned}
&\dot{x} = -b_1 z x,\\
&\dot{y} = (a-a_*)zx + b_1 yz - \nu_3\Gamma,\\
&\dot{z} = -(a-a_*)yx + b_1(x^2+y^2) + \nu_2\Gamma,\\
&\dot{\nu}_1 = a_* z\nu_2 - (a_* y + b_1 x)\nu_3,\\
&\dot{\nu}_2 = (ax + b_1 y)\nu_3 - a_* z\nu_1,\\
&\dot{\nu}_3 = (a_* y + b_1 x)\nu_1 - (ax + b_1 y)\nu_2,
\end{aligned} \tag{3.2}$$

where x, y, z are components of the kinetic moment vector in special coordinate axes; ν_1, ν_2, ν_3 are coordinates of the unit vector colinear to the direction of force field; a, a_*, b_1 are components of gyration tensor; the constant Γ characterizes intensity of the force field (action of gravity force). The following three integrals of the Euler-Poisson system of differential equations (3.2) are known

$$ax^2 + a_*(y^2+z^2) + 2b_1 yx - 2\nu_1\Gamma = 2h;$$

$$x\nu_1 + y\nu_2 + z\nu_3 = k;$$
$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1.$$

Besides, system (3.2) possesses the invariant Hess manifold $x = 0$.

Theorem 3.2 [18] *System of differential equations (3.2) possesses an additional integral of the form $I = xV$, where V is a solution of the differential equation $L_\varphi V = b_1 z V$. Partial cases of this integral are the Euler and Lagrange integrals and the Hess and Dokshevich solutions.*

In the above theorem L_φ means the operator of function differentiation along the trajectories of system (3.2). In the paper cited first approximation of the integral I is also obtained in the neighborhood of the uniform rotations curve belonging to the invariant Hess manifold.

In [17] the author developed the results by V.V. Kozlov and V.N. Koshlyakov on the application of the Rodrigues–Hamilton parameters in the motion investigation of a rigid body possessing a fixed point. By introducing in a special way a fixed system of coordinates a new form was obtained for motion equations of a rigid body which have a symmetric form and are quadratic in main variables. By means of these equations linear and nonlinear vibrations of a rigid body are studied in the Rodrigues–Hamilton parameters. To study stability of stationary motions of the Hamiltonian systems reducible to the two-dimensional ones a theorem generalizing the known Arnold–Moser result on stability of the equilibrium state of two-dimensional Hamiltonian system was proved. Application of this theorem to stability investigation of uniform rotations of a heavy rigid body with a fixed point allowed closing with this classical problem which has attracted the attention of investigators since the beginning of the 20-th century [15].

4 Stability, Control, and Stabilization of Infinite-Dimensional Systems

To study the motion of distributed parameter mechanical systems, the property of asymptotic stability with respect to a continuous functional is analyzed in [27] for generalized dynamical systems on a metric space. In particular, dynamical systems whose evolution is described by differential equations in some Banach space E are considered. Let X be a closed subset of E containing a sphere $B_R = \{x \in E \,|\, \|x\| \le R\}$, $R > 0$, and let $F : D(F) \to E$ be a nonlinear closed operator with dense in X domain of definition $D(F)$. For initial conditions $x_0 \in X$, we consider the abstract Cauchy problem

$$\frac{dx(t)}{dt} = Fx(t), \quad t \in \mathbb{R}_+ = [0, +\infty),\ x(0) = x_0. \tag{4.1}$$

We assume that the operator F is the infinitesimal generator of a continuous semigroup of nonlinear operators $\{S(t)\}_{t\ge0}$ in X, therefore the Cauchy problem (4.1) is well-posed and its mild solutions are written in the form $x(t) = S(t)x_0$.

Definition 4.1 Let $y : X \to \mathbb{R}_+$ be a continuous functional, $F(0) = 0$. The singular point $x = 0$ of differential equation (4.1) is called asymptotically stable with respect to y if

(i) for arbitrary given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $y(S(t)x_0) < \varepsilon$ for all $t \in \mathbb{R}_+$;

(ii) there exists $\Delta > 0$ such that $\|x_0\| < \Delta$ implies

$$\lim_{t\to\infty} y(S(t)x_0) = 0. \tag{4.2}$$

The above definition of partial stability is associated with the development of abstract approach to the definition of stability in two metrics. The absence of the condition of positive definiteness of the functional y enables one to consider Definition 4.1 as a generalization of the notion of asymptotic stability with respect to a part of variables in the sense of Lyapunov and Rumyantzev for the case of infinite-dimensional systems.

Let $V : E \to \mathbb{R}$ be a Fréchet differentiable functional. Then the time derivative of V along the trajectories of (4.1) can be written as

$$\dot{V}(x(t)) = (Fx(t), \nabla_{x(t)} V), \tag{4.3}$$

where $(\cdot, \cdot) : E \times E^* \to \mathbb{R}$ denotes the duality pairing of E and E^*, i.e. $(\xi, \nabla_x V)$ is the value of linear functional $\nabla_x V \in E^*$ at the point $\xi \in E$.

In order to formulate partial stability conditions, we use the class of Hahn functions $\mathcal{K}$ consisting of all continuous strictly increasing functions $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ possessing the property $\alpha(0) = 0$.

Theorem 4.1 [27] *Let F be the infinitesimal generator of a continuous semigroup $\{S(t)\}$ of nonlinear operators in X, $F(0) = 0$, and let $y : X \to \mathbb{R}_+$ be a continuous functional. We assume that there exists a Frechet differentiable functional $V : E \to \mathbb{R}$ satisfying the following conditions:*

1) For some functions $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}$, the inequality

$$\alpha_1(y(x)) \leq V(x) \leq \alpha_2(\|x\|), \forall x \in X.$$

is satisfied.

2) $\dot{V}(x) \leq 0$ for all $x \in D(F)$.

3)There exists a $\Delta > 0$ such that, for any $\|x_0\| < \Delta$, the corresponding set

$$\bigcup_{t \geq 0} \{S(t)x_0\}$$

is precompact in X.

4) The set $\operatorname{Ker} y = \{x \in X \mid y(x) = 0\}$ is invariant for (4.1), i.e. if $y(S(\tau)x_0) = 0$, $\tau \geq 0$ then $y(S(t)x_0) = 0$ for all $t \in \mathbb{R}_+$.

5) The set

$$M = \overline{\{x \in D(F) \mid \dot{V}(x) = 0\}} \setminus \operatorname{Ker} y$$

does not contain any semitrajectory of system (4.1) defined for $t \in \mathbb{R}_+$.

Then the singular point $x = 0$ of differential equation (4.1) is asymptotically stable with respect to y.

This theorem generalizes results by C. Risito and V.V. Rumyantzev for the case of partial stability of infinite-dimensional system. Theorem 4.1 is used for the synthesis of control functionals for mathematical models of hybrid mechanical systems. Such mechanical systems consisting of rigid and elastic bodies are widely applied in space industry and robot technology. In[26, 28], the author considered models of rotational motion of a satellite with an arbitrary number of elastic elements, i.e. antennas in the form of the Euler–Bernoulli beams. If all the beams have the same mechanical parameters, the system under investigation is not asymptotically stable and, under these conditions, the stabilization problem with respect to the norm of some projection operator onto an infinite-dimensional subspace of the state space was solved in [26]. In the case of beams with nonresonant parameters, the approximate controllability was proved and a control functional was proposed which ensures strong asymptotic stability of the equilibrium state [28]. From the mechanical point of view, such a control implements the stabilization of the body-carrier orientation with simultaneous damping of beams vibrations. In

[25], equations of the spatial motion of an elastic robot-manipulator were studied with allowance for the telescopic displacement of its links under the effect of control forces. The Euler-Bernoulli and Timoshenko beams with mixed boundary conditions were considered as models of link deformations. A scheme of stabilization with the help of an observer in the feedback chain was proposed for the model equilibrium state. It is proved that this approach ensures asymptotic stability of the unperturbed solution of the system for an arbitrary number of generalized coordinates corresponding to the elastic beam vibrations.

