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Special Solutions to Rotating Stratified Boussinesq Equations

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Abstract: In this paper, we have obtained some special solutions of rotating stratified Boussinesq equations and reduced these equations into the system of six coupled nonlinear ODEs. Further, in absence of strain field we have proved that the reduced system of six coupled ODEs is completely integrable.

Keywords: *rotating stratified Boussinesq equation, completely integrable systems, special solutions*

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1 Introduction

The stratified Boussinesq equations form a system of PDEs modelling the movements of planetary atmospheres. It may be noted that the Boussinesq approximation in the literature is also referred to as the Oberbeck–Boussinesq approximation for which, the reader is referred to an interesting article of Rajagopal et al [1] providing a rigorous mathematical justification as perturbations of the Navier–Stokes equations. Majda & Shefter [2] have chosen certain special solutions of this system of PDEs to demonstrate onset of instability when the Richardson number is less than 1/4. In their study of instability in stratified fluids at large Richardson number, Majda & Shefter [2] have obtained the exact solutions to stratified Boussinesq equations neglecting the effects of rotations and viscosity. In his monograph Majda [3] has obtained the special solution of rotating stratified Boussinesq equations excluding the effects of viscosity and finite rotation. Whereas, in this paper we include the effect of rotation. And then we systematically deploy the procedure of Majda & Shefter [2] (as well as the procedure applied by Craik & Criminale in their paper [4]) to obtain the exact solutions of rotating stratified Boussinesq equations and derive the system of six coupled ODEs. Further, in the absence of strain field we proved that the reduced system of ODEs is completely integrable and admits the similar results obtained by Srinivasan et all [5]. For the similar kind of work reader may refer Maas [8, 9].

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2 Nondimensional Rotating Stratified Boussinesq Equations

We consider the motion of an incompressible flow of fluid in the atmosphere and in the ocean where the flow velocities are too slow to account for compressible effects, the flow of fluid is governed by the following rotating Boussinesq equations (we ignore the effects of viscosity and heat dissipation) that involves the interaction of gravity with density stratification about the reference state.

$$\begin{aligned} \frac{D\mathbf{v}}{Dt} + f(\hat{\mathbf{e_3}} \times \mathbf{v}) &= -\nabla \frac{\tilde{p}}{\rho_b} - \frac{g\rho}{\rho_b}\hat{\mathbf{e_3}}, \\ \operatorname{div} \mathbf{v} &= 0, \\ \frac{D\tilde{\rho}}{Dt} &= 0, \end{aligned} \tag{2.1}$$

where $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$, the unit vector in vertical direction is $\hat{\mathbf{e_3}} = (0, 0, 1)$, the space variable $\mathbf{x} = (x_1, x_2, x_3)$ and fluid velocity is given by $\mathbf{v} = (v^1, v^2, v^3)$. For the local behavior of incompressible fluid the density $\tilde{\rho}$ is the sum of mean density ρ_b and perturbations ρ about the mean density, that is $\tilde{\rho}(\mathbf{x}, t) = \rho_b + \rho(\mathbf{x}, t)$. The pressure is denoted by $\tilde{p}$ and f is a rotation frequency.

Now we nondimensionalize the Boussinesq equations (2.1) with the following scales for length, time, velocity, density, and pressure:

$$\begin{array}{lcl} L & : & \text{horizontal length scale,} \\ v^* & : & \text{mean advective velocity,} \\ T_e = \frac{L}{v^*} & : & \text{eddy turnover time,} \\ T_R = f^{-1} & : & \text{rotation time,} \\ T_N = N^{-1} & : & \text{buoyancy time,} \\ \rho_b & : & \text{mean density,} \\ \overline{p} & : & \text{mean pressure,} \\ N & : & \text{buoyancy frequency.} \end{array} \tag{2.2}$$

In this scale of nondimensionalization we introduce the following nondimensional variables

$$\mathbf{x}' = \frac{\mathbf{x}}{L}, \; t' = \frac{t}{T_e}, \; \mathbf{v}' = \frac{\mathbf{v}}{v^*}, \; \tilde{\rho}' = \frac{\tilde{\rho}}{\rho_b B}, \; p' = \frac{\tilde{p}}{\overline{p}}. \tag{2.3}$$

