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Complete Analysis of an Ideal Rotating Uniformly Stratified System of ODEs

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Abstract: In this paper we discuss a system of six coupled ODEs which arise in ODE reduction of the PDEs governing the motion of uniformly stratified fluid contained in rectangular basin of dimension $L \times L \times H$, which is temperature stratified with fixed zeroth order moments of mass and heat. We prove that this autonomous system of ODEs is completely integrable if Rayleigh number $Ra = 0$ and determine the stable, unstable and center manifold passing through the rest point and discuss the qualitative feature of the solutions of this system of ODEs.

Keywords: *rotating stratified Boussinesq equation; completely integrable systems.*

Mathematics Subject Classification (2000): 34A34, 37K10.

1 Introduction

In fluid dynamics, the flow of fluid in the atmosphere and in the ocean is governed by the Navier-Stokes equations. In the scale of Boussinesq approximation (i.e., flow velocities are to slow to account for compressible effect), the flow of fluid is given by rotating stratified Boussinesq equations. In the theory of basin scale dynamics Maas [1], has considered the flow of fluid contained in rectangular basin of dimension $L \times L \times H$, which is temperature stratified with fixed zeroth order moments of mass and heat. The container is assumed to be steady, uniform rotation on an f-plane. With this assumptions Maas [1] reduces the rotating stratified Boussinesq equations to an interesting six coupled system of ODEs. Our analysis is quite different from the one employed by Maas [1] in as much as we have obtained rather precise information concerning the global phase portrait of the system as well as analytical representation of the solution in terms of elliptic functions.

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The system of six coupled ODEs is completely integrable if Rayleigh number $Ra = 0$. We provide in this paper the complete analysis of this integrable system. Four functionally independent first integrals and zero divergence of vector field implying the existence of fifth first integral, thereby prove the complete integrability of the system. The four first integrals reduce the $\mathbb{R}^6$ into a family of two dimensional invariant surfaces (when rotation frequency f is less than the twice of horizontal Rayleigh damping coefficient otherwise either degenerate into a rest point or an empty surface). We observe that gluing these surfaces along a circle of transit points we get a torus of genus one. If there is a rest point which lies on the invariant surface then it is seen to be singular and one of the generating circles gets pinched to the rest point. We obtain the stable and unstable manifolds passing through the rest point. We also find the center manifold through the rest point which shows that rest point is unstable with two dimensional stable, unstable and center manifolds passing through it. In additional we carry out the complete integration of the system in terms of elliptic functions which degenerate in special case. In the last section we obtained a fifth first integral which is guaranteed by Jacobi's last integral theorem, it is quite non trivial and expressible in terms of elliptic functions.

2 An Ideal Rotating Uniformly Stratified System of ODEs

In the scale of Boussinesq approximation, the flow of fluid in the atmosphere and in the ocean is described by rotating stratified Boussinesq equations

$$\begin{aligned} &\frac{D\mathbf{v}}{Dt} + f(\hat{\mathbf{e_3}} \times \mathbf{v}) = -\nabla p + \nu(\Delta \mathbf{v}) - \frac{g\tilde{\rho}}{\rho_b}\hat{\mathbf{e_3}}, \\ &\operatorname{div} \mathbf{v} = 0, \\ &\frac{D\tilde{\rho}}{Dt} = \kappa \Delta \tilde{\rho}. \end{aligned} \tag{2.1}$$

Here $\mathbf{v}$ denotes the velocity field, ρ is the density which is the sum of constant reference density ρ_b and perturb density $\tilde{\rho}$, p the pressure, g is the acceleration due to gravity that points in $-\hat{\mathbf{e_3}}$ direction, f is the rotation frequency of earth, ν is the coefficient of viscosity, κ is the coefficient of heat conduction and $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$ is a convective derivative. For more about rotating stratified Boussinesq equations one may consult Majda [2]. In their study of onset of instability in stratified fluids at large Richardson number, Majda and Shefter [3] obtained the ODE reduction of (2.1) by neglecting the effects of rotation and viscosity, and complete analysis of that system and qualitative features of the solution are discussed by Srinivasan et al [4] in their paper. Whereas Maas [1] consider the effects of rotation to equation (2.1) in the frame of reference of an uniformly stratified fluid contained in rotating rectangular box of dimension $L \times B \times H$. In this context, Maas [1] reduces the system of equations (2.1) to six coupled system of ODEs (2.3) given below, which form a completely integrable Hamiltonian system if Rayleigh number Ra vanishes. In his study he considers a rectangular basin of size $L \times L \times H$, which is temperature-stratified with fixed zeroth order moments of mass and heat (so that there is no net evaporation or precipitation, nor any net river input or output, and neither a net heating nor cooling). The container is assumed to be in steady, uniform rotation on an f-plane (f-plane refers to the effective background rotation axis determined by the projection of the earth's rotation vector along the vertical.) Maas [1] appeals to the idea that the dynamics of the position vector of its center of mass may, to some extent, be representative of the basin scale dynamics of a mid-latitude lake or sea; in this context one may refer to Morgan [5], and Maas [6].