5 The Method of Matrix-Valued Lyapunov Functions and the Analysis of Dynamic Properties of Nonlinear Systems

Stability analysis of zero solution of nonlinear system in the normal form

$$\frac{dx}{dt} = X(t,x), \quad x(t_0) = x_0, \tag{5.1}$$

where $x \in \mathbb{R}^n,\ X \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n),\ X(t,0) = 0$ for all $t \ge t_0$, is a challenging task if the dimension of the vector x is large enough. One of the approaches to solution of this problem is the decomposition of system (5.1) to the form

$$\frac{dx_i}{dt} = f_i(t,x_i) + g_i(t,x_1,...,x_m), \quad i = 1,2,...,m, \tag{5.2}$$

where $x_i \in \mathbb{R}^{n_i},\ f_i : \mathbb{R}_+ \times \mathbb{R}^{n_i} \to \mathbb{R}^{n_i},\ g_i : \mathbb{R}_+ \times \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_m} \to \mathbb{R}^{n_i},\ \sum\limits_{i=1}^{m} n_i = n.$

The monographs [3, 4] and Chapter 5 of the monograph [5] presented the results of development of the direct Lyapunov method in terms of auxiliary matrix-valued function

$$V(t,x) = [v_{ij}(t,x)], \quad i,j = 1,2,...,m, \tag{5.3}$$

which is considered to be a suitable medium for construction of both scalar and vector Lyapunov functions solving the problem on stability of the state $x_i = 0$ of system (5.2).

It is proposed to take the elements $v_{ij}(t,x)$ of matrix-valued function (5.3) such that to satisfy the estimates

$$\underline{\gamma}_{ij}\psi_{ij}(\|x_i\|)\psi_{ji}(\|x_j\|) \le v_{ij}(t,x) \le \overline{\gamma}_{ij}\psi_{ij}(\|x_i\|)\psi_{ji}(\|x_j\|),$$

where $\underline{\gamma}_{ii},\ \underline{\gamma}_{ij} > 0,\ \overline{\gamma}_{ij},\ \overline{\gamma}_{ij}$ are constants for $i \ne j,\ (\psi_{ij}, \psi_{ji}) \in K(KR)$-Hahn class for all $i,j = 1,2,...,m$. If conditions (5.4) are satisfied, then for the function

$$V(t,x,y) = y^T U(t,x) y, \quad y \in \mathbb{R}^m_+, \tag{5.5}$$

the bilateral estimate

$$\psi_1^T(\|x\|)Y^T\underline{G}Y\psi_1(\|x\|) \le V(t,x,y) \le \psi_2^T(\|x\|)Y^T\overline{G}Y\psi_2(\|x\|), \tag{5.6}$$

is valid, where

$\psi_1(\|x\|) = (\psi_{11}(\|x_1\|),\dots,\psi_{1m}(\|x_m\|))^T,\ \psi_2(\|x\|) = (\psi_{21}(\|x_1\|),\dots,\psi_{2m}(\|x_m\|))^T,$
$Y = \mathrm{diag}(y_1,\dots,y_m),\ \underline{G} = [\underline{\gamma}_{ij}],\ \overline{G} = [\gamma_{ij}],\ i,j = 1,2,\dots,m.$

For function (5.5) the total derivative

$$D^+V(t,x,y) = y^T D^+U(t,x)y, \tag{5.7}$$

is considered, where $D^+U(t,x) = [D^+v_{ij}(t,x)]$, $i,j = 1,2,...,m$, and $D^+v_{ij}(t,x) = \limsup\{[v_{ij}(t+\theta, x+\theta(f_i(t,x_i)+g_i(t,x_1,...,x_m))]\theta^{-1} : \theta \to 0^+\}$.

For certain restrictions on function (5.5) and its total derivative (5.7) by virtue of system (5.2) sufficient conditions are established for various types of stability of zero solution to system (5.2)((5.1) respectively).

Theorem 5.1 *Let the vector-function X in system (5.1) be continuous on $\mathbb{R} \times N$ ($N \subset \mathbb{R}^n$) and admit decomposition of system (5.1) to the form (5.2).*

If for function (5.5) estimates (5.6) are valid and

$$D^+V(t,x,y) \le \psi_3^T(\|x\|)A_3(y)\psi(\|x\|), \tag{5.8}$$

for all $(t,x) \in \mathbb{R}_+ \times N$, where $A_3(y)$ is an $m \times n$-constant matrix then:

(1) the state $x = 0$ of system (5.1) is stable if the matrices $A_1 = Y^T\underline{G}Y$ u $A_2 = Y^T\overline{G}Y$ are positive definite and the matrix $A_3(y)$ is negative definite;

(2) the state $x = 0$ of system (5.1) is uniformly stable if the matrices A_1, A_2 are positive and the matrix A_3 is negative semidefinite.

Similarly to Theorem 5.1 the results on asymptotic stability, exponential stability and instability of the state $x = 0$ of system (5.1) are formulated and proved.

For polystability analysis of the state $x = 0$ of system (5.2) it is proposed to apply the vector function

$$L(t,x,b) = AU(t,x)b, \tag{5.9}$$

where A is a constant $m \times m$-matrix, $b \in \mathbb{R}^m_+$, $U \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^{m\times m})$, and its total derivative

$$D^+L(t,x,b) = AD^+U(t,x)b \tag{5.10}$$

by virtue of system (5.2). A detailed polystability analysis for system (5.2) was carried out in the cases $m = 2,3,4$, and sufficient conditions were established for various types of polystability of the state $x = 0$ of system (5.2).

Solution of the problem of constructing a suitable matrix-valued function (5.3) is considered in the following cases:

Case 1. The elements $v_i(t,x_i)$, $i = 1,2,...,m$ are put in correspondence with the independent subsystems

$$\frac{dx_i}{dt} = f_i(t,x_i), \quad i = 1,2,...,m, \tag{5.11}$$

of system (5.2) and the elements $v_{ij}(t,x_i,x_j)$, $i \ne j$, $i,j = 1,2,...,m$ are put in correspondence with the (i,j)-pairs of the independent subsystems

$$\frac{dx_i}{dt} = q_i(t,x_i,x_j),$$
$$\frac{dx_j}{dt} = q_j(t,x_i,x_j), \quad (i \ne j) \in [1,m],$$

where $x_i \in \mathbb{R}^{n_i}$, $x_j \in \mathbb{R}^{n_j}$, $q_i \in (\mathbb{R}_+ \times \mathbb{R}^{n_i} \times \mathbb{R}^{n_j}, \mathbb{R}^{n_j})$.

Case 2. Subsystems (5.11) are decomposed into M_i second level subsystems

$$\frac{dx_{ij}}{dt} = f_{ij}(t, x_{ij}) + h_{ij}(t, x_i), \quad j = 1, 2, ..., m_i \tag{5.12}$$

where $x_{ij} \in \mathbb{R}^{n_{ij}}$, $f_{ij} \in C(\mathbb{R} \times \mathbb{R}^{n_{ij}}, \mathbb{R}^{n_{ij}})$, $h_{ij} \in C(\mathbb{R} \times \mathbb{R}^{n_i}, \mathbb{R}^{n_{ij}})$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n_i$. The elements $v_{ii}(t, x_i)$ are put in correspondence with the free subsystems of the second level of decomposition

$$\frac{dx_{ij}}{dt} = f_{ij}(t, x_{ij}), \quad j = 1, 2, ..., m, \tag{5.13}$$

and the elements $v_{ij}(t, x)$, $(i \neq j) \in [1, m]$, are constructed with allowance for the interconnection functions $h_{ij}(t, x_i)$ in system (5.12).

Case 3. For the class of systems of (5.2) type [30]

$$\frac{dx_i}{dt} = f_i(x_i) + g_i(t, x_1, ..., x_m),$$

$f_i \in \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $f_i(0) = 0$, $i = 1, 2, ..., m$, the elements $v_{ii}(x_i)$ are put in correspondence with the independent subsystems

$$\frac{dx_i}{dt} = f_i(x_i), \quad i = 1, 2, ..., m, \tag{5.14}$$

and the elements $v_{ij}(t, x_i, x_j)$ are found by the equations

$$\begin{gathered} D_t v_{ij}(t, x_i, x_j) + (D_{x_i} v_{ij}(t, x_i, x_j))^T f_i(x_i) + (D_{x_j} v_{ij}(t, x_i, x_j))^T f_j(x_j) + \\ + \frac{y_i}{2y_j}(D_{x_i} v_{ii}(x_i))^T g_{ij}(t, x_i, x_j) + \frac{y_j}{2y_i}(D_{x_j} v_{jj}(x_j))^T g_{ji}(t, x_i, x_j) = 0, \quad (i \neq j) \in [1, m], \end{gathered}$$

where $g_{ij}(t, x_i, x_j) = g_i(t, 0, ..., x_i, ..., x_j, ..., 0)$, $i \neq j$, $i, j = 1, 2, ..., m$.

