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On the Minimum Free Energy for a Rigid Heat Conductor with Memory Effects

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Abstract: A general closed expression for the minimum free energy, related to a state of a rigid heat conductor with memory, is derived in terms of Fourier-transformed functions, by using the coincidence of this quantity with the maximum recoverable work obtainable from that state. The linearized constitutive equations both for the internal energy and for the heat flux consider the effects of the actual values of the temperature and of its gradient, together with the ones of the integrated histories of such quantities, which are chosen to characterize the states of the material. An equivalent formulation for the minimum free energy is given and also used to derive explicit formulae for a discrete spectrum model.

Keywords: *fading memory; heat conduction; thermodynamics.*

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1 Introduction

A general nonlinear theory of rigid heat conductors with memory effects was proposed by Gurtin and Pipkin in [21], by using Coleman's results for materials with memory [10]. The constitutive equation for the heat flux, derived in [21] for isotropic media when small variations of the temperature and of its gradient are studied, is well known. Such a relation is expressed by a linear functional of the history of the temperature gradient and gives a generalization of the Cattaneo–Maxwell equation [9], which, therefore, is a special case of the theory in [21]. In the framework proposed in [21], later on, a linear theory of rigid heat conductors has been considered in particular in [14].

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Nunziato in [25] has subsequently developed a slightly different memory theory, which had to include the effects of the present value of the temperature gradient in the constitutive equation of the heat flux, together with the ones due to the history of the temperature gradient [11, 22]. Such a theory yields a linearized constitutive equation, for the heat flux in isotropic materials, characterized by two terms, one of which is similar to the corresponding Gurtin–Pipkin's relation, i.e. a linear functional of the past history of the temperature gradient with an integral kernel $k'(s)$, while the other term is expressed by Fourier's law, that is a term proportional to the present value of the temperature gradient with a coefficient $k_0 = k(0)$. Thus, Nunziato's linearized constitutive equation, in absence of memory effects, that is when $k'(s) = 0$, coincides with Fourier's law; moreover, if $k_0 = 0$, it reduces to Gurtin–Pipkin's linearized equation.

It is well known that the thermodynamic principles impose restrictions on any constitutive equation; the constraints related to Nunziato's relation have been derived in [19] and there used to prove a theorem of existence, uniqueness and stability of solutions to the heat equation.

In [3] this constitutive equation has been considered to derive explicit formulae for the minimum free energy for a rigid heat conductor, thus, generalizing previous articles [2, 4] related to the use of Gurtin–Pipkin's relation.

The problem of finding explicit forms for the minimum free energy associated with a given state of a material is particularly important, since it coincides with the maximum recoverable work, i.e. it allows us to determine the amount of energy available from that state. In many papers this subject has been considered, particularly for linear viscoelastic solids, see [6]-[7], [12]–[13], [15]–[17] and especially, [20], [18] and [26].

In this work we use again Nunziato's general constitutive equation to solve the analogous problems of [3], but, instead of the histories of temperature and of its gradient assumed in [3], we now choose the integrated histories of these two quantities to characterize the state of the material. The integrated histories of the temperature gradient, already introduced in the pioneer work [21], has been preferred by some authors, see for example [23]; thus, it seems interesting to study the said problems with this point of view. Therefore, in the present article the material states of the rigid body are characterized by the actual values of the temperature, as in [3], and by the integrated histories of the temperature and of its gradient.

To study these problems, in this paper, we shall refer to [14], for the linearization of the Clausius–Duhem inequality, and to [19], for the thermodynamic constraints on the constitutive equations of the internal energy and of the heat flux; finally, to derive the expression for the minimum free energy we shall follow the procedure used in [20] and [18].

As we have already observed in [3], contrary to what occurs for viscoelastic solids, for which the method used to evaluate the minimum free energy yields a Wiener–Hopf integral equation of the first kind, the use of Nunziato's relation for a rigid heat conductor yields two Wiener–Hopf integral equations but of the second kind. These two integral equations of second kind can be easily solved in the frequency domain by virtue of the thermodynamic properties of the kernels related to the expressions for the internal energy and for the heat flux, together with some theorems on factorization. Hence, an explicit expression for the minimum free energy is derived.

Another different but equivalent expression is also deduced for the minimum free energy and used to study the discrete spectrum model material response.

The layout of the paper is as follows. In Sect. 2, the linear theory, the linearized form of the Second Law of Thermodynamics and the thermodynamic constraints on the constitutive equations of the internal energy and of the heat flux are examined. In Sect. 3, we introduce the notions of states and processes together with the prolongation of histories; two particular histories are also considered. Then, in Sect. 4, we define an equivalence relation between states. After giving the expression for the thermal work in Sect. 5, in Sect. 6 we consider another equivalence relation between states by means of work and we prove its equivalence with the previous one. In Sect. 7, we derive an explicit expression for the minimum free energy. Another equivalent expression for this minimum free energy is given in Sect. 8, and, in Sect. 9, it is used to obtain explicit results for a discrete spectrum model material response.

2 Fundamental Relationships

We denote by $\mathcal{B}$ a homogeneous and isotropic rigid heat conductor, endowed of memory effects, which occupies a fixed bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. If we are concerned only with small variations of the temperature ϑ, relative to a uniform absolute temperature Θ_0, and of the temperature gradient $\mathbf{g} = \nabla\vartheta$, we can consider the linearization of the theory developed in [14]. Thus, we consider the following constitutive equations

$$\begin{aligned} e(\mathbf{x},t) &= e_0 + \alpha_0\vartheta(\mathbf{x},t) + \int_0^{+\infty} \alpha'(s)\, {}_r\vartheta^t(\mathbf{x},s)ds, && (2.1)\\ \mathbf{q}(\mathbf{x},t) &= -k_0\mathbf{g}(\mathbf{x},t) - \int_0^{+\infty} k'(s)\, {}_r\mathbf{g}^t(\mathbf{x},s)ds && (2.2)\end{aligned}$$

for the internal energy e and the heat flux $\mathbf{q}$ of $\mathcal{B}$ [25].

Here, we have denoted by $\mathbf{x} \in \Omega$ the vector position and by ${}_r\vartheta^t(\mathbf{x},s) = \vartheta(\mathbf{x},t-s)$ and ${}_r\mathbf{g}^t(\mathbf{x},s) = \mathbf{g}(\mathbf{x},t-s)$ $\forall s \in \mathbb{R}^{++} \equiv (0,+\infty)$ the past histories of ϑ and $\mathbf{g}$. The history up to time t of any function f can be expressed by means of the couple $(f(t),\, {}_rf^t)$, where $f(t)$ is the present value of f and ${}_rf^t$ its past history.

The kernels $\alpha' : \mathbb{R}^+ \to \mathbb{R}$ and $k' : \mathbb{R}^+ \to \mathbb{R}$ are the relaxation functions such that α', α'', $\alpha''' \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ and k', $k'' \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$. Moreover,

$$\alpha(t) = \alpha_0 + \int_0^t \alpha'(s)ds, \qquad k(t) = k_0 + \int_0^t k'(s)ds \tag{2.3}$$

are the heat capacity and the thermal conductivity of $\mathcal{B}$, the asymptotic values of which

$$\alpha_\infty = \lim_{t\to+\infty} \alpha(t), \qquad k_\infty = \lim_{t\to+\infty} k(t) \tag{2.4}$$

are said to be the equilibrium heat capacity and the thermal conductivity of $\mathcal{B}$.

The physical consideration that the internal energy increases, if the temperature of a body, constant up a time $t = 0$, instantaneously increases, justifies the assumption

$$\alpha_0 > 0. \tag{2.5}$$

If, for any function f, we introduce its integrated history,

$$\bar{f}^t(\mathbf{x},s) = \int_0^s f^t(\mathbf{x},\xi)d\xi = \int_{t-s}^t f(\mathbf{x},\lambda)d\lambda, \tag{2.6}$$

the internal energy (2.1) and the heat flux (2.2), by integrating by parts, assume the following forms

$$\begin{aligned} e(\mathbf{x},t) &= e_0 + \alpha_0 \vartheta(\mathbf{x},t) - \int_0^{+\infty} \alpha''(s)\bar{\vartheta}^t(\mathbf{x},s)ds, \qquad (2.7) \\ \mathbf{q}(\mathbf{x},t) &= -k_0\mathbf{g}(\mathbf{x},t) + \int_0^{+\infty} k''(s)\bar{\mathbf{g}}^t(\mathbf{x},s)ds. \qquad (2.8) \end{aligned}$$

In order to give the restrictions imposed on the constitutive equation by the thermodynamic principles, derived in [19], we recall that the Fourier transform of any function $f:\mathbb{R}\to\mathbb{R}^n$ is

$$f_F(\omega) = \int_{-\infty}^{+\infty} f(s)e^{-i\omega s}ds = f_-(\omega) + f_+(\omega) \qquad \forall\omega\in\mathbb{R}, \tag{2.9}$$

where

$$f_-(\omega) = \int_{-\infty}^{0} f(s)e^{-i\omega s}ds, \qquad f_+(\omega) = \int_0^{+\infty} f(s)e^{-i\omega s}ds. \tag{2.10}$$

Moreover, the half-range Fourier cosine and sine transforms are defined by

$$f_c(\omega) = \int_0^{+\infty} f(s)\cos\omega s\, ds, \qquad f_s(\omega) = \int_0^{+\infty} f(s)\sin\omega s\, ds; \tag{2.11}$$

we observe that they hold even if f is defined only on $\mathbb{R}^+$, as it occurs also for f_+.

Any function f, which is defined only on $\mathbb{R}^+$, can be extended on $\mathbb{R}$; thus, we recall that the Fourier transform of the new function is given by

$$f_F(\omega) = 2f_c(\omega), \quad f_F(\omega) = -2if_s(\omega), \quad f_F(\omega) = f_c(\omega) - if_s(\omega), \tag{2.12}$$

when, respectively, the extension is made with an even function ($f(\xi) = f(-\xi)\forall\xi<0$), or an odd one ($f(\xi) = -f(-\xi)\forall\xi<0$), or by using the causal extension ($f(\xi)=0\ \forall\xi<0$). Moreover, we remember that, if f and f' belong to $L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)$, we obtain

$$f'_s(\omega) = -\omega f_c(\omega) \tag{2.13}$$

and, if $f''\in L^1(\mathbb{R}^+)$, we have

$$\omega f'_s(\omega) = f'(0) + f''_c(\omega). \tag{2.14}$$

The thermodynamic constraints on (2.1)-(2.2) are expressed by the following inequalities [19]

$$\omega\alpha'_s(\omega) > 0 \quad \forall\omega\neq 0, \qquad k_0 + k'_c(\omega) > 0 \quad \forall\omega\in\mathbb{R}. \tag{2.15}$$

Hence, useful relations have been deduced, always in [19].

For α, by using (2.13), (2.14), $(2.15)_1$ and the inverse half-range Fourier transforms, we have

$$\alpha''_c(\omega) = \omega\alpha'_s(\omega) - \alpha'(0), \quad \alpha(t) - \alpha_0 = \frac{2}{\pi}\int_0^{+\infty}\frac{\alpha'_s(\omega)}{\omega}(1-\cos\omega t)d\omega > 0, \tag{2.16}$$

under the further hypothesis that $\alpha''\in L^1(\mathbb{R}^+)$. Thus, we also have

$$\alpha_\infty - \alpha_0 = \frac{2}{\pi}\int_0^{+\infty}\frac{\alpha'_s(\omega)}{\omega}d\omega > 0, \qquad \lim_{\omega\to+\infty}\omega\alpha'_s(\omega) = \alpha'(0) \geq 0 \tag{2.17}$$

and hence

$$\alpha_0 < \alpha(t) < 2\alpha_\infty - \alpha_0. \tag{2.18}$$

For k, we have

$$k_0 \geq 0, \qquad k_\infty = k_0 + k_c'(0) > 0. \tag{2.19}$$

We shall assume

$$k_0 > 0, \qquad \alpha'(0) > 0. \tag{2.20}$$

We note that the functions $f_\pm(\omega)$, given by (2.10), can be extended to the complex z-plane $\mathbb{C}$. Thus, they become analytic functions in the subsets $\mathbb{C}^{(\mp)}$ so defined

$$\mathbb{C}^{(-)} = \left\{z \in \mathbb{C};\ \mathrm{Im}\, z \in \mathbb{R}^{--}\right\}, \quad \mathbb{C}^{(+)} = \left\{z \in \mathbb{C};\ \mathrm{Im}\, z \in \mathbb{R}^{++}\right\}, \tag{2.21}$$

where $\mathbb{R}^{--} = (-\infty, 0)$ and $\mathbb{R}^{++} = (0, +\infty)$. Following [20], we assume the analyticity of the Fourier transforms on $\mathbb{R}$; therefore, $f_\pm(z)$ become analytic on an open region containing $\mathbb{C}^\mp$, given by

$$\mathbb{C}^{-} = \left\{z \in \mathbb{C};\ \mathrm{Im}\, z \in \mathbb{R}^{-}\right\}, \quad \mathbb{C}^{+} = \left\{z \in \mathbb{C};\ \mathrm{Im}\, z \in \mathbb{R}^{+}\right\}. \tag{2.22}$$

Finally, will shall use the notation $f_{(\pm)}(z)$ to denote that the zeros and the singularities of f are in $\mathbb{C}^\pm$.

Since we are concerned with a linear theory of rigid heat conductors, the linearization of the Clausius-Duhem inequality is also required. Such a linearization, derived in [14], involves the first order approximation for e and $\mathbf{q}$ and the second order one for the free energy and the entropy. By introducing the pseudo-free energy $\psi = \Theta_0(e - \Theta_0\eta)$, whose properties closely resemble those of the canonical free energy, the factor $1/\Theta_0$, involved in the linearization of the term $\mathbf{q} \cdot \mathbf{g}$, is eliminated by the presence of the factor Θ_0 in the definition of ψ; thus, the authors have derived the following linearized local form of the Second Law of Thermodynamics

$$\dot{\psi}(\mathbf{x}, t) \leq \dot{e}(\mathbf{x}, t)\vartheta(\mathbf{x}, t) - \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{g}(\mathbf{x}, t). \tag{2.23}$$

Henceforth, since our attention shall be fixed on a specific point $\mathbf{x} \in \Omega$, the dependence on such a vector will be omitted in the equations.

