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Oscillation of Solutions and Behavior of the Nonoscillatory Solutions of Second-order Nonlinear Functional Equations

J. Tyagi *

TIFR Centre For Applicable Mathematics, Post Bag No.-6503, Sharda Nagar, Chikkabommasandra, Bangalore-560065, Karnataka, India.

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Abstract: The aim of this study is to present new oscillation theorems for certain classes of second-order nonlinear functional differential equations of the type

$$x''(t) + p(t)f(x(t),\, x(\tau(t))) = 0, \qquad (*)$$
$$x''(t) + p_1(t)f_1(t,\, x(t),\, x'(t))x'(t) + q(t)g_1(x(\tau(t)) = 0,\ t \in [t_0,\, \infty),\, t_0 > 0.$$

In the study of Eq. $(*)$, no sign condition on $p(t)$ is explicitly assumed. Also, we study the behavior of the nonoscillatory solution of Eq. $(*)$.

Keywords: *nonlinear; functional differential equations; oscillatory solution; nonoscillatory solution.*

Mathematics Subject Classification (2000): 34K11, 34K12, 34C10.

1 Introduction

Over the last three decades, many studies have dealt with the oscillation theory for functional differential equations. For an excellent bibliography and later developments of this theory, we refer to the books by Agarwal, Bohner and Wan–Tong Li [1], Erbe, Kong and Zhang [3], Gopalsamy [4], Györi and Ladas [6], Ladde, Lakshmikantham and Zhang [10]. In this note, we consider the second-order nonlinear functional differential equations of the form

$$x''(t) + p(t)f(x(t),\, x(\tau(t))) = 0, \tag{1.1}$$
$$x''(t) + p_1(t)f_1(t,\, x(t),\, x'(t))x'(t) + q(t)g_1(x(\tau(t)) = 0,\ t \in [t_0,\, \infty), \tag{1.2}$$

* Corresponding author: jtyagi1@gmail.com

where $p \in C([t_0, \infty), \mathbb{R}), p_1, q \in C([t_0, \infty), \mathbb{R}^+), f \in C(\mathbb{R}^2, \mathbb{R}), f_1 \in C([t_0, \infty) \times \mathbb{R}^2, \mathbb{R}^+), g_1 \in C(\mathbb{R}, \mathbb{R}), yg_1(y) > 0, \forall 0 \neq y \in \mathbb{R}, \tau \in C^1([t_0, \infty), \mathbb{R}^+), \tau'(t) > 0$ for all large t and $\tau(t) \to \infty$ as $t \to \infty$. In case $p(t)$ is positive, the oscillation criteria for Eq. (1.1) and its special case

$$x''(t) + p(t)F_1(x(\tau(t))) = 0, \ t \in [t_0, \infty)$$

is extensively studied by many investigators in this area (see, [7, 8, 13–15] and the references cited therein). All of them restrict the sign condition on $p(t)$; i.e., $p(t) \geq 0, \ \forall t \in [t_0, \infty)$. For the oscillation of Eq. (1.1), our study is free from such restriction. Also, as far as the author knows there is no oscillation result in literature for Eq. (1.2). The ideas of [2] are used to extend the oscillation results for Eq. (1.2). Let $\psi : [\tau(t_0), t_0] \to \mathbb{R}$ is a continuous function. By a solution of Eq. (1.1) (resp. Eq. (1.2)), we mean a continuously differentiable function $x : [\tau(t_0), \infty] \to \mathbb{R}$ such that $x(t) = \psi(t)$ for $\tau(t_0) < t_0$ and x satisfies Eq. (1.1) (resp. Eq. (1.2)) $\forall t \geq t_0$. We restrict our discussion to the nontrivial solutions of Eq. (1.1) (resp. Eq. (1.2)). A nontrivial solution of Eq. (1.1) (resp. Eq. (1.2)) is said to be oscillatory if it has arbitrarily large zeros, i.e., for any $T_1 > t_0, \ \exists t \geq T_1$ such that $x(t) = 0$, otherwise the solution is said to be nonoscillatory.