In all the above cases new conditions are established for various types of stability of the state $x = 0$ of system (5.1), without assuming on exponential stability of the state $x = 0$ of subsystems (5.11), (5.13) or (5.14). As is known this condition is necessary for the application of the vector Lyapunov function and appropriate comparison system.

Also, the method of matrix-valued Lyapunov functions was developed for:

— time discrete systems in terms of semidefinite positive functions (5.3), whose elements are linear forms, and hierarchical matrix Lyapunov functions;

— large-scale impulse systems of the form

$$\begin{aligned} \frac{dx_j}{dt} &= f_j(t, x_j) + f_j^*(t, x), \quad t \neq \tau_k(x), \quad j = 1, 2, ..., m, \\ \Delta x_j &= I_{kj}(x_j) + I_{kj}^*(x), \quad t = \tau_k(x), \quad k = 1, 2, ... \end{aligned}$$

in terms of auxiliary functions satisfying conditions (5.4), and also in terms of hierarchical matrix Lyapunov functions whose method of construction is indicated;

— systems with random parameters in the Ito form and Katz–Krasovsky form in terms of stochastic matrix-valued function;

— singularly perturbed systems of the form

$$\begin{aligned} \frac{dx}{dt} &= f(t, x, y, \mu), \\ \mu\frac{dy}{dt} &= g(t, x, y, \mu), \end{aligned} \tag{5.15}$$

where $(x^T, y^T)^T$ is a state vector of system (5.15), $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times M, \mathbb{R}^n)$, $g \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \times M, \mathbb{R}^m)$, $\mu \in [0,1] = M$, in terms of the matrix-valued function

$$V(t,x,y,\mu) = \begin{pmatrix} v_{11}(t,x) & v_{12}(t,x,y,\mu) \\ v_{21}(t,x,y,\mu) & v_{22}(t,y,\mu) \end{pmatrix}.$$

Stability conditions of the state $x = y = 0$ of system (5.15), and large-scale system of Lurie–Postnikov type are obtained in terms of sign-definiteness of special matrices. Moreover, the upper bound μ^* of the values of parameter μ is calculated for which an appropriate type of stability of slow variables and boundary layer takes place.

The developed technique is illustrated by numerous examples and applications to the problems of mechanics, electric power industry, population biology, etc.

6 Generalization of the Direct Lyapunov Method and Comparison Method for Non-classical Stability Theories

The classical stability theory developed by A.M. Lyapunov is based on three fundamental concepts:

(1) deviations of perturbed motion from the nominal one should be infinitely small;

(2) in the course of motion perturbing forces are absent;

(3) motion is considered on unbounded interval.

We refer all other stability theories which are based on other concepts to the non-classical ones. One of such theories is the theory of practical stability based on the following concepts:

(1) initial and further deviations of perturbing motion from the nominal one are final;

(2) system motion is performed under persistent perturbations;

(3) interval of system functioning is unbounded.

In the monograph [6] general theory of practical stability of motion is presented with the applications in mechanics. The system of perturbed motion equations

$$\frac{dx}{dt} = X(t,x) + R(t,x), \tag{6.1}$$

is considered, where $x \in \mathbb{R}^n$; $X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$; $R : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and it is not assumed that $R(t,0) \neq 0$, i.e. $x = 0$ is not a solution of system (6.1), but it is a solution of the system

$$\frac{dx}{dt} = X(t,x). \tag{6.2}$$

For given estimates of the domains ($S_0(t)$, $S(t)$, $\Pi(t)$, $\mathbb{R}_+$) unperturbed motion of system (6.2) is practically stable under persistent perturbations if for $t_0 \in \mathbb{R}_+$ and any

$$x(t_0) \in S_0(t_0), \quad R(t,x) \in \Pi(t),$$

the solution $x(t,t_0,x_0)$ of system (6.1) remains inside the domain $S(t)$, i.e. $x(t) \in \mathrm{int}S(t)$ for all $t \geq t_0$.

Practical stability of unperturbed motion of system (6.2) is determined as a motion property opposite to practical stability.

To solve the problem on practical stability of systems of (6.1), (6,2) type or their partial form three approaches were developed in the monograph [6]:

Approach 1 is based on the representation of general solution to system (6.2) as series of special form.

Approach 2 is based on the application of the direct Lyapunov method and locally large auxiliary function.

Approach 3 is based on the reduction of system (6.1) or (6.2) to the other one called a comparison system with further analysis of its solutions. Here both scalar and vector Lyapunov functions are applied as a nonlinear transformation of the initial system.

In the framework of Approach 1 practical stability conditions are established for system (6.2) with uniformly bounded and uniformly analytic right-hand side and for a system with integrable approximation of the form

$$\frac{dx}{dt} = A(t)x + g(t,x), \tag{6.3}$$

where $x \in \mathbb{R}^n$; $A(t)$ is an $n \times n$-continuous and bounded matrix, $g(t,x)$ satisfies the estimate

$$\|g(t,x)\| \le b(t)\|x\|^{\alpha}, \quad \alpha > 0, \quad \text{for all} \quad (t,x) \in \mathbb{R}_+ \times S(t).$$

These conditions are based on representation of solution to system (6.2) by series of the form

$$x(t) = x_0 + \sum_{m=1}^{\infty} C_m(x_0)\psi^m,$$

where $\psi = \{\exp[\lambda(t-t_0)] - 1\}\{\exp[\lambda(t-t_0)] + 1\}^{-1}$, λ is a positive number, with further application of the Schur theorem on convergence of series (6.4). Practical stability conditions for the state $x = 0$ of system (6.3) are based on the estimates associated with nonlinear integral inequality.

The results obtained are employed for the analysis of dynamics of large scale systems with integrable approximation.

In the framework of Approach 2 the direct Lyapunov method is applied with necessary modifications. For locally large function $V(t,x)$ the quantitative estimates

$$V_M^{\widehat{S}}(t) = \sup(V(t,x) \quad \text{for} \quad x \in \partial S(t)),$$
$$V_m^{\widehat{S}_0}(t) = \inf(V(t,x) \quad \text{for} \quad x \in \partial S_0(t)),$$

are introduced, where $S_0(t) \subset S(t)$ and $\partial S_1 \cap \partial S_0 = \emptyset$ for all $t \in \mathbb{R}_+$.

Theorem 6.1 *Assume that*

(1) $V(t,x) \in C(\mathbb{R}_+ \times S(t), \mathbb{R}_+)$*,* $V(t,x)$ *is locally large and locally Lipschitz in* x*;*

(2) $D^+V(t,x) < D^+\eta(t)$ *for* $(t,x) \in \mathbb{R}_+ \times S(t)$*, where* $\eta \in C(\mathbb{R}_+, (0,\infty))$ *and* $\eta(t)$ *is nondecreasing in* $t \in \mathbb{R}_+$*;*

(3) for some $t_0 \in \mathbb{R}_+$ *the estimate* $\eta(t_0) \le V_M^{S_0}(t_0)$ *is valid and* $\eta(t) \le V_m^{\partial S}(t)$ *for all* $t \ge t_0$*.*

Then the unperturbed motion of system (6.2) is practically stable with respect to the domains $(S_0(t), S(t))$*.*

Similar theorems are proved for various types of practical stability and instability of solutions for systems (6.1) and (6.2) with respect to different domains $S_0(t)$, $S(t)$.

In the realization of Approach 3 scalar (vector) comparison equations are incorporated which satisfy quasimonotonicity condition. Practical stability conditions are expressed in the form of quantitative restrictions on variation of solutions to comparison equation.

Moreover, alongside systems (6.1) and (6.2) the systems with first integrals are considered. General concept of practical stability is formulated in terms of extended system (6.1) and stability with respect to a part of variables.

With regard to practical stability the problems of stabilization of controlled systems are solved for some classes of linear and nonlinear systems on the basis of the principle of comparison with mixed monotonicity of comparison system. For the axiomatically determined system of processes conditions of practical stability are established with respect to two vector measures whose components may take negative values.