3 States and Processes of the Rigid Heat Conductor

The behaviour of our rigid heat conductor $\mathcal{B}$ is characterized by the constitutive equations (2.1)-(2.2) in the form given by (2.7)-(2.8), which allow us to consider $\mathcal{B}$ as a simple material, that we can describe in terms of states and processes.

Thus, we assume that the material state of the conductor at time t is expressed by

$$\sigma(t) = (\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t), \tag{3.1}$$

where $\vartheta(t)$ is the instantaneous value of ϑ, $\bar{\vartheta}^t$and $\bar{\mathbf{g}}^t$ are the integrated histories of the temperature and of its gradient.

The thermodynamic process of the conductor is given by a piecewise continuous map $P : [0, d) \to \mathbb{R} \times \mathbb{R}^3$ defined by

$$P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau)) \qquad \forall\tau \in [0, d), \tag{3.2}$$

which can be applied to the body at any time $t \geq 0$ and generally has a finite duration d.

The sets of the states and of the processes, which are possible for the material, are denoted by Σ and Π. We can introduce the state transition function $\rho : \Sigma \times \Pi \to \Sigma$, which maps any initial state $\sigma^i \in \Sigma$ and any process $P \in \Pi$ into the final state $\sigma^f = \rho(\sigma^i, P) \in \Sigma$. We can consider any restriction of a given $P \in \Pi$ to a subset $[\tau_1, \tau_2) \subset [0, d)$, denoted by $P_{[\tau_1,\tau_2)}$, as well as the composition of two processes $P_j \in \Pi$ $(j = 1, 2)$ with durations d_j $(j = 1, 2)$ so defined

$$P_1 * P_2(\tau) = \begin{cases} P_1(\tau) & \forall \tau \in [0, d_1), \\ P_2(\tau - d_1) & \forall \tau \in [d_1, d_1 + d_2); \end{cases} \tag{3.3}$$

both also belong to Π. In particular, the restriction $P_{[0,\tau)}$ applied to $\sigma^i = \sigma(0)$ yields the final state $\sigma(\tau) = \rho(\sigma(0), P_{[0,\tau)})$. Finally, the pair (σ,P) is said to be a cycle if $\sigma(d) = \rho(\sigma(0), P) = \sigma(0)$.

Let $\sigma(0) = (\vartheta_*(0), \bar{\vartheta}^0_*, \bar{\mathbf{g}}^0_*)$ be the initial state at time $t = 0$, when a process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ is applied for any $\tau \equiv t \in [0, d)$. In particular, we have

$$\vartheta(t) = \vartheta_*(0) + \int_0^t \dot{\vartheta}_P(s)ds, \quad {}_r\vartheta^t(s) = \begin{cases} \vartheta(t-s) & \forall s \in (0, t], \\ \vartheta^0_*(s-t) & \forall s > t \end{cases} \tag{3.4}$$

and hence the subsequent states are expressed by

$$\bar{\vartheta}^t(s) = \begin{cases} \int_{t-s}^t \vartheta_P(\xi)d\xi & \forall s \in [0, t), \\ \int_0^t \vartheta_P(\lambda)d\lambda + \bar{\vartheta}^0_*(s-t) & \forall s \geq t, \end{cases} \tag{3.5}$$

$$\bar{\mathbf{g}}^t(s) = \begin{cases} \int_{t-s}^t \mathbf{g}_P(\xi)d\xi & \forall s \in [0, t), \\ \int_0^t \mathbf{g}_P(\lambda)d\lambda + \bar{\mathbf{g}}^0_*(s-t) & \forall s \geq t, \end{cases} \tag{3.6}$$

together with the values $\vartheta\,(t)$ of the temperature given by $(3.4)_1$.

When $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ $\forall \tau \in [0, d)$ is applied at time $t > 0$ to the initial state $\sigma^i(t) = (\vartheta_i(t), \bar{\vartheta}^t_i, \bar{\mathbf{g}}^t_i)$, we must consider the continuations of the integrated histories of ϑ and $\mathbf{g}$ to express the ensuing states. Now, we have

$$\vartheta_P(\tau) \equiv \vartheta(t+\tau) = \vartheta_i(t) + \int_0^\tau \dot{\vartheta}_P(\eta)d\eta, \tag{3.7}$$

$$\vartheta(t+d-s) = (\vartheta_P * \vartheta_i)^{t+d}(s) = \begin{cases} \vartheta_P(d-s) & \forall s \in [0, d), \\ \vartheta_i(t+d-s) & \forall s \geq d, \end{cases} \tag{3.8}$$

and hence

$$\bar{\vartheta}(t+d-s) = (\vartheta_P * \bar{\vartheta}_i)^{t+d}(s) = \begin{cases} \int_{d-s}^d \vartheta_P(s)ds = \bar{\vartheta}^d_P(s) & \forall s \in [0, d), \\ \bar{\vartheta}^d_P(d) + \bar{\vartheta}^t_i(s-d) & \forall s \geq d, \end{cases} \tag{3.9}$$

$$\bar{\mathbf{g}}(t+d-s) = (\mathbf{g}_P * \bar{\mathbf{g}}_i)^{t+d}(s) = \begin{cases} \int_{d-s}^d \mathbf{g}_P(\xi)d\xi = \bar{\mathbf{g}}^d_P(s) & \forall s \in [0, d), \\ \bar{\mathbf{g}}^d_P(d) + \bar{\mathbf{g}}^t_i(s-d) & \forall s \geq d. \end{cases} \tag{3.10}$$

These prolongations $(\vartheta_P * \bar{\vartheta}_i)^{t+d}$ and $(\mathbf{g}_P * \bar{\mathbf{g}}_i)^{t+d}$ given in (3.9)-(3.10) allow us to evaluate the final values of the internal energy (2.7) and the heat flux (2.8) at the end of a process.

Let the restriction $P_{[0,\tau)}$ be applied at time $t > 0$ to the state $\sigma^i(t) = (\vartheta_i(t), \bar{\vartheta}_i^t, \bar{\mathbf{g}}_i^t)$. Then, the expressions (2.7) and (2.8) for e and $\mathbf{q}$, by replacing d with τ in (3.9)-(3.10), become

$$\begin{aligned} e(t+\tau) &= e_0 + \alpha_0 \vartheta_P(\tau) - \int_0^\tau \alpha''(s)\bar{\vartheta}_P^\tau(s)ds - \int_\tau^{+\infty} \alpha''(s)[\bar{\vartheta}_P^\tau(\tau) \\ &\qquad + \bar{\vartheta}_i^t(s-\tau)]ds, \qquad (3.11) \\ \mathbf{q}(t+\tau) &= -k_0 \mathbf{g}_P(\tau) + \int_0^\tau k''(s)\bar{\mathbf{g}}_P^\tau(s)ds + \int_\tau^{+\infty} k''(s)[\bar{\mathbf{g}}_P^\tau(\tau) \\ &\qquad + \bar{\mathbf{g}}_i^t(s-\tau)]ds. \qquad (3.12) \end{aligned}$$

We observe that in these last two relations, in particular, we have

$$\bar{\vartheta}_P^\tau(\tau) = \int_0^\tau \vartheta_P(\xi)d\xi, \qquad \bar{\mathbf{g}}_P^\tau(\tau) = \int_0^\tau \mathbf{g}_P(\xi)d\xi, \tag{3.13}$$

which immediately follow from (2.6).

These formulae are now applied to the particular cases of a static continuation of histories and of constant histories.

Firstly, we examine the static continuation of two assigned histories $(\vartheta(t),\, _r\vartheta^t)$ and $(\mathbf{g}(t),\, _r\mathbf{g}^t)$, with a finite duration $a \in \mathbb{R}^{++}$, defined by

$$\vartheta^{t_{(a)}}(s) = \begin{cases} \vartheta(t) & \forall s \in [0,a], \\ \vartheta^t(s-a) & \forall s > a, \end{cases} \qquad \mathbf{g}^{t_{(a)}}(s) = \begin{cases} \mathbf{g}(t) & \forall s \in [0,a], \\ \mathbf{g}^t(s-a) & \forall s > a; \end{cases} \tag{3.14}$$

hence, the corresponding integrated histories of ϑ and $\mathbf{g}$ are expressed by

$$\bar{\vartheta}^{t+a}(s) = \begin{cases} \int_{a-s}^a \vartheta(t)d\xi = s\vartheta(t) & \forall s \in [0,a], \\ \int_0^a \vartheta(t)d\xi + \int_{t-(s-a)}^t \vartheta(\xi)d\xi = a\vartheta(t) + \int_0^{s-a} \vartheta^t(\rho)d\rho & \forall s > a, \end{cases} \tag{3.15}$$

$$\bar{\mathbf{g}}^{t+a}(s) = \begin{cases} \int_{a-s}^a \mathbf{g}(t)d\xi = s\mathbf{g}(t) & \forall s \in [0,a], \\ \int_0^a \mathbf{g}(t)d\xi + \int_{t-(s-a)}^t \mathbf{g}(\xi)d\xi = a\mathbf{g}(t) + \int_0^{s-a} \mathbf{g}^t(\rho)d\rho & \forall s > a. \end{cases} \tag{3.16}$$

Thus, (3.11) and (3.12), by substituting (3.15) and (3.16), give

$$e(t+a) = e_0 + \alpha(a)\vartheta(t) - \int_0^{+\infty} \alpha''(\xi+a)\bar{\vartheta}^t(\xi)d\xi, \tag{3.17}$$

$$\mathbf{q}(t+a) = -k(a)\mathbf{g}(t) + \int_0^{+\infty} k''(\xi+a)\bar{\mathbf{g}}^t(\xi)d\xi. \tag{3.18}$$

Now, let $\vartheta(t-s) = \vartheta^\dagger(s) = \vartheta$ and $\mathbf{g}(t-s) = \mathbf{g}^\dagger(s) = \mathbf{g}$ $\forall s \in \mathbb{R}^+$ be two given constant histories; the internal energy and the heat flux at time t can be evaluated directly from (2.7)-(2.8) and are expressed by

$$e(t) = e_0 + \alpha_\infty \vartheta, \qquad \mathbf{q}(t) = -k_\infty \mathbf{g}, \tag{3.19}$$

where the asymptotic values (2.4) of α and of k are involved; in particular, we note that the heat flux has the opposite versus of the temperature gradient.

Taking account of the constitutive equations (2.7) and (2.8), we can introduce the functionals $\tilde{e} : \mathbb{R} \times \Gamma_\alpha \to \mathbb{R}$ and $\tilde{\mathbf{q}} : \Gamma_k \to \mathbb{R}^3$ defined by

$$e(\sigma(t)) = \tilde{e}(\vartheta(t), \bar{\vartheta}^t), \qquad \mathbf{q}(\sigma(t), P_t) = \tilde{\mathbf{q}}(\mathbf{g}(t), \bar{\mathbf{g}}^t), \tag{3.20}$$

where Γ_α and Γ_k denote the function spaces of the integrated histories of ϑ and $\mathbf{g}$ up to time t, which, by virtue of (3.17)-(3.18), are so defined

$$\Gamma_\alpha = \left\{ \bar{\vartheta}^t : [0,+\infty) \to \mathbb{R}; \left| \int_0^{+\infty} \alpha''(\eta+\tau)\bar{\vartheta}^t(\eta)d\eta \right| < +\infty \quad \forall \tau \in \mathbb{R}^+ \right\}, \tag{3.21}$$

$$\Gamma_k = \left\{ \bar{\mathbf{g}}^t : [0,+\infty) \to \mathbb{R}^3; \left| \int_0^{+\infty} k''(\eta+\tau)\bar{\mathbf{g}}^t(\eta)d\eta \right| < +\infty \quad \forall \tau \in \mathbb{R}^+ \right\}. \tag{3.22}$$

4 An Equivalence Relation Between States

An equivalence relation can be introduced in the state space Σ with this definition.

Definition 4.1 Two states $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t) \in \Sigma$ $(j = 1,2)$ of a rigid heat conductor, characterized by the constitutive equations (2.7) and (2.8), are said to be equivalent if, for every process $P_\tau \in \Pi$ and for every $\tau > 0$,

$$e(\rho(\sigma_1(t), P_\tau)) = e(\rho(\sigma_2(t), P_\tau)), \quad \mathbf{q}(\rho(\sigma_1(t), P_\tau), P_\tau) = \mathbf{q}(\rho(\sigma_2(t), P_\tau), P_\tau). \tag{4.1}$$

Such a definition of equivalence requires the coincidence of the response of the material, expressed by the values of e and $\mathbf{q}$; this implies some consequences, which are shown in the following theorem.

Theorem 4.1 *For a conductor, characterized by the constitutive equations (2.7) and (2.8), two states $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t) \in \Sigma$ $(j = 1,2)$ are equivalent if and only if*

$$\vartheta_1(t) = \vartheta_2(t), \qquad \int_0^{+\infty} \alpha''(\xi+\tau)\left[\bar{\vartheta}_1^t(\xi) - \bar{\vartheta}_2^t(\xi)\right] d\xi = 0, \tag{4.2}$$

$$\int_0^{+\infty} k''(\xi+\tau)\left[\bar{\mathbf{g}}_1^t(\xi) - \bar{\mathbf{g}}_2^t(\xi)\right] d\xi = \mathbf{0} \tag{4.3}$$

for every $\tau > 0$.

Proof If $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t)$ $(j = 1,2)$ are two equivalent states, then the equalities (4.1) are satisfied for every $P_\tau \in \Pi$ and every $\tau > 0$; thus, we have

$$\tilde{e}(\vartheta_{P_1}(\tau), (\vartheta_{P_1} * \bar{\vartheta}_1)^{t+\tau}) = \tilde{e}(\vartheta_{P_2}(\tau), (\vartheta_{P_2} * \bar{\vartheta}_2)^{t+\tau}), \tag{4.4}$$

$$\tilde{\mathbf{q}}(\mathbf{g}_P(\tau), (\mathbf{g}_P * \bar{\mathbf{g}}_1)^{t+\tau}) = \tilde{\mathbf{q}}(\mathbf{g}_P(\tau), (\mathbf{g}_P * \bar{\mathbf{g}}_2)^{t+\tau}). \tag{4.5}$$

These two equalities, by using (3.11), (3.12) and (3.7), (3.9), (3.10), yield

$$\alpha(\tau)\left[\vartheta_1(t) - \vartheta_2(t)\right] - \int_\tau^{+\infty} \alpha''(s)\left[\bar{\vartheta}_1^t(s-\tau) - \bar{\vartheta}_2^t(s-\tau)\right] ds = 0, \tag{4.6}$$

$$\int_\tau^{+\infty} k''(s)\left[\bar{\mathbf{g}}_1^t(s-\tau) - \bar{\mathbf{g}}_2^t(s-\tau)\right] ds = \mathbf{0}, \tag{4.7}$$

which must be satisfied for arbitrary values of τ.