The paper is organized as follows. Section 2 deals with the oscillation theorems for Eqs. (1.1) and (1.2). The behavior of nonoscillatory solution of Eq. (1.1) is discussed in Section 3. In Section 4, we construct some examples for the illustration of these results.

2 Oscillation Theorems

We begin this section with the list of hypotheses:

(H1) $p(t) > 0$ for t sufficiently large.

(H2) $f(y_1, y_2) > 0$ if $y_i > 0$; $f(y_1, y_2) < 0$ if $y_i < 0, \forall i = 1, 2$.

(H3) $f(y_1, y_2)$ is a continuously differentiable function w. r. t. y_1 and y_2 and

suppose there exists $\alpha > 0$ such that $\frac{\partial}{\partial y_i} f(y_1, y_2) \geq \alpha$ for $y_i \neq 0, \forall i = 1, 2$.

(H4) There exist a C^1 function u defined on $[t_0, \infty)$, a C^1 function F on $\mathbb{R}$ and

a continuous function J on $\mathbb{R}$ such that $F'(u) = \sqrt{\alpha} J(u), \ F(u) \geq \frac{(J(u))^2}{4}$.

(H5) $\liminf\limits_{t \to \infty} \frac{1}{t} \int\limits_{t_0}^{t} [(u'(s))^2 - p(s)F(u(s))] ds < 0$.

(H6) Let $U = \{(t, s) \in [t_0, \infty) \times [t_0, \infty) \text{ such that } t > s \geq 0\}$.

There exists a function $G \in C(U, \mathbb{R})$ such that $G(t, s) > 0$,

$\frac{\partial}{\partial s} G(t, s) \leq 0$ on U and $G(t, t) = 0, \ \forall t \geq t_0$.

(H7) Let there exist $h \in C^1(([t_0, \infty), (0, \infty))$ such that $h'(t) \leq 0, \forall t \in [t_0, \infty)$ and

(i) $\int\limits_{t_0}^{\infty} q(s)h(s)ds = \infty$.

(ii) $\limsup\limits_{t\to\infty} \frac{1}{G(t,\,t^*)} \int\limits_{t_0}^{t} G(t,\,s)q(s)h(s)ds = \infty,\ \ \forall\, t^* \geq t_0$.

(H8) Let there exist $h \in C^1([t_0,\,\infty),\,(0,\,\infty))$ such that $-\infty < \int\limits_{t_0}^{\infty} \frac{h'(t)}{h(t)}dt < \infty$ and $\int\limits_{t_0}^{\infty} q(t)h(t)\exp^{-\int\limits_{t^*}^{t} \frac{h'(s)}{h(s)}ds} dt = \infty$ for some $t^* > t_0$.

(H9) $g_1 \in C^1(B,\,\mathbb{R})$ such that $yg_1(y) > 0,\ \forall\, 0 \neq y \in \mathbb{R}$ and $\exists\, \beta > 0$ such that $g_1'(y) \geq \beta > 0,\ \forall\, 0 \neq y \in B,$ where $B = (-\infty,\,-N) \cup (N,\,\infty),\ N > 0.$

(H10) $\int\limits_{t_0}^{\infty} \left(\int\limits_{u}^{\infty} q(s)ds \right) du = \infty.$

Remark 2.1 Hypotheses (H4), (H5) are the extension of the conditions introduced by V. Komkov [9] and (H9), (H10) are given by Baculíková [2].