The numerical factor B in the density equation is positive. Applying equations (2.3) to equations (2.1) and dropping the primes finally we get the nondimensional rotating stratified Boussinesq equations

$$\begin{aligned} \frac{D\mathbf{v}}{Dt} + \frac{1}{R_0}\mathbf{u} &= -\overline{P}\nabla p - \Gamma\rho\hat{\mathbf{e_3}}, \\ \operatorname{div} \mathbf{v} &= 0, \\ \frac{D\tilde{\rho}}{Dt} &= 0. \end{aligned} \tag{2.4}$$

Here, we have $\mathbf{u} = (u^1, u^2, u^3) = \hat{\mathbf{e_3}} \times \mathbf{v}$, $\Gamma = \frac{BgL}{v^{*2}}$ the nondimensional number, $R_0 = \frac{v^*}{Lf}$ the Rossby number and $\overline{P} = \frac{\overline{p}}{\rho_b v^{*2}}$ the Euler number. Nondimensional density function is $\tilde{\rho}(\mathbf{x}, t) = \rho_b + \rho(\mathbf{x}, t)$. The more elaborate discussion about the nondimensional analysis of rotating stratified Boussinesq equations is given by Majda in his monograph [3]. In the following section we have obtained the special solutions to (2.4).

3 Special Solutions

In this section we investigate the special solutions to (2.4) in large scale part. We are looking for the local behavior of an incompressible fluid, and we expand the smooth velocity field and density function in a Taylor's series about some point $\mathbf{x_0}$:

$$\begin{aligned} \mathbf{v}(\mathbf{x},t) &= \mathbf{v}(\mathbf{x}_0,t) + \nabla\mathbf{v}|_{(\mathbf{x}_0,t)}(\mathbf{x}-\mathbf{x}_0) + O(|\mathbf{x}-\mathbf{x}_0|^2), \\ \tilde{\rho}(\mathbf{x},t) &= \rho_b + \nabla\tilde{\rho}|_{(\mathbf{x}_0,t)}(\mathbf{x}-\mathbf{x}_0) + O(|\mathbf{x}-\mathbf{x}_0|^2), \end{aligned} \tag{3.1}$$

where $\nabla\mathbf{v}$ is a 3×3 matrix whose $(i,j)^{th}$ entry is $\frac{\partial v^i}{\partial x_j}$, $i = 1,2,3$, $j = 1,2,3$. The following equation (3.2) is the decomposition of the matrix $\nabla\mathbf{v}$ as a sum of symmetric and skew-symmetric matrices and such kind of decomposition is unique:

$$\begin{aligned} \nabla\mathbf{v}|_{(\mathbf{x}_0,t)} &= \left(\tfrac{\nabla\mathbf{v}+(\nabla\mathbf{v})^T}{2}\right) + \left(\tfrac{\nabla\mathbf{v}-(\nabla\mathbf{v})^T}{2}\right) \\ &= \mathcal{D}(\mathbf{x}_0,t) + \Omega(\mathbf{x}_0,t), \end{aligned} \tag{3.2}$$

where $\mathcal{D}$ is the symmetric part of $\nabla\mathbf{v}$ and is called the deformation matrix, it has the property that the trace of matrix $\mathcal{D}$ is equal to the divergence of vector field $\mathbf{v}$. Whereas, Ω is a skew symmetric part of matrix $\nabla\mathbf{v}$ and satisfy the following equation (3.3).

$$\Omega\mathbf{h} = \frac{1}{2}\mathbf{w}\times\mathbf{h}, \tag{3.3}$$

for any vector $\mathbf{h}\in\mathbb{R}^3$. The vector $\mathbf{w}$ is vorticity vector that is $\mathbf{w} = \nabla\times\mathbf{v} = (w_1,w_2,w_3)$. Hence, from equation (3.2) we get

$$\nabla\mathbf{v}|_{(\mathbf{x}_0,t)}\mathbf{h} = \mathcal{D}(\mathbf{x}_0,t)\mathbf{h} + \frac{1}{2}\mathbf{w}(\mathbf{x}_0,t)\times\mathbf{h}. \tag{3.4}$$

The decomposition of a vector $\mathbf{v}$ as in equations (3.1) by mean of equation (3.4) has a simple physical interpretation namely, every incompressible velocity field is a sum of translation, stretching and rotation. We may deprive the translation part by a Galilean transformation, for this one may refer to Majda & Bertozzi [10]. We assume that $\mathbf{v}(\mathbf{x}_0,t) = 0$.