Taking the limit $\tau \to +\infty$ in (4.6), we have

$$\alpha_\infty\left[\vartheta_1(t) - \vartheta_2(t)\right] = 0 \tag{4.8}$$

and hence $(4.2)_1$ follows. Thus, (4.6) reduces to its integral, which, by changing the variable of integration, coincides with $(4.2)_2$. An analogous change in (4.7) gives (4.3).

Obviously, the converse follows from these same relations. □

Consequently, using (3.11)-(3.12), it follows that a state $\sigma(t) = (\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t)$ is equivalent to the zero state $\sigma_0(t) = (0, \bar{0}^\dagger, \bar{\mathbf{0}}^\dagger)$, where $\bar{0}^\dagger(s) = \bar{\vartheta}^t(s) = 0\ \forall s \in \mathbb{R}^+$ and $\bar{\mathbf{0}}^\dagger(s) = \bar{\mathbf{g}}^t(s) = \mathbf{0}\ \forall s \in \mathbb{R}^+$ denote the zero integrated histories of ϑ and $\mathbf{g}$, if

$$\vartheta(t) = 0, \qquad \int_\tau^{+\infty} \alpha''(s)\bar{\vartheta}^t(s-\tau)ds \equiv \int_0^{+\infty} \alpha''(\xi+\tau)\bar{\vartheta}^t(\xi)d\xi = 0, \tag{4.9}$$

$$\int_\tau^{+\infty} k''(s)\bar{\mathbf{g}}^t(s-\tau)ds \equiv \int_0^{+\infty} k''(\xi+\tau)\bar{\mathbf{g}}^t(\xi)d\xi = \mathbf{0}. \tag{4.10}$$

Moreover, it follows that the difference of two given equivalent states $\sigma_j(t)$ $(j = 1, 2)$, i.e. $\sigma_1(t) - \sigma_2(t) = (\vartheta_1(t) - \vartheta_2(t), \bar{\vartheta}_1^t - \bar{\vartheta}_2^t, \bar{\mathbf{g}}_1^t - \bar{\mathbf{g}}_2^t)$ is a state equivalent to the zero state $\sigma_0(t) = (0, \bar{0}^\dagger, \bar{\mathbf{0}}^\dagger)$.

5 Thermal Work

The linearized form of the Clausius-Duhem inequality, expressed by (2.23), gives for the thermal power the following form [14]

$$w(t) = \dot{e}(t)\vartheta(t) - \mathbf{q}(t) \cdot \mathbf{g}(t). \tag{5.1}$$

Therefore, during the application of a process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))\ \forall \tau \in [0, d)$, starting at time $t > 0$ when $\sigma(t) = (\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t)$ is the initial state, the thermal work done on the material is

$$W(\sigma, P) = \tilde{W}(\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t; \dot{\vartheta}_P, \mathbf{g}_P) = \int_0^d [\dot{e}(t+\tau)\vartheta_P(\tau) - \mathbf{q}(t+\tau) \cdot \mathbf{g}_P(\tau)]d\tau. \tag{5.2}$$

To evaluate the derivative of the internal energy, which appears in (5.2), we observe that (3.11), with an integration, can be written as follows

$$\begin{aligned} e(t+\tau) \quad &= \quad e_0 + \alpha_0\vartheta_P(\tau) + \alpha'(\tau)\bar{\vartheta}_P^\tau(\tau) - \int_0^\tau \alpha''(s)\bar{\vartheta}_P^\tau(s)ds \\ &\quad - \int_0^{+\infty} \alpha''(\xi+\tau)\bar{\vartheta}^t(\xi)d\xi. \end{aligned} \tag{5.3}$$

Thus, we can differentiate with respect to τ this expression (5.3); by using (3.13) and the relation $\frac{d}{d\tau}\bar{\vartheta}_P^\tau(s) \equiv \dot{\bar{\vartheta}}^\tau(s) = \vartheta_P(\tau) - \vartheta_P^\tau(s)$, we obtain

$$\begin{aligned} \dot{e}(t+\tau) \quad &= \quad \alpha_0\dot{\vartheta}_P(\tau) + \alpha'(0)\vartheta_P(\tau) + \int_0^\tau \alpha''(s)\vartheta_P^\tau(s)ds \\ &\quad - \int_0^{+\infty} \alpha'''(\xi+\tau)\bar{\vartheta}^t(\xi)d\xi. \end{aligned} \tag{5.4}$$

Moreover, the expression (3.12) for $\mathbf{q}(t+\tau)$, by replacing $\bar{\mathbf{g}}_i^t$ with the integrated history $\bar{\mathbf{g}}^t$ of the initial state $\sigma(t)$, with two integrations, can be rewritten as

$$\mathbf{q}(t+\tau) = -k_0\mathbf{g}_P(\tau) - \int_0^\tau k'(s)\mathbf{g}_P^\tau(s)ds + \int_0^{+\infty} k''(\xi+\tau)\bar{\mathbf{g}}^t(\xi)d\xi. \tag{5.5}$$

We firstly consider the particular case when the process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ of duration $d < +\infty$ is applied at time $t = 0$ to the initial state $\sigma_0(0) = (0, \bar{0}^\dagger, \bar{\mathbf{0}}^\dagger)$, in order to derive the amount of work due only to P. Denoting the ensuing fields by $(\vartheta_0, \bar{\vartheta}_0^t, \bar{\mathbf{g}}_0^t)$ and using (3.5)-(3.6), we have

$$\vartheta_0(t) = \int_0^t \dot{\vartheta}_P(s)ds, \quad \bar{\vartheta}_0^t(s) = (\vartheta_P * \bar{0}^\dagger)^t(s) = \begin{cases} \bar{\vartheta}_0^t(s) & \forall s \in [0,t), \\ \bar{\vartheta}_0^t(t) & \forall s \geq t, \end{cases} \tag{5.6}$$

$$\bar{\mathbf{g}}_0^t(s) = (\mathbf{g}_P * \bar{\mathbf{0}}^\dagger)^t(s) = \begin{cases} \bar{\mathbf{g}}_0^t(s) & \forall s \in [0,t), \\ \bar{\mathbf{g}}_0^t(t) & \forall s \geq t. \end{cases} \tag{5.7}$$

Thus, (5.4) and (5.5) become

$$\dot{e}(t) = \alpha_0\dot{\vartheta}_0(t) + \alpha'(0)\vartheta_0(t) + \int_0^t \alpha''(s)\vartheta_0^t(s)ds, \tag{5.8}$$

$$\mathbf{q}(t) = -k_0\mathbf{g}_0(t) - \int_0^t k'(s)\mathbf{g}_0^t(s)ds. \tag{5.9}$$

By substituting (5.8)-(5.9) into (5.2), we have

$$\begin{aligned} W(\sigma_0(0), P) = \tilde{W}(0, \bar{0}^\dagger, \bar{\mathbf{0}}^\dagger; \dot{\vartheta}_P, \mathbf{g}_P) = \frac{1}{2}\alpha_0\vartheta_0^2(d) \\ +\alpha'(0)\int_0^d \vartheta_0^2(t)dt + \int_0^d \left[\int_0^t \alpha''(s)\vartheta_0^t(s)ds\right]\vartheta_0(t)dt \\ +k_0\int_0^d \mathbf{g}_0^2(t)dt + \int_0^d \left[\int_0^t k'(s)\mathbf{g}_0^t(s)ds\right]\cdot \mathbf{g}_0(t)dt. \end{aligned} \tag{5.10}$$

Definition 5.1 A process $P = (\dot{\vartheta}_P, \mathbf{g}_P)$ with a duration d, applied at time $t = 0$ and related to (5.8)-(5.9), is said to be a finite work process if

$$\tilde{W}(0, \bar{0}^\dagger, \bar{\mathbf{0}}^\dagger; \dot{\vartheta}_P, \mathbf{g}_P) < +\infty. \tag{5.11}$$

Theorem 5.1 *The work done during the application of any finite work process is positive.*

Proof Let the process P be extended to $\mathbb{R}$ by assuming that $P(t) = (0, \mathbf{0})$ $\forall t > d$ and $\vartheta_0(t) = 0$ $\forall t > d$; the expression of the work, by applying Plancherel's theorem and using * to denote the complex conjugate, can be written as follows

$$\begin{aligned} W(\sigma_0(0), P) = \frac{1}{2}\alpha_0\vartheta_0^2(d) + \frac{\alpha'(0)}{2\pi}\int_{-\infty}^{+\infty} \vartheta_{0_F}(\omega)\left[\vartheta_{0_F}(\omega)\right]^* d\omega \\ +\frac{1}{2\pi}\int_{-\infty}^{+\infty} \alpha''_F(\omega)\vartheta_{0_F}(\omega)\left[\vartheta_{0_F}(\omega)\right]^* d\omega + \frac{k_0}{2\pi}\int_{-\infty}^{+\infty} \mathbf{g}_{0_F}(\omega)\left[\mathbf{g}_{0_F}(\omega)\right]^* d\omega \\ +\frac{1}{2\pi}\int_{-\infty}^{+\infty} k'_F(\omega)\mathbf{g}_{0_F}(\omega)\cdot\left[\mathbf{g}_{0_F}(\omega)\right]^* d\omega. \end{aligned} \tag{5.12}$$

Using $(2.12)_3$ for the Fourier transforms of any function which vanishes on $\mathbb{R}^{--}$, it follows that $\alpha''_F(\omega) = \alpha''_c(\omega) - i\alpha''_s(\omega)$ and $k'_F(\omega) = k'_c(\omega) - ik'_s(\omega)$ are expressed in terms of the cosine and the sine transforms, which are even and odd functions, respectively; therefore, (5.12) can be rewritten as

$$
\begin{aligned}
W(\sigma_0(0),P) &= \frac{1}{2}\alpha_0\vartheta_0^2(d) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\alpha'(0)+\alpha_c''(\omega)\right]\left[\vartheta_{0_c}^2(\omega)+\vartheta_{0_s}^2(\omega)\right]d\omega \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[k_0+k_c'(\omega)\right]\left[\mathbf{g}_{0_c}^2(\omega)+\mathbf{g}_{0s}^2(\omega)\right]d\omega \\
&= \frac{1}{2}\alpha_0\vartheta_0^2(d) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\omega\alpha_s'(\omega)\left[\vartheta_{0_c}^2(\omega)+\vartheta_{0_s}^2(\omega)\right]\right. \\
&\quad \left. + \left[k_0+k_c'(\omega)\right]\left[\mathbf{g}_{0_c}^2(\omega)+\mathbf{g}_{0s}^2(\omega)\right]\right\}d\omega > 0,
\end{aligned}
\tag{5.13}
$$

by virtue of $(2.16)_1$ and (2.15). □

Thus, the work $W(\sigma_0(0),P)$ depends on the ensuing field of the temperature $\vartheta_0(t)$, which is related to P through $\dot{\vartheta}_P$ by means of (3.4) or (3.5), and on the temperature gradient $\mathbf{g}_P(t)\equiv\mathbf{g}_0(t)$ assigned with P. Consequently, we can characterize the finite work processes by introducing the following function spaces [18]

$$
\tilde{H}_\alpha(\mathbb{R}^+,\mathbb{R}) = \left\{\vartheta:\mathbb{R}^+\to\mathbb{R};\ \int_{-\infty}^{+\infty}\omega\alpha_s'(\omega)\vartheta_+(\omega)\left[\vartheta_+(\omega)\right]^* d\omega < +\infty\right\}, \tag{5.14}
$$

$$
\tilde{H}_k(\mathbb{R}^+,\mathbb{R}^3) = \left\{\mathbf{g}:\mathbb{R}^+\to\mathbb{R}^3;\ \int_{-\infty}^{+\infty}\left[k_0+k_c'(\omega)\right]\mathbf{g}_+(\omega)\cdot\left[\mathbf{g}_+(\omega)\right]^* d\omega < +\infty\right\}, \tag{5.15}
$$

which, with the completions with respect to the norms corresponding to the following two inner products $(\vartheta_1,\vartheta_2)_\alpha = \int_{-\infty}^{+\infty}\omega\alpha_s'(\omega)\vartheta_{1+}(\omega)\left[\vartheta_{2+}(\omega)\right]^* d\omega$ and $(\mathbf{g}_1,\mathbf{g}_2)_k = \int_{-\infty}^{+\infty}\left[k_0+k_c'(\omega)\right]\mathbf{g}_{1+}(\omega)\cdot\left[\mathbf{g}_{2+}(\omega)\right]^* d\omega$, respectively, yield two Hilbert spaces $H_\alpha(\mathbb{R}^+,\mathbb{R})$ and $H_k(\mathbb{R}^+,\mathbb{R}^3)$.