Maas [1] reduces the system of equations (2.1) into the following system of six coupled ODEs:

$$\begin{aligned} &Pr^{-1}\frac{d\mathbf{w}}{dt} + f'\hat{\mathbf{e_3}} \times \mathbf{w} = \hat{\mathbf{e_3}} \times \mathbf{b} - (w_1,\, w_2,\, rw_3) + \hat{T}\mathbf{T}, \\ &\frac{d\mathbf{b}}{dt} + \mathbf{b} \times \mathbf{w} = -(b_1,\, b_2,\, \mu b_3) + Ra\mathbf{F}. \end{aligned} \tag{2.2}$$

In these equations, $\mathbf{b} = (b_1,\, b_2,\, b_3)$ is the center of mass, $\mathbf{w} = (w_1,\, w_2,\, w_3)$ is the basin's averaged angular momentum vector, $\mathbf{T}$ is the differential momentum, $\mathbf{F}$ are buoyancy fluxes, $f' = f/2r_h$ is the earth's rotation, $r = r_v/r_h$ is the friction ($r_{v,h}$ are the Rayleigh damping coefficients), Ra is the Rayleigh number, Pr is the Prandtl number, μ is the diffusion coefficient and $\hat{T}$ is the magnitude of the wind stress torque.

Neglecting diffusive and viscous terms, Maas [1] considers the dynamics of an ideal rotating, uniformly stratified fluid in response to forcing. He assumes this to be due solely to differential heating in the meridional (y) direction $\mathbf{F} = (0,\, 1,\, 0)$; the wind effect is neglected i.e. $\mathbf{T} = 0$. For Prandtl number, Pr, equal to one the system of equations (2.2) reduces to the following an ideal rotating, uniformly stratified system of six coupled ODEs.

$$\begin{aligned} &\frac{d\mathbf{w}}{dt} + f'\hat{\mathbf{e_3}} \times \mathbf{w} = \hat{\mathbf{e_3}} \times \mathbf{b}, \\ &\frac{d\mathbf{b}}{dt} + \mathbf{b} \times \mathbf{w} = Ra\mathbf{F}. \end{aligned} \tag{2.3}$$

We see the system of equations (2.3) is divergence free and, when $Ra = 0$, admits the following four functionally independent first integrals

$$|\mathbf{b}|^2 = c_1, \quad \hat{\mathbf{e_3}} \cdot \mathbf{w} = c_2, \quad |\mathbf{w}|^2 + 2\hat{\mathbf{e_3}} \cdot \mathbf{b} = c_3, \quad \mathbf{b} \cdot \mathbf{w} + f'\hat{\mathbf{e_3}} \cdot \mathbf{b} = c_4. \tag{2.4}$$

Hence, by using Liouville theorem on integral invariants and theorem of Jacobi [7] there exists an additional first integral. Also we see from (2.4) that $|\mathbf{b}|$ and $|\mathbf{w}|$ remain bounded so that the invariant surface (2.4) is compact and the flow of the vector field ($\mathbf{w}$, $\mathbf{b}$) is complete. Therefore, the system of equations (2.3) is completely integrable for $Ra = 0$. Maas [1] took $f' = 1$ and equations (2.3) show that the horizontal circulation (w_3) is constant hence without loss of generality he took $w_3 = 0$ which is one of the first integral of the system (2.3). Using the first integral $\frac{|\mathbf{w}|^2}{2} + b_3 = B\,(constant)$, he obtained the Hamiltonian

$$H = \frac{1}{2}\Big(r^2 + s^2 + \{B - (w_1^2 + w_2^2)/2\}^2\Big) + Ra\, w_1\,, \tag{2.5}$$

where $r = \dot{w_1}$ and $s = \dot{w_2}$. With this Hamiltonian H, Maas [1] has shown that the system of equations (2.3) is completely integrable if $Ra = 0$.

Here we see that if $Ra = 0$, the system of equations (2.3) is completely integrable and we can rewrite it as follows

$$\begin{aligned} &\dot{\mathbf{w}} = -f'\hat{\mathbf{e_3}} \times \mathbf{w} + \hat{\mathbf{e_3}} \times \mathbf{b}, \\ &\dot{\mathbf{b}} = \mathbf{w} \times \mathbf{b}. \end{aligned} \tag{2.6}$$

It is easy to see that the critical points (rest points) of the system (2.6) are $(\lambda_1\hat{\mathbf{e_3}},\, \lambda_2\hat{\mathbf{e_3}})$, $(\lambda_1\hat{\mathbf{e_3}},\, 0)$,

$(0,\, \lambda_2\hat{\mathbf{e_3}})$, $(0,\, 0)$, $(\mathbf{w},\, f'\mathbf{w})$ and $(\frac{1}{f'}\mathbf{b},\, \mathbf{b})$ where λ_1, λ_2 are arbitrary scalars. Of these critical points, $(\hat{\mathbf{e_3}},\, \hat{\mathbf{e_3}})$ is the only one lying on the invariant surface

$$|\mathbf{b}|^2 = 1, \quad \hat{\mathbf{e_3}} \cdot \mathbf{w} = 1, \quad |\mathbf{w}|^2 + 2\hat{\mathbf{e_3}} \cdot \mathbf{b} = 3, \quad \mathbf{w} \cdot \mathbf{b} + f'\hat{\mathbf{e_3}} \cdot \mathbf{b} = 1 + f'. \tag{2.7}$$

We give the details of the analysis of the system (2.6) in the following section.