Lemma 2.1 *Let x be a nonoscillatory solution of (1.1) on* $[T,\,\infty)$ *and let* (H1)–(H3) *hold. Then for all large t, we have $x(t)x'(t) > 0$.*

Proof Without any loss of generality, this solution can be supposed to be such that $x(t) > 0$ for $t \geq T_1 \geq T$. Further, we observe that the substitution $u = -x$ transforms (1.1) into the Eq.

$$u''(t) + p(t)\bar{f}(u(t),\,u(\tau(t))) = 0, \tag{2.1}$$

where $\bar{f}(u_1,\,u_2) = -f(-u_1,\,-u_2)$. The function $\bar{f}$ is subject to the same conditions as f. So, there is no loss of generality to restrict our discussion to the case when the solution x is positive on $[T_1,\,\infty)$. If this lemma is not true, then either $x'(t) < 0$ for all large t or $x'(t)$ oscillates. By (H1), we choose T_1 sufficiently large so that $p(t) > 0,\ x'(t) < 0,\ \forall\, t \geq T_1$. This implies that

$$\int_{T_1}^{t} p(s)ds \geq 0, \text{ and } x'(\tau(t)) < 0,\ \forall\, t \geq T_1.$$

Hence, we have

$$\int_{T_1}^{t} p(s)f(x(s),\,x(\tau(s)))ds = f(x(t),\,x(\tau(t)))\int_{T_1}^{t} p(s)ds - \int_{T_1}^{t} \Big(\frac{\partial}{\partial x(s)} f(x(s),\,x(\tau(s))))x'(s) + \frac{\partial}{\partial x(\tau(s))} f(x(s),\,x(\tau(s))))x'(\tau(s))\tau'(s)\Big) \left(\int_{T_1}^{s} p(\sigma)d\sigma \right) ds \geq 0, \forall\, t \geq T_1.$$

Now integrating (1.1), we get

$$x'(t) \leq x'(T_1) < 0,\ \ \forall\, t \geq T_1,$$

which contradicts the fact that $x(t)$ is nonoscillatory.

If $x'(t)$ is oscillatory. Then $\exists\,\{t_n\} \subset [t_0,\,\infty)$ such that $t_n \to \infty$ as $n \to \infty$ and $x'(t_n) = 0, \forall\, n \in \mathbb{N}$. Let $\hat{t} > T_1$ be the zero of x'. This implies that $x'(\hat{t}) = 0,\ x''(\hat{t}) < 0$, from which one can prove that x' can not have another zero after it vanishes for large t, which is a contradiction. This completes the proof of the lemma.

Remark 2.2 For a lemma, similar to Lemma 2.1 under a similar hypothesis, we refer the reader to [11].

Theorem 2.1 *Under the hypotheses* (H1)–(H5)*, Eq. (1.1) is oscillatory.*

Proof Suppose on the contrary, (1.1) has a nonoscillatory solution $x(t)$. Then there exists some $t_1 \geq t_0$ such that either $x(t) > 0$ or $x(t) < 0, \ \forall t \geq t_1$.

Case 1. $x(t) > 0, \forall t \geq t_1$. By Lemma 2.1, we have $x(t)x'(t) > 0,$ for all large t. So, we choose a T sufficiently large such that $x(t)x'(t) > 0, \ \forall t \geq T$. This implies that $x'(\tau(t)) > 0, \ \forall t \geq T$. Now we note that the following identity is valid on $[T, \infty)$:

$$\begin{aligned}
&(u'(t))^2 - p(t)F(u(t)) + \frac{F(u(t))}{f(x(t),\, x(\tau(t)))}[x''(t) + p(t)f(x(t),\, x(\tau(t)))] \\
&= \left(\frac{x'(t)F(u(t))}{f(x(t),\, x(\tau(t)))}\right)' + \frac{(\frac{\partial}{\partial x(\tau(t))} f(x(t),\, x(\tau(t))))\ x'(t)x'(\tau(t))\tau'(t)F(u(t))}{(f(x(t),\, x(\tau(t)))^2} \\
&+ \frac{(\frac{\partial}{\partial x(t)} f(x(t),\, x(\tau(t))))x'(t)x'(t)F(u(t))}{(f(x(t),\, x(\tau(t))))^2} - \left(\frac{x'(t)F'(u(t))u'(t)}{f(x(t),\, x(\tau(t)))}\right) + (u'(t))^2.
\end{aligned}$$