Now, we consider the general case when the initial state of $\mathcal{B}$ at time $t>0$ is $\sigma(t) = (\vartheta(t),\bar{\vartheta}^t,\bar{\mathbf{g}}^t)$, where $\bar{\vartheta}^t$ and $\bar{\mathbf{g}}^t$, belonging to the function spaces Γ_α and Γ_k, introduced in (3.21)-(3.22), are possible integrated histories, which yield a finite work during any process P, characterized by $\mathbf{g}_P\in H_k(\mathbb{R}^+,\mathbb{R}^3)$ and related to $\vartheta_P\in H_\alpha(\mathbb{R}^+,\mathbb{R})$. If, as above, we extend any of these processes $P=(\dot{\vartheta}_P,\mathbf{g}_P)$ with a finite duration $d<+\infty$ to $\mathbb{R}^+$, by assuming that $P(\tau)=(0,\mathbf{0})\ \forall\tau\geq d$ and that $\vartheta_P(\tau)=0\ \forall\tau>d$, the work done during the application of any of these processes can be derived by means of (5.2), where $\dot{e}(t+\tau)$ has the form (5.4) and $\mathbf{q}(t+\tau)$ is given by (5.5). Thus, we obtain

$$
\begin{aligned}
& W(\sigma(t),P) = \tilde{W}(\vartheta(t),\bar{\vartheta}^t,\bar{\mathbf{g}}^t;\dot{\vartheta}_P,\mathbf{g}_P) \\
& = \frac{1}{2}\alpha_0[\vartheta_P^2(d)-\vartheta_P^2(0)] + \alpha'(0)\int_0^{+\infty}\vartheta_P^2(\tau)d\tau + k_0\int_0^{+\infty}\mathbf{g}_P^2(\tau)d\tau \\
& + \int_0^{+\infty}\left[\int_0^\tau \alpha''(\tau-\eta)\vartheta_P(\eta)d\eta - \int_0^{+\infty}\alpha'''(\xi+\tau)\bar{\vartheta}^t(\xi)d\xi\right]\vartheta_P(\tau)d\tau \\
& + \int_0^{+\infty}\left[\int_0^\tau k'(\tau-\eta)\mathbf{g}_P(\eta)d\eta - \int_0^{+\infty}k''(\xi+\tau)\bar{\mathbf{g}}^t(\xi)d\xi\right]\cdot\mathbf{g}_P(\tau)d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\alpha_0[\vartheta_P^2(d) - \vartheta_P^2(0)] + \alpha'(0)\int_0^{+\infty} \vartheta_P^2(\tau)d\tau + k_0 \int_0^{+\infty} \mathbf{g}_P^2(\tau)d\tau \\
&+ \int_0^{+\infty} \left[\frac{1}{2}\int_0^{+\infty} \alpha''(\mid \tau-\eta \mid)\vartheta_P(\eta)d\eta - I_{(\alpha)}^t(\tau,\bar{\vartheta}^t)\right]\vartheta_P(\tau)d\tau \\
&+ \int_0^{+\infty} \left[\frac{1}{2}\int_0^{+\infty} k'(\mid \tau-\eta \mid)\mathbf{g}_P(\eta)d\eta - \mathbf{I}_{(k)}^t(\tau,\bar{\mathbf{g}}^t)\right]\cdot \mathbf{g}_P(\tau)d\tau, \qquad (5.16)
\end{aligned}
$$

where

$$
I_{(\alpha)}^t(\tau,\bar{\vartheta}^t) = \int_0^{+\infty} \alpha'''(\xi+\tau)\bar{\vartheta}^t(\xi)d\xi, \ \mathbf{I}_{(k)}^t(\tau,\bar{\mathbf{g}}^t) = \int_0^{+\infty} k''(\xi+\tau)\bar{\mathbf{g}}^t(\xi)d\xi \ \forall\tau\geq 0. \quad (5.17)
$$

Contrary to what occurs for $\mathbf{I}_{(k)}^t$ in (3.18), the quantity $I_{(\alpha)}^t$ is not present in (3.17), which gives the value of the internal energy after a static continuation with a fixed duration, $\tau = a$. The reason for such a result is due to the fact that in the expression (5.2) we have the presence of $\dot{e}$, instead of e, as already observed in [4]. We only observe that such quantities are related to the minimal state of the rigid heat conductor (see, for example, [8]).

6 The Equivalence Between States by Means of the Work

In Section 4 we have called equivalent two states $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t)$ $(j = 1, 2)$ if the application of the same process to each of them yields the same response of the material, that is the final values of the internal energy and of the heat flux, corresponding to the two cases, coincide.

A new but equivalent definition of this relation can be given in terms of the work.

Definition 6.1 Two states $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t)$ $(j = 1, 2)$ are said to be w-equivalent if

$$
W(\sigma_1(t), P_\tau) = W(\sigma_2(t), P_\tau) \qquad (6.1)
$$

for every process $P : [0, \tau) \to \mathbb{R} \times \mathbb{R}^3$ and for every $\tau > 0$.

The two definitions, we have introduced, are equivalent by virtue of the following theorem.

Theorem 6.1 *Two states are equivalent in the sense of Definition 4.1 if and only if they are w-equivalent.*

Proof Let $\sigma_j(t) = (\vartheta_j(t), \bar{\vartheta}_j^t, \bar{\mathbf{g}}_j^t)$ $(j = 1, 2)$ be two states equivalent in the sense of Definition 4.1, then (4.1) and hence (4.2)-(4.3) hold for every process P_τ and for every $\tau > 0$. Therefore, it follows that

$$
\int_0^d [\dot{e}_1(t+\tau)\vartheta_{P_1}(\tau) - \mathbf{q}_1(t+\tau)\cdot\mathbf{g}_P(\tau)]d\tau = \int_0^d [\dot{e}_2(t+\tau)\vartheta_{P_2}(\tau) - \mathbf{q}_2(t+\tau)\cdot\mathbf{g}_P(\tau)]d\tau. \quad (6.2)
$$

In fact, on account of $(4.1)_1$, the derivatives with respect to τ of e_1 and e_2 coincide; $\vartheta_{P_j}(\tau)$ $(j = 1, 2)$ are expressed by (3.7), where we have $\vartheta_1(t) = \vartheta_2(t)$, by virtue of $(4.2)_1$, and the same $\dot{\vartheta}_P(\tau)$; finally, $\mathbf{q}_j(t + \tau)$ $(j = 1, 2)$, given by (3.12) or equivalently by (5.5), where we have the same $\mathbf{g}_P$ in $[0, \tau)$ and the last integrals related to $j = 1$ and

$j = 2$ coincide because (4.3) holds by hypothesis, assume the same value. Since such an equality expresses the equality of the two works done during the application of the same process to $\sigma_j(t)$ $(j = 1, 2)$, (6.1) is satisfied.

Let us now suppose that two states $\sigma_j(t)$ $(j = 1, 2)$ are w-equivalent; then, they satisfy (6.1) for any P with any arbitrary duration $d > 0$. From (6.1), taking account of the expression for the work (5.2) in the form $(5.16)_2$, it follows that

$$\begin{aligned}
&\alpha_0 \int_0^d \dot{\vartheta}_P(\tau)\left[\vartheta_{P_1}(\tau) - \vartheta_{P_2}(\tau)\right] d\tau + \alpha'(0) \int_0^d \left[\vartheta_{P_1}^2(\tau) - \vartheta_{P_2}^2(\tau)\right] d\tau \\
&+\frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha''(\mid \tau - \eta \mid)\left[\vartheta_{P_1}(\eta)\vartheta_{P_1}(\tau) - \vartheta_{P_2}(\eta)\vartheta_{P_2}(\tau)\right] d\eta d\tau \\
&- \int_0^{+\infty} \left[I_{(\alpha)}^t(\tau, \bar{\vartheta}_1^t)\vartheta_{P_1}(\tau) - I_{(\alpha)}^t(\tau, \bar{\vartheta}_2^t)\vartheta_{P_2}(\tau)\right] d\tau \\
&- \int_0^{+\infty} \left[\mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_1^t) - \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_2^t)\right] \cdot \mathbf{g}_P(\tau) d\tau = 0,
\end{aligned} \tag{6.3}$$

where the integrals with the factor k_0 and the one with the kernel k' have been eliminated since expressed by means of the same $\mathbf{g}_P$. From (3.7) we have $\vartheta_{P_j}(\tau) = \vartheta_j(t) + \int_0^\tau \dot{\vartheta}_P(\eta) d\eta$ $(j = 1, 2)$, which allow us to rewrite (6.3) as

$$\begin{aligned}
&\alpha_0 \left[\vartheta_1(t) - \vartheta_2(t)\right] \int_0^d \dot{\vartheta}_P(\tau) d\tau + \alpha'(0) \int_0^d \left\{ \left[\vartheta_1^2(t) - \vartheta_2^2(t)\right] \right. \\
&\left. +2\left[\vartheta_1(t) - \vartheta_2(t)\right] \int_0^\tau \dot{\vartheta}_P(\xi) d\xi \right\} d\tau + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha''(\mid \tau - \eta \mid) \left\{ \left[\vartheta_1^2(t) - \vartheta_2^2(t)\right] \right. \\
&\left. +\left[\vartheta_1(t) - \vartheta_2(t)\right] \left[\int_0^\eta \dot{\vartheta}_P(\nu) d\nu + \int_0^\tau \dot{\vartheta}_P(\rho) d\rho\right] \right\} d\eta d\tau - \int_0^{+\infty} \left[I_{(\alpha)}^t(\tau, \bar{\vartheta}_1^t)\vartheta_1(t) \right. \\
&\left. -I_{(\alpha)}^t(\tau, \bar{\vartheta}_2^t)\vartheta_2(t)\right] d\tau - \int_0^{+\infty} \left[I_{(\alpha)}^t(\tau, \bar{\vartheta}_1^t) - I_{(\alpha)}^t(\tau, \bar{\vartheta}_2^t)\right] \left[\int_0^\tau \dot{\vartheta}_P(\xi) d\xi\right] d\tau \\
&- \int_0^{+\infty} \left[\mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_1^t) - \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_2^t)\right] \cdot \mathbf{g}_P(\tau) d\tau = 0.
\end{aligned} \tag{6.4}$$

Since this relation must hold for any P and any $d > 0$, we can choose the arbitrary quantities $\dot{\vartheta}_P$ and $\mathbf{g}_P$ equal to zero; thus, the sum of the remaining terms must vanish. Consequently, (6.4) reduces to

$$\begin{aligned}
&\left[\vartheta_1(t) - \vartheta_2(t)\right] \left\{ \alpha_0 \int_0^d \dot{\vartheta}_P(\tau) d\tau + 2\alpha'(0) \int_0^d \left[\int_0^\tau \dot{\vartheta}_P(\xi) d\xi\right] d\tau \right. \\
&\left. +\frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha''(\mid \tau - \eta \mid) \left[\int_0^\eta \dot{\vartheta}_P(\nu) d\nu + \int_0^\tau \dot{\vartheta}_P(\rho) d\rho\right] d\eta d\tau \right\} \\
&- \int_0^{+\infty} \left[I_{(\alpha)}^t(\tau, \bar{\vartheta}_1^t) - I_{(\alpha)}^t(\tau, \bar{\vartheta}_2^t)\right] \left[\int_0^\tau \dot{\vartheta}_P(\xi) d\xi\right] d\tau \\
&- \int_0^{+\infty} \left[\mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_1^t) - \mathbf{I}_{(k)}^t(\tau, \bar{\mathbf{g}}_2^t)\right] \cdot \mathbf{g}_P(\tau) d\tau = 0.
\end{aligned} \tag{6.5}$$

We now observe that

$$\alpha''(\mid s_1 - s_2 \mid) = -2\delta(s_1 - s_2)\alpha'(\mid s_1 - s_2 \mid) - \alpha_{12}(\mid s_1 - s_2 \mid), \tag{6.6}$$

where $\alpha_{12} = \partial^2\alpha/\partial s_1 \partial s_2$; hence, substituting it into (6.5) and recalling that both $\dot{\vartheta}_P(\tau)$ and $\vartheta_P(\tau)$ are equal to zero for any $\tau > d$, we obtain

$$\begin{aligned}
&[\vartheta_1(t) - \vartheta_2(t)] \left\{ \alpha_0 f(d) - \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha_{12}(\mid \tau - \eta \mid) \left[f(\eta) + f(\tau) \right] d\eta d\tau \right\} \\
&- \int_0^{+\infty} \left[I^t_{(\alpha)}(\tau, \bar{\vartheta}^t_1) - I^t_{(\alpha)}(\tau, \bar{\vartheta}^t_2) \right] f(\tau) d\tau \\
&- \int_0^{+\infty} \left[\mathbf{I}^t_{(k)}(\tau, \bar{\mathbf{g}}^t_1) - \mathbf{I}^t_{(k)}(\tau, \bar{\mathbf{g}}^t_2) \right] \cdot \mathbf{g}_P(\tau) d\tau = 0,
\end{aligned} \tag{6.7}$$

where we have put

$$f(\tau) \equiv \int_0^\tau \dot{\vartheta}_P(\xi) d\xi, \qquad f(\tau) = f(d) \quad \forall \tau > d, \tag{6.8}$$

whence $f'(\tau) \equiv \dot{\vartheta}_P(\tau)$.