We take advantage of the local representation to determine certain special solutions to the rotating stratified Boussinesq equation (2.4). We derive now an equation for gradient of velocity

$$(v^i_{x_k})_t + \sum_j v^j (v^i_{x_k})_{x_j} + \sum_j \frac{\partial v^j}{\partial x_k}\frac{\partial v^i}{\partial x_j} + \frac{1}{R_0}(u^i_{x_k}) = -\overline{P}(p_{x_i})_{x_j} - \Gamma\frac{\partial\rho}{\partial x_k}\delta_{k3}, \tag{3.5}$$

where δ is the Kronecker delta. Then, we introduce the notations $V = (v^i_{x_k})$ and $\hat{P} = \overline{P}(p_{x_i})_{x_k}$ for the Hessian matrix of the pressure $\hat{P}$. With this notation we can rewrite equation (3.5) in the matrix form as follows

$$\frac{DV}{Dt} + V^2 + \frac{1}{R_0}(u^i_{x_k}) = -\hat{P} - \Gamma\hat{\mathbf{e}}_\mathbf{3}(\nabla\rho)^T. \tag{3.6}$$

A matrix $(u^i_{x_k})$ can uniquely be expressed as $(u^i_{x_k}) = S + Q$, where the symmetric matrix

S and skew symmetric matrix Q are as given below

$$S = \frac{1}{2}\begin{pmatrix} -2\frac{\partial v^1}{\partial x_1} & \frac{\partial v^1}{\partial x_1} - \frac{\partial v^2}{\partial x_2} & -\frac{\partial v^2}{\partial x_3} \\ \frac{\partial v^1}{\partial x_1} - \frac{\partial v^2}{\partial x_2} & 2\frac{\partial v^1}{\partial x_2} & \frac{\partial v^1}{\partial x_3} \\ -\frac{\partial v^2}{\partial x_3} & \frac{\partial v^1}{\partial x_3} & 0 \end{pmatrix}, \; Q = \frac{1}{2}\begin{pmatrix} 0 & -\frac{\partial v^2}{\partial x_2} - \frac{\partial v^1}{\partial x_1} & -\frac{\partial v^2}{\partial x_3} \\ \frac{\partial v^2}{\partial x_2} + \frac{\partial v^1}{\partial x_1} & 0 & \frac{\partial v^1}{\partial x_3} \\ \frac{\partial v^2}{\partial x_3} & -\frac{\partial v^1}{\partial x_3} & 0 \end{pmatrix}. \tag{3.7}$$

For any $\mathbf{h} \in \mathbb{R}^3$, a skew symmetric matrix Q satisfies the equation

$$Q\mathbf{h} = -\frac{1}{2}\frac{\partial \mathbf{v}}{\partial x_3} \times \mathbf{h}. \tag{3.8}$$

From equation (2.4), we have the density function $\tilde{\rho} = \rho_b + \rho$ and $\frac{D\tilde{\rho}}{Dt} = 0$. Therefore, we have $\nabla\tilde{\rho} = \nabla\rho$. Now differentiating the density equation partially with respect to x_k we get

$$\frac{\partial}{\partial t}(\tilde{\rho}_{x_k}) + \sum_j \frac{\partial v^j}{\partial x_k}\frac{\partial \tilde{\rho}}{\partial x_j} + \sum_j v^j \frac{\partial^2 \tilde{\rho}}{\partial x_k \partial x_j} = 0 \tag{3.9}$$

which may be recast as

$$\frac{D}{Dt}(\nabla\tilde{\rho}) + V^T(\nabla\tilde{\rho}) = 0. \tag{3.10}$$

Since $\mathcal{D}$ and Ω are symmetric and skew symmetric parts of ∇v a simple calculation gives

$$V^2 = \mathcal{D}^2 + \Omega^2 + \mathcal{D}\Omega + \Omega\mathcal{D}. \tag{3.11}$$

