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Book: [2] Liapunov, A.M. *Title of the book.* Name of the Publishers, Town, Year.

Proceeding: [3] Bellman, R. Title of the article. In: *Title of the book.* (Eds.). Name of the Publishers, Town, Year, Pages. [Language]

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NONLINEAR DYNAMICS AND SYSTEMS THEORY

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PERSONAGE IN SCIENCE

Professor V.I. Zubov

to the 80th Birthday Anniversary

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The paper presents a biographical sketch and a review of scientific achievements of Vladimir Ivanovich Zubov (1930–2000), the outstanding researcher in Control and Stability Theory of the 20th century.

1 Brief Outline of Zubov's Life

Vladimir Ivanovich Zubov was born on April 14, 1930 in the town of Kashira, Moscow region, Russia. In 1945 he finished a secondary school.

At the age of 14, Vladimir suffered in the explosion accident happened when he and his playfellows found a hand grenade. His eyes were injured and he failed eyesight soon. In 1949 he finished the Leningrad special school for blind and visually impaired children being the winner of the 15th Leningrad mathematical Olympiad for graduates. The same year he entered the Mathematical and Mechanical Faculty of the Leningrad State University. In 1953 he graduated from the university with honors and received his MSc degree in Mathematics. In the same year he began his post-graduate studies.

Zubov was an active participant of the seminar held under the supervision of Professor N. P. Erugin at the Department of Differential and Integral Equations of the Leningrad State University. When discussing the state of the theory of motion stability N. P. Erugin formulated a set of problems requiring constructive solutions. In particular, very important problems were those of the Lyapunov theorems inversion and representation of the general solution for the differential equations system in the asymptotic stability region. V. I. Zubov obtained a number of profound results in these directions which laid the foundation to his PhD thesis titled "Boundaries of the Asymptotic Stability Region" defended in 1955 (with N. P. Erugin as an advisor and professors E.A. Barbashin and N. N. Krasovskij as official opponents).

In December 1955 Zubov joined the Institute of Mathematics and Mechanics of the Leningrad State University as a leading researcher.

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V. I. Zubov defended his Doctor of Science thesis in April 1960 at the Leningrad Polytechnic Institute. The thesis was based on his book "The methods of A. M. Lyapunov and their application" published in 1957. This book manifested new ideas and fundamental results on the Lyapunov methods and gave rise to constructive approaches for solving various practical problems.

Zubov was affiliated with the Scientific Research Institute of Mathematics and Mechanics of the Leningrad State University as a chief researcher since 1955 till 1962. In 1962 he became the chief of the Laboratory of Control Devices and since 1967 till the end of his life he was the head of the Control Theory Department of the Faculty of Applied Mathematics of the Leningrad (St. Petersburg) State University.

V.I. Zubov married Alexandra Zubova in 1953. His wife is DSc and professor. The Zubovs have 6 children and 21 grandchildren.

2 Basic Trends of His Scientific Work

2.1 Region of asymptotic stability

One of the well known Zubov's results is his theorem on the region of asymptotic stability. This theorem not only solves the stated problem but is also of immense practical value for engineers and specialists in control theory. The starting point for Zubov's investigations was the monograph of A. M. Lyapunov "General problem on the stability of motion". In the 50es of the last century Zubov and other scientists proved the existence of the Lyapunov functions in the cases of stability, asymptotic stability and instability of unperturbed motions. These results ground the possibility of finding the Lyapunov functions for solving the stability problem for various classes of differential equation systems.

V. I. Zubov was the first to solve the problem on estimation of the set of initial values belonging to the attraction region of the asymptotically stable zero solution of ordinary differential equations system. He deduced the equation for the Lyapunov function which allowed the boundary of an asymptotic stability region to be found. In the analytical case the solution for this equation can be obtained in the form of series. On this base the numerical methods were developed for the estimation of an asymptotic stability region. For controllable dynamical systems it was shown that the region of asymptotic stability would be maximized when the optimal stabilizing control was used.

The development of these results was described in his monographs [1, 2, 3, 5, 31].

2.2 Stability of nonlinear systems in critical cases

V. I. Zubov proved that if the zero solution of a system of differential equations with the homogeneous right-hand sides was asymptotically stable, then for this system there existed an homogeneous Lyapunov function satisfying the conditions of the Lyapunov asymptotic stability theorem. He showed that this function could be found as a solution of a special system of partial differential equations [1, 5].

Using the results obtained he estimated the time of transients for asymptotically stable homogeneous systems. Besides, he determined the stability and the ultimate boundedness criteria for nonlinear systems based on the the first homogeneous approximation. Furthermore, new stability conditions were established in the critical cases of several zero roots and of several pairs of pure imaginary roots of characteristic equation. Moreover, he extended the results above to the systems with generally homogeneous right-hand sides and to the problem of stability by generally homogeneous first approximation [1, 2, 3, 5].

V. I. Zubov also stated a problem of the stability by the first, in the broad sense, approximation, and obtained a number of results for solution of this problem. He investigated the conditions for stability of the zero solution for the arbitrary admissible functions included in the first approximation [27, M].

V. I. Zubov suggested an approach for the construction of solutions of systems of nonlinear equations in a neighborhood of a regular critical point. This problem was stated in the works by Briot and Bouquet with its special cases investigated by Poincaré and Picard. V. I. Zubov solved completely this problem in a classical statement [1, 5].

2.3 Control theory

Zubov's results on the theory of optimal control systems and on solution of the corresponding theoretical and numerical problems of optimal stabilization are presented in his books [2, 3, 6, 13, 17, 18, 20]. He suggested constructive analytical methods for finding the optimal controls. V. I. Zubov established the relationship between the Lyapunov direct method and the theory of optimal control. He introduced the notion of optimal control with respect to the damping of a deviation measure of a transient from the preassigned motion. He solved the problem of a synthesis of optimal control. In particular, the problem on synthesis of linear controls with the aftereffect and in the presence of intermediate control points was solved.

Zubov's theorem on the canonical decomposition of nonlinear force fields into the potential component and the gyroscopic one should be mentioned especially [D]. He applied this result to the control problems for finite-dimensional holonomic mechanical systems.

2.4 Asymptotic equilibrium states and asymptotic auto-oscillations

V. V. Nemytskij stated the problem of studying the solutions of differential equations systems for which limit manifolds exist under unbounded increase and decrease of time, but these limit sets are not invariant sets of the systems under consideration. V. I. Zubov showed that in a number of cases such a behavior of motions resulted in the appearance of asymptotic equilibrium states and investigated the conditions for this. It was proved that asymptotic equilibrium states could occur in the systems of differential equations subjected to perturbations tending to zero under the increase of time. Besides that, he established that the forced oscillations arising in perturbed system could be damped if the perturbation was characterized as an oscillatory process with frequency growing in time. In this case the amplitudes of these perturbations can remain finite and, moreover, they can be arbitrary large.

V. I. Zubov also investigated the problem on conservation of auto-oscillations under the action of perturbations formulated by A. A. Andronov. He determined the conditions under which trajectories of perturbed systems tended asymptotically to auto-oscillating modes of the initial systems. He referred to these limiting operating modes as the asymptotic auto-oscillations.

The results obtained along this topic have been presented in monographs [24, 27].

2.5 Nonlinear oscillations and stability of orbits

One of the main directions of Zubov's investigations was the analysis of stationary oscillations of nonlinear systems. He studied the problems of the existence of stationary modes, developed the methods for the construction of these modes and for the analysis of the integral curves behavior in their neighborhoods.

In Zubov's works the approaches for proper and forced oscillations construction for multiple degrees-of-freedom systems were developed. In addition to the well known small parameter method he also introduced the method of successive approximations applied for these investigations. In a number of cases the latter enabled one to obtain more complete results on the appearance of periodic and almost periodic motions and their interconnection.

V. I. Zubov established a qualitative criterion of periodic and almost periodic convergence for nonlinear systems. The constructive approach for the verification of conditions of this criterion was based on the usage of special functions similar to those introduced by A. M. Lyapunov for the stability analysis.

V. I. Zubov also developed a new method for the investigation of integral curves behavior in the neighborhood of a periodical orbit. This method is based on the transformation of the original system into a special form describing the behavior of a mapping point on the hyperplane normal to the periodical orbit. In the framework of this approach new results on the Lyapunov stability of periodical solutions were obtained. Furthermore, the necessary and sufficient conditions were found for the prescribed periodical solution to be the auto-oscillating one. Application of the Zubov method to the differential equation systems possessing several periodical orbits allowed one to simplify the solution of analytical problems in various applications. For instance, new equations of the celestial mechanics were deduced.

Numerous results obtained along this line were presented in monographs [2, 3, 4, 16, 24, 28].

2.6 Development of the dynamical systems theory and analytical representation of stationary oscillations

V. I. Zubov was the first to introduce the concept of a general dynamical system in metric space. He extended the problem of stability investigation for individual trajectories to the problem of stability analysis of invariant sets of dynamical systems. In his works the qualitative structure of a neighborhood of a stable invariant set was studied. Also, he extended the direct Lyapunov method to solution of the problems of stability of invariant sets for general dynamical systems. He also developed a method for estimating the distance from the motion to the invariant set and proved the theorem on the asymptotic stability region for uniformly asymptotically stable and uniformly attracting invariant sets. Furthermore, the method for the determination of boundary of asymptotic stability region was suggested. A special attention was paid to the construction of the theory of periodical dynamical systems. The results obtained were applied for the stability analysis of partial differential equations systems.

One of the most important problems of the theory of dynamical systems is the analysis of stationary oscillations. G. D. Birkhoff proved that the most general class of stationary oscillations of dynamical systems could be described by recurrent functions. V. I. Zubov developed the analytical theory for the representation of the ergodic classes of recurrent functions. He showed that the space of recurrent functions was complete, but neither linear nor transitive. The approach suggested by Zubov was based on the decomposition of the recurrent functions space into the isolated classes of functions possessing relatively dense sets of common quasiperiods. To implement such a decomposition he generalized the Kronecker theorem on the existence of common solutions for inequalities systems. For the constructed ergodic classes of recurrent functions V. I. Zubov proved the theorem on the approximation of functions from the given class by the trigonometric polynomials of a special form. This theorem generalizes the well known Weierstrass theorem. Moreover, he developed the mathematical methods for the analytical representation of the stable Poisson motions.

The results obtained in this area appeared in [1, 4, 5, 16, 19, 20, 23, 24, 28, B, C, J].

2.7 Systems with aftereffect. Quantization of orbits

Another direction of Zubov's investigations deals with the estimation of the finite velocity of interactions extension or the allowance for the control signal delay in the feedback channel [13, 18, A]. This necessitates the stability analysis of delay-differential systems. V. I. Zubov established the representation of solutions of linear delay-differential systems whose right-hand sides were given by the Stieltjes integrals in the form of asymptotic series. He obtained the root criteria of exponential stability for delay systems.

V. I. Zubov formulated a universal law for the orbits quantization [22, 25, K]. This law is based on taking into account the finiteness of interactions extension velocity by the introduction of delays in the force fields determining the motions of the mass points system. He showed that finiteness of the velocity of the interactions and perturbations extension caused the quantization of orbits of individual mass points and of their configurations. Moreover, the quantization also occurs for the energy levels and for the momenta of momentum.

2.8 Conservative methods of numerical integration

V. I. Zubov developed conservative methods for numerical integration of differential equations systems [15, 17, L, N]. These methods are based on the construction of finite-difference schemes preserving certain properties of motions such as integrals of motion, integral invariants and other physical and qualitative characteristics. Zubov's approach consists in a modification of the known numerical methods by introducing the control in the computation process. This control is constructed with the aim to provide convergence, required precision and stability for the modified numerical method and, in addition, to preserve the given properties of motion on the discrete trajectories. Although the finite-difference equations obtained are the nonlinear and implicit ones, the advantage of such schemes over the known schemes by Euler, Runge and Kutta, and Adams et al. consists in the opportunity they provide for the qualitative behavior analysis of trajectories of the generating differential equations.

2.9 Investigation of rotation motion of a rigid body

V. I. Zubov established that in the Euler and Lagrange cases all motions of a rigid body are periodical or almost periodical with the exception of the motions occurring in a special integral manifold. He determined the precise bounds of nutational oscillations of the proper rotation axis for the dynamically asymmetric rigid body moving inertially around a fixed point. Furthermore, he established stability and instability conditions of the rigid body motions with respect to axes orientation in the space.

V. I. Zubov developed the methods for the rotational motion control solving the problem of the system transfer from one state to the other. In particular, the problem of the body orientation in a prescribed direction and the problem of scanning the body axes in accordance with the pre-specified program were solved. These methods are based on finding the motions of the carried bodies which create Coriolis forces moments providing the prescribed motion of the carrying body.

For the bodies with the liquid-filled cavities and bodies with the flexible constructions the mathematical models based on the ordinary differential equations were suggested. For such models the analytical constructions of controls providing given rotational motions of the carrier were also obtained.

The development of these methods was described in [7, 13, 15, 20, 22, E].

2.10 Investigation of free and forced oscillations of gyroscopic systems

V. I. Zubov developed a precise method for the analysis of equations of gyroscopic systems motions based on the construction of convergent functional series expansions in powers of the angular momentum inverse. By the use of such series the solutions of linear and nonlinear differential equations systems describing free and forced oscillations of gyroscopic systems were obtained. Furthermore, the stability and asymptotic stability conditions for equilibrium states of gyroscopic systems were deduced and the approach for the numerical solution of stability problem was suggested.

The results obtained in terms of the precise Zubov method were compared with those obtained with the aid of the approximate precession theory. The cases were detected where the latter yielded qualitatively false conclusions on the behavior of oscillations in gyroscopic systems. For the cases where the precession theory results were correct, the method was suggested for the successive refinement of the quantitative results obtained in the framework of this theory.

The results obtained along this topic have been presented in [8, 22].

2.11 Theory of charged particles beams and relativity principles

V. I. Zubov solved the inverse problem of electrodynamics: for the given velocity field of charged particles he proposed a method for the determination of the electric and magnetic fields strengths providing this field. He found the equations for the various fields of such a type and established the theorem on universality of electrodynamics equations. The results obtained were used for the design of various types of electro-physical equipment.

V. I. Zubov treated an arbitrary vector wave as a superposition of a finite number of simple waves. From this point of view the only characteristic of a simple wave turns out to be its phase depending on the time and space coordinates and satisfying the wave equation. He extended Einstein's notion of the equivalence of two coordinate systems based on the relativity principle. This extension allowed a set of relativity principles to be obtained.

Numerous results obtained along this line have been presented in [16, 20, 24, 27, 28, G].

2.12 Distribution of resources and funds

V. I. Zubov developed the theory of a support plan resulted in a subsequent software implementation [9, 21]. The mathematical model construction permits to connect initial, intermediate and final states of the developing branches of a national economy and to solve problems of initial distribution of investments in urban branches of a national economy with an opportunity for redistribution in emergency cases.

As a member of the inter-branch council of scientists V. I. Zubov created the theory of the balanced co-development of various branches of live-stock farming with allowance for the soil and climatic conditions and density of population in a region [20, F].

2.13 Investigation of distribution functions spaces

V. I. Zubov established that any continuous distribution function could be approximated in the real axis with an arbitrarily given precision in the uniform metric by the mixture of normal distributions with the distinct expected values and variances. Furthermore, he showed that the normal distribution law is not of a unique nature. It was proved that any continuous distribution law gave a set of sliding sums with weight coefficients defining an everywhere dense subset in the space of all continuous distribution functions [H, I, O].

3 Applied Investigations

Zubov's investigations were always aimed at applications. Since 1957 he was efficiently contributing to the development of modern technologies in the following fields:

(1) inertial navigation systems for which he solved the problem on deviation of the gyro-system axes depending on nutational oscillations and kinematic moment of inertia of gyro-rotors;

(2) self-guidance of cruise missiles;

(3) precision control systems of spacecraft position for the "Proton" system;

(4) control systems for the rotational motion of spacecrafts for the precision orientation of sensitive axes of devices on the base of magneto-hydrodynamic control systems with the use of conducting fluids in feedback contours;

(5) control problems for beams of charged particles to be transported in a given physical channel;

(6) noise stability of the information transmission methods;

(7) tactical scheme constructions for the USSR Navy to oppose aircraft carriers of a potential enemy.

In all the above mentioned pure applied directions Zubov obtained fundamental results in control and stability theory.

On the occasion of awarding V. I. Zubov by the USSR State Prize, the president of the Academy of Sciences of the Soviet Union M. V. Keldysh noted: "Zubov's works are well known in the Soviet Union and abroad. The profound researches carried out by him on the theory of motion stability, theory of automatic control and theory of optimal processes allow one to solve the important applied problems, in particular, in the field of design of controlled automatic devices and stabilization of program motions. Zubov's methods are also effective in the application to control problems arising in industry, mathematical economics, biology, medicine and navigation".

4 Science Management and Teaching Activity

Fundamental successes in the investigation of new branches of applied mathematics and control theory resulted in the creation of the Laboratory of Control Devices in 1962, the Control Theory Department in 1967, the Faculty of Applied Mathematics and Control Theory in 1969 and the Research Institute of Computational Mathematics and Control Processes at the Leningrad (St. Petersburg) State University in 1971. Furthermore, V. I. Zubov organized the Center of Applied Mathematics and Control Processes. In his time V. I. Zubov headed these institutions . Also, he was a permanent Chief of the faculty Curriculum Committee and the Special Council for DSc Dissertation defenses. He was an advisor for 20 DSc and about 100 PhD dissertations. Under Zubov's supervision a worldwide known school in control theory was developed in St. Petersburg.

5 Editorial Activity and International Scientific Activity

For many years V. I. Zubov was a member of the Editorial Board of the Journal of "Differential equations".

He was chair of the Program Committees of the International Seminars "Beam Dynamics & Optimization", the International Symposium "Hydrogen Energetic, Theoretical and Engineering Solutions", the 11th International IFAC Seminar "Control Applications of Optimization".

6 Awards

In 1968 V. I. Zubov became the USSR State Prize winner for his pioneer works in Control Theory. Twice, in 1962 and 1996 he received the Leningrad (St. Petersburg) University Prize for scientific achievements. In 1981 he was elected as corresponding member of the Soviet Union Academy of Sciences and in 1998 he was conferred with a title of Honored Scientist of the Russian Federation.

In 1996 Zubov's scientific school of "Processes of control and stability" was the winner of the competition for the state support of the leading scientific schools of Russia.

In 2001, the Research Institute of Computational Mathematics and Control Processes of St.Petersburg State University was named after him.

For his outstanding scientific merits Zubov's name was perpetuated as the name of minor planet 'ZUBOV 10022'. This asteroid has the size of 6 km, the brightness of 13.8 magnitude, and the largest orbit semi-axis of 2.369 astronomical units.

7 Public Activities

In addition to his intensive scientific researches and tuition duties, V. I. Zubov was involved in public social activity. He was the President of the St. Petersburg Charity Foundation for blind and visually impaired children.

His poetic talent is evident in his books "Behest of the past generations", St. Petersburg: "Mobil'nost Plus" Publisher, 1993 and "Poetry. Sonnets. Behest of the past generations", St. Petersburg: St. Petersburg State University Publisher, 2000.

V. I. Zubov is the author of about 200 publications including 31 monographs and books. Four of his monographs were republished abroad in English and French.

8 List of Monographs and Books by V. I. Zubov

[1] Methods of A. M. Lyapunov and their Application. Leningrad, Leningrad State University, 1957, 242 p. (Russian)

[2] Mathematical Methods of the Study of Automatic Control Systems. Leningrad, Sudpromgiz, 1959, 324 p. (Russian)

[3] Mathematical Methods of the Study of Automatic Control Systems. New York etc.: Pergamon Press; Yerusalem: Academic Press, 1962, 327 p.

[4] Oscillations in Nonlinear and Controlled Systems. Leningrad, Sudpromgiz, 1962, 630 p. (Russian)

[5] Methods of A. M. Lyapunov and Their Applications, Groningen: NoordHoff Ltd., 1964, 263 p.

[6] Theory of Optimal Control of Ships and Other Moving Objects. Leningrad, Sudpromgiz, 1966, 352 p. (Russian)

[7] Dynamics of Free Rigid Body and Determination of Its Orientation in the Space. Leningrad, Leningrad State University, 1968, 208 p. (Russian, with V. S. Ermolin et al.)

[8] Analytical Dynamics of Gyroscopic Systems. Leningrad, Sudostroenie, 1970, 317 p. (Russian)

[9] The Problem of Optimal Distribution of Capital Investments. Leningrad, Leningrad State University, 1971, 26 p. (Russian, with L. A. Petrosyan)

[10] Lectures on Control Theory, Part 1. Leningrad, Leningrad State University, 1972, 203 p. (Russian)

[11] Stability of Motion. Methods of Lyapunov and their Application. Moscow, Vysshaya Shkola, 1973, 271 p. (Russian)

[12] Mathematical Methods of the Study of Automatic Control Systems, 2nd ed. Leningrad, Mashinostroenie, 1974, 335 p. (Russian)

[13] Lectures on Control Theory. Moscow, Nauka, 1975, 496 p. (Russian)

[14] Theorie de la Commande. Moscow, Mir, 1978, 470 p. (French)

[15] Control of Rotational Motion of a Rigid Body. Leningrad, Leningrad State University, 1978, 200 p. (Russian, with V. S. Ermolin et al.)

[16] Oscillations Theory. Moscow, Vysshaya Schkola, 1979, 400 p. (Russian)

[17] Stability Problem of Control Processes. Leningrad, Sudostroenie, 1980, 253 p. (Russian)

[18] Theory of Equations of Controlled Motion. Leningrad, Leningrad State University, 1980, 288 p. (Russian)

[19] Stability of Invariant Sets of Dynamical Systems. Saransk, Mordovian State University, 1980, 80 p. (Russian)

[20] Dynamics of Controlled Systems. Moscow, Vysshaya Schkola, 1982, 285 p. (Russian)

[21] Mathematical Methods in Planning. Leningrad, Leningrad State University, 1982, 80 p. (Russian, with L. A. Petrosyan)

[22] Analytical Dynamics of Systems of Bodies. Leningrad, Leningrad State University, 1983, 343 p. (Russian)

[23] Periodical Dynamical Systems. Saransk, Mordovian State University, 1983, 88 p. (Russian)

[24] Oscillations and Waves. Leningrad, Leningrad State University, 1989, 416 p. (Russian)

[25] Mathematical Problem of Quantization. Saransk, Saransk Division of Saratov University, 1989, 56 p. (Russian)

[26] Mathematical Theory of Motion Stability. St. Petersburg, AO "Mobil'nost Plus", 1997, 340 p.

[27] Processes of Control and Stability. St. Petersburg, St. Petersburg State University, 1999, 325 p. (Russian)

[28] Theory of Oscillations. Singapore etc., World Scientific, 1999, 400 p.

[29] Stability Problem of Control Processes, 2nd ed. St. Petersburg, St. Petersburg State University, 2001, 354 p. (Russian)

[30] Dynamics of Controlled Systems, 2nd ed. St. Petersburg, St. Petersburg State University, 2004, 380 p. (Russian)

[31] Boundaries of the Asymptotic Stability Region. St. Petersburg, AOOT "Mobil'nost Plus", 2007, 85 p. (Russian)

9 List of the Zubov Selected Papers

[A] On the theory of linear stationary systems with delay. *Izvestija Vuzov. Matematica* (6) (1958) 86–95 (Russian).

[B] On the ergodic classes of recurrent motions. *Doklady Akademii Nauk SSSR* **132**(3) (1960) 507–509. (Russian)

[C] On the recurrent functions theory. *Sibirskij Matematicheskij Zhurnal* **3**(4) (1962) 532–560. (Russian)

[D] Canonical structure of a vector force field. In: *Problems of Mechanics of a Solid Deformable Body.* Leningrad, Sudostroenie, 1970, 167–170. (Russian)

[E] On the active control of rigid body rotation. *Differentsial'nye Uravneniya* **6**(11) (1970) 2086–2087. (Russian)

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Decoupled-natural-dynamics Model for the Relative Motion of two Spacecraft without and with J2 Perturbation

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Abstract: This paper presents the analytical steps for decoupling the natural dynamics representing the relative motion of two spacecraft flying in close orbits, both without and with the inclusion of the J2 perturbation. Linear mathematical models with constant coefficients are available in literature for representing such dynamics. In both cases two modes can be highlighted through the eigenvalue analysis of the state matrix: a double integrator, representing the secular part of the spacecraft relative motion, and a harmonic part, related to the typical oscillations present in spacecraft relative dynamics. In this work we introduce a rigorous two-step state vector transformation, based on a Jordan form, in order to decouple the two modes and be able to focus on either of them independently. The obtained results give a deep insight to the control designer, allowing for easy stabilization of the two spacecraft relative dynamics, i.e. canceling out the double integrator mode, which implies a constant drift taking the two spacecraft apart. On the other hand, one could desire an immediate control on the harmonic part of the dynamics, which is here made possible thanks to the decoupled form of the final equations. Furthermore, the obtained decoupled equations of motion present an analytical solution when only along-track control is applied to the spacecraft. This solution is here presented. The phase planes behavior for the controlled cases is reported.