From the relation (6.7), with an integration of each term of the integral with the kernel α_{12}, we have

$$\begin{aligned}
&-\frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha_{12}(\mid \tau - \eta \mid) \left[f(\eta) + f(\tau) \right] d\eta d\tau \\
&\qquad = \int_0^{+\infty} \alpha'(\tau) f(\tau) d\tau = - \int_0^{+\infty} \alpha(\tau) f'(\tau) d\tau + \alpha_\infty f(d).
\end{aligned} \tag{6.9}$$

By substituting this result into (6.7), after putting $\mathbf{g}_P(\tau) = \mathbf{0}$, we obtain

$$\begin{aligned}
&[\vartheta_1(t) - \vartheta_2(t)] \left[(\alpha_0 + \alpha_\infty) f(d) - \int_0^{+\infty} \alpha(\tau) f'(\tau) d\tau \right] \\
&\qquad = \int_0^{+\infty} \left[I^t_{(\alpha)}(\tau, \bar{\vartheta}^t_1) - I^t_{(\alpha)}(\tau, \bar{\vartheta}^t_2) \right] f(\tau) d\tau,
\end{aligned} \tag{6.10}$$

which also implies

$$\int_0^{+\infty} \left[\mathbf{I}^t_{(k)}(\tau, \bar{\mathbf{g}}^t_1) - \mathbf{I}^t_{(k)}(\tau, \bar{\mathbf{g}}^t_2) \right] \cdot \mathbf{g}_P(\tau) d\tau = 0. \tag{6.11}$$

In (6.10), with an integration by parts and taking account of $(6.8)_2$, we have

$$\begin{aligned}
\mathcal{I} &\equiv \int_0^{+\infty} \left[I^t_{(\alpha)}(\tau, \bar{\vartheta}^t_1) - I^t_{(\alpha)}(\tau, \bar{\vartheta}^t_2) \right] f(\tau) d\tau \\
&= - \int_0^{+\infty} \left\{ \int_0^\tau \left[I^t_{(\alpha)}(\beta, \bar{\vartheta}^t_1) - I^t_{(\alpha)}(\beta, \bar{\vartheta}^t_2) \right] d\beta \right\} f'(\tau) d\tau \\
&\qquad + f(d) \int_0^{+\infty} \left[I^t_{(\alpha)}(\rho, \bar{\vartheta}^t_1) - I^t_{(\alpha)}(\rho, \bar{\vartheta}^t_2) \right] d\rho.
\end{aligned} \tag{6.12}$$

After substituting the expression $(5.17)_1$ for $I^t_{(\alpha)}$, which has α''' as kernel, into this relation, we can integrate with respect to β in $(0, \tau)$ and to ρ in $(0, +\infty)$; thus, $\mathcal{I}$ reduces to

$$\mathcal{I} = - \int_0^{+\infty} \int_0^{+\infty} \alpha''(\xi + \tau) \left[\bar{\vartheta}^t_1(\xi) - \bar{\vartheta}^t_2(\xi) \right] f'(\tau) d\xi d\tau. \tag{6.13}$$

Therefore, (6.10) can be written as

$$\int_0^{+\infty} \{ [\vartheta_1(t) - \vartheta_2(t)] [\alpha_0 + \alpha_\infty - \alpha(\tau)] + \int_0^{+\infty} \alpha''(\xi + \tau) \left[\bar{\vartheta}_1^t(\xi) - \bar{\vartheta}_2^t(\xi) \right] d\xi \Big\} f'(\tau) d\tau = 0. \tag{6.14}$$

Hence, the arbitrariness of $f'(\tau) = \dot{\vartheta}_P(\tau)$ yields

$$[\vartheta_1(t) - \vartheta_2(t)] [\alpha_0 + \alpha_\infty - \alpha(\tau)] = - \int_0^{+\infty} \alpha''(\xi + \tau) \left[\bar{\vartheta}_1^t(\xi) - \bar{\vartheta}_2^t(\xi) \right] d\xi. \tag{6.15}$$

Hence the limit as $\tau \to +\infty$, by virtue of (2.5), yields $\vartheta_1(t) = \vartheta_2(t)$, that is $(4.2)_1$.

This result implies that also the right-hand side of (6.15) vanishes so that $(4.2)_2$ is satisfied.

Finally, from (6.11), which must hold for any non-zero $\mathbf{g}_P(\tau)$, it follows that

$$\mathbf{I}^t_{(k)}(\tau, \bar{\mathbf{g}}^t_1) = \mathbf{I}^t_{(k)}(\tau, \bar{\mathbf{g}}^t_2), \tag{6.16}$$

which, by virtue of the definition $(5.17)_2$, yields (4.3).

Thus, all the equalities required by Theorem 4.1 hold; consequently, the two w-equivalent $\sigma_j(t)$ $(j = 1, 2)$ are also equivalent in the sense of Definition 4.1. □

7 A First Expression for the Minimum Free Energy

Let $\psi_m(t)$ denote the minimum free energy and Π be the set of finite work processes of $\mathcal{B}$, we have

$$\psi_m(t) \equiv W_R(\sigma) = \sup \{ -W(\sigma, P) : \; P \in \Pi \}, \tag{7.1}$$

where $W_R(\sigma)$ is the maximum recoverable work from a given state σ of the body [15, 18, 20].

The work $W_R(\sigma)$ is a non-negative function of the state, since in Π there exists the null process, for which the work done on the body starting from σ vanishes; moreover, by virtue of thermodynamic considerations, it follows that $W_R(\sigma) < +\infty$.

Let $\sigma(t) = (\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t)$ be the initial state at time $t > 0$, when a process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ is applied to the body for any $\tau \in [0, d)$. By extending P on $\mathbb{R}^+$ by means of $P(\tau) = (0, \mathbf{0})$ $\forall \tau \in [0, +\infty)$, the work done on the body has the expression given by $(5.16)_2$, which, if we assume that $\vartheta_P(d) \equiv \vartheta(t + d) = 0$, can be written as

$$\begin{aligned} W(\sigma, P) = & -\frac{1}{2}\alpha_0 \vartheta^2(t) + \alpha'(0) \int_0^{+\infty} \vartheta_P^2(\tau) d\tau \\ & + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \alpha''(| \tau - \eta |) \vartheta_P(\eta) \vartheta_P(\tau) d\eta d\tau - \int_0^{+\infty} I^t_{(\alpha)}(\tau, \bar{\vartheta}^t) \vartheta_P(\tau) d\tau \\ & + k_0 \int_0^{+\infty} \mathbf{g}_P^2(\tau) d\tau + \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} k'(| \tau - \eta |) \mathbf{g}_P(\eta) \cdot \mathbf{g}_P(\tau) d\eta d\tau \\ & - \int_0^{+\infty} \mathbf{I}^t_{(k)}(\tau, \bar{\mathbf{g}}^t) \cdot \mathbf{g}_P(\tau) d\tau. \end{aligned} \tag{7.2}$$

In order to obtain the maximum recoverable work, we must evaluate the maximum of $-W(\sigma, P)$, which will correspond to an opportune process, denoted by $P^{(m)}(\tau) = (\dot{\vartheta}^{(m)}(\tau), \mathbf{g}^{(m)}(\tau))$, where $\dot{\vartheta}^{(m)}(\tau)$ will be related to the optimal temperature $\vartheta^{(m)}$, which characterizes the maximum together with $\mathbf{g}^{(m)}$. We can consider the ensuing field ϑ_P with $\mathbf{g}_P$ expressed by means of the quantities $\vartheta^{(m)}$ and $\mathbf{g}^{(m)}$ by assuming

$$\vartheta_P(\tau) = \vartheta^{(m)}(\tau) + \gamma\varphi(\tau), \quad \mathbf{g}_P(\tau) = \mathbf{g}^{(m)}(\tau) + \delta\mathbf{e}(\tau) \quad \forall\tau \in \mathbb{R}^+, \tag{7.3}$$

with γ and δ being two real parameters, φ and $\mathbf{e}$ being two arbitrary smooth functions such that $\varphi(0) = 0$ and $\mathbf{e}(0) = \mathbf{0}$.

Substitution of (7.3) into (7.2) gives

$$\begin{aligned}
-W(\sigma, P) &= -\tilde{W}(\vartheta(t), \bar{\vartheta}^t, \bar{\mathbf{g}}^t; \dot{\vartheta}^{(m)} + \gamma\dot{\varphi}, \mathbf{g}^{(m)} + \delta\mathbf{e}) \\
&= \frac{1}{2}\alpha_0\vartheta^2(t) - \alpha'(0)\int_0^{+\infty} \left\{ \left[\vartheta^{(m)}(\tau)\right]^2 + 2\vartheta^{(m)}(\tau)\varphi(\tau)\gamma + \varphi^2(\tau)\gamma^2 \right\} d\tau \\
&\quad -\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} \alpha''(|\,\tau-\eta\,|)\left\{\vartheta^{(m)}(\eta)\vartheta^{(m)}(\tau) + [\vartheta^{(m)}(\eta)\varphi(\tau) \right. \\
&\quad \left. +\varphi(\eta)\vartheta^{(m)}(\tau)]\gamma + \varphi(\eta)\varphi(\tau)\gamma^2 \right\} d\eta d\tau + \int_0^{+\infty} I^t_{(\alpha)}(\tau, \bar{\vartheta}^t)[\vartheta^{(m)}(\tau) \\
&\quad +\varphi(\tau)\gamma]d\tau - k_0\int_0^{+\infty} \left\{ \left[\mathbf{g}^{(m)}(\tau)\right]^2 + 2\mathbf{g}^{(m)}(\tau)\cdot\mathbf{e}(\tau)\delta + \mathbf{e}^2(\tau)\delta^2 \right\} d\tau \\
&\quad -\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} k'(|\,\tau-\eta\,|)\left\{\mathbf{g}^{(m)}(\eta)\cdot\mathbf{g}^{(m)}(\tau) + [\mathbf{g}^{(m)}(\eta)\cdot\mathbf{e}(\tau) \right. \\
&\quad \left. +\mathbf{e}(\eta)\cdot\mathbf{g}^{(m)}(\tau)]\delta + \mathbf{e}(\eta)\cdot\mathbf{e}(\tau)\delta^2 \right\} d\eta d\tau \\
&\quad + \int_0^{+\infty} \mathbf{I}^t_{(k)}(\tau, \bar{\mathbf{g}}^t)\cdot[\mathbf{g}^{(m)}(\tau) + \mathbf{e}(\tau)\delta]d\tau,
\end{aligned} \tag{7.4}$$

whence, the derivatives with respect to γ and δ yield

$$\left\{\begin{aligned}
&\tfrac{\partial}{\partial\gamma}[-W(\sigma,P)]\,|_{\gamma=0} = \textstyle\int_0^{+\infty}\varphi(\tau)\left[-2\alpha'(0)\vartheta^{(m)}(\tau)\right. \\
&\qquad\qquad \left. - \textstyle\int_0^{+\infty}\alpha''(|\,\tau-\eta\,|)\vartheta^{(m)}(\eta)d\eta + I^t_{(\alpha)}(\tau,\bar{\vartheta}^t)\right]d\tau = 0 \\
&\tfrac{\partial}{\partial\delta}[-W(\sigma,P)]\,|_{\delta=0} = \textstyle\int_0^{+\infty}\mathbf{e}(\tau)\cdot\left[-2k_0\mathbf{g}^{(m)}(\tau)\right. \\
&\qquad\qquad \left. - \textstyle\int_0^{+\infty}k'(|\,\tau-\eta\,|)\mathbf{g}^{(m)}(\eta)d\eta + \mathbf{I}^t_{(k)}(\tau,\bar{\mathbf{g}}^t)\right]d\tau = 0.
\end{aligned}\right. \tag{7.5}$$

From the arbitrariness of φ and $\mathbf{e}$ in (7.5) it follows that

$$\left\{\begin{array}{l}
\int_0^{+\infty}\alpha''(|\,\tau-\eta\,|)\vartheta^{(m)}(\eta)d\eta + 2\alpha'(0)\vartheta^{(m)}(\tau) = I^t_{(\alpha)}(\tau,\bar{\vartheta}^t) \\
\int_0^{+\infty}k'(|\,\tau-\eta\,|)\mathbf{g}^{(m)}(\eta)d\eta + 2k_0\mathbf{g}^{(m)}(\tau) = \mathbf{I}^t_{(k)}(\tau,\bar{\mathbf{g}}^t)
\end{array}\right. \quad \forall\tau\in\mathbb{R}^+. \tag{7.6}$$

In this system we have two Wiener-Hopf integral equations of the second kind, which are solvable by virtue of the thermodynamic properties of the kernels and of some theorems on factorization; thus, we are able to derive the solutions $\vartheta^{(m)}$ and $\mathbf{g}^{(m)}$, which give the maximum recoverable work.

Such a work, by substituting the expressions of $I^t_{(\alpha)}$ and $\mathbf{I}^t_{(k)}$, given by (7.6), into

(7.2), assumes the form

$$W_R(\sigma) = \frac{1}{2}\alpha_0\vartheta^2(t) + \alpha'(0)\int_0^{+\infty}\left[\vartheta^{(m)}(\tau)\right]^2 d\tau + k_0\int_0^{+\infty}\left[\mathbf{g}^{(m)}(\tau)\right]^2 d\tau$$
$$+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\alpha''(|\,\tau-\eta\,|)\vartheta^{(m)}(\eta)\vartheta^{(m)}(\tau)d\eta d\tau$$
$$+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}k'(|\,\tau-\eta\,|)\mathbf{g}^{(m)}(\eta)\cdot\mathbf{g}^{(m)}(\tau)d\eta d\tau. \tag{7.7}$$

This relation can be expressed in terms of Fourier's transform, by using Plancherel's theorem and $(2.16)_1$, as follows

$$\begin{aligned} W_R(\sigma) &= \frac{1}{2}\alpha_0\vartheta^2(t) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\omega\alpha'_s(\omega)\vartheta_+^{(m)}(\omega)\left[\vartheta_+^{(m)}(\omega)\right]^* d\omega \\ &+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[k_0 + k'_c(\omega)\right]\mathbf{g}_+^{(m)}(\omega)\cdot\left[\mathbf{g}_+^{(m)}(\omega)\right]^* d\omega. \end{aligned} \tag{7.8}$$

It remains to solve the Wiener-Hopf integral equations in (7.6). To do this we introduce

$$r^{(\alpha)}(\tau) = \begin{cases} \int_{-\infty}^{+\infty}\alpha''(|\,\tau-\eta\,|)\vartheta^{(m)}(\eta)d\eta & \forall\tau\in\mathbb{R}^-, \\ 0 & \forall\tau\in\mathbb{R}^{++}, \end{cases} \tag{7.9}$$

$$\mathbf{r}^{(k)}(\tau) = \begin{cases} \int_{-\infty}^{+\infty}k'(|\,\tau-\eta\,|)\mathbf{g}^{(m)}(\eta)d\eta & \forall\tau\in\mathbb{R}^-, \\ \mathbf{0} & \forall\tau\in\mathbb{R}^{++}, \end{cases} \tag{7.10}$$

which allow us to give to (7.6) the following form

$$\begin{cases} \int_{-\infty}^{+\infty}\alpha''(|\,\tau-\eta\,|)\vartheta^{(m)}(\eta)d\eta + 2\alpha'(0)\vartheta^{(m)}(\tau) = I^t_{(\alpha)}(\tau,\bar{\vartheta}^t) + r^{(\alpha)}(\tau) \\ \int_{-\infty}^{+\infty}k'(|\,\tau-\eta\,|)\mathbf{g}^{(m)}(\eta)d\eta + 2k_0\mathbf{g}^{(m)}(\tau) = \mathbf{I}^t_{(k)}(\tau,\bar{\mathbf{g}}^t) + \mathbf{r}^{(k)}(\tau) \end{cases} \quad \forall\tau\in\mathbb{R}. \tag{7.11}$$

Hence, using the Fourier transform, we obtain

$$\begin{cases} 2\left[\alpha''_c(\omega) + \alpha'(0)\right]\vartheta_+^{(m)}(\omega) = I^t_{(\alpha)+}(\omega,\bar{\vartheta}^t) + r_-^{(\alpha)}(\omega), \\ 2\left[k'_c(\omega) + k_0\right]\mathbf{g}_+^{(m)}(\omega) = \mathbf{I}^t_{(k)+}(\omega,\bar{\mathbf{g}}^t) + \mathbf{r}_-^{(k)}(\omega). \end{cases} \tag{7.12}$$

Since, in particular, $\alpha''_c(\omega) + \alpha'_0 = \omega\alpha'_s(\omega)$ by virtue of $(2.16)_1$, we can put

$$H^{(\alpha)}(\omega) = \omega\alpha'_s(\omega) \geq 0, \qquad H^{(k)}(\omega) = k_0 + k'_c(\omega) > 0, \tag{7.13}$$

because of the thermodynamic restrictions (2.15).