3 Analytical Details

We have six coupled autonomous system of nonlinear ODEs (2.6) with four first integrals (2.4). We now proceed to analyzing the system (2.6). With nonzero values of c_1, c_2, c_3 and c_4 the possible critical points of the system (2.6) are $(\lambda_1\hat{\mathbf{e_3}},\, \lambda_2\hat{\mathbf{e_3}})$. With $c_1 = 1$, and $\mathbf{w} = \pm\hat{\mathbf{e_3}}$, c_3 may assume the value -1 or 3 (not both). Now take $c_3 = 3$ so that the possible critical points are $(\hat{\mathbf{e_3}},\, \pm\hat{\mathbf{e_3}})$ and at these critical points the value of c_2 is ± 1. Note that the case $c_2 = -1$ will be a surface disjoint from $\hat{\mathbf{e_3}} \cdot \mathbf{b} = 1$ so with the specific values of $c_1 = 1$, $c_2 = 1$, and $c_3 = 3$ we have only one critical point $(\hat{\mathbf{e_3}},\, \hat{\mathbf{e_3}})$. At this critical point the fourth first integral assumes the value $c_4 = 1 + f'$.

We find the eigenvalues of the matrix of linearized part of the system (2.6) at this critical point and these are given below

$$0,\, 0,\, \pm\frac{\sqrt{1 - f'^2 \pm (-1 + f')^{3/2}\sqrt{3 + f'}}}{\sqrt{2}}, \tag{3.1}$$

the double eigenvalue zero implying the critical point is degenerate. With all four possible distributions of sign and for $0 < f' < 1$, we see that among these six eigenvalues, two of them have positive real parts and two of them have negative real parts and the remaining of two eigenvalues are zero. This linear analysis suggests that when $0 < f' < 1$, the rest point is degenerate and unstable. In fact the critical point $(\hat{\mathbf{e_3}},\, \hat{\mathbf{e_3}})$ is unstable with two dimensional stable, unstable and center manifolds. For $f' = 1$ the system degenerates with all the six eigenvalues being zero possessing four linearly independent eigen vectors $(0,\, \hat{\mathbf{e_3}})$, $(\hat{\mathbf{e_2}},\, \hat{\mathbf{e_2}})$, $(\hat{\mathbf{e_1}},\, \hat{\mathbf{e_1}})$, $(\hat{\mathbf{e_3}},\, 0)$. We shall now bifurcate the analysis in two parts. (i) When a critical point lies on the invariant surface determine by equations (2.7). (ii) When no critical point lies on the invariant surface (2.7).

3.1 Critical point lying on the invariant surface

Now we set up the local coordinates on the two dimensional invariant surface (2.7), we get $w_3 = 1$. The general solution of the inhomogeneous equation $\mathbf{w} \cdot \mathbf{b} + f'\hat{\mathbf{e_3}} \cdot \mathbf{b} = 1 + f'$ is given below.

$$w_1 = \frac{-b_2 k}{1 - b_3} + \frac{(1 + f')b_1}{1 + b_3}, \quad w_2 = \frac{b_1 k}{1 - b_3} + \frac{(1 + f')b_2}{1 + b_3}, \quad w_3 = 1, \tag{3.2}$$

where k is arbitrary. To determine the k, substitute (3.2) in $|\mathbf{w}|^2 + 2\hat{\mathbf{e_3}} \cdot \mathbf{b} = 3$ to get

$$k^2 = \Big(\frac{1 - b_3}{1 + b_3}\Big)^2 \left[1 + 2b_3 - 2f' - (f')^2\right] = k(b_3). \tag{3.3}$$

From above equation and for $|\mathbf{b}|^2 = 1$, we see that k is real if and only if

$$0 \leq f' \leq 1\,. \tag{3.4}$$

Note that when $f' = 0$, the system of equations (2.6) disregards rotation. For $f' = 1$ the invariant set (2.7) degenerates into the critical point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ whereas for $f' > 1$ the invariant set (2.7) is empty. By use of the first integral $|\mathbf{b}|^2 = 1$ we can introduce the spherical polar coordinates in our system

$$b_1 = \cos\theta \sin\phi, \quad b_2 = \sin\theta \sin\phi, \quad b_3 = \cos\phi, \tag{3.5}$$

with this help of spherical polar coordinates we get k as a function of ϕ as given below

$$k^2 = \tan^4\left(\frac{\phi}{2}\right)\left[4\cos^2\frac{\phi}{2} - (1+f')^2\right]$$

or

$$k = \pm\tan^2\left(\frac{\phi}{2}\right)\left[4\cos^2\frac{\phi}{2} - (1+f')^2\right]^{1/2} \tag{3.6}$$

and

$$\begin{aligned} w_1 &= \tan\left(\frac{\phi}{2}\right)\left((1+f')\cos\theta \mp \sin\theta\sqrt{4\cos^2\frac{\phi}{2} - (1+f')^2}\right), \\ w_2 &= \tan\left(\frac{\phi}{2}\right)\left((1+f')\sin\theta \pm \cos\theta\sqrt{4\cos^2\frac{\phi}{2} - (1+f')^2}\right). \end{aligned} \tag{3.7}$$

To obtain an ODE for ϕ we observe that

$$\frac{d}{dt}(b_1^2 + b_2^2) = b_3(w_2 b_1 - w_1 b_2).$$

Substituting (3.5) and (3.7) into this we get

$$\dot{\phi} = \pm\tan\left(\frac{\phi}{2}\right)\sqrt{4\cos^2\frac{\phi}{2} - (1+f')^2}\,. \tag{3.8}$$