In [31, 32] practical stability of some classes of hybrid systems consisting of time-continuous and discrete components is studied. In [32] a nonlinear system of differential equations of perturbed motion with impulsive effect

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), \quad t \neq \tau_k, \\ \Delta x &= I_k(x), \quad t = \tau_k, \end{aligned} \tag{6.4}$$

is considered, where $x \in \mathbb{R}^n$, $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) = 0$ for all $t \in \mathbb{R}_+$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $I_k(0) = 0$, $k = 1, 2, \ldots$, $0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots \to +\infty$ for $k \to \infty$. It is assumed that the solution $x(t) = x(t; t_0, x_0)$ of the Cauchy problem (1) exists and is unique and the length of the maximal interval $[t_0, t_0 + J(t_0, x_0))$ of existence of solution to the Cauchy problem for system of equations (6.4) when the impulse effect is absent satisfies the inequality $J(t_0, x_0) > \theta_2$ for all $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = [0, +\infty)$.

In the space $\mathbb{R}^n$ let the sets $S_0 = \{x| \quad x \in \mathbb{R}^n, \quad \|x\| < \lambda\}$; $S = \{x| \quad x \in \mathbb{R}^n, \quad \|x\| < A\}$ be defined for given constants A, $\lambda > 0$, $\lambda < A$.

Let $G \subset \mathbb{R}_+ \times \mathbb{R}^n$ and for any $t \in \mathbb{R}_+$ we define the set $G(t) = \{x \in \mathbb{R}^n \quad | (t,x) \in G\}$ and the set $\mathcal{G} = \bigcup\limits_{i=1}^{\infty} G(\tau_i)$.

Practical stability is studied by means of the Lyapunov function for which the following assumptions are made:

a) function $v(t,x)$ is continuous and differentiable in $(t,x) \in [t_0, \infty) \times S$;

b) function $v(t,x)$ is locally large in the domain of values $(t,x) \in [t_0, \infty) \times S$, i.e. there exists a positive constant N such that for any c, $0 < c < N$, $t_0 \in \mathbb{R}_+$ there exists a positive number $\delta(t_0, c)$ such that outside the sphere $K_\delta = \{x : \|x\| \le \delta\}$ the inequality $v(t,x) > c$ is satisfied for all $t \in [t_0, \infty)$;

c) total derivative $\frac{dv}{dt}\Big|_{(6.4)}$ of function $v(t,x)$ along solutions of system (6.4)

$$\frac{dv}{dt}\Big|_{(6.4)} = \frac{\partial v}{\partial t} + \Big(\frac{\partial v}{\partial x}\Big)^T f(t,x)$$

vanishes together with the function $v(t,x)$ for $x = 0$;

d) function $v(t,x)$ is positive definite in the domain $\mathbb{R}_+ \times S$ in the sense of Lyapunov;

e) $a(\|x\|) \le v(t,x) \le b(t, \|x\|)$, for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, где $a(.)$ is a function of Khan class $\mathcal{K}$, $b(t,.)$ is a function continuous and nondecreasing in the second argument.

Theorem 6.2 *Let system of equations (6.4) be such that:*

1) there exists a function $v(t,x)$ for which conditions (a)–(e) are satisfied;

2) there exist an invariant set G^+ and functions $\varphi_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$, $p_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $\psi_1(.)$ is a nondecreasing function such that the *estimates below are satisfied*

$$\frac{dv}{dt}\Big|_{(6.4)} \le p_1(t)\varphi_1(v) \quad \text{for all} \quad (t,x) \in \overline{G}^+,$$
$$v(\tau_k, x + I_k(x)) \le \psi_1(v(t,x)) \quad \text{for all} \quad x \in \overline{\mathcal{G}}^+;$$

3) there exist functions $\varphi_2 \in C(\mathbb{R}_+,\mathbb{R}_+)$,$\psi_2 \in C(\mathbb{R}_+,\mathbb{R}_+)$, $p_2 \in C(\mathbb{R}_+,\mathbb{R}_+)$ *such that the estimates below are satisfied*

$$\frac{dv}{dt}\Big|_{(6.4)} \le -p_2(t)\varphi_2(v) \quad \text{for all} \quad (t,x) \in \overline{G}^-,$$
$$v(\tau_k, x + I(x)) \le \psi_2(v(t,x)) \quad \text{for all} \quad x \in \overline{\mathcal{G}}^-,$$

where $G^- = \operatorname{ext} G^+$*;*
4) constants $\lambda, A > 0, \lambda < A < A^0$ *satisfy the estimates:*
a) for all $\eta \in [0, b(t_0,\lambda))$, $k = 0,1,2,\dots,\tau_0 = t_0$,

$$\int\limits_{\eta}^{\psi_2(\eta)} \frac{ds}{\varphi_2(s)} \le \int\limits_{\tau_k}^{\tau_{k+1}} p_1(t)d\,t,$$

b) $\int\limits_{b(t_0,\lambda)}^{a(A)} \frac{ds}{\varphi_1(s)} \ge \int\limits_{\tau_k}^{\tau_{k+1}} p_2(t)d\,t$*;*
c) $\psi_1(a(A)) < b(t_0,\lambda)$ *.*
Then system (6.4) is $(S_0, S, [t_0,\infty))$*-stable.*

Theorem 6.2 generalizes the results of the paper [30] where conditions of Lyapunov stability were established in terms of two auxiliary functions. Conditions of Lyapunov stability for linear differential perturbed motion equations with impulse effect obtained in [34] and motion stability conditions for nonlinear system of perturbed motion equations of (6.4) type obtained in [30] enable one to investigate stability of the system in the case when continuous and discrete components of the system are not stable.

In [31] a hybrid system of the form

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g(t,x) + B_k u(k), \quad t \in [\tau_k, \tau_{k+1}), \\ u(k+1) &= C_k u(k) + D_k x(\tau_k), \end{aligned} \tag{6.5}$$

is considered, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in C([0,\infty), \mathbb{R}^{n\times n})$, $B_k \in \mathbb{R}^{n\times m}, C_k \in \mathbb{R}^{m\times m}, D_k \in \mathbb{R}^{m\times n}$, $g \in C([0,\infty)\times\mathbb{R}^n;\mathbb{R}^n)$. Here $\{\tau_k\}_{k=1}^{\infty}$ is a sequence of switching moments possessing a unique limiting point at infinity.

By means of the methods of the theory of integral inequalities practical stability conditions with respect to a part of variables and with respect to all variables of system (6.5) are established in terms of estimates of the Cauchy matrix of linear approximation of system (6.5).

In the monograph [5] stability conditions are obtained for systems with small parameters of the following types: systems standard by Bogoliubov, systems with slow and quick variables, systems with small perturbing forces. These conditions are based on the ideas of the direct Lyapunov method, the averaging technique and the method of comparison for auxiliary scalar functions.

7 Boundary-Value Problems in the Nonlinear Oscillations Theory

This part of the paper involves the theory of nonlinear oscillations which is one of the most important branches of nonlinear mechanics. For the first time in the field the most complete theory of the (Fredholm) boundary-value problems was constructed for the systems of differential equations with impulse effect in which the number of boundary conditions does not coincide with the number of unknowns. Most complicated and not well-studied resonance boundary-value problems, both underdetermined and overdetermined ones, are considered:

$$\dot{z} = A(t)z + f(t) + \varepsilon Z(z, t, \varepsilon), \quad t \neq \tau_i, \quad t, \tau_i \in [a, b],$$
$$\Delta z\Big|_{t=\tau_i} - S_i z(\tau_i - 0) = a_i + \varepsilon J_i(z(\tau_i - 0, \varepsilon), \varepsilon), \ i = 1, ..., p, \tag{7.1}$$
$$lz = \alpha + \varepsilon J(z(\cdot, \varepsilon), \varepsilon).$$

Here $A(t)$ и $f(t) \in C([a, b] \setminus \{\tau_i\}_I)$ are $n \times n$-dimensional matrix functions and n-dimensional vector functions respectively; $Z(z, t, \varepsilon)$ is a nonlinear n-dimensional vector function continuously differentiable with respect to the first argument in the neighbourhood of solutions to generating boundary-value problem

$$\dot{z} = A(t)z + f(t) \quad t \neq \tau_i, \quad t, \tau_i \in [a, b], \tag{7.2}$$
$$\Delta z\Big|_{t=\tau_i} - S_i z(\tau_i - 0) = a_i, \ i = 1, ..., p, \quad lz = \alpha,$$

$Z(z, t, \varepsilon)$ is continuous or piece-wise continuous in the second argument with first kind discontinuities for $t = \tau_i$ and continuous in $\varepsilon \in [0, \varepsilon_0]$; $\Delta z\Big|_{t=\tau_i} = z(\tau_i + 0) - z(\tau_i - 0)$, $S_i are (n \times n)$- constant matrices: $\det(E + S_i) \neq 0$, $a_i \in \mathbb{R}^n$; l is a linear continuous m-dimensional vector functional; $J(z(\cdot, \varepsilon), \varepsilon)$, $J_i(z(\tau_i - 0, \varepsilon), \varepsilon)$ are m-dimensional nonlinear vector functionals continuously differentiable (by Frechet) in z in the neighbourhood of solution of generating boundary-value problem (7.2)continuous in $\varepsilon \in [0, \varepsilon_0]$.