Keywords: *spacecraft relative motion; linear dynamics transformation; Jordan form.*

Mathematics Subject Classification (2000): 37N05, 70F10.

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1 Nomenclature

α = Free parameter in generalized eigenvector calculation	J_2 = Second order harmonic of Earth gravitational potential field (Earth flattening) [108263×10^{-8}, [1]]
a = First non zero and non unity value in the state matrix	λ = Vector of the eigenvalues of A
A = State matrix	LVLH = Local Vertical Local Horizontal
A' = State matrix after first transformation	ω = Reference orbit angular velocity
$\hat{A}$ = State matrix decoupled (after second transformation)	r_{ref} = Reference orbit radius
β = First free parameter in second transformation matrix T_2	R_e = Earth mean radius [$6378.1363\, km$, [1]]
b = Second non zero and non unity value in the state matrix	T = Transformation matrix
B = Control distribution matrix	T_1 = First transformation matrix
$\hat{B}$ = Control distribution matrix after transformations	T_2 = Second transformation matrix
γ = Second free parameter in second transformation matrix T_2	t = Time
i = Complex unity	u = Control vector
i_{ref} = Reference orbit inclination	$w_i,\ i = 1,..,4$ = Eigenvectors of A
	x = Spacecraft relative state vector in LVLH frame
	x, y = Spacecraft relative position components in LVLH frame
	z = Transformed spacecraft relative state vector
	$(...)_0$ = Initial value ($t = 0$)

2 Introduction

A formal state vector transformation is presented in order to separate the two modes characterizing the relative motion between a chaser spacecraft and a target spacecraft in circular orbit, for both the well known unperturbed Hill–Clohessy–Wiltshire [2] model and the more recent Schweighart–Sedwick [3] model which includes the J_2 perturbation are used. Only the in-plane part of the relative motion is here considered, being the out-of-plane dynamics decoupled.

Our work is built upon the work of Leonard [4] who separates the dynamic of the Hill–Clohessy–Wiltshire model by averaging the evolution in time of the state variables, without developing a formal state transformation.

In particular, we employ a two-steps transformation into a Jordan form [5, 6] and then into a new decoupled-natural-dynamics form by using a chain of generalized eigenvectors in order to cope with the defectiveness of the state matrix. We obtain two transformed system models (for the cases with and without J_2) with the natural dynamics decoupled into a double integrator and a harmonic oscillator. The present work embodies the results of Leonard ([4], moving-ellipse formulation of Hill–Clohessy–Wilshire model) as a particular case.

The obtained results add further insight to the description of spacecraft relative motion, and, in particular, enables the control designer to focus on either one of two critical goals regarding the stabilization of the chaser's motion with respect to the target: namely, either the stabilization into a closed elliptical relative orbit or into a separate circular orbit with respect to the Earth center.

Furthermore, we perform the analytical integration of the transformed dynamics by considering only along-track thrust (as proposed in recent literature to simplify mission design, [7]–[9]).

The decoupled dynamics here obtained, and in particular the analytical nature of the obtained results, have been used by Bevilacqua and Romano [10, 12] for developing a completely analytical differential drag controller for multiple spacecraft assembly.

The paper is organized as follows: Section 3 introduces the linear models without and with J_2 perturbation. Section 4 is dedicated to the state vector transformations. Section 5 gives the analytical solution for the time evolution of the state vector for the case of constant along-track control. Finally Section 6 concludes the paper.

3 Spacecraft Relative Motion Dynamics

The in-plane part of the motion of a chaser spacecraft with respect to a target spacecraft in circular orbit can be represented by the following general equation, encompassing both the Hill–Clohessy–Wiltshire [2] unperturbed model and the Schweighart–Sedwick [3] model which includes J_2 perturbation

$$\dot{x} = A\,x + Bu, \quad x = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{pmatrix}, \tag{3.1}$$

where x is the "R-bar" axis, pointing from the Earth's center to the LVLH frame's origin at the target spacecraft, y is the "V-bar" axis in the direction of the velocity of the target along a circular orbit.

For the Hill-Clohessy–Wiltshire model it is

$$a = 2\omega, \quad b = 3\omega^2. \tag{3.2}$$

For the Schweighart-Sedwick model it is

$$a = 2\omega\, c, \quad b = \left(5c^2 - 2\right)\omega^2, \tag{3.3}$$

where the coefficient c is given by

$$c = \sqrt{1+s}, \quad s = \frac{3J_2 R_e^2}{8r_{ref}^2}\left(1 + 3cos2i_{ref}\right). \tag{3.4}$$

The following substantial difference exists between the Hill–Clohessy–Wiltshire model and the Schweighart–Sedwick model: while the state vector of the former model describes the chaser's position and velocity with respect to either a target spacecraft or a reference point in circular orbit, the state vector of the latter model describes the chaser's position and velocity only with respect to a target spacecraft. Indeed, in the Schweighart–Sedwick case, the evolution of the state of the chaser with respect to a reference point in circular orbit is described by a more complicated expression, due to the J_2 perturbation [3].

It is immediate to see that, if we neglect the J_2 perturbation, the Schweighart–Sedwick equations reduce to the Hill–Clohessy–Wiltshire equations. Furthermore, we underline the fact that the condition $b < a^2$ holds for both models. In particular, while this is immediately obvious for the Hill–Clohessy–Wiltshire case, for the Schweighart–Sedwick model it translates onto the following condition for the variable s

$$\left(5c^2 - 2\right)\omega^2 < 4\omega^2 c^2 \rightarrow |s| < 1 \tag{3.5}$$

which is always true, because $\max\left(|s|\right) = \dfrac{3J_2 R_e^2}{2r_{ref}^2} \leq \dfrac{3J_2}{2} = 1.624 \cdot 10^{-3}$.

4 State Vector Transformation

The eigenvalues of the state matrix A in (3.1) are

$$\lambda = \left[0, 0, \sqrt{b - a^2}, -\sqrt{b - a^2},\right]^T. \tag{4.1}$$

Being $b < a^2$, the third and fourth eigenvalues in (4.1) are complex conjugated.

By observing (4.1), it is clear that a double integrator and a harmonic oscillator are the two modes composing the natural dynamics.

Only the following three independent eigenvectors exist for the matrix A

$$w_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; w_3 = \begin{bmatrix} -\dfrac{\sqrt{b-a^2}}{a} \\ 1 \\ \dfrac{a^2 - b}{a} \\ \sqrt{b - a^2} \end{bmatrix}, \; w_4 = \begin{bmatrix} \dfrac{\sqrt{b-a^2}}{a} \\ 1 \\ \dfrac{a^2 - b}{a} \\ -\sqrt{b - a^2} \end{bmatrix}, \tag{4.2}$$

where w_1 corresponds to the two multiple zero eigenvalues (Eq. (4.1)), and w_3 and w_4 correspond to the two complex conjugated eigenvalues. Since the state matrix A is defective (there are only three independent eigenvectors for the system which is of fourth order), it cannot be diagonalized. As an alternative to diagonalization, we look for a similarity transformation aiming to possibly represent the system with the state matrix in the following form

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\Omega^2 & 0 \end{pmatrix}. \tag{4.3}$$

This new form of the system matrix, inspired by the developments of [4], is useful because it decouples the natural dynamics into a double integrator and a harmonic oscillator. In (4.3), Ω represents the frequency of the harmonic oscillator.

As a first step of the transformation, we build a transformation of A into the modified-diagonal form (or Jordan form, see [5],[6]). Let us write

$$x = T_1 z', \tag{4.4}$$

where z' is the corresponding new state. The transformation matrix T_1 is obtained as follows

$$T_1 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix}, \tag{4.5}$$

where w_2 is the generalized eigenvector found by solving the following "Jordan chain" equation ([5])

$$(A - \lambda(1) I) w_2 = w_1 \to A w_2 = w_1 \to A^2 w_2 = A w_1, \tag{4.6}$$

where $\lambda(1) = 0$, from (4.1), leading to

$$w_2 = \left[-a/b, \alpha, 0, 1 \right]^T, \tag{4.7}$$

where α is an arbitrary complex parameter which is obtained form the "Jordan chain" procedure and can be conveniently chosen, as shown in the following.

The transformation of Eq. (4.5) results in the following Jordan-form

$$A' = T_1^{-1} A T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{b-a^2} & 0 \\ 0 & 0 & 0 & -\sqrt{b-a^2} \end{pmatrix}. \tag{4.8}$$

As a second step of the transformation of the system matrix toward the desired form of Eq. (4.3), we use the following complex transformation matrix

$$T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\beta\sqrt{a^2-b} & i\beta \\ 0 & 0 & \gamma\sqrt{a^2-b} & i\gamma \end{pmatrix}, \tag{4.9}$$

where β and γ are arbitrary complex parameters which can be conveniently selected, as explained later.

The final expression for the state matrix is calculated as

$$\hat{A} = T_2^{-1} A' T_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & b-a^2 & 0 \end{pmatrix}. \tag{4.10}$$

This transformed system matrix is indeed in the desired form of Eq. (4.3) with $\Omega = \sqrt{a^2 - b}$.

The overall transformation is given by

$$x = Tz, T = T_2 T_1 = \begin{pmatrix} 0 & -\dfrac{a}{b} & i\dfrac{\left(a^2-b\right)(\beta+\gamma)}{a} & \dfrac{(\beta-\gamma)\sqrt{a^2-b}}{a} \\ 1 & \alpha & -(\beta-\gamma)\sqrt{a^2-b} & i(\beta+\gamma) \\ 0 & 0 & \dfrac{(\beta-\gamma)\sqrt{\left(a^2-b\right)^3}}{a} & i\dfrac{\left(a^2-b\right)(\beta+\gamma)}{a} \\ 0 & 1 & -i\left(a^2-b\right)(\beta+\gamma) & -(\beta-\gamma)\sqrt{a^2-b} \end{pmatrix}. \tag{4.11}$$

5 Analytical Solution of the Transformed Equations in Case of Constant Along-Track Control

We here focus the attention on the case of a single control thrust acting along the y axis. In this case, the initial and transformed control distribution matrices are

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \hat{B} = T^{-1}B = \begin{bmatrix} \dfrac{\alpha b}{a^2 - b} \\ -\dfrac{b}{a^2 - b} \\ \dfrac{1}{4}\dfrac{i\,(\beta+\gamma)\,a^2}{\beta\gamma\,(a^2-b)^2} \\ -\dfrac{1}{4}\dfrac{(\beta-\gamma)\,a^2}{\beta\gamma\,(a^2-b)^{\frac{3}{2}}} \end{bmatrix}. \tag{5.1}$$

In order to have a control distribution matrix with real values, α, $\dfrac{i\,(\beta+\gamma)}{\beta\gamma}$ and $\dfrac{(\beta-\gamma)}{\beta\gamma}$ must all be real. The last two conditions are satisfied only if $\gamma = -\beta$, yielding to (5.2)

$$\hat{B} = \begin{bmatrix} \dfrac{\alpha b}{a^2 - b} \\ -\dfrac{b}{a^2 - b} \\ -\dfrac{1}{2}\dfrac{Im(\beta)\,a^2}{\|\beta\|^2\,(a^2-b)^2} \\ \dfrac{1}{2}\dfrac{Re(\beta)a^2}{\|\beta\|^2\,(a^2-b)^{\frac{3}{2}}} \\ \\ \end{bmatrix}. \tag{5.2}$$

At this stage, looking at (5.2), we are able to impose convenient values for the arbitrary parameters α and β. We choose those values to be $\alpha = 0$, $\beta = -\dfrac{1}{a}$. Therefore, the matrices in (4.11) and (5.2) become

$$T = T_2T_1 = \begin{pmatrix} 0 & -\dfrac{a}{b} & 0 & -\dfrac{2\sqrt{a^2-b}}{a^2} \\ 1 & 0 & \dfrac{2\sqrt{a^2-b}}{a} & 0 \\ 0 & 0 & -\dfrac{2\sqrt{(a^2-b)^3}}{a^2} & 0 \\ 0 & 1 & 0 & \dfrac{2\sqrt{a^2-b}}{a} \\ \\ \end{pmatrix}, \hat{B} = T^{-1}B = \begin{bmatrix} 0 \\ -\dfrac{b}{a^2-b} \\ 0 \\ \dfrac{a^3}{2\,(a^2-b)^{\frac{3}{2}}} \end{bmatrix}. \tag{5.3}$$

The expressions of (5.3) are expanded in the Appendix as functions of ω and c. Moreover, we have

$$x = Tz = \begin{bmatrix} -\dfrac{a^3 z_2 + 2b\sqrt{a^2-b}\,z_4}{a^2 b} \\ \dfrac{a z_1 + 2\sqrt{a^2-b}\,z_3}{a} \\ -\dfrac{2\sqrt{a^2-b}\,z_3}{a^2} \\ \dfrac{a z_2 + 2\sqrt{a^2-b}\,z_4}{a} \end{bmatrix}, \quad z = T^{-1}x = \begin{bmatrix} \dfrac{a^2 y - by - a\dot{x}}{a^2-b} \\ -\dfrac{b\,(ax+\dot{y})}{a^2-b} \\ -\dfrac{a^2\dot{x}}{2\,(a^2-b)^{\frac{3}{2}}} \\ -\dfrac{a^2\,(bx+a\dot{y})}{2\,(a^2-b)^{\frac{3}{2}}} \end{bmatrix}. \tag{5.4}$$

In particular, for the Hill–Clohessy–Wiltshire dynamic model, the transformed system with control along y is obtained by substituting the values of a and b given in (3.2) into (4.10) and (5.3)

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega^2 & 0 \end{pmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 4 \end{bmatrix}. \tag{5.5}$$

Eq. (5.5), corresponding to our new state $\begin{bmatrix} z_1 & z_2 & z_3 & z_4 \end{bmatrix}^T$, reproduces the results of Leonard ([4], where the state, in Leonard's notation, is $\begin{bmatrix} \bar{y} & \dot{\bar{y}} & \beta & \dot{\beta} \end{bmatrix}^T$, with β having a different meaning with respect to our notation).

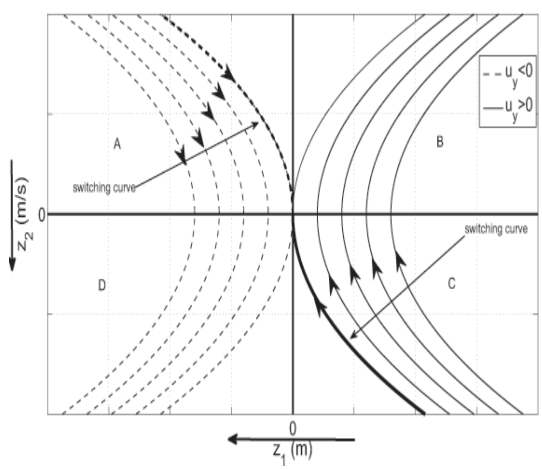

Figure 5.1: Qualitative shape of the curves on the phase plane of the double integrator subsystem (z_1 vs. z_2) in case of constant thrust along the y axis for both the Hill–Clohessy–Wiltshire and the Schweighart–Sedwick models.

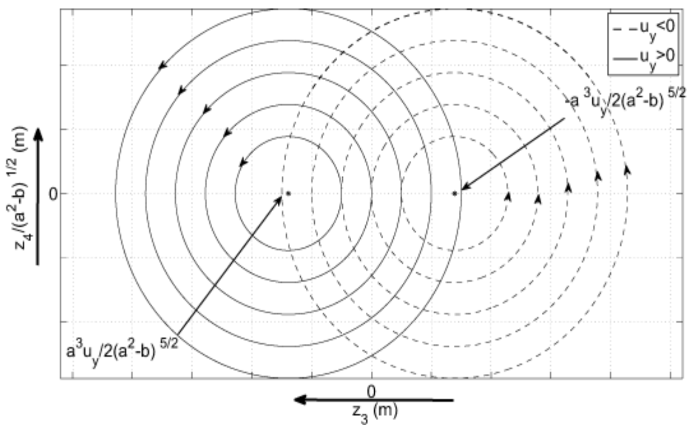

Figure 5.2: Qualitative shape of the curves on the phase plane of the harmonic oscillator subsystem (z_3 vs. z_4) in case of constant thrust along the y axis for both the Hill–Clohessy–Wiltshire and the Schweighart–Sedwick models.

Analytical integration of the transformed dynamics, taking into account only a constant controlling thrust along y, leads to

$$
\begin{gathered}
z_1 = -\frac{b}{a^2-b}u_y\frac{t^2}{2} + z_{2_0}t + z_{1_0}, \quad z_2 = -\frac{b}{a^2-b}u_y t + z_{2_0}, \\
z_3 = \left(z_{3_0} - \frac{a^3 u_y}{2\left(a^2-b\right)^{\frac{5}{2}}}\right)\cos\left[\left(\sqrt{a^2-b}\right)t\right] + \frac{z_{4_0}}{\sqrt{a^2-b}}\sin\left[\left(\sqrt{a^2-b}\right)t\right] + \frac{a^3 u_y}{2\left(a^2-b\right)^{\frac{5}{2}}}, \\
z_4 = z_{4_0}\cos\left[\left(\sqrt{a^2-b}\right)t\right] - \sqrt{a^2-b}\left(z_{3_0} - \frac{a^3 u_y}{2\left(a^2-b\right)^{\frac{5}{2}}}\right)\sin\left[\left(\sqrt{a^2-b}\right)t\right].
\end{gathered}
\tag{5.6}
$$

The assumption of continuous constant thrust reflects the state of the art for space thrusters, where only a regime value for the control is available [12]. Figure 5.1 and Figure 5.2 show the phase planes for the two types of forced motion (the forced double integrator represented by state variables z_1 and z_2, and the forced harmonic oscillator represented by state variables z_3 and z_4) with either positive or negative constant control along y. Arrows are indicating the paths directions according to the sign of the control. The curves on phase plane z_1 vs. z_2 are parabolas with symmetry about the z_2 axis for both the Hill–Clohessy–Wiltshire and the Schweighart–Sedwick models (only the curvature changes in the two cases, being in particular equal to $-\frac{3\omega^2}{8u_y}$ for the Hill–Clohessy–Wiltshire model and $-\frac{a^2-b}{2bu_y}$ for the Schweighart–Sedwick model). The curves on the phase plane $z_3 \frac{z_4}{\sqrt{a^2-b}}$ are circles for both the Hill–Clohessy–Wiltshire and Schweighart-Sedwick models. The z_3 coordinates for the centers of the circles in Figure 5.2 are given by

$$
\pm\frac{a^3 u_y}{2\left(a^2-b\right)^{\frac{5}{2}}} \tag{5.7}
$$

as calculated through the analytical solution in (5.6). The position of those centers is positive or negative according to the control sign.

Eq. (5.6) also gives the state vector evolution for coasting (control off) phases, by simply imposing $u_y = 0$. In particular, when the control is off, a drift parallel to the z_1 axis is experienced in the z_1 vs. z_2 phase plane, whose direction is related to the sign of z_2 (see Eq. (5.6)), while the circles in Figure 5.2 simply evolve around the origin. Again, the phase planes reproduce the results of [4] when the values for Hill–Clohessy–Wiltshire equations are used for a and b.

Eq. (5.4) and Eq. (5.6) together show how the spacecraft relative motion can be seen as an oscillation, represented by the states z_3 and z_4, around a virtual point, whose evolution is given by z_1 and z_2 in (5.6).

6 Conclusion

We developed a linear transformation of both the Hill–Clohessy–Wiltshire model for spacecraft relative motion nearby a circular orbit and the more recent Schweighart–Sedwick including the J2 effect. The proposed transformation highlights the superposition of double integrator and harmonic oscillator modes. Previous results in literature, regarding the traveling-ellipse formulation of the Hill–Clohessy–Wiltshire equations are included as a particular case of our state vector transformation. In particular we give analytical solution and a description of the phase planes when only along-track control is used. The achieved dynamic separation via state transformation allows the control designer to focus directly on either one of two critical goals regarding the stabilization of the chaser's motion with respect to the target: namely, either the stabilization into a closed elliptical relative orbit or into a separate circular orbit with respect to the Earth center.

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7 APPENDIX

Substitution of (3.3) into (5.3) leads to

$$T = \begin{pmatrix} 0 & -2\dfrac{c}{\omega\left(5c^2-2\right)} & 0 & \dfrac{i\sqrt{\omega^2\left(c^2-2\right)}}{2\omega^2 c^2} \\ 1 & 0 & \dfrac{-i\sqrt{\omega^2\left(c^2-2\right)}}{\omega c} & 0 \\ 0 & 0 & \dfrac{i\left(c^2-2\right)\sqrt{\omega^2\left(c^2-2\right)}}{2c^2} & 0 \\ 0 & 1 & 0 & \dfrac{i\sqrt{\omega^2\left(c^2-2\right)}}{\omega c} \end{pmatrix},$$

$$\hat{B} = \begin{bmatrix} 0 \\ -\dfrac{\left(5c^2-2\right)\omega^2}{4\omega^2c^2-\left(5c^2-2\right)\omega^2} \\ 0 \\ \dfrac{-4i\omega^3c^3}{\left(\left(5c^2-2\right)\omega^2-4\omega^2c^2\right)^{\frac{3}{2}}} \end{bmatrix}. \tag{7.1}$$

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On the PLF Construction for the Absolute Stability Study of Dynamical Systems with Non-Constant Gain

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Abstract: This paper deals with the absolute stability analysis of uncertain systems formulated in linear differential inclusion. It presents an approach based on the representation of a polyhedral positively invariant set by its vertices, allowing to construct the associated Lyapunov function. Efficiency of the method is discussed through a numerical example, where the absolute stability of a third order system has been analyzed via the construction of a Polyhedral Lyapunov Function (PLF). The flexibility of the proposed mesh and the check procedure of Molchanov–Pyatintskii conditions give a larger parameterized absolute stability domain than the one obtained by others existing in the literature.

Keywords: *absolute stability; uncertain systems; polyhedral Lyapunov function; sphere triangulation; linear programming.*

Mathematics Subject Classification (2000): 93D20, 93C10.

1 Introduction

Complex systems have always been difficult in their modeling and stability analysis since they may present nonlinearities and/or uncertainties. The problem has to do with nonlinear systems formulated in differential inclusions, where it is worth to decide about their largest parameterized domain of variation of the non-constant gain without loss of their stability [19]. Several criteria have been developed as a solution of this problem such as the circle criterion [1], Popov criterion [2] and Borne and Gentina criterion [3]. However all these criteria give sufficient but not necessary conditions of stability.
The Second Lyapunov method is a powerful tool of the stability analysis for nonlinear or uncertain system. However, its implementation is dependent on the choice and the way

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of construction of the Lyapunov function. The well-known class of quadratic functions is the most common one [4]. However this kind of functions doesn't lead usually to the best solution, since the existence of a quadratic Lyapunov function is not a necessary condition of stability. Recently, a generalization of quadratic functions have been introduced in the context of constrained control and they are called composite quadratic Lyapunov functions [5]. Some classes of non-quadratic Lyapunov functions are introduced such as polynomial homogenous functions [6]. The class of piecewise linear functions [7] which is a universal class, since their construction represents necessary and sufficient condition of stability, was introduced for stability analysis and control [8]. A sub-class is the one of polyhedral Lyapunov functions, a set-induced functions, have positively invariant polyhedral sub-level sets [20]. Therefore, their construction is based on an operation of scaling of the set boundary.

Several approaches have been established for the construction of polyhedral Lyapunov functions, the plane representation of the sub-level set have been considered to determine the absolute stability boundary of a second order system [9, 18]. The symmetric representation of the set by its vertices is used to construct a polyhedral Lyapunov function for third order uncertain system [10]. The technique of Ray-gridding is another issue for scaling [11, 12] based on uniform partitions of the state space in terms of ray directions allowing stability analysis of linear switched systems.

This paper is devoted to the stability analysis of third order uncertain systems by constructing a polyhedral Lyapunov function. We propose to represent the positively invariant set by its vertices obtained by a surface sphere triangulation [16]. This kind of representation with an associated algorithm enables to enhance the set of parameters variations, the obtained boundaries of the uncertainty are larger than those obtained by existent approaches in the literature. The paper is organized as follows: First, we remind some properties of the polyhedral sets and of their associated Lyapunov functions. Then the procedure used for the computation of the Polyhedral Lyapunov Function is presented, the efficiency of the approach is illustrated by an example. Conclusions are summarized in the end.

2 Polyhedral Lyapunov Function

Our interest in this study is the construction of polyhedral Lyapunov functions, which are induced by polyhedral positively invariant sets. These sets present several theoretical and practical advantages over the ellipsoids, but they suffer from the problem of complexity of their representation.