Finally using this in the equations for $\dot{b_1}$ and $\dot{b_2}$ in (2.6) we get the equation for θ namely,

$$\dot{\theta} = \frac{(1 - f'\cos\phi)}{2\cos^2\frac{\phi}{2}}. \tag{3.9}$$

Equations (3.8)-(3.9) admit solutions in terms of elementary functions implying the complete integrability of the system (2.6). The solutions of the more general equations (3.22)-(3.26) below involve elliptic integrals. We record these results below for this special case. Corresponding to the plus sign in (3.8) we get for an arbitrary constants of integration $C_1 > 0$ and C_2,

$$\begin{aligned} \phi(t) &= 2\sin^{-1}\left[\frac{C_1\sqrt{4-(1+f')^2}\,e^{-\frac{t}{2}\sqrt{4-(1+f')^2}}}{1 + C_1^2\,e^{-t\sqrt{4-(1+f')^2}}}\right], \\ \theta(t) &= C_2 + \frac{(1-f')}{2}\left[t + \frac{2(3+4f'+f'^2)\tan^{-1}\left(\frac{2e^{t\sqrt{3-2f'-f'^2}} - (1-2f'-f'^2)C_1^2}{\sqrt{(1+f')^2(3-2f'-f'^2)C_1^4}}\right)}{(1+f')(3-2f'-f'^2)}\right]. \end{aligned} \tag{3.10}$$

Corresponding to the negative sign in (3.8) we get

$$\phi(t) = 2\sin^{-1}\left[\frac{C_1\sqrt{4-(1+f')^2}\, e^{\frac{t}{2}\sqrt{4-(1+f')^2}}}{1+C_1^2\, e^{t\sqrt{4-(1+f')^2}}}\right],$$

$$\theta(t) = C_2 + \frac{(1-f')}{2}\left[t + \frac{2(3+4f'+f'^2)\tan^{-1}\left(\frac{2C_1^2\, e^{t\sqrt{3-2f'-f'^2}}-(1-2f'-f'^2)}{\sqrt{(1+f')^2(3-2f'-f'^2)}}\right)}{(1+f')(3-2f'-f'^2)}\right]. \tag{3.11}$$

To settle the ambiguity in sign in (3.8) note that the first integrals (2.4) except $\mathbf{w}\cdot\mathbf{b} + f'\hat{\mathbf{e_3}}\cdot\mathbf{b}$ are invariant under reflection

$$(b_1,\, b_2,\, b_3) \mapsto (-b_1,\, -b_2,\, b_3), \tag{3.12}$$

whereas the integral $\mathbf{w}\cdot\mathbf{b} + f'\hat{\mathbf{e_3}}\cdot\mathbf{b}$ remains invariant when (3.12) is simultaneously applied with the transformation $k \mapsto -k$.

From (3.6) we see that ϕ is constrained by the relation

$$0 \le \phi \le 2\cos^{-1}\left(\frac{1+f'}{2}\right), \tag{3.13}$$

and k vanishes at both extreme values. The critical point $(\hat{\mathbf{e_3}},\, \hat{\mathbf{e_3}})$ is correspond to $\phi = 0$ and at other end of extreme value of $\phi = 2\cos^{-1}\left(\frac{1+f'}{2}\right)$ the system of ODEs, (3.8) has a periodic trajectory given by

$$\phi = 2\cos^{-1}\left(\frac{1+f'}{2}\right), \quad \dot{\theta} = \frac{2-f'(1+f')}{(1+f')}\,. \tag{3.14}$$

However, this does not correspond to a periodic solution of the original system (2.6) since the parametrization (3.5)-(3.7) fails to be Lipschitz along the locus given by (3.14). The locus (3.14) consists of *transit points*, which separate the stable and unstable manifolds. The locus given by (3.14) is a periodic orbit of the system (2.6) in a special case that we identify in section 3.2.

3.1.1 Stable and unstable manifolds

Let us denote by S the portion of sphere $|\mathbf{b}|^2 = 1$ defined by

$$\left\{(b_1,\, b_2,\, b_3) | b_1^2 + b_2^2 + b_3^2 = 1;\, 0 \le \phi \le 2\cos^{-1}\left(\frac{1+f'}{2}\right)\right\} \tag{3.15}$$

which is a closed spherical cap as shown in Figure 3.1 For each choice of the sign for $k(b_3)$ we denote the graph of function $\mathbf{w} = (w_1,\, w_2,\, w_3)$, as a function of $\mathbf{b}$ on S), by $\Gamma_\pm$ namely,

$$\Gamma_\pm = \left\{(\mathbf{w}(\mathbf{b}),\, \mathbf{b}) | k = \pm\tan^2\left(\frac{\phi}{2}\right)\left[4\cos^2\frac{\phi}{2} - (1+f')^2\right]^{1/2}\right\}. \tag{3.16}$$

Note that $\mathbf{w} = (w_1,\, w_2,\, w_3)$ is defined in (3.2). Define functions $f_\pm : S \mapsto \Gamma_+$ as

$$\begin{aligned} f_+(\mathbf{b}) &= (\mathbf{w}(\mathbf{b}),\, \mathbf{b}), \quad k \ge 0, \\ f_-(\mathbf{b}) &= (\mathbf{w}(\mathbf{b}),\, \mathbf{b}), \quad k \le 0. \end{aligned} \tag{3.17}$$