For the first time a problem was solved on establishing the existence (branching) conditions for solutions $z = z(t, \varepsilon) \ : \ z(\cdot, \varepsilon) \in C^1([a, b] \setminus \{\tau_i\}_I)$, $z(t, \cdot) \in C[0, \varepsilon_0]$ of the problems which, for $\varepsilon = 0$, become one of the solutions $z_0(t, c_r) \ : \ z(t, 0) = z_0(t, c_r)$, $c_r \in \mathbb{R}^r$ of generating boundary-value problem (7.2) and algorithms for their obtaining are proposed.

Theorem 7.1 (on branching of solutions) *Let boundary-value problem (7.1) be such that the critical (resonance) case (* $\operatorname{rank}[Q := lX(\cdot)] < m$*), takes place and generating problem (7.2) has r-parametric family of linearly independent solutions* $z_0(t, c_r)$, $(r = n - \operatorname{rank} Q)$. *Then for every value of the vector* $c_r = c_r^0 \in \mathbb{R}^r$, *which is a simple real root of the equation*

$$P_{Q^*}\Big\{J(z_0(\cdot, c_r^0), 0) - l\int_a^b K(\cdot, \tau)Z(z_0(\tau, c_r^0), \tau, 0)d\tau - l\sum_{i=1}^p \bar{K}(\cdot, \tau_i)J_i(z_0(\tau_i - 0, c_r^0), 0)\Big\} = 0, \tag{7.3}$$

boundary-value problem (7.1) has at least one solution $x(t, \varepsilon) : \ x(\cdot, \varepsilon) \in C^1([a, b] \backslash \{\tau_i\}_I)$, $x(t, \cdot) \in C[0, \varepsilon]$ *which becomes generating with the vector constant* $c_r^0 \ : x(t, 0) = z_0(t, c_r^0)$. *This solution can be found with the help of the iteration process convergent on* $[0, \varepsilon_*]$.

Here $X(t)$ is a normal fundamental matrix of homogeneous differential system (7.2), $K(t,\tau)$ is a Cauchy matrix, P_{Q^*} is an orthoprojector on the co-kernel of matrix Q.

In the case of periodic boundary-value problem (7.1) without impulses [7, 35] Theorem 7.1 yields the classical result of A. Lyapunov and I. Malkin. If equation (7.3) has a physical meaning then the constants c_r^0 are the amplitudes of generating solutions and, therefore, in the periodical case this equation is called the equation for generating amplitudes. In the case when generating boundary-value problem (7.2) has no solutions bifurcation conditions were established for solutions to linearly perturbed (Fredholm) boundary-value problem

$$\dot{z} = A(t)z + \varepsilon A_1(t)z + f(t), \quad t \neq \tau_i, \tag{7.4}$$

$$\Delta z\Big|_{t=\tau_i} - S_i z(\tau_i - 0) = a_i + \varepsilon A_{1i} z(\tau_i - 0), \quad lz = \alpha + \varepsilon l_1 z.$$

Theorem 7.2 (on bifurcation of solutions) *Let boundary-value problem (7.2) generating for (7.4) have no solutions for arbitrary functions $f(t) \in C([a,b]\setminus\{\tau_i\}_I)$, $a_i \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^m$. Then under the condition*

$$\operatorname{rank}\Big[B_0 := P_{Q^*}\Big[l_1 X_r(\cdot) - l\int_a^b K(\cdot,\tau)A_1(\tau)X_r(\tau)d\tau -$$

$$-l\sum_{i=1}^{p} \bar{K}(\cdot,\tau_i)A_{1i}X_r(\tau_i - 0)\Big] = m - \operatorname{rank} Q, \tag{7.5}$$

for arbitrary nonhomogeneities $f(t) \in C([a,b]\setminus\{\tau_i\}_I)$, $a_i \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^m$ boundary-value problem (7.4) has a parametric family $\rho = m - n$ of linear independent solutions in the form of a part of the Laurent series

$$z(t,\varepsilon) = \sum_{i=k}^{\infty} \varepsilon^i z_i(t) + P_\rho c_\rho, \ \forall c_\rho \in \mathbb{R}^\rho, \ k = -1, \tag{7.6}$$

which converges for fixed sufficiently small $\varepsilon \in (0, \varepsilon_*]$.

Similar results were obtained in the investigation of boundary-value problems for systems of ordinary differential equations with delaying argument [8, pp. 170–194], [38] and for difference systems [8, pp. 93–96], [36], as well as for systems with boundary conditions at infinity [8, pp. 257–304], [37] when the appropriate homogeneous differential system is exponentially dichotomous on semi-axes. These results complete and generalize essentially the known results of R.J. Sacker and K.J. Palmer.

8 Methods of Matrix Equations and Cone Comparisons in the Stability Theory

8.1 Analogues of Matrix Lyapunov Equation and Their Application ([9], [39])

The method of Lyapunov functions for linear differential and difference systems is formulated in terms of positive definite solutions to the matrix equations

$$-AX - XA^* = Y, \quad X - AXA^* = Y.$$

The known Lyapunov theorem provides criteria for placing a spectrum of such systems inside the left-hand half-plane and a unit disk. Matrix algebraic and differential Lyapunov equations are widely applied in the theory of qualitative systems and control theory.

The monograph [9] deals with the methods of constructing, investigating and applying in motion stability theory the analogues of the matrix Lyapunov equations and their generalizations of the form

$$\sum_{i,j} \gamma_{ij} A_i X A_j^* = Y, \tag{8.1}$$

where A_i is a set of matrices, in particular, $A_i = f_i(A)$ are the analytic functions of the matrix A.

The monograph presents criteria of localization and distribution of the matrix spectrum with respect to the sets

$$\Lambda_f^+ = \left\{\lambda : f(\lambda, \bar\lambda) > 0\right\}, \quad \Lambda_f^- = \left\{\lambda : f(\lambda, \bar\lambda) < 0\right\}, \quad \Lambda_f^0 = \left\{\lambda : f(\lambda, \bar\lambda) = 0\right\}.$$

The Lyapunov theorem and the inertia theorem of Ostrovsky–Schneider are generalized for the maximal possible classes of analytical Hermitian functions $f \in \mathcal{H}_0^m$ and $f \in \mathcal{H}_2^m$ determined by the corresponding conditions

$$\|1/f(\mu_i, \overline{\mu}_j)\|_{i,j=1}^m \geq 0, \ \forall\, \mu_1, \dots, \mu_m \in \Lambda_f^+; \quad i_\pm\left(\|f(\mu_i, \bar\mu_j)\|_1^m\right) \leq 1, \ \forall\, \mu_1, \dots, \mu_m \notin \Lambda_f^0;$$

where $i_\pm(\cdot)$ are the inertia indices of the Hermitian matrix that equal to the number of its positive and negative eigenvalues. If $f(\lambda, \bar\lambda) = \sum_{ij} \gamma_{ij} f_i(\lambda) \overline{f_j(\lambda)}$ then $f \in \mathcal{H}_0^m$ and $f \in \mathcal{H}_2^m$ under the corresponding restrictions $i_+(\Gamma) = 1$ and $i_\pm(\Gamma) \leq 1$.

Theorem 8.1 [9] *Let the matrix $A \in C^{n\times n}$, the function $f \in \mathcal{H}_0^m$ and the arbitrary positive definite matrix $Y = Y^* > 0$ be given. Then the spectrum $\sigma(A)$ is located in the domain Λ_f^+ if and only if there exists a unique positive definite solution $X = X^* > 0$ of the matrix equation*

$$L_f X \triangleq -\frac{1}{4\pi^2} \oint_{\omega_1} \oint_{\omega_2} f(\lambda, \bar\mu)(A - \lambda I)^{-1} X (A - \mu I)^{-1^*} d\lambda\, d\bar\mu = Y, \tag{8.2}$$

where ω_1 (ω_2) is a closed contour embracing and not intersecting $\sigma(A)$ ($\sigma(A)$).