We remind here that a polyhedral set can be represented by:

$$\mathcal{P}(F) = \{x : Fx \leq \bar{1}\} \tag{2.1}$$

or by its dual form:

$$\mathcal{V}(X) = \{\ x = Xz, \bar{1}^T z \leq 1, z \geq 0\}, \tag{2.2}$$

where $\bar{1} = [1, 1, ..., 1]^T$, F and X are $N \times n$-matrices.

The polyhedral set can be also represented by its rays, we denote by $R_N(\lambda), 0 < \lambda \leq 1$ the ray-polytope which is a scaled version of

$$R_N(\bar{1}) = \text{conv}\Big\{\cos(\frac{2\pi k}{N}), \sin(\frac{2\pi k}{N}), 0 \leq k \leq N\Big\},$$

where $\text{conv}\{V\}$ denotes the convex hull of a set of vertices V.

Given a C-set $S \subset \mathbb{R}^n$ (a convex and compact subset of $\mathbb{R}^n$ including the origin as interior point), it is always possible to define a function, named Minkowski function, which is essentially the function whose sub-level sets are achieved by linearly scaling the set S.

Definition 2.1 [8] Given a C-set S, its Minkowski function is defined by :

$$\psi_S(x) = \inf\{\lambda \geq 0 : x \in \lambda S\}. \tag{2.3}$$

The Minkowski function ψ_S satisfies the following properties [13]:

- It is positive definite : $0 \leq \psi_S(x) \leq \infty$ and $\psi_S(x) > 0$ for all $x \neq 0$.
- It is positively homogeneous of order 1: $\psi_S(\lambda x) = \lambda \psi_S(x)$ for $\lambda \geq 0$.
- It is sub-additive: $\psi_S(x_1 + x_2) \leq \psi_S(x_1) + \psi_S(x_2)$.
- It is continuous.
- Its unit ball is $S = \{x : \psi_S(x) \leq 1\}$.
- It is convex.

If a polyhedral C-set is considered, the Minkowski functions deriving from the representations (2.1) and (2.2) are:

$$\psi_{\mathcal{P}(F)}(x) = \max\{Fx\} = \max_i\{F_i x\} \tag{2.4}$$

and

$$\psi_{\mathcal{V}(X)}(x) = \min\{\bar{1}\mu, x = X\mu, \mu \geq 0\}. \tag{2.5}$$

Consider a system (possibly resulting from a feedback connection) of the form:

$$\dot{x}(t) = f(x(t)). \tag{2.6}$$

For a convex (possibly non-differentiable) Lyapunov function $\psi(x)$, its Lyapunov derivative is defined by [14]:

$$D^+\psi(x) = \max_{i \in \mathcal{I}(x)} F_i f(x), \tag{2.7}$$

where $\mathcal{I} = \{i : F_i(x) = \psi_{\mathcal{P}(F)}(x)\}$ and D^+ denotes the upper-right Dini derivative defined by:

$$D^+\psi(x) = \limsup_{h \to 0^+} \frac{\psi(x + hf(x)) - \psi(x)}{h}.$$

3 Absolute Stability Theorem

We consider the following Linear Differential Inclusion (LDI) given by:

$$\dot{x} \in \Big\{Ax, A = \sum_{i=1}^{K} \alpha_i A_i, \alpha_i \geq 0, \sum_{i=1}^{K} \alpha_i = 1\Big\}. \tag{3.1}$$

The matrices $A_1, A_2, ...A_K \in \mathbb{R}^{n \times n}$ are vertices of the matrix polytope.

Theorem 3.1 [15] *The function $\psi_S(x)$ (2.3) induced by S represented as in (2.2) is a Lyapunov function for the system (3.1) which guarantees its absolute stability (respectively the polytope S is a positively invariant set) if and only if there exists K matrices $H_i \in \mathbb{R}^{N\times N}$, $i = 1, 2, ..., K$, each of them verifies:*

$$h_{kk}^{(i)} + \Sigma_{j=1_{j\neq k}}^{N} h_{kj}^{(i)} < 0 \tag{3.2}$$

for all $1 \leq k \leq N$, $h_{kj}^{(i)}$ denotes the elements of the matrix H_i,

$$A_i X = X H_i \tag{3.3}$$

where $X = [x_1, x_2, ..., x_N] \in \mathbb{R}^{n\times N}$ is the matrix containing the vertices of S.

4 Polyhedral Lyapunov Function Construction for Third Order System

First, the computation of Polyhedral Lyapunov Function needs the definition of an arbitrary set. The scaling of its vertices allows to get a positively invariant set which defines a sub-level set of the Lyapunov function.

4.1 Representation of the polyhedral set

The plane representation of the set for a third order system needs a tedious computation complexity. We propose to represent the set by its vertices, which are obtained by a surface triangulation of the unit sphere [16]. This triangulation is obtained by a Matlab function which uses recursive subdivision. The first approximation is a platonic solid, an octahedron (Figure 4.1).

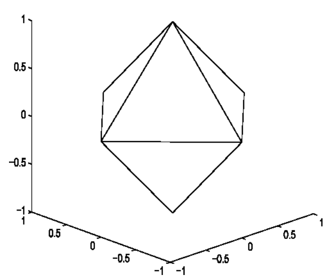

Figure 4.1: Octahedron.

This shape is defined by the vertices $[1, 0, 0]$, $[-1, 0, 0]$, $[0, 1, 0]$, $[0, -1, 0]$, $[0, 0, 1]$ and $[0, 0, -1]$. Each level of refinement subdivides each triangle face by a factor of 4 (Figure 4.2).

At each level of refinement, the vertices are projected to the sphere surface. Thus we define the arbitrary set S^A (Figure 4.3).

4.2 Generation of the Lyapunov function

After the definition of the arbitrary polytope S^A, the determination of the positively invariant set (level set of the associated Lyapunov function) is based on checking the

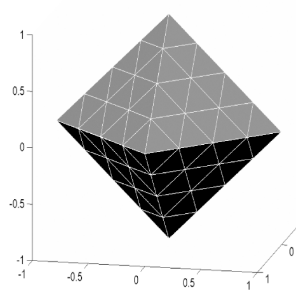

Figure 4.2: The octahedron obtained after two levels of refinement.

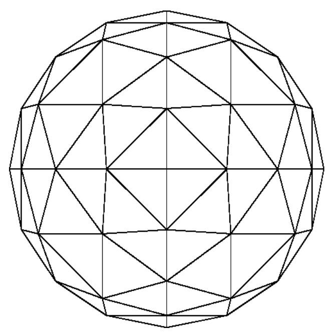

Figure 4.3: The polytope S^A obtained by a Surface Triangulation of the unit sphere.

two conditions (3.2) and (3.3) of theorem 3.1. Thus, the following linear program is formulated:

- For each vertex x_k, for all $k = 1, 2, ..., N$ we denote by $V(k)$ the matrix obtained by the neighbored vertices

$$V(k) = [-x_k, x_k, x_{1(k)}, x_{2(k)}, ..., x_{L(k)}] \tag{4.1}$$

  for all $k = 1, 2..., N$, where $x_{l(k)}$, for all $l(k) = 1, 2, ..., L(k)$ are the neighbored vertices of x_k.

- We resolve the following linear program:

$$\begin{array}{c} \max \quad FA_i x_k \\ FV(k) \leq \mathcal{J}^T \end{array} \tag{4.2}$$

  for all $k = 1, 2, ..., N$, where $\mathcal{J} = [-1, 1, 1...., 1]^T$ is a $\mathbb{R}^{L(k)+2}$ vector. The dual of the linear program (4.2) can be written:

$$\begin{array}{c} \min \quad \mathcal{J}^T \lambda(k), \\ V(k)\lambda(k) = A_i x_k \end{array} \tag{4.3}$$

where $\lambda(k) \in \mathbb{R}^{L(k)+2}$ is a vector containing the Lagrange multipliers relative to the linear program (4.3). We construct each column of the matrix H_i, for all $i = 1, 2, ..., K$ from the elements of the vectors $\lambda(k)$, $k = 1, 2, ..., N$:

$$h_{kk}^{(i)} = -\lambda_1(k), \quad h_{l(k)k}^{(i)} = \lambda_{l(k)+2}(k). \tag{4.4}$$

All the other components of H_i are equal to zero. With such a construction of H_i, $i = 1, 2, ..., K$, the condition (3.3) is well satisfied.

The computation of the matrices H_i, for all $i = 1, 2, ..., K$ followed by an operation of scaling the vertices of S^A leads to the construction of the modified polytope S^D with vertices contained in X^D. This operation consists in replacing the matrix X by $X^D = XD^{-1}$ where $D = \mathrm{diag}(d_1, d_2, ..., d_N)$ is a diagonal matrix. The vector $d = [d_1, d_2, d_3, ..., d_N]^T$ is obtained by solving the following linear program:

$$\begin{gathered} \min \quad z, \\ \begin{bmatrix} |H_1|^T \\ |H_2|^T \\ \vdots \\ |H_K|^T \end{bmatrix} d - \mathbf{1}z \leq 0, \; d \geq 0, \; z \geq -100 \end{gathered} \tag{4.5}$$

where $|H_i|$ is the matrix obtained from H_i by replacing only the off-diagonal elements by their absolute values. $\mathbf{1}$ denotes the vector of appropriate dimension, of which all entries are equal to one.

5 Numerical Example

We consider the following system with nonlinear feedback gain defined by Figure 5.1. If we consider an output linear gain, we can prove that the stability condition is a positive unlimited gain. But where the gain is non-constant, we have to determine the largest domain $[k_{\min}, k_{\max}]$ in which the nonlinear gain $\frac{\sigma(y,t)}{y}$ may vary without loss of the system stability:

$$k_{\min} \leq \frac{\sigma(y,t)}{y} \leq k_{\max}, \quad y \neq 0. \tag{5.1}$$

The absolute stability of the considered system is equivalent to that of the Linear Differential Inclusion defined by the two vertices of the matrix polytopes:

$$A_1 = \begin{bmatrix} -10 & -10k_{\min} & -10k_{\min} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -10 & -10k_{\max} & -10k_{\max} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5.2}$$

Let us set $k_{\min} = 0.2$. The problem is to determine $k_{\max}$ such that the system is absolutely stable. As long as the linear program (4.5) is feasible, we get an optimal solution $z_{opt} = -100$, which gives the associate scaling vector $d > 0$. Then the associated Lyapunov function is $\psi_{S^D}(x) = \inf\{\mu \geq 0 : x \in \mu S^D\}$.

With $N = 66$ vertices, the obtained upper boundary is $k_{\max} = 2.24$ which is upper than the values obtained by other developed criteria and approaches. Indeed the Circle

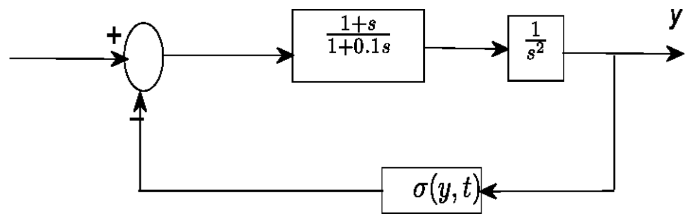

Figure 5.1: The studied system.

criterion leads to $k_{\max} = 0.5467$. The representation of the polytope with 6402 vertices [10], gives $k_{\max} = 1$ and the application of the ray-gridding technique [12] provides $k_{\max} = 1.5$. This comparison study shows the importance of the proposed procedure of PLF construction from the point of view of the width of the absolute stability domain and the reduction of the number of vertices which simplifies the computation complexity.

6 Conclusion

In this paper, we have dealt with the problem of the construction of a polyhedral lyapunov function for the absolute stability analysis of uncertain systems formulated on linear differential inclusion. It has been proved that the choice of a flexible representation of the polytope and the application of a suitable technique of scaling adjust its shape to some demands. The representation of the polytope by its vertices obtained by the proposed surface triangulation of the unit sphere associated with a suitable technique of scaling allows a convenient application of the Molchanov–Pyatintskii theorem. The comparison of the proposed procedure with other criteria and approaches has shown its availability and its efficiency.

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Special Solutions to Rotating Stratified Boussinesq Equations

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Abstract: In this paper, we have obtained some special solutions of rotating stratified Boussinesq equations and reduced these equations into the system of six coupled nonlinear ODEs. Further, in absence of strain field we have proved that the reduced system of six coupled ODEs is completely integrable.

Keywords: *rotating stratified Boussinesq equation, completely integrable systems, special solutions*

Mathematics Subject Classification (2000): 34A05, 35J35.

1 Introduction

The stratified Boussinesq equations form a system of PDEs modelling the movements of planetary atmospheres. It may be noted that the Boussinesq approximation in the literature is also referred to as the Oberbeck–Boussinesq approximation for which, the reader is referred to an interesting article of Rajagopal et al [1] providing a rigorous mathematical justification as perturbations of the Navier–Stokes equations. Majda & Shefter [2] have chosen certain special solutions of this system of PDEs to demonstrate onset of instability when the Richardson number is less than 1/4. In their study of instability in stratified fluids at large Richardson number, Majda & Shefter [2] have obtained the exact solutions to stratified Boussinesq equations neglecting the effects of rotations and viscosity. In his monograph Majda [3] has obtained the special solution of rotating stratified Boussinesq equations excluding the effects of viscosity and finite rotation. Whereas, in this paper we include the effect of rotation. And then we systematically deploy the procedure of Majda & Shefter [2] (as well as the procedure applied by Craik & Criminale in their paper [4]) to obtain the exact solutions of rotating stratified Boussinesq equations and derive the system of six coupled ODEs. Further, in the absence of strain field we proved that the reduced system of ODEs is completely integrable and admits the similar results obtained by Srinivasan et all [5]. For the similar kind of work reader may refer Maas [8, 9].

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2 Nondimensional Rotating Stratified Boussinesq Equations

We consider the motion of an incompressible flow of fluid in the atmosphere and in the ocean where the flow velocities are too slow to account for compressible effects, the flow of fluid is governed by the following rotating Boussinesq equations (we ignore the effects of viscosity and heat dissipation) that involves the interaction of gravity with density stratification about the reference state.

$$\begin{aligned} \frac{D\mathbf{v}}{Dt} + f(\hat{\mathbf{e_3}} \times \mathbf{v}) &= -\nabla \frac{\tilde{p}}{\rho_b} - \frac{g\rho}{\rho_b}\hat{\mathbf{e_3}}, \\ \operatorname{div} \mathbf{v} &= 0, \\ \frac{D\tilde{\rho}}{Dt} &= 0, \end{aligned} \tag{2.1}$$

where $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$, the unit vector in vertical direction is $\hat{\mathbf{e_3}} = (0, 0, 1)$, the space variable $\mathbf{x} = (x_1, x_2, x_3)$ and fluid velocity is given by $\mathbf{v} = (v^1, v^2, v^3)$. For the local behavior of incompressible fluid the density $\tilde{\rho}$ is the sum of mean density ρ_b and perturbations ρ about the mean density, that is $\tilde{\rho}(\mathbf{x}, t) = \rho_b + \rho(\mathbf{x}, t)$. The pressure is denoted by $\tilde{p}$ and f is a rotation frequency.

Now we nondimensionalize the Boussinesq equations (2.1) with the following scales for length, time, velocity, density, and pressure:

$$\begin{array}{lcl} L & : & \text{horizontal length scale,} \\ v^* & : & \text{mean advective velocity,} \\ T_e = \frac{L}{v^*} & : & \text{eddy turnover time,} \\ T_R = f^{-1} & : & \text{rotation time,} \\ T_N = N^{-1} & : & \text{buoyancy time,} \\ \rho_b & : & \text{mean density,} \\ \overline{p} & : & \text{mean pressure,} \\ N & : & \text{buoyancy frequency.} \end{array} \tag{2.2}$$

In this scale of nondimensionalization we introduce the following nondimensional variables

$$\mathbf{x}' = \frac{\mathbf{x}}{L},\ t' = \frac{t}{T_e},\ \mathbf{v}' = \frac{\mathbf{v}}{v^*},\ \tilde{\rho}' = \frac{\tilde{\rho}}{\rho_b B},\ p' = \frac{\tilde{p}}{\overline{p}}. \tag{2.3}$$

The numerical factor B in the density equation is positive. Applying equations (2.3) to equations (2.1) and dropping the primes finally we get the nondimensional rotating stratified Boussinesq equations

$$\begin{aligned} \frac{D\mathbf{v}}{Dt} + \frac{1}{R_0}\mathbf{u} &= -\overline{P}\nabla p - \Gamma\rho\hat{\mathbf{e_3}}, \\ \operatorname{div} \mathbf{v} &= 0, \\ \frac{D\tilde{\rho}}{Dt} &= 0. \end{aligned} \tag{2.4}$$

Here, we have $\mathbf{u} = (u^1, u^2, u^3) = \hat{\mathbf{e_3}} \times \mathbf{v}$, $\Gamma = \frac{BgL}{v^{*2}}$ the nondimensional number, $R_0 = \frac{v^*}{Lf}$ the Rossby number and $\overline{P} = \frac{\overline{p}}{\rho_b v^{*2}}$ the Euler number. Nondimensional density function is $\tilde{\rho}(\mathbf{x}, t) = \rho_b + \rho(\mathbf{x}, t)$. The more elaborate discussion about the nondimensional analysis of rotating stratified Boussinesq equations is given by Majda in his monograph [3]. In the following section we have obtained the special solutions to (2.4).

3 Special Solutions

In this section we investigate the special solutions to (2.4) in large scale part. We are looking for the local behavior of an incompressible fluid, and we expand the smooth velocity field and density function in a Taylor's series about some point $\mathbf{x_0}$:

$$\begin{aligned} \mathbf{v}(\mathbf{x},t) &= \mathbf{v}(\mathbf{x}_0,t) + \nabla\mathbf{v}|_{(\mathbf{x}_0,t)}(\mathbf{x}-\mathbf{x}_0) + O(|\mathbf{x}-\mathbf{x}_0|^2), \\ \tilde{\rho}(\mathbf{x},t) &= \rho_b + \nabla\tilde{\rho}|_{(\mathbf{x}_0,t)}(\mathbf{x}-\mathbf{x}_0) + O(|\mathbf{x}-\mathbf{x}_0|^2), \end{aligned} \tag{3.1}$$

where $\nabla\mathbf{v}$ is a 3×3 matrix whose $(i,j)^{th}$ entry is $\frac{\partial v^i}{\partial x_j}$, $i = 1,2,3$, $j = 1,2,3$. The following equation (3.2) is the decomposition of the matrix $\nabla\mathbf{v}$ as a sum of symmetric and skew-symmetric matrices and such kind of decomposition is unique:

$$\begin{aligned} \nabla\mathbf{v}|_{(\mathbf{x}_0,t)} &= \left(\tfrac{\nabla\mathbf{v}+(\nabla\mathbf{v})^T}{2}\right) + \left(\tfrac{\nabla\mathbf{v}-(\nabla\mathbf{v})^T}{2}\right) \\ &= \mathcal{D}(\mathbf{x}_0,t) + \Omega(\mathbf{x}_0,t), \end{aligned} \tag{3.2}$$

where $\mathcal{D}$ is the symmetric part of $\nabla\mathbf{v}$ and is called the deformation matrix, it has the property that the trace of matrix $\mathcal{D}$ is equal to the divergence of vector field $\mathbf{v}$. Whereas, Ω is a skew symmetric part of matrix $\nabla\mathbf{v}$ and satisfy the following equation (3.3).

$$\Omega\mathbf{h} = \frac{1}{2}\mathbf{w}\times\mathbf{h}, \tag{3.3}$$

for any vector $\mathbf{h}\in\mathbb{R}^3$. The vector $\mathbf{w}$ is vorticity vector that is $\mathbf{w} = \nabla\times\mathbf{v} = (w_1,w_2,w_3)$. Hence, from equation (3.2) we get

$$\nabla\mathbf{v}|_{(\mathbf{x}_0,t)}\mathbf{h} = \mathcal{D}(\mathbf{x}_0,t)\mathbf{h} + \frac{1}{2}\mathbf{w}(\mathbf{x}_0,t)\times\mathbf{h}. \tag{3.4}$$

The decomposition of a vector $\mathbf{v}$ as in equations (3.1) by mean of equation (3.4) has a simple physical interpretation namely, every incompressible velocity field is a sum of translation, stretching and rotation. We may deprive the translation part by a Galilean transformation, for this one may refer to Majda & Bertozzi [10]. We assume that $\mathbf{v}(\mathbf{x}_0,t) = 0$.

We take advantage of the local representation to determine certain special solutions to the rotating stratified Boussinesq equation (2.4). We derive now an equation for gradient of velocity

$$(v^i_{x_k})_t + \sum_j v^j(v^i_{x_k})_{x_j} + \sum_j \frac{\partial v^j}{\partial x_k}\frac{\partial v^i}{\partial x_j} + \frac{1}{R_0}(u^i_{x_k}) = -\overline{P}(p_{x_i})_{x_j} - \Gamma\frac{\partial\rho}{\partial x_k}\delta_{k3}, \tag{3.5}$$

where δ is the Kronecker delta. Then, we introduce the notations $V = (v^i_{x_k})$ and $\hat{P} = \overline{P}(p_{x_i})_{x_k}$ for the Hessian matrix of the pressure $\hat{P}$. With this notation we can rewrite equation (3.5) in the matrix form as follows

$$\frac{DV}{Dt} + V^2 + \frac{1}{R_0}(u^i_{x_k}) = -\hat{P} - \Gamma\mathbf{\hat{e}_3}(\nabla\rho)^T. \tag{3.6}$$

A matrix $(u^i_{x_k})$ can uniquely be expressed as $(u^i_{x_k}) = S + Q$, where the symmetric matrix

S and skew symmetric matrix Q are as given below

$$S = \frac{1}{2}\begin{pmatrix} -2\frac{\partial v^1}{\partial x_1} & \frac{\partial v^1}{\partial x_1} - \frac{\partial v^2}{\partial x_2} & -\frac{\partial v^2}{\partial x_3} \\ \frac{\partial v^1}{\partial x_1} - \frac{\partial v^2}{\partial x_2} & 2\frac{\partial v^1}{\partial x_2} & \frac{\partial v^1}{\partial x_3} \\ -\frac{\partial v^2}{\partial x_3} & \frac{\partial v^1}{\partial x_3} & 0 \end{pmatrix}, \; Q = \frac{1}{2}\begin{pmatrix} 0 & -\frac{\partial v^2}{\partial x_2} - \frac{\partial v^1}{\partial x_1} & -\frac{\partial v^2}{\partial x_3} \\ \frac{\partial v^2}{\partial x_2} + \frac{\partial v^1}{\partial x_1} & 0 & \frac{\partial v^1}{\partial x_3} \\ \frac{\partial v^2}{\partial x_3} & -\frac{\partial v^1}{\partial x_3} & 0 \end{pmatrix}. \tag{3.7}$$

For any $\mathbf{h} \in \mathbb{R}^3$, a skew symmetric matrix Q satisfies the equation

$$Q\mathbf{h} = -\frac{1}{2}\frac{\partial \mathbf{v}}{\partial x_3} \times \mathbf{h}. \tag{3.8}$$

From equation (2.4), we have the density function $\tilde{\rho} = \rho_b + \rho$ and $\frac{D\tilde{\rho}}{Dt} = 0$. Therefore, we have $\nabla\tilde{\rho} = \nabla\rho$. Now differentiating the density equation partially with respect to x_k we get

$$\frac{\partial}{\partial t}(\tilde{\rho}_{x_k}) + \sum_j \frac{\partial v^j}{\partial x_k}\frac{\partial \tilde{\rho}}{\partial x_j} + \sum_j v^j \frac{\partial^2 \tilde{\rho}}{\partial x_k \partial x_j} = 0 \tag{3.9}$$

which may be recast as

$$\frac{D}{Dt}(\nabla\tilde{\rho}) + V^T(\nabla\tilde{\rho}) = 0. \tag{3.10}$$

Since $\mathcal{D}$ and Ω are symmetric and skew symmetric parts of ∇v a simple calculation gives

$$V^2 = \mathcal{D}^2 + \Omega^2 + \mathcal{D}\Omega + \Omega\mathcal{D}. \tag{3.11}$$

The symmetric part of V^2 is $\mathcal{D}^2 + \Omega^2$ and $\mathcal{D}\Omega + \Omega\mathcal{D}$ is the skew-symmetric part. We proceed to decompose equation (3.6) into symmetric and skew symmetric parts. The symmetric part is easily seen to be

$$\frac{D\mathcal{D}}{Dt} + \mathcal{D}^2 + \Omega^2 + \frac{1}{R_0}S = -\hat{P} - \frac{\Gamma}{2}\Big[\hat{\mathbf{e_3}}(\nabla\tilde{\rho})^T + (\nabla\tilde{\rho})\hat{\mathbf{e_3}}^T\Big]. \tag{3.12}$$

The skew symmetric part of equation (3.6) is discussed in the following Proposition 3.1. Before proceeding to the proposition here we insert a simple lemma and one may find the proof of this lemma in the monograph of Majda [3].

Lemma 3.1 $\mathbf{w} \cdot \nabla\mathbf{v} = \mathbf{w} \cdot (\nabla\mathbf{v})^T$.