$$\begin{aligned}
&(u'(t))^2 - p(t)F(u(t)) + \frac{F(u(t))}{f(x(t),\, x(\tau(t)))}\,[x''(t) + p(t)f(x(t),\, x(\tau(t)))] \\
&\geq \left(\frac{x'(t)F(u(t))}{f(x(t), x(\tau(t)))}\right)' - \left(\frac{x'(t)\sqrt{\alpha}\, J(u(t))u'(t)}{f(x(t),\, x(\tau(t)))}\right) + \frac{\alpha(x'(t))^2(J(u(t)))^2}{4(f(x(t),\, x(\tau(t))))^2} + (u'(t))^2 \\
&\geq \left(\frac{x'(t)F(u(t))}{f(x(t),\, x(\tau(t)))}\right)' + \left[u'(t) - \frac{x'(t)\sqrt{\alpha}\, J(u(t))}{2f(x(t),\, x(\tau(t)))}\right]^2.
\end{aligned}$$

Since x being a solution of (1.1), so, we get

$$(u'(t))^2 - p(t)F(u(t)) \geq \left(\frac{x'(t)F(u(t))}{f(x(t),\, x(\tau(t)))}\right)' + \left[u'(t) - \frac{x'(t)\sqrt{\alpha}\ J(u(t))}{2f(x(t),\, x(\tau(t)))}\right]^2.$$

An integration over $[T, \infty)$ yields

$$\begin{aligned}
&\int_T^t [(u'(s))^2 - p(s)F(u(s))]ds \\
&\geq \int_T^t \left(\frac{x'(s)F(u(s))}{f(x(s),\, x(\tau(s)))}\right)' ds \\
&\geq \frac{x'(t)F(u(t))}{f(x(t),\, x(\tau(t)))} - \frac{x'(T)F(u(T))}{f(x(T),\, x(\tau(T)))}.
\end{aligned}$$

So,

$$\frac{1}{t}\int_T^t [(u'(s))^2 - p(s)F(u(s))]ds \geq -\frac{1}{t}\frac{x'(T)F(u(T))}{f(x(T),\, x(\tau(T)))} \to 0 \ \text{ as } \ t \to \infty,$$

which contradicts to (H5).

Case 2. $x(t) < 0, \forall t \geq t_1$. For large t, we have, $x(t) < 0, \ x(\tau(t)) < 0, \ \forall t \geq T$, where T is sufficiently large. By Lemma 2.1, we have $x'(t) < 0, \ \forall t \geq T$. Now the rest of the proof of case 2 is similar to the proof of case 1 and we omit the proof for brevity. This completes the proof of the theorem.

The next lemma is used in the proof of the next theorems.

Lemma 2.2 *Let $p_1(t) \geq 0$ and $q(t)$ be continuous non-negative and not identically zero on any ray of the form $[t^*, \infty)$, $t^* \geq t_0$ and assume that*

(i) $f_1(t, x, y) \leq | y |^\lambda$, $-\infty < x, y < \infty$, $t \geq t_0$ *and some constant* $\lambda \geq 0$.

(ii) $\left(1 + \int_{t_0}^{t} p_1(s)ds\right)^{-\frac{1}{\lambda}} \notin L(t_0, \infty)$, *if* $\lambda > 0$,

$$\int_{t_0}^{\infty} \exp\left(\int_{t_0}^{s} -p_1(\sigma)d\sigma\right) ds = \infty, \text{ if } \lambda = 0.$$

If $x(t)$ is a non-oscillatory solution of Eq. (1.2), then $x(t)x'(t) > 0$ for all large t.

For the proof of this lemma, we refer the reader to [5].

Theorem 2.2 *Let $p_1(t) \geq 0$ and $q(t)$ be continuous non-negative and not identically zero on any ray of the form $[t^*, \infty)$, $t^* \geq t_0$. Let $\tau(t) < t$, for large t. Let the conditions* (i), (ii) *hold. Then under the hypotheses* (H8)–(H10), *Eq. (1.2) is oscillatory.*

Proof Suppose on the contrary, (1.2) has a nonoscillatory solution $x(t)$. Then there exists some $t_1 \geq t_0$ such that either $x(t) > 0$ or $x(t) < 0$, $\forall\, t \geq t_1$.