All known results in the direction are the partial cases of Theorems 1–3 set out in the monograph. Also, correlations of the type of linear system controllability conditions are constructed which extend essentially the possibilities of the method of generalized Lyapunov equation in spectrum localization problems. Equation (8.2) which can be represented in the form of (8.1) was used for the first time in the problems of linear system optimization with respect to output [39].

$$\dot x = Ax + Bu,\ y = Cx,\ u = -Ky,\ J(u) = \int_\Delta \rho(x_0) \int_0^\infty (x^* Q x + u^* R u)\, dt\, dx_0 \to \min_u. \tag{8.3}$$

In terms of generalized Lyapunov theorem and matrix Atans–Levine system a relationship of the quadratic quality functional and the domain of desirable location of closed loop system spectrum is established. Optimization algorithms are constructed controlling the system spectrum location in complex domain.

A general technique of constructing the analogues of Lyapunov equation is developed for polynomial and analytic matrix functions. Operators of such equations are presented in the form of the Cauchy type integrals of logarithmic derivative and also by means of special algebraic systems of spectrum splitting and the so-called right-hand and left-hand pairs of matrix functions. We shall formulate an analogue of Lyapunov theorem with the application of the left-hand eigen pairs of ($U \in C^{m\times m}$, $T \in C^{m\times n}$) of the matrix function $F(\lambda)$ of controllability index r determined by the conditions

$$TF(\lambda) \equiv (\lambda I - U)\Phi(\lambda), \quad \operatorname{rank} E = r, \quad E = [T, UT, ..., U^{m-1}T].$$

In this case $\sigma(U) = \sigma_r(F) \subseteq \sigma(F)$ and, besides, the spectra $\sigma(U)$ and $\sigma(F)$ coincide if rank $\begin{bmatrix} F(\lambda) \\ \Phi(\lambda) \end{bmatrix} = n$, $\forall\lambda \in \sigma(F)$. We introduce a set of matrices $\mathcal{K} = \{X : EXE^* \geq 0\}$.

Theorem 8.2 [9] *If the matrices $X \in \mathcal{K}$ and $X \in \mathcal{K}$ satisfy the correlations*

$$\sum_{i,j} \gamma_{ij} f_i(U) EXE^* f_j^*(U) = EYE^*, \tag{8.4}$$

$$S_\lambda = EYE^* + (\lambda I - U)EE^*(\lambda I - U)^* \geq 0, \quad \operatorname{rang} S_\lambda \equiv m,$$

then $\sigma_r(F) \subset \Lambda_f^+$, where $f(\lambda, \overline{\lambda}) = \sum_{i,j} \gamma_{ij} f_i(\lambda)\overline{f_j(\lambda)}$. Conversely, if $\sigma_r(F) \subset \Lambda_f^+$ and $f \in \mathcal{H}_0^m$, then for any matrix $Y \in \mathcal{K}$ equation (8.4) has the solution $X \in \mathcal{K}$.

The eigen pairs (U, T) of the matrix functions $F(\lambda)$ are also employed in the construction and investigation of solutions to dynamical systems of the type of $F(D)x = g$, where D is an operator of differentiation or displacement in time t.

For the linear descriptor systems $B\dot{x} = Ax$, $Bx_{k+1} = Ax_k$ and second order differential systems

$$Ax + B\dot{x} + C\ddot{x} = g, \tag{8.5}$$

modelling the dynamics of many objects of mechanics and physics new methods are developed for stability analysis, Lyapunov function construction and estimation of spectrum location with respect to algebraic curves. In particular, for the rotative system of the Lavale rotor type described as (8.5) with the matrix coefficients

$$A = K + iS, \quad B = D + iG, \quad C = M,$$

necessary and sufficient stability conditions are constructed in analytical form in terms of the corresponding mechanical parameters. Here $M = M^T > 0$ is a mass matrix, $D = D^T = D_0 + D_1 \geq 0$, $G = G^T = \omega G_0 \geq 0$ is a gyroscopic matrix, $K = K^T > 0$ is a rigidity matrix, $S = S^T \geq 0$ is a circulation matrix, D_0 and D_1 are constituents of the internal and external dampings, ω is the angular velocity of rotor rotation. The proposed technique refines the known estimate of the critical frequency of rotor rotation at which stability is lost. Also a regulator of the type of $g = Ru$, $u = K_0x + K_1\dot{x}$, is constructed which stabilizes closed loop system.

For the linear differential-difference systems

$$\dot{x} = Ax + \sum_i A_i x(t - \tau_i)$$

an analogue of the Lyapunov equation is constructed and in terms of its solutions absolute stability conditions are formulated (see [9], Chapters 2 and 3).

The theory of linear equations and operators in the matrix space is developed ([9], Chapter 4). Systems of matrix equation transformations are constructed allowing the description of their solvability conditions and inertia properties of the Hermitian solutions. A class of equations with special families of matrix coefficients is indicated and the Hill and Schneider theorems on inertia of their Hermitian solutions are generalized. A class of linear equations in the space with cone is studied [9], Appendix 2). The method of successive approximations is used to estimate solutions and their characteristics of the type of Hermitian matrix inertia. Structure of positive and positive invertible operators in the matrix space is studied ([9], Appendix 2).

8.2 Cone Inequalities in the Stability Theory([[40], [41], [42], [43]])

For the modelling of physical objects differential and difference systems of equations are employed the phase space of which contains invariant sets, in particular, cones. The peculiarities of the systems such as positiveness and monotonicity should be taken into account in stability and control analysis problems. Examples of the positive systems with respect to a cone of symmetric negative definite matrices are the differential Lyapunov and Riccati equations and second moments equation for stochastic systems of Ito type. Positive and monotone systems appear, also due to the application of the comparison technique as a generalization of the Lyapunov functions methods in stability theory.

The main results of the paper [40] are positiveness conditions and algebraic criteria of asymptotic stability of linear systems in the Banach space $\mathcal{E}$ with normal generating cone $\mathcal{K}$

$$\dot{X} + M(t)X = 0, \quad t \geq t_0 \geq 0, \quad \mathcal{K} \subset \mathcal{E}. \tag{8.6}$$

These conditions are formulated in terms of positive and positive invertible operators.

Theorem 8.3 [40] *Positive stationary system (8.6) is exponentially stable iff the operator M is positive invertible. If the operator $M + \gamma E$ is positive invertible for any $\gamma \geq 0$ then system (8.6) is positive and exponentially stable.*

Stability investigation of linear positive reducible systems and nonstationary systems with functional commutative operators is reduced to solution of algebraic equations and cone comparison of their solutions: $MX = Y$, $X \overset{\mathcal{K}}{\geq} 0$, $Y \overset{\mathcal{K}}{>} 0$. A method of robust stability analysis is proposed as well as analogues of the known comparison systems in the space with cone.

Generalizations of the class of nonlinear monotone systems in partially ordered space are introduced:

$$\dot{X} = F(X, t), \quad t \geq t_0 \geq 0, \tag{8.7}$$

their characterization by means of linear positive functionals is presented and analogues of the Lyapunov theorem on stability of equilibrium state of such systems in first approximation are formulated. Comparison methods are developed for the solutions of differential systems with the use of constant and variable cones. As a corollary robust stability conditions are formulated for the families of systems of (8.7) type described by the cone inequalities [41, 42]

$$\underline{F}(X,t) \overset{\mathcal{K}}{\leq} F(X,t) \overset{\mathcal{K}}{\leq} \overline{F}(X,t), \quad \underline{F} \in \underline{\mathcal{F}}_1, \ \overline{F} \in \overline{\mathcal{F}}_1, \quad t \geq 0,$$

where $\underline{\mathcal{F}}_1, \overline{\mathcal{F}}_1$ are generalized classes of upper and lower systems of comparison with respect to the cone $\mathcal{K}$.