Proof For any $\mathbf{h} \in \mathbb{R}^3$, we have by identification (3.3)

$$0 = \frac{1}{2}\mathbf{w} \cdot (\mathbf{w} \times \mathbf{h}) = \frac{1}{2}\mathbf{w} \cdot \big(((\nabla\mathbf{v}) - (\nabla\mathbf{v})^T)\mathbf{h}\big) = \frac{1}{2}\mathbf{w} \cdot \big((\nabla\mathbf{v}) - (\nabla\mathbf{v})^T\big)\,\mathbf{h}$$

from which the result follows since $\mathbf{h}$ is arbitrary. □

Proposition 3.1 *The evolution of the vorticity* $\mathbf{w} = \nabla \times \mathbf{v}$ *is governed by the equation*

$$\frac{D\mathbf{w}}{Dt} = \mathbf{w} \cdot \nabla\mathbf{v} + \Gamma\begin{pmatrix} -\frac{\partial\tilde{\rho}}{\partial x_2} \\ \frac{\partial\tilde{\rho}}{\partial x_1} \\ 0 \end{pmatrix} + \frac{1}{R_0}\frac{\partial \mathbf{v}}{\partial x_3}. \tag{3.13}$$

Proof Equating the skew symmetric part of equation (3.6) we get

$$\frac{D\Omega}{Dt} + \mathcal{D}\Omega + \Omega\mathcal{D} + \frac{1}{R_0}Q = -\frac{\Gamma}{2}\begin{pmatrix} 0 & 0 & -\frac{\partial\tilde{\rho}}{\partial x_1} \\ 0 & 0 & -\frac{\partial\tilde{\rho}}{\partial x_2} \\ \frac{\partial\tilde{\rho}}{\partial x_1} & \frac{\partial\tilde{\rho}}{\partial x_2} & 0 \end{pmatrix}. \tag{3.14}$$

So that for arbitrary $\mathbf{h} \in \mathbb{R}^3$

$$\frac{1}{2}\frac{D\mathbf{w}}{Dt} \times \mathbf{h} + (\mathcal{D}\Omega + \Omega\mathcal{D})\mathbf{h} - \frac{1}{2R_0}\frac{\partial\mathbf{v}}{\partial x_3} \times \mathbf{h} = \frac{\Gamma}{2}\begin{pmatrix} -\frac{\partial\tilde{\rho}}{\partial x_2} \\ \frac{\partial\tilde{\rho}}{\partial x_1} \\ 0 \end{pmatrix} \times \mathbf{h}. \tag{3.15}$$

Here, Ω and $\mathcal{D}$ are given by

$$\Omega = \frac{1}{2}\begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{pmatrix}$$

and the elements d_{ij} of matrix $\mathcal{D}$ are expressible in terms of partial derivatives $\partial_k v^l$ with the relation $d_{11} + d_{22} + d_{33} = 0$. A simple calculation gives

$$\mathcal{D}\Omega + \Omega\mathcal{D} = \frac{1}{2}\begin{pmatrix} 0 & -c_{12} & c_{13} \\ c_{12} & 0 & -c_{23} \\ -c_{13} & c_{23} & 0 \end{pmatrix},$$

$$\mathbf{c} = \begin{pmatrix} c_{23} \\ c_{13} \\ c_{12} \end{pmatrix} = \begin{pmatrix} -w_1 d_{11} - w_2 d_{12} - w_3 d_{33} \\ -w_1 d_{12} - w_2 d_{22} - w_3 d_{23} \\ -w_1 d_{13} - w_2 d_{23} - w_3 d_{33} \end{pmatrix} = -\mathcal{D}\mathbf{w} = -\mathbf{w} \cdot \mathcal{D}.$$

Therefore,

$$(\mathcal{D}\Omega + \Omega\mathcal{D})\mathbf{h} = \frac{1}{2}\begin{pmatrix} 0 & -c_{12} & c_{13} \\ c_{12} & 0 & -c_{23} \\ -c_{13} & c_{23} & 0 \end{pmatrix}\mathbf{h} = \frac{1}{2}\begin{pmatrix} c_{23} \\ c_{13} \\ c_{12} \end{pmatrix} \times \mathbf{h}.$$

Hence, we can recast equation (3.15) as

$$\frac{1}{2}\frac{D\mathbf{w}}{Dt} \times \mathbf{h} - \frac{1}{2}\mathbf{w} \cdot \mathcal{D} \times \mathbf{h} - \frac{1}{2R_0}\frac{\partial\mathbf{v}}{\partial x_3} \times \mathbf{h} = \frac{\Gamma}{2}\begin{pmatrix} -\frac{\partial\tilde{\rho}}{\partial x_2} \\ \frac{\partial\tilde{\rho}}{\partial x_1} \\ 0 \end{pmatrix} \times \mathbf{h}. \tag{3.16}$$

Now $\mathbf{w} \cdot \mathcal{D} = \frac{1}{2}(\mathbf{w} \cdot (\nabla\mathbf{v}) + \mathbf{w} \cdot (\nabla\mathbf{v})^T) = \mathbf{w} \cdot \nabla\mathbf{v}$, substituting this into (3.16) and simplifying we get (3.13). Hence the proof of the proposition. □

Remark 3.1 From equation (3.13) we see that the infinitesimal vorticity elements are advected with the fluid and get amplified with interaction of velocity gradients and density gradients and also with addition term of rate of change of fluid velocity in vertical direction, which causes the effects of rotation. Due to this additional term caused by effect of rotation we have proved in the following Theorem 3.1 the component of vorticity along with the density gradient advected with fluid is increased according to the rate of change of velocity in vertical direction and get amplified with density gradient. This is the extension of Ertel's theorem allowing the forcing term due to the rotation effect and neglecting the dissipation. One may refer to Majda ([3], p. 14) for the details of Ertel's theorem.

Theorem 3.1 *The advective rate of change of vorticity component along with density gradient is given by*

$$\frac{D}{Dt}(\mathbf{w}\cdot\nabla\tilde{\rho}) = \frac{1}{R_0}\frac{\partial \mathbf{v}}{\partial x_3}\cdot\nabla\tilde{\rho}. \tag{3.17}$$

Proof Consider the advective rate of change of vorticity along with density gradient. We get

$$\frac{D}{Dt}(\mathbf{w}\cdot\nabla\tilde{\rho}) = \frac{D\mathbf{w}}{Dt}\cdot\nabla\tilde{\rho} + \mathbf{w}\cdot\frac{D(\nabla\tilde{\rho})}{Dt}. \tag{3.18}$$

Applying equations (3.10), (3.13) and lemma (3.1) to equation (3.18) we get the result. □

As we claim earlier we have obtained the special solutions to (2.4), these solutions are given in the form of the following Theorem 3.2. The more interesting part of these solutions is that it reduces the PDEs of rotating stratified Boussinesq equations (2.4) into the system of six coupled nonlinear ODEs.

Theorem 3.2 *The rotating stratified Boussinesq equations* (2.4) *admit the special solutions of the form*

$$\left.\begin{array}{rcl} \mathbf{v}(\mathbf{x},t) &=& \mathcal{D}(t)\mathbf{x} + \frac{1}{2}\mathbf{w}(t)\times\mathbf{x}, \\ \tilde{\rho} &=& \rho_b + \mathbf{b}(t)\cdot\mathbf{x}, \\ \overline{P}p &=& \frac{1}{2}\hat{P}(t)\mathbf{x}\cdot\mathbf{x}, \end{array}\right\} \tag{3.19}$$

where $\overline{P}$ *is a nondimensional number as defined in* (2.4), $\mathcal{D}(t)$ *is a symmetric matrix with zero trace; when* $\mathbf{w}(t) = \nabla\times\mathbf{v}$ *and* $\mathbf{b}(t) = \nabla\tilde{\rho}$ *satisfy the ODEs*

$$\left.\begin{array}{rcl} \dfrac{d\mathbf{w}}{dt} &=& \mathcal{D}(t)[\mathbf{w}(t) + \frac{1}{R_0}\hat{\mathbf{e_3}}] + \Gamma\hat{\mathbf{e_3}}\times\mathbf{b}(t) - \frac{1}{2R_0}\hat{\mathbf{e_3}}\times\mathbf{w}(t), \\ \dfrac{d\mathbf{b}}{dt} &=& -\mathcal{D}(t)\mathbf{b}(t) + \frac{1}{2}\mathbf{w}(t)\times\mathbf{b}(t), \end{array}\right\} \tag{3.20}$$

and matrix $\hat{P}(t)$ *is given by*

$$-\hat{P} = \frac{d\mathcal{D}}{dt} + \mathcal{D}^2 + \Omega^2 + \frac{1}{R_0}S + \frac{\Gamma}{2}(\hat{\mathbf{e_3}}\mathbf{b}^T + \mathbf{b}\hat{\mathbf{e_3}}^T), \tag{3.21}$$

where the matrix Ω *is as defined in* (3.2) *through the linear map given by* (3.3) *and the matrix* S *is given by (3.7).*

Proof We proceed to show that the Ansatz (3.19), (3.20) does indeed furnish solutions to (2.4). The condition $\operatorname{div}\mathbf{v} = 0$ follows from the fact that matrix $\mathcal{D}$ has zero *trace*. To verify that the momentum equation, note that $\mathbf{v}$ is linear in $\mathbf{x}$ say $\mathbf{v} = V\mathbf{x}$, where $V = \mathcal{D} + \Omega$ is a function of time alone. Therefore, $\nabla\tilde{\rho} = \nabla\rho = \mathbf{b}(t)$ and advection term is

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = (V\mathbf{x}\cdot\nabla)V\mathbf{x} = V(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3})V\mathbf{x} = V^2\mathbf{x},$$

so that $\frac{D\mathbf{v}}{Dt} = \frac{d\mathcal{D}}{dt}\mathbf{x} + \frac{d\Omega}{dt}\mathbf{x} + V^2\mathbf{x}$. Also, equation (3.12) can be recast as

$$\frac{d\mathcal{D}}{dt} + \mathcal{D}^2 + \Omega^2 + \frac{1}{R_0}S = -\hat{P} - \frac{\Gamma}{2}(\hat{\mathbf{e_3}}\mathbf{b}^T + \mathbf{b}\hat{\mathbf{e_3}}^T) \tag{3.22}$$

and equation (3.13) that is equation for vorticity is equivalent to the first equation in (3.20). The equation for skew symmetric part equivalent to (3.14) is as given below

$$\frac{d\Omega}{dt} + \mathcal{D}\Omega + \Omega\mathcal{D} + \frac{1}{R_0}Q = -\frac{\Gamma}{2}\begin{pmatrix} 0 & 0 & -b_1 \\ 0 & 0 & -b_2 \\ b_1 & b_2 & 0 \end{pmatrix}, \tag{3.23}$$

where Q is skew symmetric matrix as defined in (3.7). Inserting (3.23) and eliminating $\frac{d\mathcal{D}}{dt}$ using (3.21) we find that

$$\begin{aligned} \frac{D\mathbf{v}}{Dt} &= -\hat{P}\mathbf{x} - \mathcal{D}^2\mathbf{x} - \Omega^2\mathbf{x} - \tfrac{\Gamma}{2}\begin{pmatrix} 0 & 0 & b_1 \\ 0 & 0 & b_2 \\ b_1 & b_2 & 2b_3 \end{pmatrix}\mathbf{x} - \tfrac{1}{R_0}S\mathbf{x} \\ & \quad -(\mathcal{D}\Omega + \Omega\mathcal{D})\mathbf{x} + \tfrac{\Gamma}{2}\begin{pmatrix} 0 & 0 & b_1 \\ 0 & 0 & b_2 \\ -b_1 & -b_2 & 0 \end{pmatrix}\mathbf{x} - \tfrac{1}{R_0}Q\mathbf{x} + V^2\mathbf{x}. \end{aligned} \tag{3.24}$$

Since the term $(V^2 - \mathcal{D}^2 - \Omega^2 - \mathcal{D}\Omega - \Omega\mathcal{D})\mathbf{x}$ vanishes, (3.24) simplifies as

$$\frac{D\mathbf{v}}{Dt} = -\hat{P}\mathbf{x} - \Gamma\begin{pmatrix} 0 \\ 0 \\ \mathbf{b}\cdot\mathbf{x} \end{pmatrix} - \frac{1}{R_0}(S+Q)\mathbf{x}. \tag{3.25}$$

As the fluid velocity is defined by (3.19), then we see that

$$\frac{1}{R_0}(S+Q)\mathbf{x} = \frac{1}{R_0}(\hat{\mathbf{e_3}} \times \mathbf{v}).$$

The pressure term in (3.19) enables us to write (3.25) as

$$\frac{D\mathbf{v}}{Dt} + \frac{1}{R_0}\mathbf{u} = -\overline{P}\nabla p - \Gamma\rho\hat{\mathbf{e_3}}.$$

Finally we verify the Boussinesq equation for density

$$\begin{aligned} \frac{D\tilde{\rho}}{Dt} &= \frac{D}{Dt}(\mathbf{b}\cdot\mathbf{x}) \\ &= \frac{d\mathbf{b}}{dt}\cdot\mathbf{x} + (\mathbf{v}\cdot\nabla)(\mathbf{b}\cdot\mathbf{x}) \\ &= \frac{d\mathbf{b}}{dt}\cdot\mathbf{x} + \mathbf{v}\cdot\mathbf{b}. \end{aligned} \tag{3.26}$$

Using (3.20), we substitute $\frac{d\mathbf{b}}{dt}$ and $\mathbf{v} = (\mathcal{D}+\Omega)\mathbf{x}$ into (3.26), we get

$$\begin{aligned} \frac{D\tilde{\rho}}{Dt} &= -(\mathcal{D}\mathbf{b})\cdot\mathbf{x} + \frac{1}{2}(\mathbf{w}\times\mathbf{b})\cdot\mathbf{x} + [(\mathcal{D}+\Omega)\mathbf{x}]\cdot\mathbf{b} \\ &= -(\mathcal{D}\mathbf{b})\cdot\mathbf{x} + \frac{1}{2}(\mathbf{w}\times\mathbf{b})\cdot\mathbf{x} + (\mathcal{D}\mathbf{x})\cdot\mathbf{b} + \frac{1}{2}(\mathbf{w}\times\mathbf{x})\cdot\mathbf{b} \\ &= 0 \end{aligned} \tag{3.27}$$

completing the proof of the theorem. □

Following are the examples of special solutions of rotating stratified Boussinesq equations (2.4) in the form of (3.19).

Example 3.1 Consider a two dimensional time independent flow for which the constant vorticity vector $\mathbf{w} = (0, 0, w_0)$, density gradient vector $\mathbf{b} = (0, 0, b_0)$ and deformation matrix is given by

$$\mathcal{D} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we see that vectors $\mathbf{w}$ and $\mathbf{b}$ satisfy the system of ODEs (3.20) and a solution of the system of PDEs (2.4) is given below.

$$\begin{aligned} \mathbf{v}(\mathbf{x}, t) &= (\lambda x_1 - \tfrac{w_0}{2} x_2, \tfrac{w_0}{2} x_1 - \lambda x_2, 0), \\ \tilde{\rho} &= \rho_b + b_0 x_3, \\ \overline{P} p &= \tfrac{1}{2} \left[(-\lambda^2 + \tfrac{w_0^2}{4} + \tfrac{w_0}{2R_0})(x_1^2 + x_2^2) + \Gamma b_0 x_3^2 - \tfrac{2\lambda}{R_0} x_1 x_2 \right]. \end{aligned}$$

Example 3.2 Now we consider a two dimensional time dependent flow; let the vorticity vector be $\mathbf{w}(t) = \Big(w_{10} \cos(t/R_0) + w_{20} \sin(t/R_0), -w_{10} \sin(t/R_0) + w_{20} \cos(t/R_0), 0 \Big) = (-a_2(t), a_1(t), 0)$ with the initial condition $\mathbf{w}(0) = (w_{10}, w_{20}, 0)$ and the density gradient vector be $\mathbf{b}(t) = (0, 0, b_0)$, where b_0 is an arbitrary constant. The deformation matrix $\mathcal{D}$ is

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & a_1(t) \\ 0 & 0 & a_2(t) \\ a_1(t) & a_2(t) & 0 \end{pmatrix}.$$

Then, we see that the vectors $\mathbf{w}, \mathbf{b}$ satisfy the system of ODEs (3.20) with initial conditions $\mathbf{w}(0), \mathbf{b}(0)$. The velocity and density are then given by $\mathbf{v}(\mathbf{x}, t) = (a_1(t)x_3, a_2(t)x_3, 0)$, $\tilde{\rho} = \rho_b + b_0 x_3$. The pressure p will be computed by using equations (3.19) and (3.21).

4 Integrable System

In the above section we see that the rotating stratified Boussinesq equations (2.4) admit the special solutions in the form of (3.19) provided that $\mathbf{w}$ and $\mathbf{b}$ satisfy the system of ODEs (3.20). Further, in the absence of strain field $\mathcal{D} = 0$ we have the following reduced system of six coupled nonlinear ODEs

$$\left. \begin{aligned} \dot{\mathbf{w}} &= \Gamma \hat{\mathbf{e_3}} \times \mathbf{b} - \tfrac{1}{2R_0} \hat{\mathbf{e_3}} \times \mathbf{w}, \\ \dot{\mathbf{b}} &= \tfrac{1}{2} \mathbf{w} \times \mathbf{b}. \end{aligned} \right\} \tag{4.1}$$

We see the system of equations (4.1) is divergence free and admits the following four functionally independent first integrals

$$|\mathbf{b}|^2 = c_1, \quad \hat{\mathbf{e_3}} \cdot \mathbf{w} = c_2, \quad |\mathbf{w}|^2 + 4\Gamma(\hat{\mathbf{e_3}} \cdot \mathbf{b}) = c_3, \quad \mathbf{w} \cdot \mathbf{b} + \frac{1}{R_0} \hat{\mathbf{e_3}} \cdot \mathbf{b} = c_4. \tag{4.2}$$

Hence, by Liouville's theorem on integral invariant and theorem of Jacobi [11] there exists an additional first integral. That is an autonomous system of six coupled ODEs admitting the five global functionally independent first integrals proving the complete integrability of the system (4.1). Also, we see from (4.2) that $|\mathbf{b}|$ and $|\mathbf{w}|$ remain bounded so that the invariant surface (4.2) is compact and flow of vector field $(\mathbf{w}, \mathbf{b})$ is complete. It is easy to verify that the system (4.1) admits all the similar kind of results obtained by Srinivasan et all in their paper [5]. Also, we find that the system of equations (4.1) is similar to the system discussed by Desale [6]. For the bifurcation analysis near the degenerate critical point one may refer to [7].

5 Conclusion

In Section 1, we gave a brief introduction to the work and put up a literature survey. Then in Section 2, we present the rotating stratified Boussinesq equations (2.1) and consequently we put it into the nondimensional form (2.4). In Section 3, we obtained the special solutions to the system (2.4) in the form of (3.19). Due to the inclusion of rotating term in the equations (2.4), the special solutions obtained here are the improvement of the solutions obtained by Majda & Shefter [2]. In this link we present the Proposition 3.1, Theorem 3.1 and in Theorem 3.2, we present special solutions provided that the vorticity and density gradients satisfy the system of ODEs (3.20). Also, in that section we gave the examples of two dimensional flows. In the last Section 4, we proved that the system of six coupled nonlinear ODEs (4.1), which is obtained by neglecting the strain field is an integrable system.

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Operational Calculus in Noncooperative Stochastic Games †

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Abstract: We continue investigating an antagonistic game of two players modeled by stochastic processes describing mutual casualties. The game is observed at some random epochs of time. We consider the paths of the game in which one player loses the game. A related functional in our recent work was expressed in terms of the inverse of two-dimensional Laplace-Carson transform. Using operational calculus we manage to find explicitly inverse transforms of the exit time and casualties to both players upon the exit from the game in terms of modified Bessel functions. All are concluded by numerical examples.

Keywords: *noncooperative stochastic games; fluctuation theory; marked point processes; Poisson process; ruin time; exit time; first passage time; Bessel functions.*

Mathematics Subject Classification (2000): 82B41, 60G51, 60G55, 60G57, 91A10, 91A05, 91A60, 60K05.

1 Introduction

We continue our studies initiated in [3] in which we modeled an antagonistic stochastic game by two marked Poisson processes

$$\mathcal{A} := \sum_{j\geq 1} d_j \varepsilon_{r_j} \text{ and } \mathcal{B} := \sum_{k\geq 1} z_k \varepsilon_{w_k} \tag{1.1}$$

on a filtered probability space $(\Omega, \mathcal{F}(\Omega), \mathfrak{F}_t, P)$ specified by

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$$Ee^{-u\mathcal{A}(\cdot)} = e^{\lambda_A|\cdot|[h_A(u)-1]},\ h_A(u) = Ee^{-ud_1},\ Re(u) \geq 0, \tag{1.2}$$

$$Ee^{-v\mathcal{B}(\cdot)} = e^{\lambda_B|\cdot|[h_B(v)-1]},\ h_B(v) = Ee^{-vz_1},\ Re(v) \geq 0, \tag{1.3}$$

representing casualties incurred to players A and B. The game starts with hostile actions initiated by one of the players A or B at times r_1 or w_1 (whichever comes first). The players can exchange with several more strikes before the information is first noticed by an observer at time $t_0 \geq \max\{r_1, w_1\}$. The observation of the game continues after t_0 in accordance with a point renewal process $S = \sum_{i\geq 1} \varepsilon_{t_i}$ and it is further extrapolated to the past moment $t_{-1} := \min\{r_1, w_1\}$ thereby forming an extended observation process $\mathcal{T} = \{t_{-1}, t_0, t_1, \ldots\}$. The entire information on the game is available only upon $\mathcal{T}$ and thus the game is reduced to its embedding $\mathcal{A}_{\mathcal{T}} \otimes \mathcal{B}_{\mathcal{T}}$. We stop the game when one of the players is ruined, and of all paths of the game we focused on those where player A loses to player B.

Given independent sub-σ-algebras $\mathcal{F}_{\mathcal{A}}, \mathcal{F}_{\mathcal{B}}, \mathcal{F}_S \subseteq \mathcal{F}(\Omega)$ we assume that the processes $\mathcal{A}$, $\mathcal{B}$, and S are, respectively, measurable. Let ξ_i and η_i be the corresponding iid increments of damages to A and B and $\Delta_j = t_j - t_{j-1} \in [\Delta]$ (an equivalent class of r.v.'s), $j = 1, 2, \ldots$, with

$$g(u, v, \theta) := Ee^{-u\xi_j - v\eta_j - \theta\Delta_j},\ Re(u) \geq 0,\ Re(v) \geq 0,\ Re(\theta) \geq 0,\ j \geq 1, \tag{1.4}$$

presumably known. The initial observation is defined as $t_0 = \max\{r_1, w_1\} + \Delta_0$, where $\Delta_0 \in [\Delta]$ and Δ_0 is independent from the rest of the Δ's. The random *exit indices* are

$$\mu := \inf\{j \geq 0 : \alpha_j = \alpha_0 + \xi_1 + \ldots + \xi_j > M\}, \tag{1.5}$$

$$\nu := \inf\{k \geq 0 : \beta_k = \beta_0 + \eta_1 + \ldots + \eta_k > N\}, \tag{1.6}$$

with α_0 and β_0 being the casualties to A and B at t_0, and M and N are respective tolerance thresholds. Related on μ and ν are the following r.v.'s:

t_μ is the nearest observation epoch when player A's damages exceed threshold M.

t_ν is the first observation of $\mathcal{T}$ when player B's damages exceed threshold N.