Case 1. $x(t) > 0$, $\forall\, t \geq t_1$. By Lemma 2.2, we have $x(t)x'(t) > 0$, $\forall\, t \geq T$, where $T > t_0$ is sufficiently large. We define

$$w(t) = \frac{x'(t)h(t)}{g_1(x(\tau(t)))}, \ \forall\, t \geq T, \tag{2.2}$$

where h is appearing in (H8). Differentiating $w(t)$ and by Eq. (1.2), we get

$$\begin{aligned} w'(t) &= \frac{-h(t)p_1(t)x'(t)f_1(t, x(t), x'(t))}{g_1(x(\tau(t)))} - q(t)h(t) + \frac{x'(t)h'(t)}{g_1(x(\tau(t)))} \\ &\quad - \frac{x'(t)g_1'(x(\tau(t)))x'(\tau(t))\tau'(t)h(t)}{(g_1(x(\tau(t))))^2} \\ &\leq -q(t)h(t) - \frac{w(t)g_1'(x(\tau(t)))x'(\tau(t))\tau'(t)}{g_1(x(\tau(t)))} + \frac{h'(t)w(t)}{h(t)}. \end{aligned}$$

Since x' is a decreasing function for $t \geq T$ and $\tau(t) < t$. So,

$$w'(t) \leq -q(t)h(t) - \frac{(w(t))^2 g_1'(x(\tau(t)))\tau'(t)}{h(t)} + \frac{h'(t)w(t)}{h(t)}. \tag{2.3}$$

Now we claim that $x(t) \to \infty$ as $t \to \infty$. Suppose not, then $0 < x(t) \leq M < \infty$, as $t \to \infty$. We may also assume that $0 < x(\tau(t)) \leq M < \infty$, as $t \to \infty$. Since $x'(t)$ is positive and decreasing, so $\lim_{t\to\infty} x'(t)$ exists and is finite. An integration of Eq. (1.2) from t to ∞, yields

$$\int_t^{\infty} x''(s)ds = -\int_t^{\infty} p(s)f_1(s, x(s), x'(s))x'(s)ds - \int_t^{\infty} q(s)g_1(x(\tau(s)))ds, \ t \geq T.$$

This implies that $x'(\infty) - x'(t) \leq -\int_t^{\infty} q(s)g_1(x(\tau(s)))ds$ or

$$x'(t) \geq \int_t^{\infty} q(s)g_1(x(\tau(s)))ds, \ t \geq T. \tag{2.4}$$

Let

$$\delta = \min_{u \in [L,\, M]} g_1(u)$$

for some $L > 0$. Then $0 < \delta \leq g_1(x(\tau(s)))$. From inequality (2.4), we get

$$x'(t) \geq \delta \int_t^{\infty} q(s)ds.$$

An integration over $(t_0,\, t)$ of the above inequality yields

$$x(t) \geq x(0) + \delta \int_{t_0}^{t} \left(\int_u^{\infty} q(s)ds \right) du.$$

Letting $t \to \infty$ in above inequality, we get a contradiction from (H10). So, our claim is true and hence $x(\tau(t)) \in B$ for all large t. Now from (2.3) and (H9), we get

$$w'(t) \leq -q(t)h(t) - \frac{(w(t))^2 \beta \tau'(t)}{h(t)} + \frac{h'(t)w(t)}{h(t)} \leq -q(t)h(t) + \frac{h'(t)w(t)}{h(t)}. \tag{2.5}$$

From inequality (2.5), we get

$$w(t) \leq w(T_1) \exp^{-\int\limits_T^{T_1} \frac{h'(s)}{h(s)} ds} \exp^{\int\limits_T^{t} \frac{h'(s)}{h(s)} ds} - \exp^{\int\limits_T^{t} \frac{h'(s)}{h(s)} ds} \int\limits_{T_1}^{t} q(s)h(s) \exp^{-\int\limits_T^{s} \frac{h'(u)}{h(u)} du} ds, \tag{2.6}$$

where $t \geq T_1 > T$. Letting $t \to \infty$, from (H8), we get $w(t) \to -\infty$, which is a contradiction as $w(t) > 0$.

