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A Short Note on Semilinear Elliptic Equations in Unbounded Domain

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Abstract: Let $\Omega \subset \mathbb{R}^n$ be a domain (not necessarily bounded) with smooth boundary $\partial\Omega$. Let $1 \leq n \leq 6$ and $f \in C^{0,\alpha}(\overline{\Omega}) \cap L^2(\Omega)$ be a given function with $f < 0$. In the present study, we prove that the following BVP

$$-\Delta u = u^2 + f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega,$$

has a solution $u \in H_0^1(\Omega)$ and satisfies $u \leq 0$ in Ω.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain (i.e open and connected) with smooth boundary $\partial\Omega$. Let $1 \leq n \leq 6$ and $f \in C^{0,\alpha}(\overline{\Omega}) \cap L^2(\Omega)$ be a nonzero given function. We consider the BVP

$$-\Delta u = u^2 + f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

The variational or the weak formulation of (1.1) and (1.2) is to find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u.\nabla v = \int_\Omega u^2 v + \int_\Omega f v, \quad \text{for all } v \in H_0^1(\Omega). \tag{1.3}$$

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For a bounded domain Ω and $f < 0$, monotone iteration technique has been used to prove the existence of a solution of (1.1) and (1.2). For details of the proof, we refer to the book by Kesavan, S. [3, p.227]. The Rellich–Kondrasov theorem has been used. The present study deals with the existence of a solution of the BVP when Ω is an unbounded domain with smooth boundary.

We assume $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, where Ω_i is a bounded domain with smooth boundary $\partial\Omega_i$, for $i = 1, 2, 3....$ with $\Omega_i \subseteq \Omega_{i+1}$. Before we proceed, as a consequence of embedding theorem, we note that $u^2 \in L^{3/2}(\Omega)$ if $u \in H_0^1(\Omega)$ for $n \leq 6$, and so

$$\Big| \int_\Omega u^2 v \Big| \;\leq\; |u|^2_{0,3,\Omega} |v|_{0,3,\Omega} \;\leq\; c\; \|u\|^2_{1,\Omega} \|v\|_{1,\Omega},$$

which shows that $u \in H^{-1}(\Omega)$. Here c is a generic constant, $\|.\|_{1,\Omega}$ denotes the norm in $H_0^1(\Omega)$ and $|.|_{0,\Omega}$ denotes the norm in $L^2(\Omega)$. Hence, the integrals on the right side of (1.3) exist.

2 The Main Results

Let G be a bounded domain in $\mathbb{R}^n$, $n \leq 6$, with smooth boundary ∂G, $f \in C^{0,\alpha}(\overline{G}) \cap L^2(\Omega)$ and $f < 0$. Let $w \in H_0^1(G)$ be the smooth solution of

$$-\Delta u = f \;\text{ in } G, \quad u = 0 \;\text{ on } \partial G. \tag{2.1}$$

The following result is proved in the book [3, p. 227].

Lemma 2.1 *Let G be a bounded domain in $\mathbb{R}^n$, $n \leq 6$ with smooth boundary ∂G . Let $f \in C^{0,\alpha}(\overline{G}) \cap L^2(\Omega)$ with $f < 0$. Then, there exists, $u \in H_0^1(G)$ satisfying*

$$\int_G \nabla u . \nabla v = \int_G u^2 v + \int_G f v, \text{ for every } v \in H_0^1(G)$$

such that $w \leq u \leq 0$ in G and

$$\|u\|_{1,G} \leq c\; (|f|_{0,G} + |w|_{0,G}). \tag{2.2}$$

Remark 2.1 By Lemma 9.17 [1, p. 242], we obtain $|w|_{0,G} \leq c\; |f|_{0,G}$, where c depends only on n and G and (2.2) reduces to

$$\|u\|_{1,G} \leq c|f|_{0,G}, \tag{2.3}$$

where $c > 0$ depends on n and G only. Let $n \leq 6$ and $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, where Ω_i is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega_i$, for each $i \geq 1$. Let $f \in C^{0,\alpha}(\overline{\Omega}) \cap L^2(\Omega)$. By Lemma 2.1, for $i \geq 1$ there exists a sequence u_i such that

$$\int_{\Omega_i} \nabla u_i . \nabla v_i = \int_{\Omega_i} u_i^2 v_i + \int_{\Omega_i} f_i v_i, \text{ for all } v_i \in H_0^1(\Omega_i), \tag{2.4}$$

$$\|u_i\|_{1,\Omega_i} \leq c|f|_{0,\Omega}, \tag{2.5}$$

where c depends on Ω_i and n. Here $f_i = f|_{\Omega_i}$ is the restriction of f on $\Omega_i, i \geq 1$.

With these preliminaries, we have the main result stated below.

Theorem 2.1 *Let* $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, $\Omega_i \subseteq \Omega_{i+1}$ *be open bounded domains in* Ω. *We suppose that* $f \in L^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$. *Then* (1.1) *and* (1.2) *have a solution* $u \in H_0^1(\Omega)$ *with* $u \leq 0$ *a.e in* Ω.

Proof A part of the following proof is similar to the one found in [2]. Let M be any fixed (but arbitrary) bounded domain such that $\overline{M} \subseteq \Omega$. Then there exists an integer i such that $\overline{M} \subseteq \Omega_j$ for $j \geq i$. Let $\tilde{u}_j$ (for $j \geq i$) denote the extension of u_j by zero outside Ω_j. We continue to denote $\tilde{u}_j$ by u_j. By a lemma [4, p.124] and (2.4),(2.5) of Remark 2.1, there exists a positive constant $k > 0$ depending on α, n and M such that

$$\|u_j\|_{H_0^1(M)} \leq k, \text{ for all } j \geq i,$$

where k is independent of $j \geq i$. Since $\{u_j\}$ is a bounded sequence in $H_0^1(M)$, $\{u_j\}$ has a weakly convergent subsequence in $H_0^1(M)$, which we still denote by $\{u_j\}$. A little computation shows that $u_j^2 \rightharpoonup u^2$ weakly in $H_0^1(M)$. Since M is an arbitrary bounded domain in Ω, we have

$$\int_\Omega \nabla u.\nabla v = \int_\Omega u^2 v + \int_\Omega f v, \text{ for every } v \in H_0^1(\Omega)$$

which completes the proof of the theorem. □

Remark 2.2 When $f > 0$, (1.1) and (1.2) may not admit a solution even in bounded domain. We refer to example 5.4.1 in the book [3, p. 230].

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