Apparently, α_μ and β_ν are the respective cumulative damages to players A and B at their ruin times. We will be concerned, however, with the ruin time of player A and thus restrict our game to the trace σ-algebra $\mathcal{F}(\Omega) \cap \{\mu < \nu\}$. Accordingly, we studied in [3], among other things,

$$\varphi_\mu = \varphi_\mu(u, v, \vartheta) = E[e^{-u\alpha_\mu - v\beta_\mu - \theta t_\mu}\mathbf{1}_{\{\mu<\nu\}}] \tag{1.7}$$

and obtained

$$\varphi_\mu = \mathcal{LC}_{xy}^{-1}\Big(\phi_0(x, 0, 0, u, v+y, \theta) - \phi_0(0, 0, 0, u+x, v+y, \theta)\frac{1 - g(u, v+y, \theta)}{1 - g(u+x, v+y, \theta)}\Big), \tag{1.8}$$

where $\mathcal{LC}^{-1}$ is the inverse of the two-dimensional Laplace–Carson transform

$$\mathcal{LC}_{pq}(\cdot)(x, y) := xy\int_{p=0}^{\infty}\int_{q=0}^{\infty} e^{-xp-yq}(\cdot)d(p, q),\ Re(x) > 0,\ Re(y) > 0, \tag{1.9}$$

According to [3],

$$\phi_0(x,0,0,u,v,\theta) = E[e^{-x\alpha_{-1}-u\alpha_0-v\beta_0-\theta t_0}]$$
$$= \frac{\lambda_A\lambda_B\delta(\theta^*)}{\theta+\lambda_A+\lambda_B}\Big(\frac{1}{\theta_A+\lambda_B}h_A(x+u)h_B(v)+\frac{1}{\theta_B+\lambda_A}h_A(u)h_B(v)\Big), \tag{1.10}$$

$$\phi_0(0,0,0,u,v,\theta) = E[e^{-u\alpha_0-v\beta_0-\theta t_0}]$$
$$= \frac{\lambda_A\lambda_B\delta(\theta^*)}{\theta+\lambda_A+\lambda_B}\Big(\frac{1}{\theta_A+\lambda_B}h_A(u)h_B(v)+\frac{1}{\theta_B+\lambda_A}h_A(u)h_B(v)\Big), \tag{1.11}$$

and

$$\theta_0^* := \theta+\lambda_A(1-h_A(u))+\lambda_B(1-h_B(v)), \delta(\theta) := Ee^{-\theta\Delta}, \tag{1.12}$$

$$\theta_A := \theta-\lambda_A(h_A(u)-1), \theta_B := \theta-\lambda_B(h_B(v)-1). \tag{1.13}$$

The involvement of the inverse of the Laplace–Carson transform in (1.8) at first does not look like the above formulas are analytically tractable. We demonstrate that this is not the case and consider a number of special cases (of independent interest) which are all Laplace–Carson invertible and thus provide the first vivid argument for analytical tractability of the results obtained in [3]. They are shown to be numerically tame and as such are rendered by trivial computational procedures. Most of them are reduced to definite integrals of the modified Bessel functions. In one case an explicit marginal probability density function is obtained. The original MATLAB routine is also attached.

2 A Special Case

We assume that the intervals $\Delta_0, \Delta_1, \ldots$ between the successive observation times $t_0, t_1, \ldots,$ are exponentially distributed with parameter δ, i.e.

$$\delta(\theta) := Ee^{-\theta\Delta_0} = \frac{\delta}{\delta+\theta}. \tag{2.1}$$

Furthermore, we assume that the marks in the processes $\mathcal{A}$ and $\mathcal{B}$ specified by h_A and h_B in (1.2) and (1.3), respectively, are exponential with parameters h and H, i.e.

$$h_A(u) = \frac{h}{h+u} \text{ and } h_B(v) = \frac{H}{H+v}. \tag{2.2}$$

(1.8) for this special case reduces to a form for which we can find the Laplace–Carson inverse explicitly. We start with the first factor, $\phi_0(x,0,0,u,v+y,\theta)$ of (1.10):

$$\begin{aligned}&\phi_0(x,0,0,u,v+y,\theta)\\ &= \frac{\lambda_A\lambda_B h_B(v+y)}{\theta+\lambda_A+\lambda_B}\times\delta(\theta+\lambda_A(1-h_A(u))+\lambda_B(1-h_B(v+y)))\\ &\quad\times\Big(\frac{1}{\theta+\lambda_B-\lambda_A(h_A(u)-1)}h_A(u+x)+\frac{1}{\theta+\lambda_A-\lambda_B(h_B(v+y)-1)}h_A(u)\Big).\end{aligned} \tag{2.3}$$

Continuing with calculations, after some algebra, we arrive at

$$
\begin{aligned}
&\phi_0(x,0,0,u,v+y,\theta)\\
&\quad=\frac{\lambda_A\lambda_B hH}{\theta+\lambda_A+\lambda_B}\\
&\quad\quad\times\frac{\delta(h+u)}{(H+v+y)[(\delta+\theta)(h+u)+\lambda_A u+\lambda_B(h+u)]-\lambda_B H(h+u)}\\
&\quad\quad\times\Big(\frac{1}{(\theta+\lambda_B)(h+u)+\lambda_A u}+\frac{-x}{(\theta+\lambda_B)(h+u)+\lambda_A u}\cdot\frac{1}{h+u+x}\\
&\quad\quad\quad+\frac{H+v+y}{(\theta+\lambda_A+\lambda_B)(H+v+y)-\lambda_B H}\cdot\frac{1}{h+u}\Big).
\end{aligned}
\tag{2.4}
$$

Now we apply the Laplace–Carson inverse to (2.4):

$$\mathcal{LC}_{xy}^{-1}(\phi_0(x,0,0,u,v+y,\theta))(p,q)$$

or proceed with $\mathfrak{L}_{xy}^{-1}$ being the two-dimensional Laplace inverse, in the form

$$=\mathfrak{L}_{xy}^{-1}(\tfrac{1}{xy}\cdot\phi_0(x,0,0,u,v+y,\theta))(p,q)$$

(by Fubini's Theorem, we can apply single-variate Laplace inverses first in x and later on in y)

$$
\begin{aligned}
&=\mathfrak{L}_y^{-1}\Big\{\frac{\lambda_A\lambda_B hH\delta(h+u)}{\widetilde{\theta}G_1}\cdot\frac{1}{y}\cdot\frac{1}{H+v+y-\frac{\lambda_B H(h+u)}{G_1}}\\
&\quad\times\Big(\frac{1}{G_2}+\frac{-1}{G_2}\cdot e^{-p(h+u)}+\frac{1}{\widetilde{\theta}(h+u)}+\frac{\lambda_B H}{\widetilde{\theta}^2(h+u)}\cdot\frac{1}{H+v+y-\frac{\lambda_B H}{\widetilde{\theta}}}\Big)\Big\}(q),
\end{aligned}
$$

then

$$
\begin{aligned}
&=\frac{\lambda_A\lambda_B hH\delta}{\widetilde{\theta}}\cdot\frac{1}{(H+v)G_1-\lambda_B H(h+u)}\\
&\quad\times\Big(\frac{h+u}{G_2}+\frac{H+v}{\widetilde{\theta}(H+v)-\lambda_B H}-\frac{h+u}{G_2}\cdot e^{-p(h+u)}\\
&\quad\quad+\Big\{\frac{-\lambda_B H}{\widetilde{\theta}}\cdot\frac{1}{\widetilde{\theta}(H+v)-\lambda_B H}+\frac{-G_1}{\widetilde{\theta}G_3}\Big\}e^{-q(H+v-\frac{\lambda_B H}{\widetilde{\theta}})}\\
&\quad\quad+\Big\{\frac{-(h+u)}{G_2}+\frac{h+u}{G_3}+\frac{h+u}{G_2}\cdot e^{-p(h+u)}\Big\}e^{-q(H+v-\frac{\lambda_B H(h+u)}{G_1})}\Big),
\end{aligned}
\tag{2.5}
$$

where

$$\widetilde{\theta}=\theta+\lambda_A+\lambda_B, \tag{2.6}$$

$$G_1=(\delta+\widetilde{\theta})(h+u)-\lambda_A h,\ G_2=\widetilde{\theta}(h+u)-\lambda_A h, \tag{2.7}$$

$$G_3=\delta(h+u)-\lambda_A h. \tag{2.8}$$

We turn to the second term $\phi_0(0,0,0,u+x,v+y,\theta)\frac{1-g(u,v+y,\theta)}{1-g(u+x,v+y,\theta)}$ of (1.8). Continuing with similar but more tedious calculations, we have its Laplace-Carson inverse in variable x :

$$
\begin{aligned}
&\mathcal{LC}_x^{-1}\Big\{\phi_0(0,0,0,u+x,v+y,\theta)\frac{1-g(u,v+y,\theta)}{1-g(u+x,v+y,\theta)}\Big\}(p)\\
&=\Lambda\big(1-g(u,v+y,\theta)\big)\cdot y\\
&\quad\times\Bigg[\Big(\frac{-1}{\lambda_A h}\cdot\frac{1}{\widetilde{\theta}(H+v)-\lambda_B H}+\{\frac{1}{\widetilde{\theta}}+\frac{G_2}{\lambda_A h\widetilde{\theta}}+\frac{h+u}{G_2}\}\frac{1}{G_2(H+v)-\lambda_B H(h+u)}\Big)\frac{1}{y}\\
&\qquad+\frac{1}{\lambda_A h}\cdot\frac{1}{\widetilde{\theta}(H+v)-\lambda_B H}\cdot\frac{1}{H+v+y-\frac{\lambda_B H}{\widetilde{\theta}}}\\
&\qquad-\{\frac{1}{\widetilde{\theta}}+\frac{G_2}{\lambda_A h\widetilde{\theta}}+\frac{h+u}{G_2}\}\frac{1}{G_2(H+v)-\lambda_B H(h+u)}\cdot\frac{1}{H+v+y-\frac{\lambda_B H(h+u)}{G_2}}\\
&\qquad+\frac{1}{\lambda_B H G_2}\cdot\frac{1}{y}\cdot e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})}\\
&\qquad+\Big[\Big(\frac{-1}{\lambda_B H G_2}+\frac{1}{\lambda_A h}\cdot\frac{1}{\widetilde{\theta}(H+v)-\lambda_B H}\\
&\qquad\quad+\{\frac{-1}{\widetilde{\theta}}+\frac{-G_2}{\lambda_A h\widetilde{\theta}}+\frac{-(h+u)}{G_2}\}\frac{1}{G_2(H+v)-\lambda_B H(h+u)}\Big)\frac{1}{y}\\
&\qquad\quad+\frac{-1}{\lambda_A h}\cdot\frac{1}{\widetilde{\theta}(H+v)-\lambda_B H}\cdot\frac{1}{H+v+y-\frac{\lambda_B H}{\widetilde{\theta}}}\\
&\qquad\quad+\Big\{\frac{1}{\widetilde{\theta}}+\frac{G_2}{\lambda_A h\widetilde{\theta}}+\frac{h+u}{G_2}\Big\}\frac{1}{G_2(H+v)-\lambda_B H(h+u)}\\
&\qquad\quad\times\frac{1}{H+v+y-\frac{\lambda_B H(h+u)}{G_2}}\Big]e^{-p(h+u+\frac{D(y)}{C'(y)})}\Bigg],
\end{aligned}
\tag{2.9}
$$

where

$$
\Lambda=\frac{\lambda_A\lambda_B hH\delta}{\widetilde{\theta}},\ C'(y)=\widetilde{\theta}(H+v+y)-\lambda_B H,\ D(y)=-\lambda_A h(H+v+y). \tag{2.10}
$$

Now, we will apply the single-variate Laplace-Carson inverse in y to (2.9). We will use the following formula for the Laplace inverse (cf. [1, 2]):

$$
\begin{aligned}
\mathfrak{L}_y^{-1}(\frac{1}{y+b_2}\cdot e^{\frac{a}{y+b_1}})(q)&=e^{-b_1 q}\mathcal{I}_0(2\sqrt{aq})\\
&\quad+(b_1-b_2)\cdot e^{-b_2 q}\int_{z=0}^{q}e^{(b_2-b_1)z}\mathcal{I}_0(2\sqrt{az})dz,
\end{aligned}
\tag{2.11}
$$

where $\mathcal{I}_0$ is the modified Bessel function of order zero. Using (2.11) in (2.9) in combination with (2.5), we finally have

$$
\begin{aligned}
&\varphi_\mu(u,v,\theta):=E\Big[e^{-u\alpha_\mu-v\beta_\mu-\theta t_\mu}\mathbf{1}_{\{\mu<\nu\}}\Big]\\
&=\mathcal{L}_{xy}^{-1}(\phi_0(x,0,0,u,v+y,\theta))(p,q)\\
&\quad-\mathcal{L}_{xy}^{-1}\Big(\phi_0(0,0,0,u+x,v+y,\theta)\frac{1-g(u,v+y,\theta)}{1-g(u+x,v+y,\theta)}\Big)(p,q)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-\lambda_A \lambda_B h H \delta}{\widetilde{\theta} G_2} \cdot \frac{h+u}{(H+v)G_1 - \lambda_B H(h+u)} \cdot e^{-p(h+u)} (1 - e^{-q(H+v-\frac{\lambda_B H(h+u)}{G_1})}) \\
&\quad + \Big(\frac{-\lambda_A h \delta}{\widetilde{\theta}} \cdot \frac{H+v}{(H+v)G_1 - \lambda_B H(h+u)} + \frac{\lambda_A \lambda_B h H \delta}{\widetilde{\theta} G_2} \cdot \frac{h+u}{(H+v)G_1 - \lambda_B H(h+u)} \Big) \\
&\qquad \times e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})} \\
&\quad + \Big(\frac{-\lambda_A h \delta}{\widetilde{\theta} G_1} + \frac{\lambda_A h \delta}{\widetilde{\theta}} \cdot \frac{H+v}{(H+v)G_1 - \lambda_B H(h+u)} \\
&\qquad - \frac{\lambda_A \lambda_B h H \delta}{\widetilde{\theta} G_2} \cdot \frac{h+u}{(H+v)G_1 - \lambda_B H(h+u)} \Big) \cdot e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})} \cdot e^{-q(H+v-\frac{\lambda_B H(h+u)}{G_1})} \\
&\quad + \frac{\lambda_A h \delta}{\widetilde{\theta} G_1} \cdot e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})} \cdot e^{-q(H+v-\frac{\lambda_B H}{\widetilde{\theta}})} \mathcal{I}_0 (2 \sqrt{\frac{\lambda_A \lambda_B h H p q}{\widetilde{\theta}^2}}) \\
&\quad + \frac{\lambda_A h \delta}{\widetilde{\theta}} \cdot \frac{(H+v)^2}{(H+v)G_1 - \lambda_B H(h+u)} \cdot e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})} \\
&\qquad \times \int_{z=0}^{q} e^{-(H+v-\frac{\lambda_B H}{\widetilde{\theta}})z} \mathcal{I}_0 (2 \sqrt{\frac{\lambda_A \lambda_B h H p z}{\widetilde{\theta}^2}}) dz \\
&\quad + \Big(\frac{\lambda_A \lambda_B h H \delta (h+u)}{\widetilde{\theta} G_1^2} + \frac{-\lambda_A \lambda_B h H \delta (h+u)}{\widetilde{\theta} G_1} \cdot \frac{H+v}{(H+v)G_1 - \lambda_B H(h+u)} \Big) \\
&\qquad \times e^{-p(h+u-\frac{\lambda_A h}{\widetilde{\theta}})} \cdot e^{-q(H+v-\frac{\lambda_B H(h+u)}{G_1})} \\
&\qquad \times \int_{z=0}^{q} e^{(\frac{\lambda_B H G_3}{\widetilde{\theta} G_1})z} \mathcal{I}_0 (2 \sqrt{\frac{\lambda_A \lambda_B h H p z}{\widetilde{\theta}^2}}) dz,
\end{aligned} \tag{2.12}
$$

where G_1, G_2, G_3 are defined in (2.7-2.8).

3 Marginal Functionals

Our next goal is to get marginal transforms. This can be directly obtained from $\varphi_\mu(u, v, \theta)$ of (2.12).

Special case 1, with $v = \theta = 0$ we have the marginal Laplace-Stieltjes transform of the amount of casualties to player A (who is supposed to lose) at the exit of the game:

$$\varphi_\mu(u, 0, 0) := E\Big[e^{-u\alpha_\mu} \mathbf{1}_{\{\mu < \nu\}}\Big]. \tag{3.1}$$

Correspondingly, we modify the above components in (2.12) to

$$\widetilde{\theta} = \lambda_A + \lambda_B, \tag{3.2}$$

$$G_1 = (\delta + \lambda_A + \lambda_B)(h + u - \frac{\lambda_A h}{\delta + \lambda_A + \lambda_B}), \tag{3.3}$$

$$G_2 = (\lambda_A + \lambda_B)(h + u - \frac{\lambda_A h}{\lambda_A + \lambda_B}), \tag{3.4}$$

$$h + u - \frac{\lambda_A h}{\widetilde{\theta}} = u + \frac{\lambda_B h}{\lambda_A + \lambda_B}, \tag{3.5}$$

$$H+v-\frac{\lambda_B H(h+u)}{G_1}=\frac{H(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}+\frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}}, \tag{3.6}$$

$$H+v+\frac{\lambda_B H}{\widetilde{\theta}}=\frac{\lambda_A H}{\lambda_A+\lambda_B}, \tag{3.7}$$

$$\begin{aligned}\frac{\lambda_B H G_3}{\widetilde{\theta}G_1}&=\frac{\lambda_B H\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}\\&\quad+\frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}},\end{aligned} \tag{3.8}$$

$$\frac{1}{(H+v)G_1-\lambda_B H(h+u)}=\frac{1}{H(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}, \tag{3.9}$$

$$\widetilde{\theta}(H+v)-\lambda_B H=\lambda_A H. \tag{3.10}$$

Substituting (3.2-3.10) into (2.12), we arrive at

$$\begin{aligned}\varphi_\mu(u,0,0)&=\Big(\frac{-\lambda_A\lambda_B h\delta}{(\lambda_A+\lambda_B)^2(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\lambda_A+\lambda_B}}\cdot\frac{h+u}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\cdot e^{-p(h+u)}\Big)\\
&\quad\times\Big[1-\exp\Big(-q\big(\frac{H(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}+\frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}}\big)\Big)\Big]\\
&\quad+\Big(\frac{-\lambda_A h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\\
&\qquad+\frac{\lambda_A\lambda_B h\delta}{(\lambda_A+\lambda_B)^2(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\lambda_A+\lambda_B}}\cdot\frac{h+u}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\Big)\cdot e^{-p(u+\frac{\lambda_B h}{\lambda_A+\lambda_B})}\\
&\quad+\Big(\frac{-\lambda_A h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A+\lambda_B}}\\
&\qquad+\frac{\lambda_A h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\\
&\qquad+\frac{-\lambda_A\lambda_B h\delta}{(\lambda_A+\lambda_B)^2(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\lambda_A+\lambda_B}}\cdot\frac{h+u}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\Big)\cdot e^{-p(u+\frac{\lambda_B h}{\lambda_A+\lambda_B})}\\
&\quad\times\exp\Big(-q\big(\frac{H(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}+\frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}}\big)\Big)\\
&\quad+\frac{\lambda_A h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A+\lambda_B}}\\
&\quad\times e^{-p(u+\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_0\big(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}}\big)\\
&\quad+\frac{\lambda_A hH\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\cdot e^{-p(u+\frac{\lambda_B h}{\lambda_A+\lambda_B})}\\
&\quad\times\int_{z=0}^{q}e^{-(\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0\big(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}}\big)dz\end{aligned}$$

$$+\Big(\frac{\lambda_A\lambda_B hH\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{h+u}{(h+u-\frac{\lambda_A h}{\delta+\lambda_A+\lambda_B})^2}$$

$$+\frac{-\lambda_A\lambda_B hH\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)(\delta+\lambda_A+\lambda_B)}\cdot\frac{1}{h+u-\frac{\lambda_A h}{\delta+\lambda_A+\lambda_B}}\cdot\frac{h+u}{h+u-\frac{\lambda_A h}{\delta+\lambda_A}}\Big)$$

$$\times e^{-p(u+\frac{\lambda_B h}{\lambda_A+\lambda_B})}\exp\Big(-q\big(\frac{H(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}+\frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}}\big)\Big)$$

$$\times\int_{z=0}^{q}\exp\Big[\big(\frac{\lambda_B H\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}+\frac{-\lambda_A\lambda_B hH}{(\delta+\lambda_A+\lambda_B)^2}\cdot\frac{1}{u+\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B}}\big)z\Big]$$

$$\times\mathcal{I}_0\big(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}}\big)dz. \tag{3.11}$$

Special case 2. Setting $u=\theta=0$ gets us to the marginal Laplace-Stiltjes transform of the casualties to player B at the exit of the game to be lost by player A:

$$\varphi_\mu(0,v,0):=E\Big[e^{-v\beta_\mu}\mathbf{1}_{\{\mu<\nu\}}\Big]. \tag{3.12}$$

The Laplace inverse formula (cf. [1, 2]) that we will use along with (2.11) is:

$$\mathcal{L}_y^{-1}\big(\frac{e^{\frac{a}{y+b}}}{(y+b)^2}\big)(q)=\sqrt{\frac{q}{a}}\cdot e^{-bq}\mathcal{I}_1(2\sqrt{aq}), \tag{3.13}$$

where $\mathcal{I}_1$ is the modified Bessel function of order one. After setting $u=\theta=0$ in (1.8), we arrive at

***(i)* Case $\delta\neq\lambda_A$,**

$$\varphi_\mu^1(0,v,0)=\frac{-\lambda_A H\delta}{\lambda_A+\lambda_B}\cdot\frac{1}{H\delta+(\delta+\lambda_B)v}\cdot e^{-ph}$$

$$+\frac{\lambda_A H\delta}{\lambda_A+\lambda_B}\cdot\frac{1}{H\delta+(\delta+\lambda_B)v}\cdot e^{-ph}\cdot e^{-q(v+\frac{H\delta}{\delta+\lambda_B})}$$

$$+\Big(\frac{-\lambda_A\delta}{\lambda_A+\lambda_B}\cdot\frac{v}{H\delta+(\delta+\lambda_B)v}$$

$$+\frac{-\lambda_A H\delta^2}{(\lambda_A+\lambda_B)(\delta+\lambda_B)}\cdot\frac{1}{H\delta+(\delta+\lambda_B)v}\cdot e^{-q(v+\frac{H\delta}{\delta+\lambda_B})}\Big)\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}$$

$$+\frac{\lambda_A\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_0\big(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}}\big)$$

$$+\frac{\lambda_A\delta}{\lambda_A+\lambda_B}\cdot\frac{(H+v)^2}{H\delta+(\delta+\lambda_B)v}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\int_{z=0}^{q}e^{-(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0\big(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}}\big)dz$$

$$+\frac{-\lambda_A\lambda_B^2H^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^2}\cdot\frac{1}{H\delta+(\delta+\lambda_B)v}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(v+\frac{H\delta}{\delta+\lambda_B})}$$

$$\times\int_{z=0}^{q}e^{(\frac{\lambda_B H(\delta-\lambda_A)}{(\lambda_A+\lambda_B)(\delta+\lambda_B)})z}\mathcal{I}_0\big(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}}\big)dz. \tag{3.14}$$

***(ii)* Case $\boldsymbol{\delta = \lambda_A}$,**

$$\begin{aligned}
\varphi_\mu^2(0,v,0) = &\Big(\frac{-\lambda_A^2 H}{\lambda_A+\lambda_B}\cdot\frac{1}{\lambda_A H+(\lambda_A+\lambda_B)v} \\
&\quad + \frac{\lambda_A^2 H}{\lambda_A+\lambda_B}\cdot\frac{1}{\lambda_A H+(\lambda_A+\lambda_B)v}\cdot e^{-q(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})}\Big)\cdot e^{-ph} \\
&+\Big(\frac{-\lambda_A^2 v}{\lambda_A+\lambda_B}\cdot\frac{1}{\lambda_A H+(\lambda_A+\lambda_B)v} \\
&\quad + \frac{-\lambda_A^3 H}{(\lambda_A+\lambda_B)^2}\cdot\frac{1}{\lambda_A H+(\lambda_A+\lambda_B)v}\cdot e^{-q(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})}\Big)\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})} \\
&+\frac{\lambda_A^2}{(\lambda_A+\lambda_B)^2}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}}) \\
&+\frac{\lambda_A^2}{\lambda_A+\lambda_B}\cdot\frac{(H+v)^2}{\lambda_A H+(\lambda_A+\lambda_B)v}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})} \\
&\times\int_{z=0}^{q} e^{-(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz \\
&+\frac{-\lambda_A^2\lambda_B^2 H^2}{(\lambda_A+\lambda_B)^2}\cdot\frac{1}{\lambda_A H+(\lambda_A+\lambda_B)v}\sqrt{\frac{q}{\lambda_A\lambda_B hHp}}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(v+\frac{\lambda_A H}{\lambda_A+\lambda_B})} \\
&\times\mathcal{I}_1(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}}).
\end{aligned} \tag{3.15}$$

Special case 3, with $u = v = 0$ look into the Laplace-Stieltjes transform of the exit time of the game to be lost by player A:

$$\varphi_\mu(0,0,\theta) := E\Big[e^{-\theta t_\mu}\mathbf{1}_{\{\mu<\nu\}}\Big]. \tag{3.16}$$