The symmetric part of V^2 is $\mathcal{D}^2 + \Omega^2$ and $\mathcal{D}\Omega + \Omega\mathcal{D}$ is the skew-symmetric part. We proceed to decompose equation (3.6) into symmetric and skew symmetric parts. The symmetric part is easily seen to be

$$\frac{D\mathcal{D}}{Dt} + \mathcal{D}^2 + \Omega^2 + \frac{1}{R_0}S = -\hat{P} - \frac{\Gamma}{2}\left[\hat{\mathbf{e}_3}(\nabla\tilde{\rho})^T + (\nabla\tilde{\rho})\hat{\mathbf{e}_3}^T\right]. \tag{3.12}$$

The skew symmetric part of equation (3.6) is discussed in the following Proposition 3.1. Before proceeding to the proposition here we insert a simple lemma and one may find the proof of this lemma in the monograph of Majda [3].

Lemma 3.1 $\mathbf{w} \cdot \nabla\mathbf{v} = \mathbf{w} \cdot (\nabla\mathbf{v})^T$.

Proof For any $\mathbf{h} \in \mathbb{R}^3$, we have by identification (3.3)

$$0 = \frac{1}{2}\mathbf{w} \cdot (\mathbf{w} \times \mathbf{h}) = \frac{1}{2}\mathbf{w} \cdot \big(((\nabla\mathbf{v}) - (\nabla\mathbf{v})^T)\mathbf{h}\big) = \frac{1}{2}\mathbf{w} \cdot \big((\nabla\mathbf{v}) - (\nabla\mathbf{v})^T\big)\,\mathbf{h}$$

from which the result follows since $\mathbf{h}$ is arbitrary. □

Proposition 3.1 *The evolution of the vorticity* $\mathbf{w} = \nabla \times \mathbf{v}$ *is governed by the equation*

$$\frac{D\mathbf{w}}{Dt} = \mathbf{w} \cdot \nabla\mathbf{v} + \Gamma\begin{pmatrix} -\frac{\partial\tilde{\rho}}{\partial x_2} \\ \frac{\partial\tilde{\rho}}{\partial x_1} \\ 0 \end{pmatrix} + \frac{1}{R_0}\frac{\partial \mathbf{v}}{\partial x_3}. \tag{3.13}$$

Proof Equating the skew symmetric part of equation (3.6) we get

$$\frac{D\Omega}{Dt} + \mathcal{D}\Omega + \Omega\mathcal{D} + \frac{1}{R_0}Q = -\frac{\Gamma}{2}\begin{pmatrix} 0 & 0 & -\frac{\partial\tilde{\rho}}{\partial x_1} \\ 0 & 0 & -\frac{\partial\tilde{\rho}}{\partial x_2} \\ \frac{\partial\tilde{\rho}}{\partial x_1} & \frac{\partial\tilde{\rho}}{\partial x_2} & 0 \end{pmatrix}. \tag{3.14}$$

So that for arbitrary $\mathbf{h} \in \mathbb{R}^3$

$$\frac{1}{2}\frac{D\mathbf{w}}{Dt} \times \mathbf{h} + (\mathcal{D}\Omega + \Omega\mathcal{D})\mathbf{h} - \frac{1}{2R_0}\frac{\partial \mathbf{v}}{\partial x_3} \times \mathbf{h} = \frac{\Gamma}{2}\begin{pmatrix} -\frac{\partial\tilde{\rho}}{\partial x_2} \\ \frac{\partial\tilde{\rho}}{\partial x_1} \\ 0 \end{pmatrix} \times \mathbf{h}. \tag{3.15}$$

Here, Ω and $\mathcal{D}$ are given by

$$\Omega = \frac{1}{2}\begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{pmatrix}$$

and the elements d_{ij} of matrix $\mathcal{D}$ are expressible in terms of partial derivatives $\partial_k v^l$ with the relation $d_{11} + d_{22} + d_{33} = 0$. A simple calculation gives

$$\mathcal{D}\Omega + \Omega\mathcal{D} = \frac{1}{2}\begin{pmatrix} 0 & -c_{12} & c_{13} \\ c_{12} & 0 & -c_{23} \\ -c_{13} & c_{23} & 0 \end{pmatrix},$$

$$\mathbf{c} = \begin{pmatrix} c_{23} \\ c_{13} \\ c_{12} \end{pmatrix} = \begin{pmatrix} -w_1 d_{11} - w_2 d_{12} - w_3 d_{33} \\ -w_1 d_{12} - w_2 d_{22} - w_3 d_{23} \\ -w_1 d_{13} - w_2 d_{23} - w_3 d_{33} \end{pmatrix} = -\mathcal{D}\mathbf{w} = -\mathbf{w}\cdot\mathcal{D}.$$