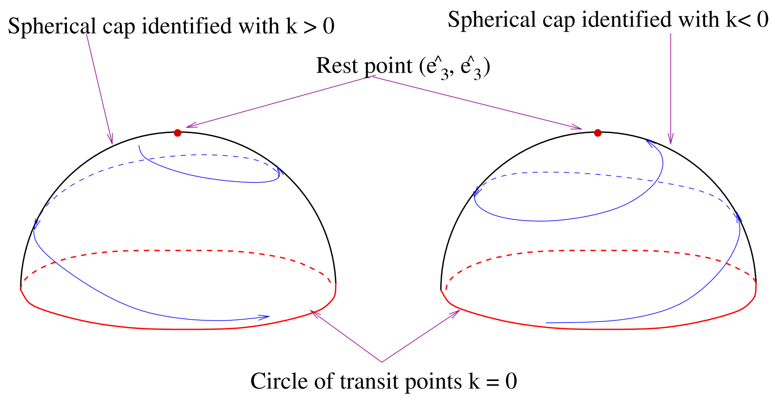

Figure 3.1: Stable and unstable manifolds.

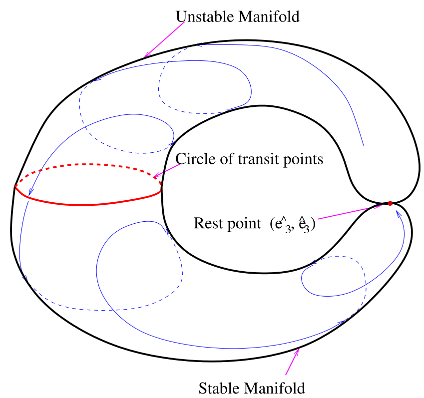

Figure 3.2: Torus pinched at critical point.

Both f_+ and f_- are homeomorphisms and they agree along the circle $k = 0$ as well as at the point $\mathbf{b} = \hat{\mathbf{e_3}}$. Thus the invariant surface is made up of the pieces $\Gamma_\pm$, each of which is homeomorphic to the closed spherical cap as shown in Figure 3.1 and given by (3.15). The invariant surface is obtained by gluing these pieces together at the critical point and the circle $k = 0$, as shown in Figure 3.2 This proves the invariant surface is a torus one of whose generating circle is pinched to a point.

Assume that for a solution starting near the critical point, $k(b_3) > 0$. Taking the plus sign in (3.8) we see that trajectories starting on Γ_+ recede away from the critical point since $\phi(t)$ monotonically increases, reaching the circle $k = 0$ in a finite time T given by

$$T = \int_\alpha^\beta \frac{\cot(\phi/2)d\phi}{\sqrt{4\cos^2(\phi/2) - (1 + f')^2}}\,. \tag{3.18}$$

Here α is the initial value of ϕ and β is the value of ϕ given by (3.8). The sign of $k(b_3)$ changes when $t > T$ whereby $\phi(t)$ decreases monotonically to zero and the trajectory, which now lies in Γ_-, approaches the critical point as $t \longrightarrow +\infty$.

On the other hand a trajectory starting on Γ_- stays in Γ_- and ultimately approaches the critical point as $t \longrightarrow +\infty$. We see that the part Γ_+ is the unstable manifold and Γ_-

the stable manifold of the system of ODEs (2.6). A trajectory starting on the unstable manifold reaches a point on (3.14) in a finite time and then enters the stable manifold.

A trajectory starting on the unstable manifold must reach a point on (3.14) in a finite time and subsequently must enter the stable manifold. This justifies the terminology "*transit points*".

3.2 When there are no critical points on the invariant surface

We perturb the initial conditions by assigning the values

$$c_1 = c_2 = 1, \quad c_4 = 1 + f', \quad c_3 = 3 + \epsilon, \tag{3.19}$$

to the first integrals (2.4). The compact invariant surface (2.4) no longer contains a rest point and so the Poincaré-Hopf index theorem shows that it is a torus. It is readily checked that the singularity $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ in the invariant surface that was initially present has smoothened out. Equations (3.2) continue to be valid except that $k(b_3)$ is now given by

$$(k(b_3))^2 = \Big(\frac{1-b_3}{1+b_3}\Big)^2\Big[2(1+b_3) - (1+f')^2\Big] + \epsilon\Big(\frac{1-b_3}{1+b_3}\Big). \tag{3.20}$$

Parameterizing the sphere as in (3.5) we get in place of (3.6) the expression

$$k^2 = \tan^2\left(\frac{\phi}{2}\right)\Big[\tan^2\left(\frac{\phi}{2}\right)\Big(4\cos^2(\phi/2) - (1+f')^2\Big) + \epsilon\Big]. \tag{3.21}$$

Now using (2.6), $\frac{d}{dt}(b_1^2 + b_2^2) = 2kb_3(1+b_3)$, which is in polar coordinates assume the form

$$\dot{\phi} = k\cot\left(\frac{\phi}{2}\right) = \pm\Big[\tan^2\left(\frac{\phi}{2}\right)\Big(4\cos^2(\phi/2) - (1+f')^2\Big) + \epsilon\Big]^{1/2}. \tag{3.22}$$

The change of variable $v = \cos^2(\phi/2)$ transforms (3.22) into an ODE for elliptic integral:

$$\left(\frac{dv}{dt}\right)^2 = (v-1)\Big[4v^2 - \Big(4 + (1+f')^2 + \epsilon\Big)v + (1+f')^2\Big] = C(v)\,. \tag{3.23}$$