Proceeding similarly as special case 1, we have

$$\begin{aligned}
\varphi_\mu(0,0,\theta) = &\frac{-\lambda_A\lambda_B\delta}{\widetilde{\theta}(\theta+\lambda_B)(\delta+\theta)}\cdot e^{-ph}+\frac{\lambda_A\lambda_B\delta}{\widetilde{\theta}(\theta+\lambda_B)(\delta+\theta)}\cdot e^{-ph}\cdot e^{-q(\frac{(\delta+\theta)H}{\delta+\theta+\lambda_B})} \\
&+\frac{-\lambda_A\delta\theta}{\widetilde{\theta}(\theta+\lambda_B)(\delta+\theta)}\cdot e^{-p(\frac{(\theta+\lambda_B)h}{\widetilde{\theta}})} \\
&+\frac{-\lambda_A\lambda_B\delta^2}{\widetilde{\theta}(\theta+\lambda_B)(\delta+\theta)(\delta+\theta+\lambda_B)}\cdot e^{-p(\frac{(\theta+\lambda_B)h}{\widetilde{\theta}})}\cdot e^{-q(\frac{(\delta+\theta)H}{\delta+\theta+\lambda_B})} \\
&+\frac{\lambda_A\delta}{\widetilde{\theta}(\delta+\theta+\lambda_B)}\cdot e^{-p(\frac{(\theta+\lambda_B)h}{\widetilde{\theta}})}\cdot e^{-q(\frac{(\theta+\lambda_A)H}{\widetilde{\theta}})}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{\widetilde{\theta}^2}}) \\
&+\frac{\lambda_A H\delta}{\widetilde{\theta}(\delta+\theta)}\cdot e^{-p(\frac{(\theta+\lambda_B)h}{\widetilde{\theta}})}\int_{z=0}^{q} e^{-(\frac{(\theta+\lambda_A)H}{\widetilde{\theta}})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{\widetilde{\theta}^2}})dz \\
&+\frac{-\lambda_A\lambda_B^2 H\delta}{\widetilde{\theta}(\delta+\theta)(\delta+\theta+\lambda_B)^2}\cdot e^{-p(\frac{(\theta+\lambda_B)h}{\widetilde{\theta}})}\cdot e^{-q(\frac{(\delta+\theta)H}{\delta+\theta+\lambda_B})} \\
&\times\int_{z=0}^{q} e^{(\frac{\lambda_B H(\delta-\lambda_A)}{\widetilde{\theta}(\delta+\theta+\lambda_B)})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{\widetilde{\theta}^2}})dz.
\end{aligned} \tag{3.17}$$

4 The Explicit Distribution of the Casualties Value to Player A

Now, we can find the pdf of the exit value of casualties to player A (special case 1) by taking the inverse Laplace transform w.r.t. variable u. We distinguish two cases which are $\delta \neq \lambda_B$ and $\delta = \lambda_B$, respectively. The Laplace inverse formulas that we will use along with (2.11) are:

$$\mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{1}{y+b})(q) = e^{-b(q-\alpha)}\mathbf{1}_{(\alpha,\infty)}(q), \tag{4.1}$$

$$\mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{1}{(y+b)^2})(q) = (q-\alpha)e^{-b(q-\alpha)}\mathbf{1}_{(\alpha,\infty)}(q), \tag{4.2}$$

$$\mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b}}}{y+b})(q) = e^{-b(q-\alpha)}\mathcal{I}_0(2\sqrt{a(q-\alpha)})\mathbf{1}_{(\alpha,\infty)}(q), \tag{4.3}$$

$$\begin{aligned} \mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b_1}}}{y+b_2})(q) &= e^{-b_1(q-\alpha)}\mathcal{I}_0(2\sqrt{a(q-\alpha)})\mathbf{1}_{(\alpha,\infty)}(q) \\ &+ (b_1-b_2) \cdot e^{-b_2(q-\alpha)}\int_{z=0}^{q-\alpha} e^{(b_2-b_1)z}\mathcal{I}_0(2\sqrt{az})dz\mathbf{1}_{(\alpha,\infty)}(q), \end{aligned} \tag{4.4}$$

$$\mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b}}}{(y+b)^2})(q) = \sqrt{\frac{q-\alpha}{a}} \cdot e^{-b(q-\alpha)}\mathcal{I}_1(2\sqrt{a(q-\alpha)})\mathbf{1}_{(\alpha,\infty)}(q), \tag{4.5}$$

$$\begin{aligned} \mathcal{L}_y^{-1}(e^{-\alpha y} \cdot \frac{e^{\frac{a}{y+b_1}}}{(y+b_2)^2})(q) &= e^{-b_2(q-\alpha)}\int_{z=0}^{q-\alpha} e^{(b_2-b_1)z}\mathcal{I}_0(2\sqrt{az})dz\mathbf{1}_{(\alpha,\infty)}(q) \\ &+ (b_1-b_2) \cdot e^{-b_2(q-\alpha)}\int_{z=0}^{q-\alpha} (q-\alpha-z) \cdot e^{(b_2-b_1)z}\mathcal{I}_0(2\sqrt{az})dz\mathbf{1}_{(\alpha,\infty)}(q). \end{aligned} \tag{4.6}$$

Equations (4.4) and (4.6) can be readily proved, while the rest of the above formulas can be found in references [1, 2].
After that, we apply the Laplace inverse in (3.11), arriving at

***(i)* Case $\boldsymbol{\delta \neq \lambda_B}$,**

$$\begin{aligned} \mathcal{L}_u^{-1}\Big\{\varphi_\mu^1(u,0,0)\Big\}(s) &= \frac{\lambda_A\lambda_B h\delta}{(\lambda_A+\lambda_B)^2(\delta-\lambda_B)} \cdot e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}}\Big(1-e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}}\Big)\mathbf{1}_{(p,\infty)}(s) \\ &+ \Big(\frac{\lambda_A\lambda_B h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)(\delta-\lambda_B)} \cdot e^{-ph} + \frac{-\lambda_A h\delta^2}{(\lambda_A+\lambda_B)(\delta+\lambda_A)(\delta-\lambda_B)} \cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\Big) \\ &\times e^{-\frac{h\delta(s-p)}{\delta+\lambda_A}}\mathbf{1}_{(p,\infty)}(s) \\ &+ \Big(\frac{\lambda_A\lambda_B h\delta}{(\lambda_A+\lambda_B)^2(\delta+\lambda_A)} \cdot e^{-ph} + \frac{-\lambda_A\lambda_B h\delta^2}{(\lambda_A+\lambda_B)^2(\delta+\lambda_A)(\delta+\lambda_A+\lambda_B)} \cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\Big) \\ &\times e^{-\frac{Hq(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}} \cdot e^{-\frac{h(\delta+\lambda_B)(s-p)}{\delta+\lambda_A+\lambda_B}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHq(s-p)}{(\delta+\lambda_A+\lambda_B)^2}})\mathbf{1}_{(p,\infty)}(s) \\ &+ \frac{\lambda_A^2\lambda_B h^2\delta^2}{(\lambda_A+\lambda_B)^3(\delta-\lambda_B)(\delta+\lambda_A+\lambda_B)} \cdot e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}} \cdot e^{-\frac{Hq(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}}\Big(e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}}-1\Big) \\ &\times \int_{w=0}^{s-p} e^{(\frac{\lambda_B h}{\lambda_A+\lambda_B}-\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B})w}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHqw}{(\delta+\lambda_A+\lambda_B)^2}})dw\mathbf{1}_{(p,\infty)}(s) \end{aligned}$$

$$+\Big(\frac{-\lambda_A^2\lambda_B^2h^2\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)^2(\delta-\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot e^{-ph}$$

$$+\frac{\lambda_A^2\lambda_B h^2\delta^2}{(\lambda_A+\lambda_B)(\delta+\lambda_A)^2(\delta-\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\Big)\cdot e^{-\frac{h\delta(s-p)}{\delta+\lambda_A}}$$

$$\times e^{-\frac{Hq(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}}\int_{w=0}^{s-p}e^{(\frac{h\delta}{\delta+\lambda_A}-\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B})w}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHqw}{(\delta+\lambda_A+\lambda_B)^2}})dw\mathbf{1}_{(p,\infty)}(s)$$

$$+\frac{\lambda_A h\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-\frac{\lambda_A Hq}{\lambda_A+\lambda_B}}\cdot e^{-\frac{h(\delta+\lambda_B)(s-p)}{\delta+\lambda_A+\lambda_B}}$$

$$\times\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}})\mathbf{1}_{(p,\infty)}(s)$$

$$+\frac{\lambda_A hH\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-\frac{h\delta(s-p)}{\delta+\lambda_A}}$$

$$\times\int_{z=0}^{q}e^{-\frac{\lambda_A Hz}{\lambda_A+\lambda_B}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz\mathbf{1}_{(p,\infty)}(s)$$

$$+\int_{z=0}^{q}\Big[\Big(\frac{-\lambda_A\lambda_B^2hH\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)(\delta+\lambda_A+\lambda_B)^2}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)(s-p)}{(\delta+\lambda_A+\lambda_B)^2}})$$

$$+\frac{\lambda_A^2\lambda_B h^2H\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)^2}\sqrt{\frac{s-p}{\lambda_A\lambda_B hH(q-z)}}$$

$$\times\mathcal{I}_1(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)(s-p)}{(\delta+\lambda_A+\lambda_B)^2}})\Big)\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-\frac{Hq(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}}$$

$$\times e^{-\frac{h(\delta+\lambda_B)(s-p)}{\delta+\lambda_A+\lambda_B}}\cdot e^{\frac{\lambda_B H\delta z}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})\mathbf{1}_{(p,\infty)}(s)$$

$$+\frac{-\lambda_A^2\lambda_B h^2H\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_A)^2(\delta+\lambda_A+\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-\frac{h\delta(s-p)}{\delta+\lambda_A}}\cdot e^{-\frac{Hq(\delta+\lambda_A)}{\delta+\lambda_A+\lambda_B}}$$

$$\times e^{\frac{\lambda_B H\delta z}{(\lambda_A+\lambda_B)(\delta+\lambda_A+\lambda_B)}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}}\int_{w=0}^{s-p}e^{(\frac{h\delta}{\delta+\lambda_A}-\frac{(\delta+\lambda_B)h}{\delta+\lambda_A+\lambda_B})w}$$

$$\times\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)w}{(\delta+\lambda_A+\lambda_B)^2}})\,dw\mathbf{1}_{(p,\infty)}(s)\Big]dz. \tag{4.7}$$

***(ii)* Case $\boldsymbol{\delta=\lambda_B}$,**

$$\mathcal{L}_u^{-1}\Big\{\varphi_\mu^2(u,0,0)\Big\}(s)=\Big(\frac{-\lambda_A^2\lambda_B h}{(\lambda_A+\lambda_B)^3}+\frac{\lambda_A^2\lambda_B^2h^2(s-p)}{(\lambda_A+\lambda_B)^4}+\frac{-\lambda_A\lambda_B^2h}{(\lambda_A+\lambda_B)^3}\cdot e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}}$$

$$+\frac{-\lambda_A^2\lambda_B^2h^2(s-p)}{(\lambda_A+\lambda_B)^4}\cdot e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}}\Big)\cdot e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}}\mathbf{1}_{(p,\infty)}(s)$$

$$+\frac{\lambda_A\lambda_B h}{(\lambda_A+\lambda_B)(\lambda_A+2\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-\frac{\lambda_A Hq}{\lambda_A+\lambda_B}}\cdot e^{-\frac{2\lambda_B h(s-p)}{\lambda_A+2\lambda_B}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}})\mathbf{1}_{(p,\infty)}(s)$$

$$
\begin{aligned}
&+\Big(\frac{\lambda_A\lambda_B^2 h}{(\lambda_A+\lambda_B)^3}\cdot e^{-ph}+\frac{-\lambda_A\lambda_B^3 h}{(\lambda_A+\lambda_B)^3(\lambda_A+2\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\Big)\\
&\quad\times e^{-\frac{Hq(\lambda_A+\lambda_B)}{\lambda_A+2\lambda_B}}\cdot e^{-\frac{2\lambda_B h(s-p)}{\lambda_A+2\lambda_B}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHq(s-p)}{(\lambda_A+2\lambda_B)^2}})\mathbf{1}_{(p,\infty)}(s)\\
&+\frac{\lambda_A\lambda_B hH}{(\lambda_A+\lambda_B)^2}\cdot e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}}\int_{z=0}^{q}e^{-\frac{\lambda_A Hz}{\lambda_A+\lambda_B}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz\mathbf{1}_{(p,\infty)}(s)\\
&+\Big(\frac{-2\lambda_A^2\lambda_B^3h^2}{(\lambda_A+\lambda_B)^4(\lambda_A+2\lambda_B)}+\frac{\lambda_A^2\lambda_B^2h^2(\lambda_A+3\lambda_B)}{(\lambda_A+\lambda_B)^4(\lambda_A+2\lambda_B)}\cdot e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}}\Big)\cdot e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}}\\
&\quad\times e^{-\frac{Hq(\lambda_A+\lambda_B)}{\lambda_A+2\lambda_B}}\int_{w=0}^{s-p}e^{(\frac{\lambda_B h}{\lambda_A+\lambda_B}-\frac{2\lambda_B h}{\lambda_A+2\lambda_B})w}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHqw}{(\lambda_A+2\lambda_B)^2}})dw\mathbf{1}_{(p,\infty)}(s)\\
&+\Big(\frac{-\lambda_A^3\lambda_B^3h^3}{(\lambda_A+\lambda_B)^5(\lambda_A+2\lambda_B)}+\frac{\lambda_A^3\lambda_B^3h^3}{(\lambda_A+\lambda_B)^5(\lambda_A+2\lambda_B)}e^{-\frac{\lambda_A hp}{\lambda_A+\lambda_B}}\Big)\\
&\quad\times e^{-\frac{\lambda_B hs}{\lambda_A+\lambda_B}}\cdot e^{-\frac{Hq(\lambda_A+\lambda_B)}{\lambda_A+2\lambda_B}}\\
&\quad\times\int_{w=0}^{s-p}(s-p-w)\cdot e^{(\frac{\lambda_B h}{\lambda_A+\lambda_B}-\frac{2\lambda_B h}{\lambda_A+2\lambda_B})w}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHqw}{(\lambda_A+2\lambda_B)^2}})dw\mathbf{1}_{(p,\infty)}(s)\\
&+\int_{z=0}^{q}\Big[\Big(\frac{-\lambda_A\lambda_B^3hH}{(\lambda_A+\lambda_B)^2(\lambda_A+2\lambda_B)^2}\cdot e^{-\frac{2\lambda_B h(s-p)}{\lambda_A+2\lambda_B}}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)(s-p)}{(\lambda_A+2\lambda_B)^2}})\\
&\qquad+\frac{\lambda_A^2\lambda_B^2h^2H}{(\lambda_A+\lambda_B)(\lambda_A+2\lambda_B)^2}\sqrt{\frac{s-p}{\lambda_A\lambda_B hH(q-z)}}\cdot e^{-\frac{2\lambda_B h(s-p)}{\lambda_A+2\lambda_B}}\\
&\qquad\times\mathcal{I}_1(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)(s-p)}{(\lambda_A+2\lambda_B)^2}})\\
&\qquad+\frac{-\lambda_A^2\lambda_B^2h^2H}{(\lambda_A+\lambda_B)^3(\lambda_A+2\lambda_B)}\cdot e^{-\frac{\lambda_B h(s-p)}{\lambda_A+\lambda_B}}\\
&\qquad\times\int_{w=0}^{s-p}e^{(\frac{\lambda_B h}{\lambda_A+\lambda_B}-\frac{2\lambda_B h}{\lambda_A+2\lambda_B})w}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hH(q-z)w}{(\lambda_A+2\lambda_B)^2}})dw\Big)\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\\
&\quad\times e^{-\frac{Hq(\lambda_A+\lambda_B)}{\lambda_A+2\lambda_B}}\cdot e^{\frac{\lambda_B^2Hz}{(\lambda_A+\lambda_B)(\lambda_A+2\lambda_B)}}\cdot\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})\Big]dz\mathbf{1}_{(p,\infty)}(s).
\end{aligned}\tag{4.8}
$$

5 The Loss Probability

A further special case is to get the probability that player A loses to player B. This can be directly obtained from

$$\varphi_\mu(u,v,\theta)=E\Big[e^{-u\alpha_\mu-v\beta_\mu-\theta t_\mu}\mathbf{1}_{\{\mu<\nu\}}\Big]\tag{5.1}$$

by setting $u=v=\theta=0$:

$$\varphi_\mu(0,0,0):=E[\mathbf{1}_{\{\mu<\nu\}}]=P\{\mu<\nu\}=P\{t_\mu<t_\nu\}.\tag{5.2}$$

With $u = v = \theta = 0$ in (1.8), we have

***(i)* Case $\delta \neq \lambda_A$,**

$$
\begin{aligned}
\varphi_\mu^1(0,0,0) &= \frac{-\lambda_A}{\lambda_A+\lambda_B}\cdot e^{-ph} + \frac{\lambda_A}{\lambda_A+\lambda_B}\cdot e^{-ph}\cdot e^{-q(\frac{H\delta}{\delta+\lambda_B})} \\
&+ \frac{-\lambda_A\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)}\cdot e^{-p(\frac{\lambda_B h}{\lambda_A+\lambda_B})}\cdot e^{-q(\frac{H\delta}{\delta+\lambda_B})} \\
&+ \frac{\lambda_A\delta}{(\lambda_A+\lambda_B)(\delta+\lambda_B)}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}}) \\
&+ \frac{\lambda_A H}{\lambda_A+\lambda_B}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\int_{z=0}^{q} e^{-(\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz \\
&+ \frac{-\lambda_A\lambda_B^2 H}{(\lambda_A+\lambda_B)(\delta+\lambda_B)^2}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-q(\frac{H\delta}{\delta+\lambda_B})} \\
&\times\int_{z=0}^{q} e^{(\frac{H\delta}{\delta+\lambda_B}-\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz.
\end{aligned}
\tag{5.3}
$$

***(ii)* Case $\delta = \lambda_A$,**

$$
\begin{aligned}
\varphi_\mu^2(0,0,0) &= \frac{-\lambda_A}{\lambda_A+\lambda_B}\cdot e^{-ph} + \frac{\lambda_A}{\lambda_A+\lambda_B}\cdot e^{-ph}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})} \\
&+ \frac{-\lambda_A^2}{(\lambda_A+\lambda_B)^2}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})} \\
&+ \frac{\lambda_A^2}{(\lambda_A+\lambda_B)^2}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}}) \\
&+ \frac{\lambda_A H}{\lambda_A+\lambda_B}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\int_{z=0}^{q} e^{-(\frac{\lambda_A H}{\lambda_A+\lambda_B})z}\mathcal{I}_0(2\sqrt{\frac{\lambda_A\lambda_B hHpz}{(\lambda_A+\lambda_B)^2}})dz \\
&+ \frac{-\lambda_A\lambda_B^2 H}{(\lambda_A+\lambda_B)^2}\sqrt{\frac{q}{\lambda_A\lambda_B hHp}}\cdot e^{-\frac{\lambda_B hp}{\lambda_A+\lambda_B}}\cdot e^{-q(\frac{\lambda_A H}{\lambda_A+\lambda_B})}\mathcal{I}_1(2\sqrt{\frac{\lambda_A\lambda_B hHpq}{(\lambda_A+\lambda_B)^2}}).
\end{aligned}
\tag{5.4}
$$

6 Numerical Results

Since the above formulas may look a little bulky, some numerical results can well illustrate them and add to their credibility. They also show how changing input parameters alters the trend of the game. For the full completion of the demonstration we bring here a detailed MATLAB routine, which can be utilized for anyone wanting to run their own input parameters such as λ_A, λ_B, h, H, p, q and δ.

```
%The probability that player A loses to player B when delta is not equal to
%lambda A.
%A=lambda A, B=lambda B, d=delta

syms z
A=18, B=20, h=14, H=16, p=20, q=24, d=500;
```

```
f11=-A/(A+B)*exp(-p*h)+A/(A+B)*exp(-p*h)*exp(-q*H*d/(d+B))
-A*d/((A+B)*(d+B))*exp(-p*B*h/(A+B))*exp(-q*H*d/(d+B))
f12=A*d/((A+B)*(d+B))*exp(-p*B*h/(A+B))*exp(-q*A*H/(A+B))
*double(besseli(0,2*sqrt(A*B*h*H*p*q/(A+B)^2)))
f13=A*H/(A+B)*exp(-p*B*h/(A+B))*double(int(exp(-A*H*z/(A+B))
*besseli(0,2*sqrt(A*B*h*H*p*z/(A+B)^2)),0,q))
f14=-A*B^2*H/((A+B)*(d+B)^2)*exp(-p*B*h/(A+B)) *exp(-q*H*d/(d+B))
*double(int(exp((H*d/(d+B)-A*H/(A+B))*z)
*besseli(0,2*sqrt(A*B*h*H*p*z/(A+B)^2)),0,q))

Probability_A_Loses_B_1=f11+f12+f13+f14

%The probability that player A loses to player B when delta is equal to
%lambda A.
%A=lambda A, B=lambda B

syms z
A=10, B=5, h=24, H=12, p=30, q=25;

f21=-A/(A+B)*exp(-p*h)+A/(A+B)*exp(-p*h)*exp(-q*A*H/(A+B))
-A^2/(A+B)^2*exp(-p*B*h/(A+B))*exp(-q*A*H/(A+B))
f22=A^2/(A+B)^2*exp(-p*B*h/(A+B))*exp(-q*A*H/(A+B))
*double(besseli(0,2*sqrt(A*B*h*H*p*q/(A+B)^2)))
f23=A*H/(A+B)*exp(-p*B*h/(A+B))*double(int(exp(-A*H*z/(A+B))
*besseli(0,2*sqrt(A*B*h*H*p*z/(A+B)^2)),0,q))
f24=-A*B^2*H/(A+B)^2*sqrt(q/(A*B*h*H*p))*exp(-p*B*h/(A+B))
*exp(-q*A*H/(A+B))*double(besseli(1,2*sqrt(A*B*h*H*p*q/(A+B)^2)))

Probability_A_Loses_B_2=f21+f22+f23+f24
```

The program utilizes (5.3) and (5.4) and the calculations are put in the tables below.

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
h	20	18	17	16	14
H	16	16	16	16	16
p	20	20	20	20	20
q	24	24	24	24	24
δ	100	100	100	100	100
Probability of A losing	0.0733	0.3448	0.5591	0.7622	0.9711

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
h	14	14	14	14	14
H	16	14	13	12	10
p	20	20	20	20	20
q	24	24	24	24	24
δ	100	100	100	100	100
Probability of A losing	0.9711	0.7474	0.5070	0.2556	0.0181

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
h	14	14	14	14	14
H	16	16	16	16	16
p	28	26	24	22	20
q	24	24	24	24	24
δ	100	100	100	100	100
Probability of A losing	0.1064	0.3060	0.6027	0.8555	0.9711

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
h	14	14	14	14	14
H	16	16	16	16	16
p	20	20	20	20	20
q	24	24	24	24	24
δ	1	2	4	10	18
Probability of A losing	0.8904	0.9432	0.9605	0.9677	0.9695

λ_A	18	18	18	18	18
λ_B	20	20	20	20	20
h	14	14	14	14	14
H	16	16	16	16	16
p	20	20	20	20	20
q	24	24	24	24	24
δ	50	100	500	1,000	10,000
Probability of A losing	0.9708	0.9711	0.9714	0.9715	0.9715

λ_A	20	20	20	20	20
λ_B	18	18	18	18	18
h	10	12	13	14	16
H	14	14	14	14	14
p	24	24	24	24	24
q	20	20	20	20	20
δ	50	50	50	50	50
Probability of A losing	0.9811	0.7389	0.4864	0.2475	0.0278

λ_A	20	20	20	20	20
λ_B	18	18	18	18	18
h	16	16	16	16	16
H	14	16	17	18	20
p	24	24	24	24	24
q	20	20	20	20	20
δ	50	50	50	50	50
Probability of A losing	0.0278	0.2332	0.4350	0.6498	0.9247

λ_A	20	20	20	20	20
λ_B	18	18	18	18	18
h	16	16	16	16	16
H	14	14	14	14	14
p	24	24	24	24	24
q	20	20	20	20	20
δ	1	2	4	10	20
Probability of A losing	0.0124	0.0175	0.0218	0.0254	0.0269

where

$$\begin{aligned}
\lambda_A^{-1} &= \text{The frequency of strikes to player A by player B,}\\
\lambda_B^{-1} &= \text{The frequency of strikes to player B by player A,}\\
h^{-1} &= \text{The average of magnitude of strikes to player A by player B,}\\
H^{-1} &= \text{The average of magnitude of strikes to player B by player A,}\\
p &= \text{The threshold of player A,}\\
q &= \text{The threshold of player B,}\\
\delta^{-1} &= \text{The observations frequency.}
\end{aligned}$$

Concluding Remarks

In this paper, we continued our studies on fully antagonistic stochastic games of two players (A and B) (initiated in [3]), modeled by two independent marked Poisson processes. We investigated the paths in which player A loses the game. In this paper, we render calculation for a variety of special cases. The latter are presented either as fully explicit Laplace–Stieltjes joint transforms of the exit time and casualties to both players upon the exit or explicit probabilities and probability density functions, mostly in terms of modified Bessel functions. The results are illustrated by many numerical examples, and a MATLAB routine for calculation is attached.

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Delay-Dependent Stability Analysis for Large Scale Production Networks of Autonomous Work Systems

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Abstract: This paper considers the problem of stability analysis for a class of production networks of autonomous work systems with delays in the capacity changes. The system under consideration does not share information between work systems and the work systems adjust capacity with the objective of maintaining a desired amount of local work in progress (WIP). Attention is focused to derive explicit sufficient delay-dependent stability conditions for the network using properties of matrix norm. Finally, numerical results are provided to demonstrate the proposed approach.