Therefore,

$$(\mathcal{D}\Omega + \Omega\mathcal{D})\mathbf{h} = \frac{1}{2}\begin{pmatrix} 0 & -c_{12} & c_{13} \\ c_{12} & 0 & -c_{23} \\ -c_{13} & c_{23} & 0 \end{pmatrix}\mathbf{h} = \frac{1}{2}\begin{pmatrix} c_{23} \\ c_{13} \\ c_{12} \end{pmatrix} \times \mathbf{h}.$$

Hence, we can recast equation (3.15) as

$$\frac{1}{2}\frac{D\mathbf{w}}{Dt} \times \mathbf{h} - \frac{1}{2}\mathbf{w}\cdot\mathcal{D} \times \mathbf{h} - \frac{1}{2R_0}\frac{\partial \mathbf{v}}{\partial x_3} \times \mathbf{h} = \frac{\Gamma}{2}\begin{pmatrix} -\frac{\partial\tilde{\rho}}{\partial x_2} \\ \frac{\partial\tilde{\rho}}{\partial x_1} \\ 0 \end{pmatrix} \times \mathbf{h}. \tag{3.16}$$

Now $\mathbf{w}\cdot\mathcal{D} = \frac{1}{2}(\mathbf{w}\cdot(\nabla\mathbf{v}) + \mathbf{w}\cdot(\nabla\mathbf{v})^T) = \mathbf{w}\cdot\nabla\mathbf{v}$, substituting this into (3.16) and simplifying we get (3.13). Hence the proof of the proposition. □

Remark 3.1 From equation (3.13) we see that the infinitesimal vorticity elements are advected with the fluid and get amplified with interaction of velocity gradients and density gradients and also with addition term of rate of change of fluid velocity in vertical direction, which causes the effects of rotation. Due to this additional term caused by effect of rotation we have proved in the following Theorem 3.1 the component of vorticity along with the density gradient advected with fluid is increased according to the rate of change of velocity in vertical direction and get amplified with density gradient. This is the extension of Ertel's theorem allowing the forcing term due to the rotation effect and neglecting the dissipation. One may refer to Majda ([3], p. 14) for the details of Ertel's theorem.

Theorem 3.1 *The advective rate of change of vorticity component along with density gradient is given by*

$$\frac{D}{Dt}(\mathbf{w}\cdot\nabla\tilde{\rho}) = \frac{1}{R_0}\frac{\partial \mathbf{v}}{\partial x_3}\cdot\nabla\tilde{\rho}. \tag{3.17}$$

Proof Consider the advective rate of change of vorticity along with density gradient. We get

$$\frac{D}{Dt}(\mathbf{w}\cdot\nabla\tilde{\rho}) = \frac{D\mathbf{w}}{Dt}\cdot\nabla\tilde{\rho} + \mathbf{w}\cdot\frac{D(\nabla\tilde{\rho})}{Dt}. \tag{3.18}$$

Applying equations (3.10), (3.13) and lemma (3.1) to equation (3.18) we get the result. □

As we claim earlier we have obtained the special solutions to (2.4), these solutions are given in the form of the following Theorem 3.2. The more interesting part of these solutions is that it reduces the PDEs of rotating stratified Boussinesq equations (2.4) into the system of six coupled nonlinear ODEs.