Note that for $\epsilon \le -[2+(1+f')]^2$ or $\epsilon \ge -[2-(1+f')]^2$, the cubic polynomial $C(v)$ has three distinct real roots namely

$$\begin{aligned}
\zeta_1 &= \tfrac{1}{8}\Big[(4+(1+f')^2+\epsilon) - \sqrt{(4+\epsilon)^2 + (1+f')^2[(1+f')^2+4+2\epsilon]}\Big],\\
\zeta_2 &= \tfrac{1}{8}\Big[(4+(1+f')^2+\epsilon) + \sqrt{(4+\epsilon)^2 + (1+f')^2[(1+f')^2+4+2\epsilon]}\Big],\\
v &= 1,
\end{aligned} \tag{3.24}$$

two of which coalesce when $\epsilon \longrightarrow 0$.

For $\epsilon > 0$, $C(v)$ has real roots ζ_1, 1 and ζ_2 where $0 < \zeta_1 < 1 < \zeta_2$ and since $0 \le v \le 1$, we see that $C(v)$ is positive only on the interval $[\zeta_1, 1]$. The point $v(t)$ attains the value ζ_1 in time T_1 given by

$$T_1 = \int_\alpha^\beta \frac{d\phi}{\sqrt{\tan^2(\phi/2)\,[4\cos^2(\phi/2) - (1+f')^2] + \epsilon}},$$

where α is initial value of ϕ and β is the value of ϕ given by (3.22). After which k becomes negative, hence by equation (3.22), ϕ is decreasing and it decreases to zero in time T_2 given by

$$T_2 = -\int_{\beta}^{0} \frac{d\phi}{\sqrt{\tan^2(\phi/2)\,[4\cos^2(\phi/2) - (1+f')^2] + \epsilon}}.$$

Here we note that the value $v = 1$ corresponding to $\mathbf{b} = \hat{\mathbf{e_3}}$. However, $k \sim \tan(\frac{\phi}{2})\sqrt{\epsilon}$ and (3.2) gives

$$w_1 = -\sqrt{\epsilon}\sin\theta, \quad w_2 = \sqrt{\epsilon}\cos\theta, \quad \omega_3 = 1, \quad \text{as} \quad t \to T_2, \tag{3.25}$$

after which the value of k again becomes positive and ϕ increases from zero to its maximum value $2\cos^{-1}(\sqrt{\zeta_1})$ and this cycle repeats itself ad infinitum. Thus the points $v = 1$ and $v = \zeta_1$ represent a pair of circles of transit points and the solution of the system of ODEs (2.6) lying on the invariant surface (3.19) continuously oscillate between these circles of transit points in $\mathbf{b}$-space.

On the other hand, for $\epsilon < 0$, equation (3.21) does not permit ϕ to approach zero. In fact the roots of the cubic polynomial $C(v)$ are real and satisfy $0 < \zeta_1 < \zeta_2 < 1$, forcing v to be in the interval $[\zeta_1, \zeta_2]$. Note that k vanishes along the pair of circles given by $2\cos^{-1}(\sqrt{\zeta_1})$ and $2\cos^{-1}(\sqrt{\zeta_2})$. These circles consist of *transit points* determining a frustum in which $\mathbf{b}$ is constrained to lie.

The equation governing θ is again (3.9) which in conjunction with (3.22) can be written as

$$\frac{d\theta}{d\phi} = \pm\frac{(1+f')\sec^2(\frac{\phi}{2}) - 2f'}{2\sqrt{\tan^2(\frac{\phi}{2})\Big(4\cos^2(\frac{\phi}{2}) - (1+f')^2\Big) + \epsilon}}\,. \tag{3.26}$$

Hence $\theta(t)$ may be expressed as an elliptic function of $\tan(\frac{\phi}{2})$.

In the special case when $\epsilon = -[2-(1+f')]^2$ the cubic polynomial $C(v)$ has two equal roots $\frac{(1+f')}{2}$, the frustum $\zeta_1 \le v \le \zeta_2$ is squeezed to a circle and the locus $k = 0$ does provide a periodic solution to the system (2.6) given by

$$\phi = 2\cos^{-1}\left(\sqrt{\frac{1+f'}{2}}\right), \quad \dot{\theta} = 1 - f'. \tag{3.27}$$

We summarize these results in the form of following theorem.

Theorem 3.1 *The solutions of the system of ODEs (2.6) lying on the two dimensional invariant surface (3.19) oscillate between circles of transit points and are expressible in terms of elliptic functions.*

3.2.1 The center manifold

We have noticed in previous section that if we perturb the initial conditions so that the first integrals assumes the values as indicated in equations (3.19), then the system admits a periodic solution lying on the invariant surface (3.19) when $\epsilon = -[2-(1+f')]^2$. This suggest the possibility of a more general perturbation that is, involving several parameters, resulting in a one parameter family of periodic solutions spanning a two dimensional invariant set that defines the center manifold.