Case 2. $x(t) < 0,\ \forall\, t \geq t_1$. The proof of case 2 is similar to the proof of case 1 and we omit the proof for brevity. This completes the proof of the theorem.

Theorem 2.3 *Let* (H8) *be replaced by* (H7(i)) *in Theorem 2.2. Then Eq. (1.2) is oscillatory.*

Proof Suppose on the contrary, (1.2) has a nonoscillatory solution $x(t)$. As in the foregoing text, there exists some $t_1 \geq 0$ such that either $x(t) > 0$ or $x(t) < 0,\ \forall\, t \geq t_1$.
Case 1. $x(t) > 0,\ \forall\, t \geq t_1$. By Lemma 2.2, we have $x(t)x'(t) > 0,\ \forall\, t \geq T$, where $T > 0$ is sufficiently large. We define

$$w(t) = \frac{x'(t)h(t)}{g_1(x(\tau(t)))},\ \forall\, t \geq T, \tag{2.7}$$

where h is appearing in (H7). As in the proof of Theorem 2.2, we have Inequality (2.5)

$$w'(t) \leq -q(t)h(t) + \frac{h'(t)w(t)}{h(t)}.$$

In view of (H7), we get

$$w'(t) \leq -q(t)h(t). \tag{2.8}$$

An integration over $(T,\, \infty)$ yields

$$w(t) \leq w(T) - \int_T^t q(s)h(s)ds.$$

Letting $t \to \infty$ in above inequality, we get a contradiction from (H7(i)).
Case 2. $x(t) < 0,\ \forall\, t \geq t_1$. The proof of case 2 is similar to the proof of case 1 and we omit the proof. This completes the proof of the theorem.

Theorem 2.4 *Let* (H6) *hold and suppose* (H8) *be replaced by* (H7(ii)) *in Theorem 2.2. Then Eq. (1.2) is oscillatory.*

Proof Suppose on the contrary, (1.2) has a nonoscillatory solution $x(t)$.
Case 1. $x(t) > 0,\ \forall\, t \geq t_1$. By Lemma 2.2, we have $x(t)x'(t) > 0,\ \ \forall\, t \geq T$, where $T > 0$ is sufficiently large. We define

$$w(t) = \frac{x'(t)h(t)}{g_1(x(\tau(t)))},\ \forall\, t \geq T,$$

where h is appearing in (H7). From (2.8), we have

$$\begin{aligned}\int_T^t G(t,\, s)q(s)h(s)ds \qquad &\leq -G(t,\, t)w(t) + G(t,\, T)w(T) + \int_T^t \frac{\partial G(t,\, s)}{\partial s} w(s)ds \\ &\leq\ G(t,\, T)w(T),\end{aligned}$$

which implies that

$$\frac{1}{G(t,\, T)} \int_T^t G(t,\, s)q(s)h(s)ds \leq w(T).$$

Letting $t \to \infty$, we get a contradiction from (H7(ii)).
Case 2. $x(t) < 0,\ \forall\, t \geq t_1$. The proof of case 2 is similar to the proof of case 1 and hence is omitted.

Remark 2.3 Theorems 2.2, 2.3 and 2.4 can be applied to sublinear and superlinear equations as the boundedness of $g_1'(y)$ is not required near zero.