Theorem 3.2 *The rotating stratified Boussinesq equations* (2.4) *admit the special solutions of the form*

$$\left.\begin{aligned} \mathbf{v}(\mathbf{x},t) &= \mathcal{D}(t)\mathbf{x} + \tfrac{1}{2}\mathbf{w}(t)\times\mathbf{x}, \\ \tilde{\rho} &= \rho_b + \mathbf{b}(t)\cdot\mathbf{x}, \\ \overline{P}p &= \tfrac{1}{2}\hat{P}(t)\mathbf{x}\cdot\mathbf{x}, \end{aligned}\right\} \tag{3.19}$$

where $\overline{P}$ *is a nondimensional number as defined in* (2.4), $\mathcal{D}(t)$ *is a symmetric matrix with zero trace; when* $\mathbf{w}(t) = \nabla\times\mathbf{v}$ *and* $\mathbf{b}(t) = \nabla\tilde{\rho}$ *satisfy the ODEs*

$$\left.\begin{aligned} \frac{d\mathbf{w}}{dt} &= \mathcal{D}(t)[\mathbf{w}(t) + \tfrac{1}{R_0}\hat{\mathbf{e_3}}] + \Gamma\hat{\mathbf{e_3}}\times\mathbf{b}(t) - \tfrac{1}{2R_0}\hat{\mathbf{e_3}}\times\mathbf{w}(t), \\ \frac{d\mathbf{b}}{dt} &= -\mathcal{D}(t)\mathbf{b}(t) + \tfrac{1}{2}\mathbf{w}(t)\times\mathbf{b}(t), \end{aligned}\right\} \tag{3.20}$$

and matrix $\hat{P}(t)$ *is given by*

$$-\hat{P} = \frac{d\mathcal{D}}{dt} + \mathcal{D}^2 + \Omega^2 + \frac{1}{R_0}S + \frac{\Gamma}{2}(\hat{\mathbf{e_3}}\mathbf{b}^T + \mathbf{b}\hat{\mathbf{e_3}}^T), \tag{3.21}$$

where the matrix Ω *is as defined in* (3.2) *through the linear map given by* (3.3) *and the matrix* S *is given by (3.7).*

Proof We proceed to show that the Ansatz (3.19), (3.20) does indeed furnish solutions to (2.4). The condition $\operatorname{div}\mathbf{v} = 0$ follows from the fact that matrix $\mathcal{D}$ has zero *trace*. To verify that the momentum equation, note that $\mathbf{v}$ is linear in $\mathbf{x}$ say $\mathbf{v} = V\mathbf{x}$, where $V = \mathcal{D} + \Omega$ is a function of time alone. Therefore, $\nabla\tilde{\rho} = \nabla\rho = \mathbf{b}(t)$ and advection term is

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = (V\mathbf{x}\cdot\nabla)V\mathbf{x} = V(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3})V\mathbf{x} = V^2\mathbf{x},$$

so that $\frac{D\mathbf{v}}{Dt} = \frac{d\mathcal{D}}{dt}\mathbf{x} + \frac{d\Omega}{dt}\mathbf{x} + V^2\mathbf{x}$. Also, equation (3.12) can be recast as

$$\frac{d\mathcal{D}}{dt} + \mathcal{D}^2 + \Omega^2 + \frac{1}{R_0}S = -\hat{P} - \frac{\Gamma}{2}(\hat{\mathbf{e_3}}\mathbf{b}^T + \mathbf{b}\hat{\mathbf{e_3}}^T) \tag{3.22}$$

and equation (3.13) that is equation for vorticity is equivalent to the first equation in (3.20). The equation for skew symmetric part equivalent to (3.14) is as given below

$$\frac{d\Omega}{dt} + \mathcal{D}\Omega + \Omega\mathcal{D} + \frac{1}{R_0}Q = -\frac{\Gamma}{2}\begin{pmatrix} 0 & 0 & -b_1 \\ 0 & 0 & -b_2 \\ b_1 & b_2 & 0 \end{pmatrix}, \tag{3.23}$$

where Q is skew symmetric matrix as defined in (3.7). Inserting (3.23) and eliminating $\frac{d\mathcal{D}}{dt}$ using (3.21) we find that

$$\begin{aligned} \frac{D\mathbf{v}}{Dt} &= -\hat{P}\mathbf{x} - \mathcal{D}^2\mathbf{x} - \Omega^2\mathbf{x} - \tfrac{\Gamma}{2}\begin{pmatrix} 0 & 0 & b_1 \\ 0 & 0 & b_2 \\ b_1 & b_2 & 2b_3 \end{pmatrix}\mathbf{x} - \tfrac{1}{R_0}S\mathbf{x} \\ & -(\mathcal{D}\Omega + \Omega\mathcal{D})\mathbf{x} + \tfrac{\Gamma}{2}\begin{pmatrix} 0 & 0 & b_1 \\ 0 & 0 & b_2 \\ -b_1 & -b_2 & 0 \end{pmatrix}\mathbf{x} - \tfrac{1}{R_0}Q\mathbf{x} + V^2\mathbf{x}. \end{aligned} \tag{3.24}$$