We now proceed to obtain the center manifold through the rest point $(R_2\hat{\mathbf{e_3}},\, R_1\hat{\mathbf{e_3}})$ as the locus of a one parameter family of periodic solutions. At the place of equation (3.19) we assign to the constants the values given by

$$c_1 = R_1^2, \quad c_2 = R_2, \quad c_3 = R_2^2 + 2R_1 + \epsilon, \quad c_4 = R_1(R_2 + f'). \tag{3.28}$$

Instead of (3.2) we get

$$w_1 = \frac{-kR_2b_2}{R_1 - b_3} + \frac{(R_2 + f')b_1}{R_1 + b_3}, \quad w_2 = \frac{kR_2b_1}{R_1 - b_3} + \frac{(R_2 + f')b_2}{R_1 + b_3}, \quad w_3 = R_2. \tag{3.29}$$

Substituting in $|\mathbf{w}|^2 + 2\hat{\mathbf{e_3}} \cdot \mathbf{b} = R_2^2 + 2R_1 + \epsilon$ and using spherical polar coordinates, we find the value of k to be

$$k^2 = R_2^{-2} \tan^2\left(\frac{\phi}{2}\right)\left[4R_1 \sin^2\left(\frac{\phi}{2}\right) - (R_2 + f')^2 \tan^2\left(\frac{\phi}{2}\right) + \epsilon\right], \tag{3.30}$$

consequently we obtain the ODE for ϕ as given below

$$\left(\frac{d\phi}{dt}\right)^2 = \left[4R_1 \sin^2\left(\frac{\phi}{2}\right) - (R_2 + f')^2 \tan^2\left(\frac{\phi}{2}\right) + \epsilon\right].$$

Using the change of variable $v = \cos^2(\phi/2)$ the above equation transforms into the following ODE for elliptic function

$$\left(\frac{dv}{dt}\right)^2 = (v-1)\left[4R_1v^2 - \left(4R_1 + (R_2 + f')^2 + \epsilon\right)v + (R_2 + f')^2\right]. \tag{3.31}$$

The two roots of the cubic polynomial on the right hand side of (3.31) coincide (keeping v real) if and only if $\epsilon = -\left(R_2 + f' - 2\sqrt{R_1}\right)^2$, and corresponding repeated root is

$$\cos^2\left(\frac{\phi_0}{2}\right) = \frac{R_2 + f'}{2\sqrt{R_1}}. \tag{3.32}$$

The condition that the system of ODEs (2.6) admits a periodic solution $\cos^2(\frac{\phi_0}{2}) =$ constant is similar to the coalescence condition. Equation for $\dot{\theta}$ is

$$\dot{\theta} = \frac{R_2 + f' - 2f' \cos^2\left(\frac{\phi}{2}\right)}{2\cos^2\left(\frac{\phi}{2}\right)},$$

hence for the periodic trajectory we get $\dot{\theta} = \frac{R_2\sqrt{R_1} - f'(R_2 + f' - \sqrt{R_1})}{R_2 + f'}$. In particular, taking $R_1 = (\omega + f')^2$ we get the family of periodic trajectories parameterized by ω:

$$\begin{aligned} &w_1 = (R_2 + f')\tan\left(\frac{\phi_0}{2}\right)\cos(\omega t),\ w_2 = (R_2 + f')\tan\left(\frac{\phi_0}{2}\right)\sin(\omega t),\ w_3 = R_2, \\ &b_1 = R_1 \sin(\phi_0)\cos(\omega t),\ b_2 = R_1 \sin(\phi_0)\sin(\omega t),\ b_3 = R_1\cos(\phi_0). \end{aligned} \tag{3.33}$$

We see that when $\omega = \left(\frac{R_2 - f'}{2}\right)$, the value of ϕ_0 vanishes and the periodic trajectory collapses to the rest point $(R_2\hat{\mathbf{e_3}},\, R_1\hat{\mathbf{e_3}})$ and the family (3.33) is the center manifold through the rest point.

We summarize our observations in the form of the following theorem.

Theorem 3.2 *The ODE reductions (2.3) of the Boussinesq equations with stratification and rotation form a completely integrable system if Rayleigh number Ra vanishes. Further, when $0 < f' < 1$, the critical point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ is degenerate with two dimensional stable, unstable and center manifolds, and when $f' = 1$, the invariant surface (2.7), which is an intersection of four first integrals, degenerates into the critical point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$, whereas for $f' > 1$, the invariant surface is empty.*

4 Missing First Integral

Here we present some details on the computation of the evasive missing first integral whose existence is guaranteed by Jacobi's theorem.

$$\begin{aligned} z_j &= w_j, \qquad j = 1, 2, 3, \\ z_4 &= |\mathbf{b}|^2, \\ z_5 &= \mathbf{w}\cdot\mathbf{b} + f'\hat{\mathbf{e_3}}\cdot\mathbf{b}, \\ z_6 &= |\mathbf{w}|^2 + 2\hat{\mathbf{e_3}}\cdot\mathbf{b} = z_1^2 + z_2^2 + z_3^2 + 2b_3\,. \end{aligned} \tag{4.1}$$

Now we determine the ODEs for z_j, $1 \le j \le 6$. From equations (2.6) and (2.4) we get

$$\dot{z_1} = f'z_2 - b_2, \quad \dot{z_2} = -f'z_1 + b_1, \quad \dot{z_j} = 0, \quad 3 \le j \le 6, \tag{4.2}$$

so that for $3 \le j \le 6$, z_j are constant and

$$z_5 = w_1b_1 + w_2b_2 + w_3b_3 + f'b_3 = z_1b_1 + z_2b_2 + (z_3 + f')b_3,$$

$$\begin{aligned} z_1b_1 + z_2b_2 &= z_5 - \frac{(z_3+f')z_6}{2} + \frac{(z_3+f')z_3^2}{2} + \frac{(z_3+f')}{2}(z_1^2+z_2^2) \\ &= A + B(z_1^2+z_2^2), \end{aligned} \tag{4.3}$$

where

$$A = z_5 - \frac{(z_3+f')}{2}(z_6 - z_3^2), \quad B = \frac{z_3+f'}{2}. \tag{4.4}$$