3 Behavior of Nonoscillatory Solutions

In this section, we study the behavior of nonoscillatory solutions of Eq. $(*)$. In fact, we study the behavior of nonoscillatory solutions of

$$x''(t) + P(t)f(x(t),\, x(\tau(t)))g(x'(t)) = 0, t \in [t_0,\, \infty), \tag{3.1}$$

where $P \in C([t_0,\, \infty),\, \mathbb{R}^+),\ f \in C(\mathbb{R}^2,\, \mathbb{R}),\ g \in C(\mathbb{R},\, \mathbb{R})$. Let there exist $k > 0, l > 0$ such that $\frac{f(x,\, y)}{x} \geq k > 0, \forall\, 0 \neq x \in \mathbb{R},\, y \in \mathbb{R}$ and $g(y) \geq l > 0,\ y \in \mathbb{R}$. Let $\tau \in C([t_0,\, \infty),\, \mathbb{R})$. Let there exists $\mu > 0$. Consider the second-order linear differential equation

$$x''(t) + \lambda P(t)x(t) = 0,\ \lambda > 0. \tag{3.2}$$

We establish that all nonoscillatory solutions $x(t)$ of Eq. (3.1) are such that $y(t) = O(x(t))$ as $t \to \infty$, where y is any oscillatory solution of Eq. (3.2), $\forall\, \lambda \in (0,\, \mu]$. The technique of Philos et al. [12] is employed to establish the following result. This result gives a new direction in the study of nonoscillatory behavior of functional differential equations.

Theorem 3.1 *Let x be any nonoscillatory solution of Eq. (3.1) and y be an oscillatory solution of Eq.* (3.2)*. Then $y(t) = O(x(t))$ as $t \to \infty$.*

Proof Since x is any nonoscillatory solution of Eq. (3.1), so there exists some $T_0 \geq t_0$ such that $x(t) \neq 0,\ \forall\, t \geq T_0$. There are two cases.

Case 1. $x(t) > 0,\ \forall\, t \geq T_0$. We define

$$v(t) = \frac{y(t)}{x(t)}, \quad \forall\, t \geq T_0.$$

We obtain

$$v'(t) = \frac{y'(t) - v(t)x'(t)}{x(t)},\ \ \forall\, t \geq T_0$$

and

$$v''(t) = \frac{y''(t) - v(t)x''(t) - 2v'(t)x'(t)}{x(t)},\ \ \forall\, t \geq T_0. \tag{3.3}$$

From Eqs. (3.1), (3.2) and (3.3), we get

$$v''(t) = -\frac{2v'(t)x'(t)}{x(t)} + \frac{-\lambda P(t)y(t)}{x(t)} + \frac{v(t)P(t)f(x(t),\, x(\tau(t)))g(x'(t))}{x(t)}. \tag{3.4}$$

Now we will show that v is bounded on the interval $[T_0,\, \infty)$. Assume on the contrary that v is unbounded on $[T_0,\, \infty)$. As $-y$ is also an oscillatory solution of Eq. (3.2) and $-v = \frac{-y}{x}$ on $[T_0,\, \infty)$. We may suppose that v is unbounded from above. Clearly, v is oscillatory. Thus, we can choose a sufficiently large $T \geq T_0$ so that

$$v'(T) = 0,\, v(T) >\mid v(t) \mid \ \text{ for } T_0 \leq t < T \tag{3.5}$$

and $v''(T) \leq 0$, (see, [Thm. 2, 12]). In view of Eq. (3.5), from Eq. (3.4), we get

$$v(T)P(T)[f(x(t),\, x(\tau(t)))g(x'(T)) - \lambda x(T)] \leq 0.$$

That is,

$$f(x(t),\, x(\tau(t)))g(x'(T)) - \lambda x(T) \leq 0. \tag{3.6}$$

From the hypotheses, we get

$$\frac{f(x(T),\, x(\tau(T)))}{x(T)} \geq k > 0, \quad \text{and } g(x'(T)) \geq l > 0. \tag{3.7}$$

That is,

$$\frac{f(x(T),\, x(\tau(T)))g(x'(T)) - klx(T)}{x(T)} \geq 0.$$

We choose $\mu = kl$, since $\lambda \in (0,\, \mu]$, we obtain

$$\frac{f(x(T),\, x(\tau(T)))g(x'(T)) - \lambda x(T)}{x(T)} \geq 0. \tag{3.8}$$

Eqs. (3.6) and (3.8) implies that $x(T) \leq 0$, which is a contradiction.