In [43] the methods for positiveness and stability investigation are developed for linear dynamic systems in partially ordered space. For stability analysis of positive systems special methods are worked out which are based on spectral properties of positive and positive invertible operators. Invariance conditions are found for the cones of circular type and their generalizations which allow, in particular, solution of the problem on positive stabilization of systems with respect to given cones by means of dynamical compensators. Invariance conditions for ellipsoidal cones and exponential stability conditions for linear differential and difference systems are formulated in terms of matrix inequalities. The notion of maximal eigen pairs of a matrix polynomial is used to establish algebraic conditions of exponential stability of linear arbitrary order differential systems.

9 Multifrequency Oscillations of Nonlinear Systems

Consider a multifrequency nonlinear system of ordinary differential equations with slow and quick variables of the form

$$\frac{dx}{d\tau} = a(x, \varphi, \tau, \varepsilon), \quad \frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + b(x, \varphi, \tau, \varepsilon), \tag{9.1}$$

where x and φ are n- and m-dimensional vectors respectively, ε is a small positive parameter, $\tau = \varepsilon t$ is a "slow" time, real functions a, b, ω belong to some classes of smooth and almost periodic in φ functions. Systems of the type appear in the investigation of oscillatory processes in many problems of mechanics, electrical engineering, biology, etc.

We write an averaged in φ system

$$\frac{d\bar{x}}{d\tau} = \bar{a}(\bar{x}, \tau, \varepsilon), \quad \frac{d\bar{\varphi}}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + \bar{b}(\bar{x}, \tau, \varepsilon), \tag{9.2}$$

where

$$\bar{a}(x, \tau, \varepsilon) = \lim_{k\to\infty} k^{-m} \int_0^k \dots \int_0^k a(x, \varphi, \tau, \varepsilon) d\varphi_1 \dots d\varphi_m,$$

and designate by $W_p(\tau)$ and $W_p^T(\tau)$ the $p \times m$-matrix

$$\left(\frac{d^{j-1}}{d\tau^{j-1}} \omega_\nu(\tau) \right)_{j,\nu=1}^{p,m}$$

and the transpose matrix respectively. Here $\omega = (\omega_1, ..., \omega_m)$.

Under the assumption that $\det \left(W_p^T(\tau) W_p(\tau) \right) > 0, \ \tau \in [0, L]$, we obtain an exact estimate with respect to the order in ε [10]

$$\|x(\tau, \varepsilon) - \bar{x}(\tau, \varepsilon)\| + \|\varphi(\tau, \varepsilon) - \bar{\varphi}(\tau, \varepsilon)\| \le c\varepsilon^{\frac{1}{p}}, \ \ \tau \in [0, L], \ \varepsilon > 0, \tag{9.3}$$

where (x, φ) and $(\bar{x}, \bar{\varphi})$ are solutions of systems (9.1) and (9.2), coinciding for $\tau = 0$. For the proof of inequality (9.3) uniform estimates of oscillation integrals are essentially used [10].

The averaging technique was applied for solution of boundary-value problems for system (9.1) with multipoint and integral boundary conditions. Moreover, in the case of integral boundary conditions the averaged problem is constructed via averaging of not only differential equations but boundary conditions as well [10].

If system (9.1) is given for $\tau \in R$ and

$$\left\| \left(W_p^T(\tau) W_p(\tau)\right)^{-1} W_p^T(\tau) \right\| \leq c_1 = \text{const}, \ \tau \in R,$$

then existence of the integral manifold $x = X(\varphi, \tau, \varepsilon)$ of system (9.1) is proved on which the equations of quick variables become

$$\frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + b(X(\varphi, \tau, \varepsilon), \varphi, \tau, \varepsilon).$$

Under the assumption that the functions a, b, ω have continuous bounded partial derivatives in all variables up to the order of $l \geq 2$ it is proved that the function X is $l-1$ times differentiable and [10]

$$\left\| D_\varphi^s \frac{\partial^q}{\partial \tau^q} \frac{\partial^r}{\partial \varepsilon^r} X(\varphi, \tau, \varepsilon) \right\| \leq c_2 \varepsilon^{\frac{1}{p} - q - 2r}, \ 1 \leq s + q + r \leq l - 1,$$

and the derivatives of $(l-1)$-th order satisfy Lipschitz condition in $\varphi, \tau, \varepsilon$. Also, conditional asymptotic stability of integral manifold is studied and decomposition of slow and quick variables is accomplished in the neighbourhood of asymptotically stable integral manifold [10].

The averaging method for initial and boundary-value problems and the method of integral manifolds are justified as well in the case of systems of (9.1) type with impulse effect at fixed moments of time $\tau_j = \varepsilon t_j$, $t_{j+1} - t_j = \theta = \text{const} > 0$ and moreover,

$$\Delta x|_{\tau=\tau_j} = \varepsilon p(x, \varphi, \tau_j), \ \ \Delta \varphi|_{\tau=\tau_j} = \varepsilon q(x, \varphi, \tau_j).$$

It should be noted that in this case the average system is smooth and not subject to the impulse effect [10]

$$\frac{d\bar{x}}{d\tau} = \bar{a}(\bar{x}, \tau) + \frac{1}{\theta} \bar{p}(\bar{x}, \tau), \ \ \frac{d\bar{\varphi}}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + \bar{b}(\bar{x}, \tau) + \frac{1}{\theta} \bar{q}(\bar{x}, \tau).$$

10 Absolute Stability, Stability and Instability by Linear Approximation and Essential Instability of Motion for Nonlinear Infinite Dimensional Systems

10.1 Absolute stability of systems with aftereffect.

In practice one have sometimes to study stability of dynamical systems at arbitrary parameter values. If the systems are stable at arbitrary values of the corresponding parameters these systems are called absolutely stable (with respect to these parameters). Mathematical models of a wide class of dynamical systems are differential delay equations and delays are the corresponding parameters.

In [11] spectral criteria of absolute stability (with respect to constant deviations of argument) are obtained for solutions of linear autonomous differential difference equations of delay and neutral type

$$\frac{dx(t)}{dt} = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - \tau_k),$$

$$B_0\frac{dx(t)}{dt}+\sum_{k=1}^{m}B_k\frac{dx(t-\delta_k)}{dt}=C_0x(t)+\sum_{k=1}^{m}C_kx(t-\tau_k)$$

in a Banach space the separate case of which is the known theorem of Yu.M. Repin. Here $A_k,\ B_k,\ C_k,\ k=\overline{0,m}$, are linear continuous operators, $\delta_k,\ \tau_k,\ k=\overline{0,m}$, are arbitrary positive or nonnegative constants. Also, classes of systems with arbitrary slowly changing operator coefficients and argument deviations are constructed whose solutions are strong absolutely asymptotically stable. Algebraic criteria of absolute asymptotic stability and instability are obtained for solutions to the scalar equation

$$\frac{d^nx(t)}{dt^n}+\sum_{k=0}^{n-1}a_k\frac{d^kx(t)}{dt^k}+\sum_{k=0}^{n}\sum_{j=1}^{m}b_{kj}\frac{d^kx(t-\tau_j)}{dt^k}=0,$$

which strengthen the known result of L.A. Zhivotovskii. It is shown that absolute exponential stability of solutions to the equations under consideration is preserved as well for small nonlinear perturbations of equations. The results of investigation are applied in stability investigation of the equilibrium states of mechanical systems. In particular, undercarriage galloping at aircraft uniform motion on an even ground air strip is studied and stability conditions are established for the equilibrium state at steady cutting at trace turning for arbitrary constant angular velocity of spindle rotation. Note that in these two examples the oscillation processes under some restrictions are described by differential difference equation of the type

$$\frac{d^2x(t)}{dt^2}+a\frac{dx(t)}{dt}+bx(t)+cx(t-\tau)=f\left(x(t),\frac{dx(t)}{dt},x(t-\tau),\frac{dx(t-\tau)}{dt}\right),$$

where $a,b,c\in R$ and $f(x_1,x_2,x_3,x_4)=o(|x_1|+|x_2|+|x_3|+|x_4|)$ for $x_k\to 0$, $k=\overline{1,4}$.

The investigations are based on the analogue of the maximum principle for the spectrum of operator holomorphic function (see [11]).