We note that $H_\infty^{(\alpha)}$ is an even function, that is

$$H^{(\alpha)}(\omega) = H^{(\alpha)}(-\omega), \tag{7.14}$$

and goes to zero at least quadratically at the origin; we assume for such a function a behaviour no stronger than the quadratic one. Moreover, using $(2.16)_1$ and $(2.20)_2$, it follows that

$$H_\infty^{(\alpha)} = \lim_{\omega\to+\infty}\omega\alpha'_s(\omega) = \alpha'(0) > 0. \tag{7.15}$$

We also observe that

$$H_{\infty}^{(k)} = \lim_{\omega\to+\infty} [k_0 + k'_c(\omega)] = k_0 > 0, \quad H^{(k)}(0) = \lim_{\omega\to 0} [k_0 + k'_c(\omega)] = k_\infty > 0, \tag{7.16}$$

by virtue of $(2.20)_1$ and $(2.19)_2$.

Hence, the functions $H^{(\alpha)}(\omega)$ and $H^{(k)}(\omega)$ can be factorized [20]

$$H^{(\alpha)}(\omega) = H^{(\alpha)}_{(+)}(\omega) H^{(\alpha)}_{(-)}(\omega), \qquad H^{(k)}(\omega) = H^{(k)}_{(+)}(\omega) H^{(k)}_{(-)}(\omega), \tag{7.17}$$

where the extensions to the complex plane $\mathbb{C}$ of $H^{(\alpha)}_{(+)}(\omega)$ and $H^{(k)}_{(+)}(\omega)$ have no singularities and zeros in $\mathbb{C}^{(-)}$ and, therefore, they are analytic in $\mathbb{C}^{-}$, while the extensions of $H^{(\alpha)}_{(-)}(\omega)$ and $H^{(k)}_{(-)}(\omega)$, without zeros and singularities in $\mathbb{C}^{(+)}$, are analytic in $\mathbb{C}^{+}$.

Thus, from (7.12), by using $(2.16)_1$, (7.13) and (7.17), we obtain

$$H^{(\alpha)}_{(+)}(\omega)\vartheta^{(m)}_{+}(\omega) = \frac{I^t_{(\alpha)+}(\omega,\bar{\vartheta}^t)}{2H^{(\alpha)}_{(-)}(\omega)} + \frac{r^{(\alpha)}_{-}(\omega)}{2H^{(\alpha)}_{(-)}(\omega)}, \tag{7.18}$$

$$H^{(k)}_{(+)}(\omega)\mathbf{g}^{(m)}_{+}(\omega) = \frac{\mathbf{I}_{(k)+}(\omega,\bar{\mathbf{g}}^t)}{2H^{(k)}_{(-)}(\omega)} + \frac{\mathbf{r}^{(k)}_{-}(\omega)}{2H^{(k)}_{(-)}(\omega)}. \tag{7.19}$$

Using the Plemelj formulae [24], we have

$$\frac{I^t_{(\alpha)+}(\omega,\bar{\vartheta}^t)}{2H^{(\alpha)}_{(-)}(\omega)} = P^t_{(\alpha)(-)}(\omega) - P^t_{(\alpha)(+)}(\omega), \tag{7.20}$$

$$\frac{\mathbf{I}^t_{(k)+}(\omega,\bar{\mathbf{g}}^t)}{2H^{(k)}_{(-)}(\omega)} = \mathbf{P}^t_{(k)(-)}(\omega) - \mathbf{P}^t_{(k)(+)}(\omega), \tag{7.21}$$

where

$$P^t_{(\alpha)}(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{I^t_{(\alpha)+}(\omega,\bar{\vartheta}^t)}{2H^{(\alpha)}_{(-)}(\omega)}}{\omega - z} d\omega, \quad P^t_{(\alpha)(\pm)}(\omega) = \lim_{\beta\to 0^{\mp}} P^t_{(\alpha)}(\omega + i\beta), \tag{7.22}$$

$$\mathbf{P}^t_{(k)}(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{\mathbf{I}^t_{(k)+}(\omega,\bar{\mathbf{g}}^t)}{2H^{(k)}_{(-)}(\omega)}}{\omega - z} d\omega, \quad \mathbf{P}^t_{(k)(\pm)}(\omega) = \lim_{\beta\to 0^{\mp}} \mathbf{P}^t_{(k)}(\omega + i\beta). \tag{7.23}$$

From (7.18)-(7.19), by virtue of (7.20)-(7.21), we obtain

$$H^{(\alpha)}_{(+)}(\omega)\vartheta^{(m)}_{+}(\omega) + P^t_{(\alpha)(+)}(\omega) = P^t_{(\alpha)(-)}(\omega) + \frac{r^{(\alpha)}_{-}(\omega)}{2H^{(\alpha)}_{(-)}(\omega)}, \tag{7.24}$$

$$H^{(k)}_{(+)}(\omega)\mathbf{g}^{(m)}_{+}(\omega) + \mathbf{P}^t_{(k)(+)}(\omega) = \mathbf{P}^t_{(k)(-)}(\omega) + \frac{\mathbf{r}^{(k)}_{-}(\omega)}{2H^{(k)}_{(-)}(\omega)}, \tag{7.25}$$

where $P^t_{(\alpha)(\pm)}(\omega)$ and $\mathbf{P}^t_{(k)(\pm)}(\omega)$, considered as functions of $z \in \mathbb{C}$, are analytic in $\mathbb{C}^{(\mp)}$ but also in $\mathbb{R}$, by virtue of the assumption on the Fourier transforms. Hence, the functions at the left-hand sides are analytic in $\mathbb{C}^-$, while the others at the right-hand sides in $\mathbb{C}^+$; moreover, they vanish at infinity. Therefore, both sides must be equal to zero and, in particular, give

$$\vartheta_+^{(m)}(\omega) = -\frac{P^t_{(\alpha)(+)}(\omega)}{H^{(\alpha)}_{(+)}(\omega)}, \quad \mathbf{g}_+^{(m)}(\omega) = -\frac{\mathbf{P}^t_{(k)(+)}(\omega)}{H^{(k)}_{(+)}(\omega)}, \tag{7.26}$$

which, substituted into (7.8), yield

$$\psi_m(t) = \frac{1}{2}\alpha_0\vartheta^2(t) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mid P^t_{(\alpha)(+)}(\omega) \mid^2 d\omega + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mid \mathbf{P}^t_{(k)(+)}(\omega) \mid^2 d\omega. \tag{7.27}$$

8 Another Equivalent Expression for the Minimum Free Energy

The expression (7.27), now derived, can be changed in an equivalent one by considering the relation between $P^t_{(\alpha)(+)}(\omega)$ and $\bar{\vartheta}^t(\omega)$ and the one between $\mathbf{P}^t_{(k)(+)}(\omega)$ and $\mathbf{g}^t_+(\omega)$.

In order to derive this new expression, firstly we consider the casual extensions of $\bar{\vartheta}^t$ and $\bar{\mathbf{g}}^t$, by assuming $\bar{\vartheta}^t(s) = 0$ and $\bar{\mathbf{g}}^t(s) = \mathbf{0}$ $\forall s \in \mathbb{R}^{--}$; then, we consider the odd extension of $\alpha'''(s)$ and $k''(s)$ on $\mathbb{R}^{--}$, denoted by $\alpha'''^{(o)}(s)$ and $k''^{(o)}(s)$, that is

$$\alpha'''^{(o)}(s) = \begin{cases} \alpha'''(s) & \forall s \geq 0, \\ -\alpha'''(-s) & \forall s < 0, \end{cases} \qquad k''^{(o)}(s) = \begin{cases} k''(s) & \forall s \geq 0, \\ -k''(-s) & \forall s < 0. \end{cases} \tag{8.1}$$

Hence, we can give to (5.17) the new forms

$$I^t_{(\alpha)}(\tau,\bar{\vartheta}^t) = \int_{-\infty}^{+\infty} \alpha'''^{(o)}(\xi+\tau)\bar{\vartheta}^t(\xi)d\xi,\ \mathbf{I}^t_{(k)}(\tau,\bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k''^{(o)}(\xi+\tau)\bar{\mathbf{g}}^t(\xi)d\xi \ \forall\tau \geq 0 \tag{8.2}$$

and, by putting

$$I^{t_{(n)}}_{(\alpha)}(\tau,\bar{\vartheta}^t) = \int_{-\infty}^{+\infty} \alpha'''^{(o)}(\xi+\tau)\bar{\vartheta}^t(\xi)d\xi,\ \mathbf{I}^{t_{(n)}}_{(k)}(\tau,\bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k''^{(o)}(\xi+\tau)\bar{\mathbf{g}}^t(\xi)d\xi \ \forall\tau < 0, \tag{8.3}$$

extend the functions (8.2) on $\mathbb{R}$ as follows

$$I^{t_{(\mathbb{R})}}_{(\alpha)}(\tau,\bar{\vartheta}^t) = \int_{-\infty}^{+\infty} \alpha'''^{(o)}(\xi+\tau)\bar{\vartheta}^t(\xi)d\xi = \begin{cases} I^t_{(\alpha)}(\tau,\bar{\vartheta}^t) & \forall\tau \geq 0, \\ I^{t_{(n)}}_{(a)}(\tau,\bar{\vartheta}^t) & \forall\tau < 0, \end{cases} \tag{8.4}$$

$$\mathbf{I}^{t_{(\mathbb{R})}}_{(k)}(\tau,\bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k''^{(o)}(\xi+\tau)\bar{\mathbf{g}}^t(\xi)d\xi = \begin{cases} \mathbf{I}^t_{(k)}(\tau,\bar{\mathbf{g}}^t) & \forall\tau \geq 0, \\ \mathbf{I}^{t_{(n)}}_{(k)}(\tau,\bar{\mathbf{g}}^t) & \forall\tau < 0. \end{cases} \tag{8.5}$$

Let us introduce the functions $\bar{\vartheta}^t_n(s) = \bar{\vartheta}^t(-s)$ and $\bar{\mathbf{g}}^t_n(s) = \bar{\mathbf{g}}^t(-s)$ for any $s \leq 0$. Then, we extend them on $\mathbb{R}$ by assuming $\bar{\vartheta}^t_n(s) = 0$ and $\bar{\mathbf{g}}^t_n(s) = \mathbf{0}$ $\forall s > 0$. We can rewrite (8.4)-(8.5) as follows

$$I^{t_{(\mathbb{R})}}_{(\alpha)}(\tau,\bar{\vartheta}^t) = \int_{-\infty}^{+\infty} \alpha'''^{(o)}(\tau-s)\bar{\vartheta}^t_n(s)ds,\ \mathbf{I}^{t_{(\mathbb{R})}}_{(k)}(\tau,\bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k''^{(o)}(\tau-s)\bar{\mathbf{g}}^t_n(s)ds. \tag{8.6}$$

Using their Fourier's transforms, given by $(2.10)_1$ and expressed by

$$\bar{\vartheta}^t_{n_F}(\omega) = \bar{\vartheta}^t_{n_-}(\omega) = \left[\bar{\vartheta}^t_+(\omega)\right]^*, \quad \bar{\mathbf{g}}^t_{n_F}(\omega) = \bar{\mathbf{g}}^t_{n_-}(\omega) = \left[\bar{\mathbf{g}}^t_+(\omega)\right]^*, \tag{8.7}$$

we obtain

$$\begin{aligned}
I^{t(\mathbb{R})}_{(\alpha)_F}(\omega,\bar{\vartheta}^t) &= -2i\alpha_s'''(\omega)\bar{\vartheta}^t_{n_F}(\omega) = 2i\omega\alpha_c''(\omega)\left[\bar{\vartheta}^t_+(\omega)\right]^* \\
&= 2i\omega[H^{(\alpha)}(\omega)-\alpha'(0)]\left[\bar{\vartheta}^t_+(\omega)\right]^*, \qquad (8.8)\\
\mathbf{I}^{t(\mathbb{R})}_{(k)_F}(\omega,\bar{\mathbf{g}}^t) &= -2ik_s''(\omega)\bar{\mathbf{g}}^t_{n_F}(\omega) = 2i\omega k_c'(\omega)\left[\bar{\mathbf{g}}^t_+(\omega)\right]^* \\
&= 2i\omega\left[H^{(k)}(\omega)-k_0\right]\left[\bar{\mathbf{g}}^t_+(\omega)\right]^*, \qquad (8.9)
\end{aligned}$$

where we have used $(2.12)_2$, (2.13) for $\alpha_s'''(\omega)$ and $k_s''(\omega)$, $(2.16)_1$ and (7.13).