Since the term $(V^2 - \mathcal{D}^2 - \Omega^2 - \mathcal{D}\Omega - \Omega\mathcal{D})\mathbf{x}$ vanishes, (3.24) simplifies as

$$\frac{D\mathbf{v}}{Dt} = -\hat{P}\mathbf{x} - \Gamma\begin{pmatrix} 0 \\ 0 \\ \mathbf{b}\cdot\mathbf{x} \end{pmatrix} - \frac{1}{R_0}(S+Q)\mathbf{x}. \tag{3.25}$$

As the fluid velocity is defined by (3.19), then we see that

$$\frac{1}{R_0}(S+Q)\mathbf{x} = \frac{1}{R_0}(\hat{\mathbf{e_3}} \times \mathbf{v}).$$

The pressure term in (3.19) enables us to write (3.25) as

$$\frac{D\mathbf{v}}{Dt} + \frac{1}{R_0}\mathbf{u} = -\overline{P}\nabla p - \Gamma\rho\hat{\mathbf{e_3}}.$$

Finally we verify the Boussinesq equation for density

$$\begin{aligned} \frac{D\tilde{\rho}}{Dt} &= \frac{D}{Dt}(\mathbf{b}\cdot\mathbf{x}) \\ &= \frac{d\mathbf{b}}{dt}\cdot\mathbf{x} + (\mathbf{v}\cdot\nabla)(\mathbf{b}\cdot\mathbf{x}) \\ &= \frac{d\mathbf{b}}{dt}\cdot\mathbf{x} + \mathbf{v}\cdot\mathbf{b}. \end{aligned} \tag{3.26}$$

Using (3.20), we substitute $\frac{d\mathbf{b}}{dt}$ and $\mathbf{v} = (\mathcal{D} + \Omega)\mathbf{x}$ into (3.26), we get

$$\begin{aligned} \frac{D\tilde{\rho}}{Dt} &= -(\mathcal{D}\mathbf{b})\cdot\mathbf{x} + \frac{1}{2}(\mathbf{w}\times\mathbf{b})\cdot\mathbf{x} + [(\mathcal{D}+\Omega)\mathbf{x}]\cdot\mathbf{b} \\ &= -(\mathcal{D}\mathbf{b})\cdot\mathbf{x} + \frac{1}{2}(\mathbf{w}\times\mathbf{b})\cdot\mathbf{x} + (\mathcal{D}\mathbf{x})\cdot\mathbf{b} + \frac{1}{2}(\mathbf{w}\times\mathbf{x})\cdot\mathbf{b} \\ &= 0 \end{aligned} \tag{3.27}$$

completing the proof of the theorem. □

Following are the examples of special solutions of rotating stratified Boussinesq equations (2.4) in the form of (3.19).

Example 3.1 Consider a two dimensional time independent flow for which the constant vorticity vector $\mathbf{w} = (0, 0, w_0)$, density gradient vector $\mathbf{b} = (0, 0, b_0)$ and deformation matrix is given by

$$\mathcal{D} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we see that vectors $\mathbf{w}$ and $\mathbf{b}$ satisfy the system of ODEs (3.20) and a solution of the system of PDEs (2.4) is given below.

$$\begin{aligned} \mathbf{v}(\mathbf{x}, t) &= (\lambda x_1 - \tfrac{w_0}{2} x_2, \tfrac{w_0}{2} x_1 - \lambda x_2, 0), \\ \tilde{\rho} &= \rho_b + b_0 x_3, \\ \overline{P}p &= \tfrac{1}{2}\left[(-\lambda^2 + \tfrac{w_0^2}{4} + \tfrac{w_0}{2R_0})(x_1^2 + x_2^2) + \Gamma b_0 x_3^2 - \tfrac{2\lambda}{R_0} x_1 x_2\right]. \end{aligned}$$