Keywords: *stability analysis; production networks; autonomous systems; delay.*

Mathematics Subject Classification (2000): 34K50, 37B55, 39A11, 90C06.

1 Introduction

Production networks are emerging as a new type of cooperation between and within companies, requiring new techniques and methods for their operation and management [1]. Coordination of resource use is a key challenge in achieving short delivery times and delivery time reliability. These networks can exhibit unfavourable dynamic behaviour as individual organizations respond to variations in orders in the absence of sufficient communication and collaboration, leading to recommendations that supply chains should be globally rather than locally controlled and that information sharing should be extensive [2, 3]. However, the dynamic and structural complexity of these emerging networks inhibits collection of the information necessary for centralized planning and control, and

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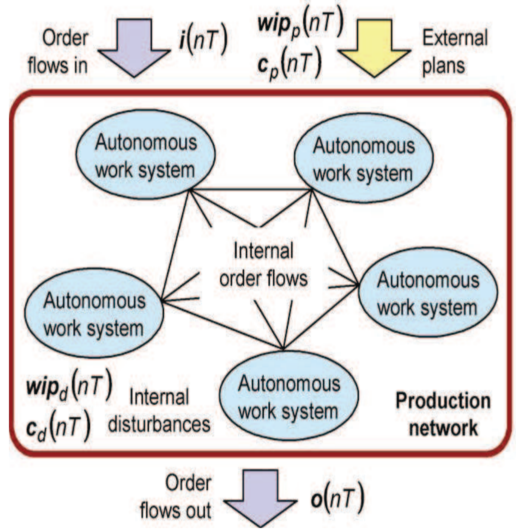

Figure 1.1: Production network consisting of a group of autonomous work systems.

decentralized coordination must be provided by logistic processes with autonomous capabilities [4].

A production network with several autonomous work systems is depicted in Figure 1.1. The behaviour of such a network is affected by external and internal order flows, planning, internal disturbances, and the control laws used locally in the work systems to adjust resources for processing orders [5]. In prior work, sharing of capacity information between work systems has been modelled [6] along with the benefits of alternative control laws and reducing delay in capacity changes [7, 8]. Several authors have described both linear and nonlinear dynamical models for control of variables such as inventory levels and work in progress (WIP), including the use of pipeline flow concepts to represent lead times and production delays [9, 10]. Delivery reliability and delivery time have established themselves as equivalent buying criteria alongside product quality and price (see [1, 11]). High delivery reliability and short delivery times for companies demand high schedule reliability and short throughput times in production. In order to manufacture economically under such conditions, it is necessary to minimize WIP levels in production and utilize operational resources in the best possible way.

Production Planning and Control (PPC) has become more challenging as manufacturing companies adapt to a fast changing market [12–14]. Current PPC methods often do not deal with unplanned orders and other types of turbulence in a satisfactory manner [15]. Assumptions such as infinite capacity and fixed lead time are often made, leading to a static view of the production system may not be valid because WIP affects lead time and performance, while capacity is finite and varies both according to plan and due to unplanned disturbances such as equipment breakdowns, worker illness, market changes etc. Understanding the dynamic nature of production systems requires new approaches for the design of PPC based on company's logistics [16]. The controllers implicitly interact to adjust capacity to eliminate backlog as the system maintains its planned WIP level [15]. A discrete closed-loop PPC model was developed and analyzed by Duffie and

Falu [17] in which two discrete controllers, one for backlog and one for WIP, with different periods between adjustments of work input and capacity, respectively, were selected and evaluated using transfer function analysis and time-response simulation. A second architecture for continuous WIP control and discrete backlog control, with delay capacity adjustment, was developed and analyzed by Ratering and Duffie for cases of high and low WIP [18].

On the other hand, delay differential systems are assuming an increasingly important role in many disciplines like economics, mathematics, science, and engineering. For instance, in economic systems, delays appear in a natural way since decisions and effects are separated by some time interval. The delay effects problem on the stability of systems is a problem of recurring interest since the delay presence may induce complex and undesired behaviors (oscillation, instability, bad performance) for the schemes [19–23]. Over the past few decades, discrete-time systems with time-delay have received little attention compared with its continuous-time counterpart [24–27]. The stability of time-delay systems is a fundamental problem because of its importance in the analysis of such systems. With regard to the stability analysis issue, Verriest and Ivanov in [28] studied the sufficient conditions for the asymptotic stability of the discrete-time state delayed systems by using an algebraic matrix inequality approach. The basic method for stability analysis is the direct Lyapunov method, and by this method, strong results have been obtained. But finding Lyapunov functions for nonautonomous delay difference systems is usually a difficult task. In contrast, many methods different from Lyapunov functions have been successfully applied to establish stability results for difference equations with delay, for example, [29–31]. Recently, in [32] a computational method was presented using Haar wavelets to determine the piecewise constant feedback controls for a finite-time linear optimal control problem of a time-varying state-delayed system.

In this paper, we contribute to the problem of stability analysis for a class of production networks of autonomous work systems with delays in the capacity changes. The system under consideration does not share information between work systems and the work systems adjust capacity with the objective of maintaining a desired amount of WIP. Attention is focused to derive explicit sufficient delay-dependent stability conditions for the network using properties of matrix norm. Finally, numerical results are provided to demonstrate the proposed approach.

2 Model of Autonomous Work Systems

A linear discrete-time dynamic approach for modeling the flow of orders into, out of, and between work systems was chosen because it promotes straightforward calculation of fundamental dynamic properties such as characteristic times and damping. Assume that there are N work systems in a production network, as shown in Figure 1.1, and that vector $i(nT)$ is the rate at which orders are input to the N work systems from sources external to the production network, which is constant over time $nT \leq t < (n+1)T$, where $n = 1, 2, \cdots$, and T is a time period between capacity adjustments (for example, 1 shop-calendar day [scd]). The total orders that have been input to the work systems up to time $(k+1)T$ then can be represented as the vector [5]

$$w_i((n+1)T) = w_i(nT) + T(i(nT) + R^T(nT)c_a(nT)), \tag{2.1}$$

where vector $c_a(nT)$ is the rate at which orders are output from the N work systems during time $nT \leq t < (n+1)T$ (the actual capacity of each work system) and R is a matrix in which element approximates the fraction of the flow out of work system j that flows into work system k.

The total number of orders that have been output by the work systems up to time $nT \leq t < (n+1)T$ can be represented by the vector

$$w_o((n+1)T) = w_o(nT) + Tc_a(nT) \tag{2.2}$$

while the rate at which orders are output from the network during time $nT \leq t < (n+1)T$ is

$$o(nT) = R_o(nT)c_a(nT), \tag{2.3}$$

where $R_o(nT)$ is a diagonal matrix in which non-zero diagonal elements represent the fraction of orders flowing out of work systems that flow out of the network during time $nT \leq t < (n+1)T$. $R_o(nT)$ is assumed to be constant during this period, and

$$R_{o_{ii}}(nT) + \sum_{j=1, j\neq i}^{N} R_{o_{ij}}(nT) = 1, \tag{2.4}$$

$R(nT)$ and $R_o(nT)$ represent the structure of order flow in the network. The WIP in the work systems is

$$wip_a(nT) = w_i(nT) - w_o(nT) + w_d(nT), \tag{2.5}$$

where $w_d(nT)$ represents local work disturbance, such as rush order, that affects the work system. Furthermore, the actual capacity of each work system depends on three components as follows:

$$c_a(nT) = c_p(nT) + c_m((n-d)T) - c_d(nT), \tag{2.6}$$

where $c_d(nT)$ represents local capacity disturbances such as equipment failures, $c_p(nT)$ denotes planned capacities of the work systems and $c_m(nT)$ represents local capacity adjustments to maintain the WIP in each work system in the vicinity of the planned levels $wip_p(nT)$ using gain k_c and is described in the form of

$$c_m(nT) = k_c(wip_a(nT) - wip_p(nT)). \tag{2.7}$$

It is assumed that a delay dT exists in the capacity changes $c_m(nT)$ for logistic reasons such as operator work rules. In this network, the work systems do not share information regarding the expected physical flow of orders between them. A capacity plan is required for each work system. For constants $R(nT)$ and $R_o(nT)$, the transfer functions relating $wip_a(z)$ and $c_a(z)$ to the inputs $i(z)$, $w_d(z)$, $wip_p(z)$, $c_p(z)$ and $c_d(z)$ are:

$$\begin{aligned} wip_a(z) &= ((1-z^{-1})I + k_cT(I-R^T)z^{-(d+1)})^{-1}(Tz^{-1}i(z) + (1-z^{-1})w_d(z) \\ &+k_cT(I-R^T)z^{-(d+1)}wip_p(z) - T(I-R^T)z^{-1}c_p(z) + T(I-R^T)z^{-1}c_d(z)) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} c_a(z) &= ((1-z^{-1})I + k_cT(I-R^T)z^{-(d+1)})^{-1}(k_cTz^{-(d+1)}i(z) + k_c(1-z^{-1})z^{-d}w_d(z) \\ &-k_c(1-z^{-1})z^{-d}wip_p(z) - (1-z^{-1})c_p(z) - (1-z^{-1})c_d(z)). \end{aligned} \tag{2.9}$$

Our purpose is to investigate the stability of the network (2.1)–(2.7) with respect to the delay parameter and the controller gain which is characterize by the roots of

$$det((1-z^{-1})I - Az^{-(d+1)}) = 0 \tag{2.10}$$

with $A = -k_cT(I-R^T)$.

3 Stability Analysis

In this section, sufficient conditions for the stability of the network (2.1)–(2.7) with respect to the delay parameter and the controller gain are proposed using characteristic equation.

The characteristic equation (2.10) can be represented in the form of

$$\det(A + Iz^d - Iz^{d+1}) = 0 \tag{3.1}$$

and (3.1) is corresponding to the characteristic equation of the following system

$$x_n = x_{n-1} + Ax_{n-d-1}. \tag{3.2}$$

Levitskaya in [30] established that (3.2) is asymptotically stable if and only if any eigenvalue of the matrix A lies inside the oval of the complex plane bounded by a curve

$$\Gamma = \left\{ z \in C : z = 2\ i\ sin\left(\frac{\varphi}{2d+1}\right)e^{i\varphi}, |\varphi| \leq \frac{\pi}{2} \right\}. \tag{3.3}$$

Remark 3.1 Let λ_i be eigenvalues of the matrix $A = -k_c T(I - R^T)$. The equation (3.2) is asymptotically stable if and only if

$$|\lambda_i| < 2\ \sin\left(\frac{\pi}{2(2d+1)}\right). \tag{3.4}$$

Theorem 3.1 *If the system* (3.2) *is asymptotically stable, then all eigenvalues of A lie inside the unit disk.*

Proof It is sufficient to consider the stability ovals (3.3) and to remark that $2\,sin(\pi/2(2d+1)) \leq 1$ for $k > 1$. □

In the sequel, we will obtain the necessary and sufficient condition in terms of the eigenvalues location of the matrix A for the asymptotic stability of the equation (3.2).

Lemma 3.1 [29] *If $\sum_{i=1}^{k} \|A_i\| < 1$, then the linear system $x_n = \sum_{i=1}^{k} A_i x_{n-i}$ is asymptotically stable.*

Theorem 3.2 *If*

$$\|A + I\| + d\,\|A\|^2 < 1, \tag{3.5}$$

then (3.2) *is asymptotically stable.*

Proof The equation (3.2) is rewritten as

$$\begin{aligned} x_n &= (A + I)x_{n-1} - A(x_{n-1} - x_{n-d-1}) \\ &= (A + I)x_{n-1} - A\sum_{i=1}^{d}(x_{n-i} - x_{n-i-1}) \\ &= (A + I)x_{n-1} - A\sum_{i=1}^{d} Ax_{n-i-d-1}. \end{aligned} \tag{3.6}$$

According to Lemma 3.1, from (3.6) we conclude (3.5). □

Now, we introduce an additional stability condition for (3.2) depending on whether the delay d is odd or even.

Theorem 3.3 *If*

$$\left\|I + (-1)^d A\right\| + d\,\|A\|\,(2 + \|A\|) < 1, \tag{3.7}$$

then (3.2) *is asymptotically stable.*

Proof If d is even the equation (3.2) is rewritten as

$$\begin{aligned} x_n &= (A+I)x_{n-1} - A(x_{n-1} - x_{n-d-1}) \\ &= (A+I)x_{n-1} - A\sum_{i=1}^{d}(-1)^{i+1}(2Ix_{n-i-1} + Ax_{n-k-i}) \end{aligned} \tag{3.8}$$

and if d is odd we have

$$\begin{aligned} x_n &= (I-A)x_{n-1} + A(x_{n-1} - x_{n-d-1}) \\ &= (I-A)x_{n-1} + A\sum_{i=1}^{d}(-1)^{i+1}(2Ix_{n-i-1} + Ax_{n-k-i}). \end{aligned} \tag{3.9}$$

Similar to the proof of Theorem 3.2, the inequality (3.7) is concluded. □

4 Numerical Results

Consider the case of a supplier of components to the automotive industry and for which production data documents orders are flowing between five work systems over a 162-day period. These work systems and the order-flow structure over this period is illustrated in Figure 4.1. In this network, all order flows are unidirectional; therefore, the fundamental dynamic properties of capacity adjustment in the individual work systems are independent. Then, the internal flow of orders is approximated using the following matrix [5],

$$R = \begin{bmatrix} 0 & 106/341 & 235/341 & 0 & 0 \\ 0 & 0 & 0 & 188/401 & 204/401 \\ 0 & 0 & 0 & 100/236 & 129/236 \\ 0 & 0 & 0 & 0 & 268/295 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

in which element R_{ij} is the total number of orders that went from work system i to work system j divided by the total number of orders that left work system i.

Consider the sampling time $T = 1$ scd. It is clear that the condition in Lemma 3.1 cannot be applied. Applying all of the Theorems derived, the conditions of maximum controller gain for the asymptotic stability of the network are shown in Table 4.1. The result from Table 4.1 guarantees the asymptotic stability of system under consideration.

5 Conclusion

The problem of stability analysis for a class of production networks of autonomous work systems with delays in the capacity changes was investigated in this paper. The system under consideration does not share information between work systems and the work systems adjust capacity with the objective of maintaining a desired amount of local work in progress (WIP). In terms of properties of matrix norm some explicit sufficient delay-dependent stability conditions were derived for the network. Finally, numerical results were provided to demonstrate the proposed approach.

	Theorem 3.1	Theorem 3.2	Theorem 3.3
$d = 1$	1.0000	0.8500	0.8650
$d = 2$	0.6180	0.6250	0.6850
$d = 3$	0.4450	0.4750	0.4875
$d = 4$	0.3473	0.3845	0.3950

Table 4.1: Controller gain k_c w.r.t. d.

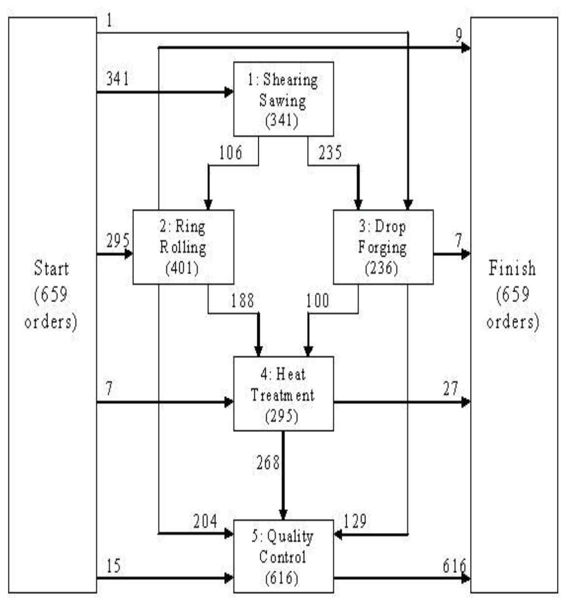

Figure 4.1: A production network consisting of five work systems.

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Appendix

$\|.\|$ is any matrix norm which satisfies the following conditions:

(i) $\|A\| \geq 0$, and $\|A\| = 0$ if and only if $A = 0$,

(ii) for each $c \in \Re$, $\|cA\| = |c| \, \|A\|$,

(iii) $\|A + B\| \leq \|A\| + \|B\|$,

(iV) $\|AB\| \leq \|A\| \, . \, \|B\|$ for all $m \times m$ matrices A, B.

In addition, matrix norm should be concordant with the vector norm $\|.\|_*$, that is,

$$\|Ax\|_* \leq \|A\| \, . \, \|x\|_*$$

for all $x \in \Re^m$ and any $m \times m$ matrix A. For real $m \times m$ matrix A, we define, as usual, $\|A\|_1 = \max\limits_{1 \leq j \leq m} \sum\limits_{i=1}^{m} |a_{ij}|$ and $\|A\|_\infty = \max\limits_{1 \leq i \leq m} \sum\limits_{j=1}^{m} |a_{ij}|$.

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Training a Neural Network Using Hierarchical Genetic Algorithm for Modeling and Controlling a Nonlinear System of Water Level Regulation

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Abstract: In this paper, we present a new approach of Hierarchical Genetics Algorithms (HGA), and the improvement brought compared to the backpropagation algorithm for the simultaneous determination of the structure and the learning of a Multilayer Perceptron (MLP). The neural model found by the two methods are employed separately in a non-linear system for water level regulation. A comparison study will therefore be presented.

Keywords: *hierarchical genetic algorithms; neural networks; backpropagation algorithm; training; multilayer perceptron; optimization; modeling and controlling; non-linear systems.*

Mathematics Subject Classification (2000): 49J35, 34A34 92C20.

1 Introduction

The use of artificial neural network is an approach that has its origins in the study of nervous tissue. In fact, the operation of an artificial neuron is by analogy with that of the nerve cell.

Neural network consists of a set of artificial neurons interconnected by weights whose values affect the behaviours of the whole structure. The rules under which the adjustment operation is carried out connections characterize the learning algorithm of network. Due to the massively parallel structure and ability to reproduce arbitrary behaviours from examples, neural networks are an interesting tool for solving various problems [1–4].

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Since this learning phase is the basis of a good run, we will focus on it. Once the architecture of a neural network has been chosen, it is necessary to make learning to determine the values of weight allowing the output of the neural network to be as close as possible to the target. This learning takes place through the minimization of a function, called cost function, based on examples of the learning basis and the neural network output. This function determines the goal. This minimization can be done through several algorithms called learning algorithms.

In this work, we will mainly focus on the backpropagation algorithm and try to improve their learning process by using a new approach called Hierarchical Genetic Algorithms. This approach will operate to leave the local minima which is the disadvantage of the backpropagation algorithm. Then we move to the implementation of such NN for modeling and controlling an unit of water level regulation. The results of their implementation are compared and the advantage of HGA over backpropagation is released.

2 Description of the Water Level Regulation Unit

The process that will be used throughout the experiments, the block diagram of which is given in Figure 2.1, is made mainly of two tanks ($T1$ and $T2$), a drain valve which is manually controlled, a sensor level placed inside tank $T2$ and a pump controlled directly through computer.

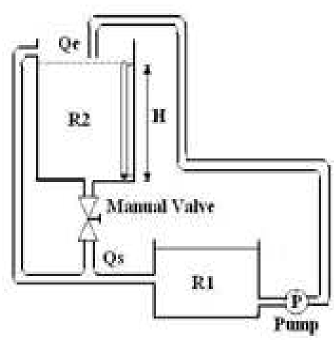

Figure 2.1: Description of the studied process.

The pump draws the liquid in the Tank to be conveyed to Tank $T2$ with a flow rate of $[0, 2.31]l/mn$. Depending on the liquid level H in Tank $T2$, the DC motor, which controls the pump, receives an order to advance the flow of entry Q_e. The command signal of the pump ranges from 0 to $12v$; the conversion from analog to digital signal produces a value between 0 and 255 [8].

3 Presentation of Training Algorithms Used

3.1 Backpropagation algorithm

The learning process of the backpropagation algorithm is an iterative procedure that aims to find the weight of connections minimizing the mean square error cost function J

committed by the network throughout the learning date; the cost function J is given by:

$$J = \frac{1}{2}\sum_{k=1}^{N}[y - y_m]^2 \tag{3.1}$$

with:
N: Number of examples,
k: samples $k, k = 1, 2, \dots, N$,
y: output of the process,
y_m: NN output.

$W1$ define the weight matrix that characterizes the connection between input and hidden layer and $W2$ the matrix of weight between the output layer and hidden layer. The neuronal model is governed by the following equation:

$$S(k) = h(W_2 g(W_1.E(k)) \tag{3.2}$$

with :
$y_m(k)$: output vector number k,
h : activation function of exit neurons layer,
g: activation function of hidden layer,
$E(k)$: entry vector or stimulus number k.
We have:
$S(k) = g(W1.E(k))$
Neural network weights are up to date as following :

$$W_{new} = W_{old} - \mu\frac{\partial J}{\partial W}, \tag{3.3}$$

where μ is the step of learning.
Variation of weight in matrix $W1$ et $W2$ is defined by :

$$\frac{\partial J}{\partial W_2} = \frac{\partial J}{\partial y_m(k)}.\frac{\partial y_m(k)}{\partial E_i(k)}.\frac{\partial E_i(k)}{\partial W_2}, \tag{3.4}$$

$$\frac{\partial J}{\partial W_2} = [y_m(k) - y(k)].\frac{\partial h(E_i(k))}{\partial E_i(k)}.\frac{\partial E_i(k)}{\partial W_2}, \tag{3.5}$$

$$\frac{\partial J}{\partial W_1} = [y_m(k) - y(k)].\frac{\partial h(E_i(k))}{\partial E_i(k)}.\frac{\partial E_i(k)}{\partial S_i}.\frac{\partial g(E_j(k))}{\partial E_j(k)}.\frac{\partial E_j(k)}{\partial W_1}. \tag{3.6}$$

This algorithm remains questionable since its convergence is not proven. Its use can lead to deadlock in a local minimum of the error surface. Its effectiveness depends mostly on a large number of parameters to be fixed by the user: the step gradient, the parameters of sigmoid functions, network architecture (number of layers, number of neurons per layer ...), initialization of weights ...

This learning method has limitations, including:

– The topology of NN must be defined firstly;
– Very sensitive to local minima.

3.2 Hierarchical genetic algorithm

The HGA is used to optimize parameters and topology of NN. The advantage of this approach is that genes of chromosome are classified into two categories (hierarchy). This approach is ideal to represent the relations between:

– NN layers number;
– neurons in hidden layers;
– synaptic weights associated with genes on a chromosome.

We start by building a HGA that selects a structure of a MLP (number of neurons in the hidden layer) and then make learning. Therefore, in contrast with RPG, the HGA will go through several different structures and will make learning. To do so, it will be an evaluation function called fitness (or cost). This function is to minimize the same criterion of the backpropagation algorithm:

$$J = \frac{1}{2}\sum_{k=1}^{N}[y - y_m]^2. \tag{3.7}$$

Figure 3.1 illustrates the principle of operation of the new strategy.

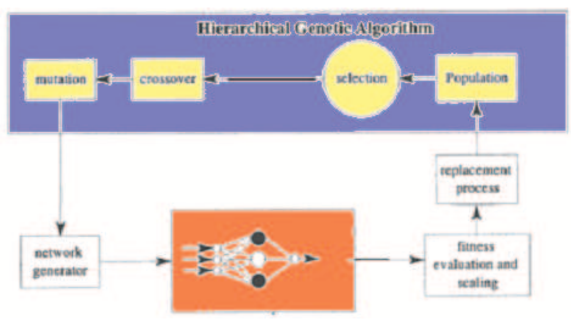

Figure 3.1: Principle of the new strategy.

Concerning the chromosome coding, we used a matrix instead of a vector. The first matrix's line is encoded with a sequence of "0" and "1" which indicate the existence or not of the neuron in the hidden layer. The remaining lines contain the real numbers that represent all the input connections and output neurons in the hidden layer (weights).

Example of a chromosome coding and its equivalent in NN:

Consider the following chromosome (Figure 3.2).

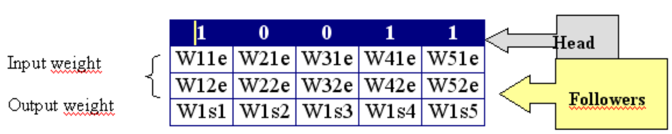

Figure 3.2: The chromosome Code.

Correspondence between Neurons network and chromosome is given by Figure 3.3.

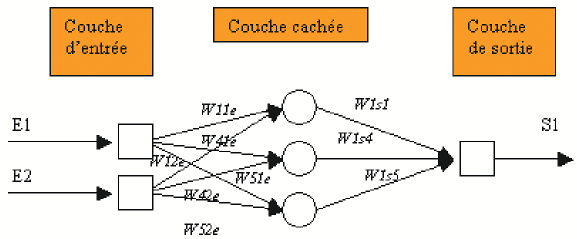

Figure 3.3: Translation of the chromosome toward NN.