These last two Fourier transforms can be evaluated also by means of (8.4)-(8.5); thus, we obtain

$$I^{t(\mathbb{R})}_{(\alpha)_F}(\omega,\bar{\vartheta}^t) = I^{t(n)}_{(\alpha)-}(\omega,\bar{\vartheta}^t) + I^t_{(\alpha)+}(\omega,\bar{\vartheta}^t), \qquad (8.10)$$

$$\mathbf{I}^{t(\mathbb{R})}_{(k)_F}(\omega,\bar{\mathbf{g}}^t) = \mathbf{I}^{t(n)}_{(k)-}(\omega,\bar{\mathbf{g}}^t) + \mathbf{I}^t_{(k)+}(\omega,\bar{\mathbf{g}}^t), \qquad (8.11)$$

which, by virtue of (7.20)-(7.21), give

$$\frac{I^{t(\mathbb{R})}_{(\alpha)_F}(\omega,\bar{\vartheta}^t)}{2H^{(\alpha)}_{(-)}(\omega)} = \frac{I^{t(n)}_{(\alpha)-}(\omega,\bar{\vartheta}^t)}{2H^{(\alpha)}_{(-)}(\omega)} + P^t_{(\alpha)(-)}(\omega) - P^t_{(\alpha)(+)}(\omega), \qquad (8.12)$$

$$\frac{\mathbf{I}^{t(\mathbb{R})}_{(k)_F}(\omega,\bar{\mathbf{g}}^t)}{2H^{(k)}_{(-)}(\omega)} = \frac{\mathbf{I}^{t(n)}_{(k)-}(\omega,\bar{\mathbf{g}}^t)}{2H^{(k)}_{(-)}(\omega)} + \mathbf{P}^t_{(k)(-)}(\omega) - \mathbf{P}^t_{(k)(+)}(\omega). \qquad (8.13)$$

Hence, using the Plemelj formulae, we also have

$$\frac{I^{t(\mathbb{R})}_{(\alpha)_F}(\omega,\bar{\vartheta}^t)}{2H^{(\alpha)}_{(-)}(\omega)} = P^t_{(\alpha)_1(-)}(\omega) - P^t_{(\alpha)_1(+)}(\omega), \qquad (8.14)$$

$$\frac{\mathbf{I}^{t(\mathbb{R})}_{(k)_F}(\omega,\bar{\mathbf{g}}^t)}{2H^{(k)}_{(-)}(\omega)} = \mathbf{P}^t_{(k)_1(-)}(\omega) - \mathbf{P}^t_{(k)_1(+)}(\omega), \qquad (8.15)$$

where the new functions $P^t_{(\alpha)_1(\pm)}(\omega)$ and $\mathbf{P}^t_{(k)_1(\pm)}(\omega)$, as in (7.22)-(7.23), are defined by

$$P^t_{(\alpha)_1}(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{I^{t(\mathbb{R})}_{(\alpha)_F}(\omega,\bar{\vartheta}^t)}{2H^{(\alpha)}_{(-)}(\omega)}}{\omega - z}\,d\omega, \qquad (8.16)$$

$$\mathbf{P}^t_{(k)_1}(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{\mathbf{I}^{t(\mathbb{R})}_{(k)_F}(\omega,\bar{\mathbf{g}}^t)}{2H^{(k)}_{(-)}(\omega)}}{\omega - z}\,d\omega. \qquad (8.17)$$

Using (8.12)-(8.15), we can consider two functions

$$\begin{aligned}
F_{(\alpha)}(\omega) &\equiv P^t_{(\alpha)(+)}(\omega) - P^t_{(\alpha)_1(+)}(\omega) \\
&= P^t_{(\alpha)(-)}(\omega) - P^t_{(\alpha)_1(-)}(\omega) + \frac{I^{t(n)}_{(\alpha)-}(\omega,\bar{\vartheta}^t)}{2H^{(\alpha)}_{(-)}(\omega)} \qquad (8.18)
\end{aligned}$$

and

$$\begin{aligned}\mathbf{F}_{(k)}(\omega) &\equiv \mathbf{P}^t_{(k)(+)}(\omega) - \mathbf{P}^t_{(k)_1(+)}(\omega) \\ &= \mathbf{P}^t_{(k)(-)}(\omega) - \mathbf{P}^t_{(k)_1(-)}(\omega) + \frac{\mathbf{I}^{t_{(n)}}_{(k)-}(\omega, \bar{\mathbf{g}}^t)}{2H^{(k)}_{(-)}(\omega)},\end{aligned} \tag{8.19}$$

defined by means of two different quantities, which are analytic in $\mathbb{C}^-$ and in $\mathbb{C}^+$, respectively, and vanish at infinity. Consequently, these functions must vanish, i. e. $F_{(\alpha)}(\omega) = 0$ and $\mathbf{F}_{(k)}(\omega) = \mathbf{0}$; thus, taking account of (8.16)-(8.17), it follows that

$$P^t_{(\alpha)(+)}(\omega) \equiv P^t_{(\alpha)_1(+)}(\omega) = \lim_{z\to\omega^-} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\frac{I^{t(\mathbb{R})}_{(\alpha)_F}(\omega', \bar{\vartheta}^t)}{2H^{(\alpha)}_{(-)}(\omega')}}{\omega' - z} d\omega', \tag{8.20}$$

$$\mathbf{P}^t_{(k)(+)}(\omega) \equiv \mathbf{P}^t_{(k)_1(+)}(\omega) = \lim_{z\to\omega^-} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\frac{\mathbf{I}^{t(\mathbb{R})}_{(k)_F}(\omega', \bar{\mathbf{g}}^t)}{2H^{(k)}_{(-)}(\omega')}}{\omega' - z} d\omega'. \tag{8.21}$$

Hence, using $(8.8)_3$, $(8.9)_3$ and (7.17), we have

$$\begin{aligned}P^t_{(\alpha)(+)}(\omega) &= \lim_{z\to\omega^-} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' H^{(\alpha)}_{(+)}(\omega') \left[\bar{\vartheta}^t_+(\omega')\right]^*}{\omega' - z} d\omega' \\ &\quad - \lim_{z\to\omega^-} \frac{\alpha'(0)}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' \frac{\left[\bar{\vartheta}^t_+(\omega')\right]^*}{H^{(\alpha)}_{(-)}(\omega')}}{\omega' - z} d\omega',\end{aligned} \tag{8.22}$$

$$\begin{aligned}\mathbf{P}^t_{(k)(+)}(\omega) &= \lim_{z\to\omega^-} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' H^{(k)}_{(+)}(\omega') \left[\bar{\mathbf{g}}^t_+(\omega')\right]^*}{\omega' - z} d\omega' \\ &\quad - \lim_{z\to\omega^-} \frac{k_0}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' \frac{\left[\bar{\mathbf{g}}^t_+(\omega')\right]^*}{H^{(k)}_{(-)}(\omega')}}{\omega' - z} d\omega',\end{aligned} \tag{8.23}$$

which yield

$$\begin{aligned}\left[P^t_{(\alpha)(+)}(\omega)\right]^* &= \lim_{w\to\omega^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' H^{(\alpha)}_{(-)}(\omega') \bar{\vartheta}^t_+(\omega')}{\omega' - w} d\omega' \\ &\quad - \lim_{w\to\omega^+} \frac{\alpha'(0)}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' \frac{\bar{\vartheta}^t_+(\omega')}{H^{(\alpha)}_{(+)}(\omega')}}{\omega' - w} d\omega' \\ &= \lim_{w\to\omega^+} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{i\omega' H^{(\alpha)}_{(-)}(\omega') \bar{\vartheta}^t_+(\omega')}{\omega' - w} d\omega',\end{aligned} \tag{8.24}$$

$$\begin{aligned}\left[\mathbf{P}^t_{(k)(+)}(\omega)\right]^* &= \lim_{w\to\omega^+}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{i\omega' H^{(k)}_{(-)}(\omega')\bar{\mathbf{g}}^t_+(\omega')}{\omega'-w}d\omega' \\ &\qquad -\lim_{w\to\omega^+}\frac{k_0}{2\pi i}\int_{-\infty}^{+\infty}\frac{i\omega'\frac{\bar{\mathbf{g}}^t_+(\omega')}{H^{(k)}_{(+)}(\omega')}}{\omega'-w}d\omega' \\ &= \lim_{w\to\omega^+}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{i\omega' H^{(k)}_{(-)}(\omega')\bar{\mathbf{g}}^t_+(\omega')}{\omega'-w}d\omega'. \end{aligned} \tag{8.25}$$

We note that, in these last two relations $(8.24)_1$ and $(8.25)_1$, the two integrals with $\alpha'(0)$ and k_0 are equal to zero, since they can be evaluated by closing the contour in $\mathbb{C}^{(-)}$, where $\bar{\vartheta}^t_+(\omega)$ with $H^{(\alpha)}_{(+)}(\omega)$ and $\bar{\mathbf{g}}^t_+(\omega)$ with $H^{(k)}_{(+)}(\omega)$ have no singularities and hence they are analytic in $\mathbb{C}^-$, by virtue of the hypothesis assumed for the Fourier transforms after (2.21); moreover, for the integral with $\alpha'(0)$ in $(8.24)_1$, we observe that the zero at the origin of $H^{(\alpha)}_{(+)}(\omega)$ is eliminated by the presence of the factor ω.

The application of the Plemelj formulae to $(8.24)_2$ and $(8.25)_2$ yields

$$\omega H^{(\alpha)}_{(-)}(\omega)\bar{\vartheta}^t_+(\omega) = Q^t_{(\alpha)(-)}(\omega) - Q^t_{(\alpha)(+)}(\omega), \tag{8.26}$$

where

$$Q^t_{(\alpha)(\pm)}(\omega) = \lim_{z\to\omega^\mp}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\omega' H^{(\alpha)}_{(-)}(\omega')\bar{\vartheta}^t_+(\omega')}{\omega'-z}d\omega' \tag{8.27}$$

and

$$\omega H^{(k)}_{(-)}(\omega)\bar{\mathbf{g}}^t_+(\omega) = \mathbf{Q}^t_{(k)(-)}(\omega) - \mathbf{Q}^t_{(k)(+)}(\omega) \tag{8.28}$$

with

$$\mathbf{Q}^t_{(k)(\pm)}(\omega) = \lim_{z\to\omega^\mp}\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\omega' H^{(k)}_{(-)}(\omega')\bar{\mathbf{g}}^t_+(\omega')}{\omega'-z}d\omega'. \tag{8.29}$$

Thus, comparison of $(8.24)_2$ and $(8.25)_2$ with (8.27) and (8.29), respectively, yields

$$\left[P^t_{(\alpha)(+)}(\omega)\right]^* = iQ^t_{(\alpha)(-)}(\omega), \qquad \left[\mathbf{P}^t_{(k)(+)}(\omega)\right]^* = i\mathbf{Q}^t_{(k)(-)}(\omega), \tag{8.30}$$

which give the required new expression

$$\psi_m(t) = \frac{1}{2}\alpha_0\vartheta^2(t) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mid Q^t_{(\alpha)(-)}(\omega)\mid^2 d\omega + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mid \mathbf{Q}^t_{(k)(-)}(\omega)\mid^2 d\omega. \tag{8.31}$$

9 The Discrete Spectrum Model

We now apply the results of the previous section to study the particular class of response functions, which characterize the discrete spectrum model.

For this purpose, we assume the following relaxation functions

$$\alpha(t) = \begin{cases} \alpha_\infty - \sum_{i=1}^n h_i e^{-\alpha_i t} & \forall t\in\mathbb{R}^+, \\ 0 & \forall t\in\mathbb{R}^{--}, \end{cases} \quad k(t) = \begin{cases} k_\infty - \sum_{i=1}^n g_i e^{-k_i t} & \forall t\in\mathbb{R}^+, \\ 0 & \forall t\in\mathbb{R}^{--}, \end{cases} \tag{9.1}$$

where n is a positive integer, the inverse decay times α_i, k_i and the coefficients h_i, g_i $(i = 1, 2, ..., n)$ are also assumed to be positive; moreover, α_i and k_i are such that $\alpha_j < \alpha_{j+1}$ and $k_j < k_{j+1}$ $(j = 1, 2, ..., n-1)$.

From (9.1) we have

$$\alpha'(t) = \sum_{i=1}^{n} \alpha_i h_i e^{-\alpha_i t}, \quad \alpha'_F(\omega) = \sum_{i=1}^{n} \frac{\alpha_i h_i}{\alpha_i + i\omega} = \sum_{i=1}^{n} \alpha_i h_i \frac{\alpha_i - i\omega}{\alpha_i^2 + \omega^2}, \tag{9.2}$$

$$k'(t) = \sum_{i=1}^{n} k_i g_i e^{-k_i t}, \quad k'_F(\omega) = \sum_{i=1}^{n} \frac{k_i g_i}{k_i + i\omega} = \sum_{i=1}^{n} k_i g_i \frac{k_i - i\omega}{k_i^2 + \omega^2}. \tag{9.3}$$

Hence, by virtue of $(2.12)_3$, it follows that

$$\alpha'_s(\omega) = \omega \sum_{i=1}^{n} \frac{\alpha_i h_i}{\alpha_i^2 + \omega^2}, \quad k'_c(\omega) = \sum_{i=1}^{n} \frac{k_i^2 g_i}{k_i^2 + \omega^2}. \tag{9.4}$$

These Fourier's transforms allow us to derive the expressions for the two functions defined in (7.13); we have

$$H^{(\alpha)}(\omega) = \omega^2 \sum_{i=1}^{n} \frac{\alpha_i h_i}{\omega^2 + \alpha_i^2} \geq 0 \, \forall \omega \in \mathbb{R}, \quad H_\infty^{(\alpha)} = \sum_{i=1}^{n} \alpha_i h_i = \alpha'(0) > 0, \tag{9.5}$$

$$H^{(k)}(\omega) = k_0 + \sum_{i=1}^{n} \frac{k_i^2 g_i}{k_i^2 + \omega^2} > 0 \, \forall \omega \in \mathbb{R}, \quad H_\infty^{(k)} = k_0 > 0. \tag{9.6}$$

We observe that the relaxation functions α and k, we have assumed in (9.1), satisfy all the conditions of Sect. 2. In fact we have

$$\alpha_\infty - \alpha_0 = \sum_{i=1}^{n} h_i > 0, \qquad k_\infty - k_0 = \sum_{i=1}^{n} g_i > 0, \tag{9.7}$$

which satisfy $(2.17)_1$, $(2.19)_2$ and (2.20), by virtue of $(9.5)_2$.

Firstly, we consider the kernel α, for which the expression $(9.5)_1$ is analogous to the one obtained in [20], where however a minus sign occurs at the right-hand side of $(9.5)_1$ and the numerators are negative; it coincides with the one derived in [4] and [3]. We recall the results deduced in these works.

Let $n \neq 1$.