Example 3.2 Now we consider a two dimensional time dependent flow; let the vorticity vector be $\mathbf{w}(t) = \Big(w_{10}\cos(t/R_0) + w_{20}\sin(t/R_0), -w_{10}\sin(t/R_0) + w_{20}\cos(t/R_0), 0\Big) = (-a_2(t), a_1(t), 0)$ with the initial condition $\mathbf{w}(0) = (w_{10}, w_{20}, 0)$ and the density gradient vector be $\mathbf{b}(t) = (0, 0, b_0)$, where b_0 is an arbitrary constant. The deformation matrix $\mathcal{D}$ is

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & a_1(t) \\ 0 & 0 & a_2(t) \\ a_1(t) & a_2(t) & 0 \end{pmatrix}.$$

Then, we see that the vectors $\mathbf{w}, \mathbf{b}$ satisfy the system of ODEs (3.20) with initial conditions $\mathbf{w}(0), \mathbf{b}(0)$. The velocity and density are then given by $\mathbf{v}(\mathbf{x}, t) = (a_1(t)x_3, a_2(t)x_3, 0)$, $\tilde{\rho} = \rho_b + b_0 x_3$. The pressure p will be computed by using equations (3.19) and (3.21).

4 Integrable System

In the above section we see that the rotating stratified Boussinesq equations (2.4) admit the special solutions in the form of (3.19) provided that $\mathbf{w}$ and $\mathbf{b}$ satisfy the system of ODEs (3.20). Further, in the absence of strain field $\mathcal{D} = 0$ we have the following reduced system of six coupled nonlinear ODEs

$$\left.\begin{aligned} \dot{\mathbf{w}} &= \Gamma \hat{\mathbf{e}_3} \times \mathbf{b} - \tfrac{1}{2R_0} \hat{\mathbf{e}_3} \times \mathbf{w}, \\ \dot{\mathbf{b}} &= \tfrac{1}{2} \mathbf{w} \times \mathbf{b}. \end{aligned}\right\} \tag{4.1}$$

We see the system of equations (4.1) is divergence free and admits the following four functionally independent first integrals

$$|\mathbf{b}|^2 = c_1, \quad \hat{\mathbf{e}_3} \cdot \mathbf{w} = c_2, \quad |\mathbf{w}|^2 + 4\Gamma(\hat{\mathbf{e}_3} \cdot \mathbf{b}) = c_3, \quad \mathbf{w} \cdot \mathbf{b} + \frac{1}{R_0} \hat{\mathbf{e}_3} \cdot \mathbf{b} = c_4. \tag{4.2}$$

Hence, by Liouville's theorem on integral invariant and theorem of Jacobi [11] there exists an additional first integral. That is an autonomous system of six coupled ODEs admitting the five global functionally independent first integrals proving the complete integrability of the system (4.1). Also, we see from (4.2) that $|\mathbf{b}|$ and $|\mathbf{w}|$ remain bounded so that the invariant surface (4.2) is compact and flow of vector field $(\mathbf{w}, \mathbf{b})$ is complete. It is easy to verify that the system (4.1) admits all the similar kind of results obtained by Srinivasan et all in their paper [5]. Also, we find that the system of equations (4.1) is similar to the system discussed by Desale [6]. For the bifurcation analysis near the degenerate critical point one may refer to [7].

5 Conclusion

In Section 1, we gave a brief introduction to the work and put up a literature survey. Then in Section 2, we present the rotating stratified Boussinesq equations (2.1) and consequently we put it into the nondimensional form (2.4). In Section 3, we obtained the special solutions to the system (2.4) in the form of (3.19). Due to the inclusion of rotating term in the equations (2.4), the special solutions obtained here are the improvement of the solutions obtained by Majda & Shefter [2]. In this link we present the Proposition 3.1, Theorem 3.1 and in Theorem 3.2, we present special solutions provided that the vorticity and density gradients satisfy the system of ODEs (3.20). Also, in that section we gave the examples of two dimensional flows. In the last Section 4, we proved that the system of six coupled nonlinear ODEs (4.1), which is obtained by neglecting the strain field is an integrable system.

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