10.2 Stability and instability in linear approximation and essential instability of evolutionary systems.

New conditions of stability and instability in linear approximation are established for solutions to differential and difference equations of the type

$$\frac{dx(t)}{dt}=Ax(t)+F(t,x(t)),\ t\geq 0,$$

$$x_{n+1}=Ax_n+G_n(x_n),\ n\geq 0,$$

and similar functional differential equations in a Banach space which generalize and strengthen the results of A.M. Lyapunov, M.G. Krein and Yu.L. Daletskii. In these equations A is a continuous linear operator and $F(t,\cdot)$ and G_n are nonlinear operators for which

$$\lim_{x\to 0}\frac{\sup\limits_{t\geq 0}\|F(t,x)\|}{\|x\|}=0\quad\text{and}\quad\lim_{x\to 0}\frac{\sup\limits_{n\geq 0}\|G_n(x)\|}{\|x\|}=0.$$

Examples of autonomous nonlinear systems with asymptotically stable solutions are set out for the linear approximations of which these solutions are unstable and are spectrum points of operators generated by linear approximations with positive real parts (differential case) or absolute values larger than one (difference case) [12]. Theorems on stability in linear approximation are applicable to the investigation of oscillation processes of a series of nonlinear mechanical systems, in particular, vibroimpact ones whose constituents are components with distributed parameters, systems with impulse loadings, etc. A mathematical apparatus is so far created for solution of a wide class of problems of the theory of nonlinear oscillations of complex mechanical systems.

A notion of essentially unstable solution to evolution equation in infinite dimensional case is introduced which is associated with the essentially approximate spectrum of operator. Such equations have the property that arbitrary absolutely continuous and some other perturbations can not influence essentially unstable solution so that it becomes stable. The notion of essentially unstable solution allowed new results on instability of solutions which have no analogues in the finite dimensional case [12].

The Belitskii–Lyubich hypothesis on smooth mapping of a convex compact subset of finite dimensional space was disproved. The hypothesis claimed that in the case when spectral radius of the Frechet derivative of the mapping at all points of the subset is smaller than a unit, the sequences generated by this iteration mapping converge to the unique point of the subset. It is shown that in general case the iteration sequences can diverge and the mapping can have an arbitrary number of cycles. Mappings of the type occur in practice in the computer investigation of oscillation processes in nonlinear mechanical systems. Additional conditions are found under which the hypothesis is true. Also, global asymptotic stability conditions are established for solutions to nonlinear differential and difference equations in a Banach space [12].

11 Concluding Remarks

The paper provides review of results obtained by the authors in the field of nonlinear mechanics. The development of the Lyapunov's methods and the averaging theory allowed solutions to a wide range of problems of the mathematical stability theory, motion control theory, dynamics of a rigid body and systems of bodies, theory of boundary-value problems and multifrequency oscillation theory to be described from a unique methodological point of view. New approaches set out in the paper are applied not only to the investigation of systems of ordinary differential equations, but also to a huge class of hybrid dynamical systems including the systems with impulse effect, delay equations and differential difference equations in a Banach space. It seems reasonable to develop further the presented methods for description of dynamical properties of complex systems in abstract spaces and to apply the obtained results to motion stability and control problems for mechanical objects with distributed parameters.

The worked out method of oriented manifolds reduced the controllability problem to the investigation of solvability of differential equations with respect to auxiliary functions under general assumptions on regularity of vector fields of controlled system. For this method to be constructively used it is of interest to develop approaches for constructing basic systems for arbitrary nonlinear control processes. The results obtained in the paper demonstrate efficiency of applying the method of trajectory set for solution of inverse problems of control theory. Generalization of theorems of the direct Lyapunov method yielded a complete description of conditions of strong and partial stabilizability of the class of plane mechanical systems with elastic beams. Meanwhile, the problem on compactness of limiting trajectory sets of nonlinear differential equations with non-monotone and unbounded right-hand sides in a Banach space should be further investigated.

Note that the method of matrix-valued Lyapunov functions allows to extend maximally the assumptions on dynamical properties of subsystems in large scale system and assumptions on interconnection functions between the subsystems. As compared with the other approaches developed in stability theory of large scale systems this method has the following advantages: it does not require the application of quasimonotone comparison systems which is a necessary condition when the vector Lyapunov functions are applied; it allows extension of the class of auxiliary functions by means of which an appropriate Lyapunov function can be constructed for the problem under consideration; it provides a possibility of taking into account the effect of interconnection functions between subsystems on the whole system dynamics; the method also allows to take into account the effect of pairs of subsystems appearing as result of first level decomposition on the whole system dynamics.

It is known that the a priori determination of the domains of initial and subsequent deviations of solutions from zero equilibrium states (or given nominal solution) and the domain of persistent perturbations is characteristic for nonclassical stability theories such as technical and practical ones. Moreover, the interval of system functioning is also fixed. An efficient application of the direct Lyapunov method in the practical stability problems by A.A. Martynyuk yielded significant extensions of this method, which are follows: an extension of the class of auxiliary functions suitable for the studying practical stability of motion; elimination of the property of having a fixed sign of the total derivative of an auxiliary function along with solutions of the system under investigation; establishing a relationship between the quantitative values of the auxiliary function in given (finite) domains of the phase space and decrement (increment) of this function along with solutions of the system under investigation.

The importance of practical application of the theory of boundary-value problems in various fields (nonlinear oscillation theory, motion stability theory, control theory, a series of economical and biological problems) attracts a great interest to the investigations in the theory of boundary-value problems for a wide class of systems of functional differential equations.

General theory of under- and over-determined resonance boundary-value problems is constructed, natural classification of the problems is worked out, efficient coefficient criteria of existence of solutions to both linear and nonlinear problems are obtained and algorithms for their construction are developed [7, 8]. Perturbation theory for such problems is constructed and bifurcation and branching conditions are established for solutions of boundary-value problems (including the problems with conditions at infinity) with the Fredholm operator in linear part. The application of the apparatus of generalized inverse operators based on classical results of A.M. Lyapunov and I.J. Malkin on nonlinear periodic oscillation theory provided the development of the qualitative theory of boundary-value problems for the systems of ordinary differential [7, 8] and difference [36] equations, systems of differential equations with delaying argument [38] and differential systems with impulse effect [35]. Further original application of this theory was to the known problem on bounded on the whole real axis solutions to differential and difference equations of appropriate homogeneous system under the dichotomy condition on semiaxes [8, pp. 257–304].

Originality and importance of the main results of the papers [9], [39]–[43] are as follows. The author generalizes the Lyapunov and Ostrovsky–Schneider theorems on localization of matrix spectrum for the classes of analytic domains including the previously known ones and being maximally admissible in the framework of the method of matrix equations. Generalized Lyapunov equation is used in the problem on quadratic optimization of linear controlled systems. The analogues of generalized Lyapunov equation constructed for analytic matrix-functions enable formulation of new algebraic methods for stability and localization analysis of spectrum of different classes of differential, difference and differential-difference systems. The elaborated transformation systems and generalized inertia theory provide new techniques for classification of linear matrix equations with respect to their solvability conditions and properties of solutions employed in the applied investigations. Stability criteria are obtained for linear dynamic systems in partially ordered space in terms of positive and positive invertible operators. New methods for stability analysis and generalized principle of comparison of nonlinear differential systems with the use of cone inequalities are formulated. The results obtained allow one to describe algebraically the classes of stable systems in the parameter space, to compare their dynamics and to construct stabilizing controls.

Scientific novelty of the results presented in the monograph [10] is as follows. New uniform estimates are obtained for oscillation integrals and parameter dependent sums. These estimates are used to substantiate the method of averaging with respect to all quick parameters on a segment and semiaxis for nonlinear oscillation systems with slowly varying frequencies in the resonance case. A new construction technique is developed for integral manifolds of resonant multifrequency systems and their smoothness and stability are studied. Solvability conditions are established for boundary-value problems of multifrequency systems with multipoint and integral boundary conditions and new error estimates are proved for the averaging method for such problems. The averaging method and the method of integral manifolds are justified for oscillation systems with slowly varying frequencies and impulse effect.

In the monographs [11, 12] functional analytical methods are developed for investigation of absolute stability of dynamic systems with aftereffect, stability, instability and essential instability of trajectories of dynamic systems in infinite dimensional phase space. These methods allow, first of all, obtaining general results on asymptotic behaviour of trajectories of nonlinear systems under investigation, constructing a mathematical apparatus for investigation of dynamic processes in complex nonlinear systems and finding out general regularities of the evolutionary processes going on in many real systems where motion occurs. Besides, they open up new possibilities for investigating oscillation of trajectories of nonlinear dynamic systems and studying invertibility of nonlinear functional operators.

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