The general solution of equation (4.3) is given by

$$b_1 = \frac{-z_2k}{z_1^2+z_2^2} + \frac{Az_1}{z_1^2+z_2^2} + Bz_1, \quad b_2 = \frac{z_1k}{z_1^2+z_2^2} + \frac{Az_2}{z_1^2+z_2^2} + Bz_2\,, \tag{4.5}$$

where k is an arbitrary parameter. On substituting this in equation (4.1) we get

$$\begin{aligned} z_4 \;=\; & \left(\frac{-z_2k}{z_1^2+z_2^2} + \frac{Az_1}{z_1^2+z_2^2} + Bz_1\right)^2 + \left(\frac{z_1k}{z_1^2+z_2^2} + \frac{Az_2}{z_1^2+z_2^2} + Bz_2\right)^2 \\ & + \left(\frac{(z_6 - z_3^2) - (z_1^2+z_2^2)}{2}\right)^2, \end{aligned}$$

which after simplification gives the value of k^2 as

$$k^2 = -A^2 + C(z_1^2+z_2^2) + D(z_1^2+z_2^2)^2 - \frac{1}{4}(z_1^2+z_2^2)^3 := \psi(z_1^2+z_2^2).$$

Here C and D are given by

$$C = z_4 - 2AB - \frac{1}{4}(z_6 - z_3^2)^2, \quad D = -B^2 + \frac{1}{2}(z_6 - z_3^2).$$

Rewriting the ODE (4.2) as

$$\frac{\dot{z_1}}{\dot{z_2}} = \frac{f' z_2 - b_2}{-f' z_1 + b_1}$$

and substituting for b_1 and b_2 from equation (4.5) we get

$$\frac{f'}{2}\frac{d(z_1^2+z_2^2)}{dt} - \left\{ \left(\frac{-z_2 k}{z_1^2+z_2^2} + \frac{A z_1}{z_1^2+z_2^2} + B z_1 \right) \dot{z_1} + \left(\frac{z_1 k}{z_1^2+z_2^2} + \frac{A z_2}{z_1^2+z_2^2} + B z_2 \right) \dot{z_2} \right\} = 0.$$

After simplification this can be written as

$$\left(\frac{f' - B}{4} \right) \frac{d}{dt}(z_1^2+z_2^2)^2 - \frac{A}{2}\frac{d}{dt}(z_1^2+z_2^2) - k(z_1\dot{z_2} - z_2\dot{z_1}) = 0\,,$$

which on integrating gives the first integral

$$\tan^{-1}(z_2/z_1) + \frac{1}{2}\int \left\{ (z_1^2+z_2^2)\sqrt{\psi(z_1^2+z_2^2)} \right\}^{-1} [A - (f'-B)(z_1^2+z_2^2)] d(z_1^2+z_2^2). \quad (4.6)$$

The integral term in equation (4.6) is an elliptic function and the term $\tan^{-1}(z_2/z_1)$ explains the spiraling of the solution curves on the surface of intersection of first integrals in equation (2.4). If $f' = 0$, then the equation (4.6) agrees with the missing first integral obtained by Srinivasan et al [4] in their study of integrable system of stratified Boussinesq equations without effects of rotation.

Note that the above first integral is singular in a neighborhood of the rest point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$. The values of A, B, C, D are given by

$$A = 0, \quad B = \frac{1+f'}{2}, \quad C = 0, \quad D = \frac{4-(1+f')^2}{4}$$

and a function ψ is given by

$$\psi(z_1^2+z_2^2) = (z_1^2+z_2^2)^2 \left[\frac{4-(1+f')^2 - (z_1^2+z_2^2)}{4} \right]$$

so (4.6) simplifies to

$$\tan^{-1}\left(\frac{z_2}{z_1}\right) + \frac{(1-f')}{2}\int \frac{d(z_1^2+z_2^2)}{(z_1^2+z_2^2)\sqrt{H-(z_1^2+z_2^2)}},$$

where $H = 4 - (1+f')^2$. It implies that the first integral (4.6) is singular at $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$.

5 Conclusion

In this paper we have incorporated the effects of rotation in a stratified Boussinesq equations in the context of dynamics of an uniformly stratified fluid contained in a rectangular basin of dimension $L \times L \times H$. The ODE reductions provide a system of six coupled equations, which is completely integrable if a Rayleigh number $Ra = 0$. For

$0 < f' = \frac{f}{2r_h} < 1$, the critical point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ of the system (2.6) is degenerate with two dimensional unstable, stable and center manifolds. For $f' = 1$ the invariant surface (2.7) degenerates into the critical point $(\hat{\mathbf{e_3}}, \hat{\mathbf{e_3}})$ whereas for $f' > 1$ the invariant surface (2.7) is empty. The two dimensional compact invariant surface on which the solution curves develop is a torus, one of whose generating circle pinched to a critical point. We have obtained the analytical solutions of the system (2.6) lying on the invariant surface. Moreover these solutions are elementary functions, if a critical point lies on this invariant surface; whereas if there are no critical points lying on the invariant surface, the solutions are expressible in terms of elliptic functions.

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