Case 2. $x(t) < 0,\ \forall\, t \geq T_0$. The proof of case 2 is similar to the proof of case 1 and we omit the proof for brevity. This completes the proof of the theorem.

Remark 3.1 As a hypothesis, $''$*Eq.* (3.2) is oscillatory $\forall\, \lambda > 0''$ is used by Lynn Erbe [11].

4 Examples

Finally, we give some examples to illustrate our results.

Example 4.1 Consider the differential equation

$$x''(t)+\left(1-\frac{\sin t}{t^2}\right)\left[x(t)+(x(t))^{2m+1}+x\left(\frac{t}{2}\right)+\left(x\left(\frac{t}{2}\right)\right)^{2n+1}\right]=0,\;\; m,n\in\mathbb{N},\, t>0. \tag{4.1}$$

Eq. (4.1) can be viewed as Eq. (1.1) with $p(t)=1-\frac{\sin t}{t^2}$, $f(y_1,\,y_2)=y_1+y_1^{2m+1}+y_2+y_2^{2n+1}$, $\tau(t)=\frac{t}{2}$. With the choice of $\alpha=1$, $F(u)=u^2$, $u(t)=t$, it is easy to see that the hypotheses of Theorem 2.1 are satisfied. An application of Theorem 2.1 implies that (4.1) is oscillatory.

Remark 4.1 Here $p(t)\ngeq 0$, $\forall\, t\in[t_0,\,\infty)$, so none of the known criteria [8, 13, 14] can obtain this result to Eq. (4.1).

Example 4.2 Consider the differential equation

$$x''(t)+\left(e^{-t}+\frac{2}{t^2}+\frac{1}{t^4}\right)\left(x(t)+x\left(\frac{t}{3}\right)+x\left(\frac{t}{3}\right)^5\right)=0,\;\; t>0. \tag{4.2}$$

Eq. (4.2) can be viewed as Eq. (1.1) with $p(t)=e^{-t}+\frac{2}{t^2}+\frac{1}{t^4}$, $f(y_1,\,y_2)=y_1+y_2+y_2^5$, $\tau(t)=\frac{t}{3}$. With the choice of $\alpha=1$, $F(u)=u^2$, $u(t)=t$, it is easy to see that the hypotheses of Theorem 2.1 are satisfied. An application of Theorem 2.1 implies that Eq. (4.2) is oscillatory, whereas none of the known criteria [8, 13, 14] can obtain this result to Eq. (4.2).

Example 4.3 Consider the differential equation

$$x''(t)+\frac{1}{t+1}x'(t)+\frac{1}{t^2}\left(\frac{\left(x\left(\frac{t}{3}\right)\right)^3}{\mid x\left(\frac{t}{3}\right)\mid +1}\right)=0,\;\; t>0. \tag{4.3}$$

Eq. (4.3) can be viewed as Eq. (1.2) with $p_1(t)=\frac{1}{t+1}$, $f_1(t,x,y)=1$, $q(t)=\frac{1}{t^2}$, $g_1(y)=\frac{y^3}{|y|+1}$, $\tau(t)=\frac{t}{3}$. With the choice of $h(t)=1$, it is easy to see that the hypotheses of Theorem 2.2 are satisfied. So, by Theorem 2.2, Eq. (4.3) is oscillatory.

Example 4.4 Consider the differential equation

$$x''(t)+(x'(t))^2+e^t\left(x\left(\frac{t}{2}\right)\right)^3=0. \tag{4.4}$$

Eq. (4.4) can be viewed as Eq. (1.2) with $p_1(t)=1$, $f_1(t,\,x,\,y)=y$, $q(t)=e^t$, $g_1(y)=y^3$, $\tau(t)=\frac{t}{2}$. Since $f_1(t,\,x,\,y)=y$, so in view of Lemma 2.2(i), $\lambda=1$. With the choice of $h(t)=e^{-t}$, it is easy to see that the hypotheses of Theorem 2.3 are satisfied and by Theorem 2.3, Eq. (4.4) is oscillatory in view of Lemma 2.2(i).

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