The function $f_{(\alpha)}(z) = H^{(\alpha)}(\omega) \mid_{z=-\omega^2}$ has n simple poles at α_i^2 $(i = 1, 2, ..., n)$ and n simple zeros at $\gamma_1 = 0$ and γ_j^2 $(j = 2, 3, ..., n)$, which are so ordered

$$\alpha_1^2 < \gamma_2^2 < \alpha_2^2 < ... < \alpha_{n-1}^2 < \gamma_n^2 < \alpha_n^2. \tag{9.8}$$

Consequently, we can give a new form to $(9.5)_1$, since it easily yields the factorization $(7.17)_1$; we have

$$H^{(\alpha)}(\omega) = H_\infty^{(\alpha)} \prod_{i=1}^{n} \left\{ \frac{\gamma_i^2 + \omega^2}{\alpha_i^2 + \omega^2} \right\} = H_\infty^{(\alpha)} \prod_{i=1}^{n} \left\{ \frac{(\omega - i\gamma_i)(\omega + i\gamma_i)}{(\omega - i\alpha_i)(\omega + i\alpha_i)} \right\}, \tag{9.9}$$

and, in particular,

$$H_{(-)}^{(\alpha)}(\omega) = h_\infty^{(\alpha)} \prod_{i=1}^{n} \left\{ \frac{\omega + i\gamma_i}{\omega + i\alpha_i} \right\} \equiv h_\infty^{(\alpha)} \left(1 + i \sum_{i=1}^{n} \frac{R_i}{\omega + i\alpha_i} \right), \tag{9.10}$$

with

$$R_i = (\gamma_i - \alpha_i) \prod_{j=1, j\neq i}^{n} \left\{ \frac{\gamma_j - \alpha_i}{\alpha_j - \alpha_i} \right\}, \quad h_\infty^{(\alpha)} = \sqrt{H_\infty^{(\alpha)}}. \tag{9.11}$$

We must consider (8.27) to derive the quantity $Q^t_{(\alpha)(-)}(\omega)$ in (8.27). For this purpose we note that, contrary to what occurs in the previous works [4] and [3], $H^{(\alpha)}_{(-)}(\omega)$ is now multiplied by ω in the integrand of (8.27).

Substitution of $(9.10)_2$ into (8.27) yields

$$Q^t_{(\alpha)(-)}(\omega) = \frac{h_\infty^{(\alpha)}}{2\pi i} \left[\int_{-\infty}^{+\infty} \frac{\omega' \bar{\vartheta}^t_+(\omega')}{\omega' - \omega^+} d\omega' + i \sum_{r=1}^{n} R_r \int_{-\infty}^{+\infty} \frac{\bar{\vartheta}^t_+(\omega') \frac{\omega'}{\omega' - \omega^+}}{\omega' - (-i\alpha_r)} d\omega' \right]. \tag{9.12}$$

The first integral of this relation vanishes since it can be extended to an infinite contour in $\mathbb{C}^{(-)}$, where the integrand function, considered as a function of $z \in \mathbb{C}$, is analytic. By closing again in $\mathbb{C}^{(-)}$ and taking account of the sense of the integrations, we can evaluate the other integrals; we have

$$Q^t_{(\alpha)(-)}(\omega) = h_\infty^{(\alpha)} \sum_{r=1}^{n} \frac{\alpha_r R_r}{\omega + i\alpha_r} \bar{\vartheta}^t_+(-i\alpha_r). \tag{9.13}$$

Hence, we obtain

$$\left[Q^t_{(\alpha)(-)}(\omega) \right]^* = h_\infty^{(\alpha)} \sum_{l=1}^{n} \frac{\alpha_l R_l}{\omega - i\alpha_l} \bar{\vartheta}^t_+(-i\alpha_l), \tag{9.14}$$

since, by virtue of $(2.10)_2$,

$$\bar{\vartheta}^t_+(-i\alpha_r) = \int_0^{+\infty} \bar{\vartheta}^t(s) e^{-\alpha_r s} ds = \left[\bar{\vartheta}^t_+(-i\alpha_r) \right]^*. \tag{9.15}$$

By closing now in $\mathbb{C}^{(+)}$, the first integral of (8.31) can be evaluated; we have

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mid Q^t_{(\alpha)(-)}(\omega) \mid^2 d\omega \\ &= \left[h_\infty^{(\alpha)} \right]^2 \sum_{r,l=1}^{n} \alpha_r \alpha_l R_r R_l \bar{\vartheta}^t_+(-i\alpha_r) \bar{\vartheta}^t_+(-i\alpha_l) \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\frac{i}{\omega + i\alpha_r}}{\omega - i\alpha_l} d\omega \\ &= H_\infty^{(\alpha)} \sum_{r,l=1}^{n} \frac{B_r B_l}{\alpha_r + \alpha_l} \bar{\vartheta}^t_+(-i\alpha_r) \bar{\vartheta}^t_+(-i\alpha_l), \end{aligned} \tag{9.16}$$

where we have put

$$B_r = \alpha_r R_r \qquad (r = 1, 2, ..., n). \tag{9.17}$$

Let $n = 1$.

From $(9.10)_1$, since $\gamma_1 = 0$, we have

$$H^{(\alpha)}_{(-)}(\omega) = h_\infty^{(\alpha)} \frac{\omega}{\omega + i\alpha_1} = h_\infty^{(\alpha)} \left(1 + i \frac{R_1}{\omega + i\alpha_1} \right), \;\; R_1 = -\alpha_1, \;\; h_\infty^{(\alpha)} = \sqrt{\alpha_1 h_1}. \tag{9.18}$$

Now, we consider the kernel k.

Case $n \neq 1$.

Taking in mind the expression $(9.6)_1$, we observe that the function $f_{(k)}(z) = H^{(k)}(\omega)\mid_{z=-\omega^2}$, such that

$$f_{(k)}(0) = k_0 + \sum_{i=1}^{n} g_i, \qquad \lim_{z \to \pm\infty} f_{(k)}(z) = k_0^{\mp}, \qquad \lim_{z \to \left(k_i^2\right)^{\mp}} f_{(k)}(z) = \pm\infty, \tag{9.19}$$

has n simple poles at k_i^2 $(i = 1, 2, ..., n)$ and n simple zeros in ν_i^2 $(i = 1, 2, ..., n)$ so ordered

$$k_1^2 < \nu_1^2 < k_2^2 < ... < \nu_{n-1}^2 < k_n^2 < \nu_n^2; \tag{9.20}$$

thus, $(9.6)_1$ can be written as

$$H^{(k)}(\omega) = H_{\infty}^{(k)} \prod_{i=1}^{n} \left\{ \frac{\nu_i^2 + \omega^2}{k_i^2 + \omega^2} \right\} = k_0 \prod_{i=1}^{n} \left\{ \frac{(\omega - i\nu_i)(\omega + i\nu_i)}{(\omega - ik_i)(\omega + ik_i)} \right\}. \tag{9.21}$$

Hence, it follows that

$$H_{(-)}^{(k)}(\omega) = k_0^{1/2} \prod_{i=1}^{n} \left\{ \frac{\omega + i\nu_i}{\omega + ik_i} \right\} \equiv k_0^{1/2} \left(1 + i \sum_{i=1}^{n} \frac{S_i}{\omega + ik_i} \right), \tag{9.22}$$

where

$$S_r = (\nu_r - k_r) \prod_{j=1, j\neq r}^{n} \left\{ \frac{\nu_j - k_r}{k_j - k_r} \right\} \quad (r = 1, 2, ..., n). \tag{9.23}$$

As above for $Q_{(\alpha)(-)}^t(\omega)$, still now in the expression for $\mathbf{Q}_{(k)(-)}^t(\omega)$ given by (8.29), the quantity $H_{(-)}^{(k)}(\omega)$ is multiplied by ω. Thus, by substituting $(9.22)_2$, with S_r given by (9.23), into (8.29), we obtain

$$\begin{aligned} \mathbf{Q}_{(k)(-)}^t(\omega) &= \frac{k_0^{1/2}}{2\pi i} \left[\int_{-\infty}^{+\infty} \frac{\omega' \bar{\mathbf{g}}_+^t(\omega')}{\omega' - \omega^+} d\omega' + i \sum_{r=1}^{n} S_r \int_{-\infty}^{+\infty} \frac{\bar{\mathbf{g}}_+^t(\omega') \frac{\omega'}{\omega' - \omega^+}}{\omega' - (-ik_r)} d\omega' \right] \\ &= k_0^{1/2} \sum_{r=1}^{n} \frac{k_r S_r}{\omega + ik_r} \bar{\mathbf{g}}_+^t(-ik_r), \end{aligned} \tag{9.24}$$

since, as we have noted above for $Q_{(\alpha)(-)}^t(\omega)$ after (9.12), still now the first integral vanishes and the other integrals can be evaluated by closing again in $\mathbb{C}^{(-)}$ and taking account of the sense of integrations.

Using $(2.10)_2$, as for $\bar{\vartheta}_+^t$ in (9.15),

$$\bar{\mathbf{g}}_+^t(-ik_r) = \left[\bar{\mathbf{g}}_+^t(-ik_r)\right]^* \tag{9.25}$$

also holds; thus, $(9.24)_2$ gives

$$\left[\mathbf{Q}_{(k)(-)}^t(\omega)\right]^* = k_0^{1/2} \sum_{r=1}^{n} \frac{k_r S_r}{\omega - ik_r} \bar{\mathbf{g}}_+^t(-ik_r). \tag{9.26}$$

Hence, the second integral of (8.31) can be evaluated by closing in $\mathbb{C}^{(+)}$; we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mid \mathbf{Q}^t_{(k)(-)}(\omega)\mid^2 d\omega \\
&= k_0 \sum_{r,l=1}^{n} k_r k_l S_r S_l^t \bar{\mathbf{g}}^t_+(-ik_r)\cdot \bar{\mathbf{g}}^t_+(-ik_l)\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\frac{i}{\omega+ik_r}}{\omega-ik_l}d\omega \\
&= k_0 \sum_{r,l=1}^{n} \frac{C_r C_l}{k_r+k_l}\bar{\mathbf{g}}^t_+(-ik_r)\cdot \bar{\mathbf{g}}^t_+(-ik_l),
\end{aligned}\tag{9.27}$$

where we have put

$$C_r = k_r S_r \qquad (r=1,2,...,n). \tag{9.28}$$

Let $n=1$.

In this particular case from $(9.6)_1$ we can evaluate the zero $\nu_1 = k_1\sqrt{1+\frac{g_1}{k_0}}$; then, $(9.22)_2$ gives

$$H^{(k)}_{(-)}(\omega) = k_0^{\frac{1}{2}}\frac{\omega+i\nu_1}{\omega+ik_1} \equiv k_0^{\frac{1}{2}}\left(1+i\frac{S_1}{\omega+ik_1}\right), \tag{9.29}$$

with

$$S_1 = \nu_1 - k_1 = k_1\left(\sqrt{1+\frac{g_1}{k_0}}-1\right). \tag{9.30}$$

Therefore, when $n\neq 1$, by using $(9.15)_1$ for $\bar{\vartheta}^t_+$ and the analogous relation (9.25) for $\bar{\mathbf{g}}^t_+$, in $(9.16)_2$ and in $(9.27)_2$, respectively, (8.31) yields the required expression

$$\begin{aligned}
\psi_m(t) &= \frac{1}{2}\alpha_0\vartheta^2(t) \\
&+\frac{1}{2}\int_0^{+\infty}\int_0^{+\infty}\left[2H^{(\alpha)}_\infty \sum_{r,l=1}^{n}\frac{B_r B_l}{\alpha_r+\alpha_l}e^{-(\alpha_r s_1+\alpha_l s_2)}\bar{\vartheta}^t(s_1)\bar{\vartheta}^t(s_2)\right. \\
&\left.+2k_0\sum_{r,l=1}^{n}\frac{C_r C_l}{k_r+k_l}e^{-(k_r s_1+k_l s_2)}\bar{\mathbf{g}}^t(s_1)\cdot\bar{\mathbf{g}}^t(s_2)\right]ds_1 ds_2.
\end{aligned}\tag{9.31}$$

In the simple case when $n=1$, (9.31), by using (9.18) and (9.29), becomes

$$\begin{aligned}
\psi_m(t) &= \frac{1}{2}\alpha_0\vartheta^2(t)+\frac{1}{2}h_1\alpha_1^4\left[\int_0^{+\infty}e^{-\alpha_1 s}\bar{\vartheta}^t(s)ds\right]^2 \\
&+\frac{1}{2}k_0k_1^3\left(\sqrt{1+\frac{g_1}{k_0}}-1\right)^2\left[\int_0^{+\infty}e^{-k_1 s}\bar{\mathbf{g}}^t(s)ds\right]^2.
\end{aligned}\tag{9.32}$$

Finally, we want to observe that, at least in this last case characterized by $n=1$, from (9.32), by means of the following integrations by parts

$$\begin{aligned}
\int_0^{+\infty}e^{-\alpha_1 s}\bar{\vartheta}^t(s)ds &= \frac{1}{\alpha_1}\int_0^{+\infty}e^{-\alpha_1 s}\,{}_r\vartheta^t(s)ds, \\
\int_0^{+\infty}e^{-k_1 s}\bar{\mathbf{g}}^t(s)ds &= \frac{1}{k_1}\int_0^{+\infty}e^{-k_1 s}\,{}_r\mathbf{g}^t(s)ds,
\end{aligned}$$

it is easy to obtain the corresponding result derived in [3] for $\psi_m(t)$, that is

$$\begin{aligned}\psi_m(t) \quad = \quad & \frac{1}{2}\alpha_0\vartheta^2(t) + \frac{1}{2}\left\{\alpha_1^2 h_1 \left[\int_0^{+\infty} e^{-\alpha_1 s}\, {}_r\vartheta^t(s)ds\right]^2\right. \\ & \left. + k_0 k_1 \left(\sqrt{1+\frac{g_1}{k_0}}-1\right)^2 \left[\int_0^{+\infty} e^{-k_1 s}\, {}_r\mathbf{g}^t(s)ds\right]^2\right\} \qquad (9.33)\end{aligned}$$

expressed in terms of $(\vartheta(t),\, {}_r\vartheta^t(s),\, {}_r\mathbf{g}^t(s))$, there assumed as the material state of the body.

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