Different genetic operators used later will allow the determination of a new structure and a new redistribution of weight.

4 Modeling Process

The use of learning techniques in process control can overcome the difficulties caused by the strong non-linearity. The main interest in neural networks for control is their ability to easily model non-linear systems by learning.

According to the control structure, direct model and inverse model of the process are necessary. We present in this section direct and inverse model of the unit water level regulation. These models will be used in determining the control law based on the internal model [2–4].

To obtain both the direct and inverse model of the regulation water level unit, we have excited the system by a rich signal frequency and with an amplitude that varies between 0 and 2 l/min in order to obtain an output that varies between 0 and 0.4 m. This is the sequence of learning.

We divide the database into two parts,one of which serves to learning and the other in the neuronal model validation, Figure 4.1 shows the sequence that has been used.[8]

4.1 Direct Neural Model (DNM)

The DNM builds a non-linear function that estimates the outputs of the process through old data of its inputs and outputs. In the following (Figure 4.2), a DNM is presented to be used in the sequel. The learning process of a DNM is presented, first through the backpropagation algorithm and secondly by adding our HGA.

For the backpropagation algorithm we are forced to give in advance the architecture of neural network. After several tests, we considered a non-looped network of 2 layers with 2 inputs, 10 hidden neurones in sigmoid activation function and a linear output neuron.

For HGA, we generated randomly some individuals for the first generation in which we injected the backpropagation solution. This injection has the primary effect of prohibit divergence and expulsion of the solution space.

Figure 4.3 shows the generalization error between the actual output of the process and the output of the model developed in the two cases considered. This error has the

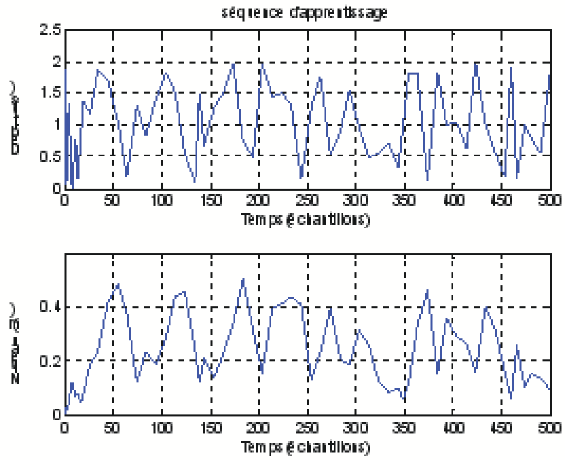

Figure 4.1: Sequence of training and test.

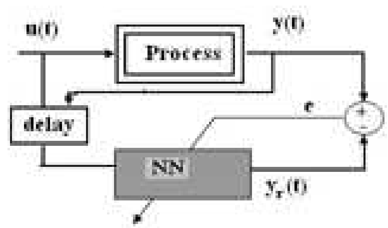

Figure 4.2: Direct model training.

maximum value 0.2 for backpropagation with fluctuations greater than that of HGA. Indeed, we can conclude that the result given by our HGA has considerably improved the backpropagation. So we will keep the direct model developed by our HGA.

4.2 Inverse neural model (INM)

The goal is to identify inverse model parameters through learning process; that is to find weights that render the behavior of the NN as close as possible to the desired control signal. The inverse NN model of the process is built using a NN made of 3 inputs, 10 hidden neurones with a sigmoid activation function and one neurone linear output. The learning algorithm used is the backpropagation algorithm. Most of the algorithms in NN used for learning the inverse neural model, determine the control error (u_{ref}-u_r), that is the difference between the desired reference u_{ref} and the obtained control for the inverse model u_r. For learning the INM of level regulation unit, we applied a technique of direct

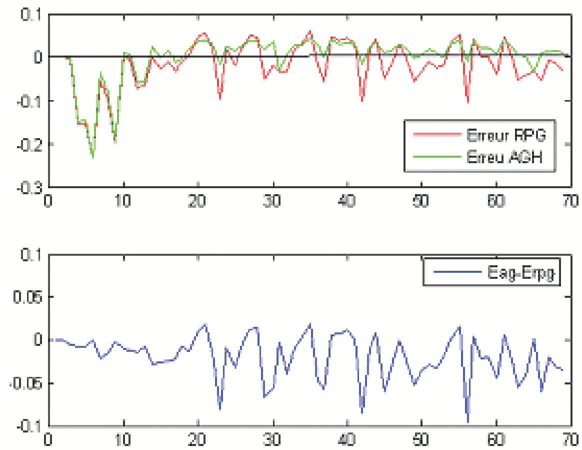

Figure 4.3: Variation of the error for the two methods of direct training.

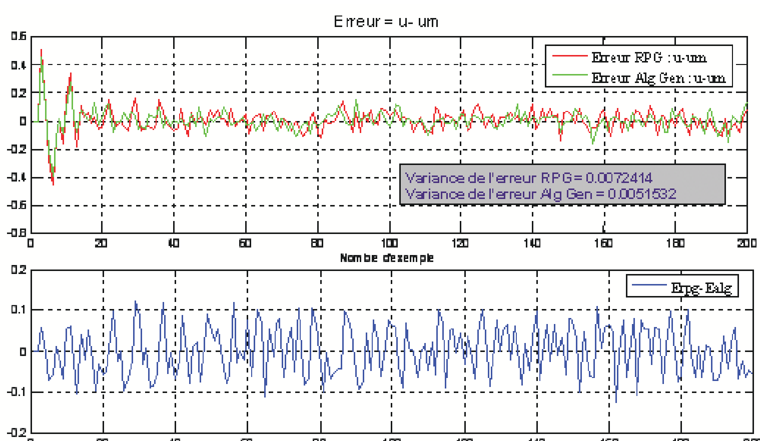

Figure 4.4: Prediction error after the use of backpropagation and HGA.

supervised training which minimizes the following cost criterion:

$$J = \frac{1}{2}\sum_{k=1}^{N}[u_{ref} - u_r]^2 \tag{4.1}$$

with
N: number of samples,
u_{ref} : control signal desired,
u_r : output of neural model.

The learning process of the NN is performed using the backpropagation algorithm. This algorithm assures a convergence to a minimum, it is however worth to notice that this minimum can not be a global one. To overcome this obstruction, we will introduce the HGA.

Validation of the INM.

In order to test the validity of the model immediately after its learning, we apply a sequence of tests to the NN then we compare the resulted tests to the desired output. Figure 4.4 shows the obtained results and the prediction error to evaluate the performance of the NN.

As for the direct model, the HGA has improved the results provided by the back-propagation.

5 System Neural Control

In this section, we aim to control the water-level system regulation by using the generated neural models. We will make use of the INM for the direct control of the inverse model and then we apply simultaneously the INM and DNM for the control of internal model.

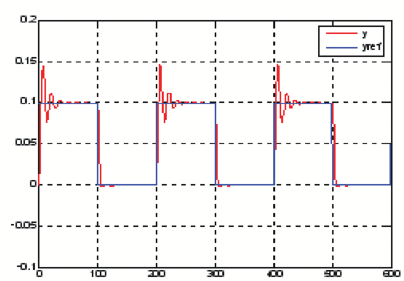

Figure 5.1: Water level control by INM (backpropagation).

5.1 Direct control of the inverse model

The principle of the control law designed for the regulation process of the water-level is to calculate every sampling step of the pump flow to reach the desired level. The inverse model, previously presented, receives reference inputs for the water level, it should then generate the appropriate control law for the pump. In the first experiment, we used the INM that we found after learning process through the method of backpropagation.

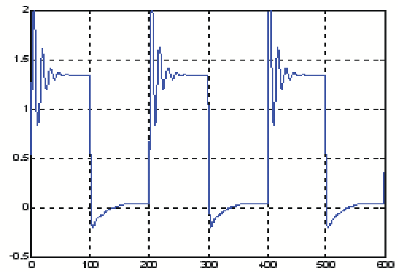

Figure 5.2: Control signal by INM (backpropagation).

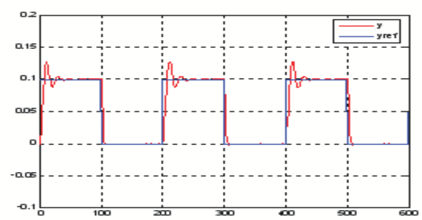

Figure 5.3: Water level control by INM (HGA).

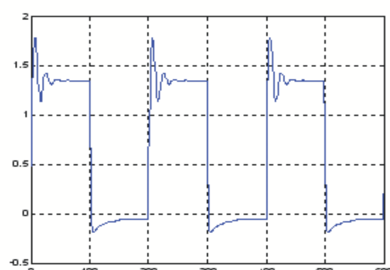

Figure 5.4: Control signal by INM (HGA).

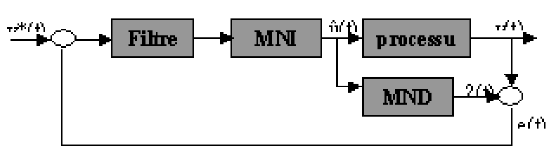

Figure 5.5: IMC structure.

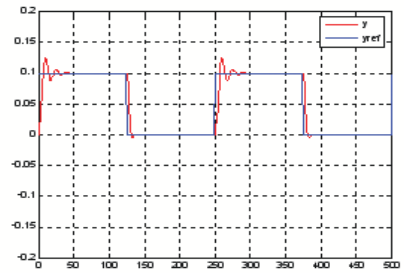

Figure 5.6: Water level control by IMC (backpropagation).

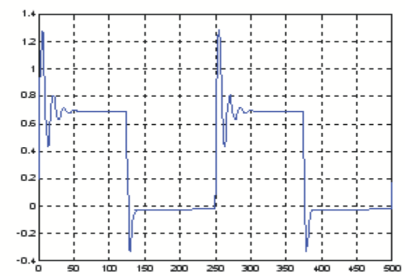

Figure 5.7: Control signal by IMC (backpropagation).

Figure 5.8: Water level control by IMC (HGA).

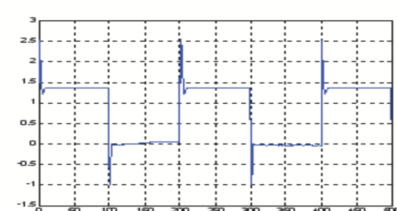

Figure 5.9: Control signal by IMC (HGA).

Figure 5.1 illustrates the evolution of the reference input and the output and Figure 5.2 gives the control signal generated for water level regulation.

In the second experiment, we used the INM that we established after learning process through the HGA. Figure 5.3 illustrates the evolution of the references input and the output and Figure 5.4 shows the control signal generated for the system to regulate water level.

This experience confirms even more improvements brought by the HGA with respect to the results given by the backpropagation algorithm. The first improvement is made in the modeling (a more appropriate model) and the second is confirmed at the control level, overshoot and fluctuations are less important (reduced).

5.2 Internal model control (IMC)

The internal model control structure, applied to our control system requires the use of inverse model as a controller and the direct model as internal model. These models are established in the previous paragraphs and are generated by two different learning algorithms, the backpropagation and HGA. When using the control law based internal model, the controller is placed in cascade with the control system, whereas the direct model is placed in parallel. The block diagram of the control law is shown in Figure 5.5.

We show the results of the simulations for the choice of the filter transfer function bellow: $F(z) = \frac{0{,}2}{z-0{,}8}$. This filter is used to eliminate fluctuations. As the first result, we present the IMC of which the INM and the DNM are generated through learning process using the backpropagation algorithm. Figure 5.6 illustrates the evolution of the reference and the output. Figure 5.7 plots the control signal generated by the IMC. Finally, we apply the inverse model control law which uses the INM and the DNM that are generated by learning through HGA. Figure 5.8 illustrates the evolution of the reference input and the output. Figure 5.9 shows the control signal generated by the IMC for our control system.

We note that the output of the control process in the case of learning through the HGA (Figure 5.8) is better than that of the backpropagation (Figure 5.6). This is evident since the HGA further minimizes the error output $(y*(t) - y(t))$ compared to backpropagation. We can say that the HGA has allowed us to leave the local minimum found by backpropagation. Indeed, we note that the response of the internal model does not oscillate, but small peak on each variation of the reference input.

6 Conclusion

This validation on a real system has allowed us to show that we have achieved our main objective, which is overcoming the defects of the backpropagation algorithm using a new algorithm and approach that is the HGA. Indeed this algorithm can take into account a large number of MLP and make the learning process by implementing its various operators. So we conclude that HGA meets our needs.

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Semilinear Hyperbolic Integrodifferential Equations with Nonlocal Conditions

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Abstract: In this paper, we consider an abstract nonlocal semilinear hyperbolic integrodifferential equation in a Banach space. Using the theory of resolvent operators, we establish the existence and uniqueness of a mild solution under local Lipschitz conditions on the nonlinear maps and an integrability condition on the kernel. The existence of a classical solution of the problem considered is proved under some additional conditions on the nonlinear maps.

Keywords: *hyperbolic problem, integrodifferential equation, nonlocal Cauchy problem, mild and classical solutions.*

Mathematics Subject Classification (2000): 34G20, 35L90, 34A12.

1 Introduction

In the present paper, we study the following semilinear integrodifferential equation with a nonlocal Cauchy problem:

$$u'(t) = A[u(t) + \int_{t_0}^{t} F(t-s)u(s)ds] + f(t, u(t)) + \int_{t_0}^{t} k(t-s)h(s, u(s))ds, \tag{1.1}$$

$$u(t_0) + g(t_1, \cdots, t_n, u(t_1), \cdots, u(t_n)) = u_0 \in E,\ t \in [t_0, T], \tag{1.2}$$

where $t_0 < t_1 < t_2 < \ldots < t_n \leq T$, $(n \in \mathbb{N})$, $A : D(A) := D \subset E \to E$ is a linear operator and generates the strongly continuous semigroup $S(t)$, the nonlinear maps f, h are defined as $f, h : [t_0, T] \times E \to E$, $g : I_T^n \times E^n \to E$, $F(t) \in B(E)$, $t \in I_T$,

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$F(t) : Y \to Y$ where Y is the Banach space $D(A)$, endowed with the graph norm, and the kernel k is defined on $[t_0, T]$ to $\mathbb{R}$.

We consider the following semilinear equation

$$
\begin{aligned}
u'(t) &= Au(t) + f(t, u(t)), \quad t \in [t_0, T], && (1.3)\\
u(t_0) &= u_0, && (1.4)
\end{aligned}
$$

in a Banach space E where $A : D(A) := D \subset E \to E$ is a linear operator and generates a strongly continuous semigroup $\{S(t) : t \geq 0\}$ and $f : [t_0, T] \times E \to E,\ u_0 \in E$ are given. Problem (1.3)-(1.4) is referred to as an abstract initial value problem, or a Cauchy problem. For applications to certain physical problems many researchers, for instance, Byszewski [5], Byszewski and Lakshmikantham [12], Jackson [15] and references therein, have considered the study of the existence and uniqueness of a mild solution and a classical solution to the following nonlocal Cauchy problem

$$
\begin{aligned}
u'(t) &= Au(t) + f(t, u(t)), && (1.5)\\
u(t_0) &+ g(t_1, t_2, \ldots, t_n, u(t_1), u(t_2), \cdots, u(t_n)) = u_0,\ t \in [t_0, T], && (1.6)
\end{aligned}
$$

where $t_0 < t_1 < t_2 < \ldots < t_n \leq T, (n \in \mathbb{N}),\ A$ is the generator of a C_0 semigroup $\{S(t) : t \geq 0\}$ on a Banach space E, $f : [t_0, T] \times E \to E$, and $g : [t_0, T]^n \times E^n \to E$ are the given functions. A possible example for a function g is

$$
g(t_1, t_2, \ldots, t_n, u(t_1), u(t_2), \cdots, u(t_n)) = \sum_{i=1}^{n} c_i u(t_i). \qquad (1.7)
$$

The main advantage to use a nonlocal condition (1.6) is that it may be applied to a physical problem with a better effect than the classical condition (1.4) as (1.6) is generally more practical for the physical measurements as compared to the classical condition (1.4).

Recently Lin and Liu [16] have dealt with the following semilinear integrodifferential equation

$$
\begin{aligned}
&u'(t) = A[u(t) + \int_0^t F(t-s)u(s))ds] + f(t, u(t)), \quad t \in [0, T], && (1.8)\\
&u(0) + g(t_1, \cdots, t_n, u(t_1), \cdots, u(t_n)) = u_0, && (1.9)
\end{aligned}
$$

in a Banach space E with A being the generator of a strongly continuous semigroup and $F(t)$ being a bounded linear operator for $t \in [0, T]$, by generalizing the results of (1.5)-(1.6). We note that the method used to study (1.5)-(1.6) is to first establish the existence of a mild solution using a fixed point theorem when f satisfies a Lipschitz condition in the second argument, where a mild solution is defined to be a solution of the following integral equation:

$$
\begin{aligned}
u(t) &= S(t - t_0)[u_0 - g(t_1, t_2, \ldots, t_n, u(t_1), u(t_2), \cdots u(t_n))]\\
&\quad + \int_{t_0}^{t} S(t-s) f(s, u(s))ds, \quad t_0 \leq t \leq T, \qquad (1.10)
\end{aligned}
$$

with $S(t)$ being the semigroup generated by A. Then a mild solution is shown to be a classic solution if $f \in C^1([t_0, T] \times E,\ E)$.

Similar approach has been used to study (1.8)–(1.9) by Lin and Liu [16] who have shown the existence and uniqueness of a mild solution by showing the existence and uniqueness of a solution of the following integral equation (generally known as variation of constants formula):

$$
\begin{aligned}
u(t) &= R(t)[u_0 - g(t_1, t_2, \dots, t_n, u(t_1), u(t_2), \cdots u(t_n))] \\
&\quad + \int_0^t R(t-s) f(s, u(s)) ds, \quad 0 \le t \le T, \qquad (1.11)
\end{aligned}
$$

where the semigroup $S(t)$ is replaced by the resolvent operator $R(t)$, the counterpart of the semigroup $S(t)$ for the integrodifferential equations. Then a mild solution is shown to be a classical solution if $f \in C^1([0,T] \times E,\ E)$.

For the initial works on existence, uniqueness and stability of various types of solutions of different kinds of differential equations, we refer to [6]–[10] and the references cited in these papers.

Our aim is to use the properties of the resolvent operator $R(t)$ studied in [13]–[17] and the techniques of Pazy [18] and Byszewski [5] for proving the existence, uniqueness, representation of solutions by variation of constants formula.

We first prove the existence and uniqueness of a mild solution to (1.1), using the fixed point argument under a Lipschitz condition on the nonlinear maps and an integrability condition on the kernel k. Where by a mild solution to (1.1) we mean a function $u \in C(I_T, E)$ satisfying the following integral equation

$$
\begin{aligned}
u(t) &= R(t-t_0)[u_0 - g(t_1 \dots t_n, u(t_1), \cdots, u(t_n)] + \int_{t_0}^t R(t-s)[f(s, u(s)) ds \\
&\quad + \int_{t_0}^s k(s-\tau) h(\tau, u(\tau)) d\tau] ds, \quad t \in [t_0, T], \qquad (1.12)
\end{aligned}
$$

where the semigroup $S(t)$ in (1.12) is replaced by the resolvent operator $R(t)$ used in (1.11). Then a mild solution is shown to be a classical solution under certain differentiability condition on the nonlinear maps.

The organization of this paper is as follows. In Section 2, we give some basic results, assumptions on the resolvent operator $R(t)$ and on the variation of parameters formula. Then in Section 3 we will study the nonlocal Cauchy problem (1.1) using the results given in Section 2.

2 Preliminaries and Assumptions

In this section we give some basic definitions, notations and results. Let E be a Banach space with the norm $\|.\|$ and let $t_0 < T \le \infty$, and throughout the paper we denote $[t_0, T]$ by I_T . We will use in this paper the following Banach spaces of functions (endowed with their usual norms):

- $C(I_T; E)$: the space of all continuous functions $u : I_T \to E$.
- $C^n(I_T; E)$: the space of all n times continuously differentiable functions $u : I_T \to E$.
- $L^p(I_T; E)$: the space of all measurable functions $u : I_T \to E$. such that $\|u(.)\| \in L^p(I_T); 1 \le p < \infty$.

In the following, for a linear operator A on a Banach space E, we denote by Y the Banach space $D(A)$ endowed with the graph norm. By $L(E,F)$, we denote the set of all linear operators from E to F. By $B(E)$, we denote the set of all bounded linear operators from E to E itself.

We will make the following assumptions used in [16], [14] and [17] :

(D1) A generates a strongly continuous semigroup in E,

(D2) $F(t) \in B(E),\ t \in I_T,\ F(t) : Y \to Y$ and for $u : I_T \to Y$ continuous, $AF(.)u(.) \in L^1(I_T, E)$. For $u \in E,\ F'(t)u$ is continuous in $t \in I_T$.

Now, we define resolvent operator for (1.1) as follows.

Definition 2.1 (see [17]) R(.) is a resolvent operator of (1.1) with $f,\ g,\ h \equiv 0$ if $R(t) \in B(E)$ for $t \in I_T$ and satisfies

1. $R(0) = I$ (the identity operator on E),
2. for all $u \in E,\ R(t)u$ is continuous for $t \in I_T$,
3. $R(t) \in B(E), t \in I_T$. For $y \in Y, R(.)y \in C^1(I_T, E) \cap C(I_T, Y)$ and

$$\begin{aligned} \frac{d}{dt}R(t-t_0)y &= A[R(t-t_0)y + \int_{t_0}^{t} F(t-s)R(s)yds] \\ &= R(t-t_0)Ay + \int_{t_0}^{t} R(t-s)AF(s)yds,\ t \in I_T. \end{aligned} \tag{2.1}$$

Definition 2.2 $u(., u_0) \in C(I_T, E)$ is a mild solution of (1.1) if it satisfies

$$\begin{aligned} u(t) &= R(t-t_0)[u_0 - g(t_1 \dots t_n, u(t_1), \cdots, u(t_n)] + \int_{t_0}^{t} R(t-s)[f(s,u(s))ds \\ &\quad + \int_{t_0}^{s} k(s-\tau)h(\tau, u(\tau))d\tau]ds, \quad t \in [0,T]. \end{aligned} \tag{2.2}$$

Definition 2.3 A classical solution $u(., u_0)$ of (1.1) is a function $u \in C(I_T, Y) \cap C^1(I_T, E)$ which satisfies (1.1) on I_T.

Now we state here some results about the existence and uniqueness of the resolvent operators, already proved in [16] and [17].

Theorem 2.1 ([16]) *Let* (D1) *and* (D2) *be satisfied. Then* (1.1) *with* $f, g, h \equiv 0$ *has a unique resolvent operator.*

We also state a result about the classical solution to the (1.1) for the particular case, i.e. $f(t,u) \equiv f(t)$.

Theorem 2.2 ([17]) *Let assumptions* (D1) *and* (D2) *be satisfied and assume that* $f(t,u) \equiv f(t),\ g, h \equiv 0,\ u_0 \in D$, *and* $f \in C^1(I_T, E)$. *Then* (1.1) *has a unique classical solution.*

Finally, we state a theorem about the variation of constants formula for (1.1).

Theorem 2.3 ([13],[17]) *Let* $f \in C(I_T,\ E)$ *and* $R(t)$ *be the resolvent operator for* (1.1) *with* $g, h \equiv 0$. *If* u *is a classical solution of* (1.1) *with* $g, h \equiv 0$, *then it satisfies the following integral equation:*

$$u(t) = R(t - t_0)u(t_0) + \int_{t_0}^{t} R(t - s)f(s)ds, \quad t \in I_T. \tag{2.3}$$

This is also known as variation of constant formula for (1.1).

3 Main Results

In this section we give the sufficient conditions for the existence and uniqueness of solutions to (1.1). We first prove the local existence and uniqueness of mild solution to (1.1) under the assumptions (D1)-(D2), $f(t, u)$, $h(t, u)$ are continuous in t and satisfy the certain local Lipschitz condition in u with Lipschitz constants depending on t and $\|u\|_E$ and $k \in L^p(I_T)$, $1 < p < \infty$. Finally, we show that (1.1) has a classical solution provided f and g are continuously differentiable from $I_T \times E \to E$.

We have the following result for a mild solution of (1.1).

Theorem 3.1 *Let* (D1) *and* (D2) *hold. Let* $f, g : I_T \times E \to E$ *be continuous in* t *on* I_T *and satisfy the following conditions.*

(H1) *There exists a constant* $L_1 > 0$ *such that*

$$\|f(t, u) - f(t, v)\| \le L_1\|u - v\|, \ u, v \in E.$$

(H2) *For almost every* $t \in I_T$ *and* $u, v \in E$ *there exists a nonnegative function* $L_2 \in L^P(I_T), 1 < P < \infty$ *such that*

$$\|h(t, u) - h(t, v)\| \le L_2\|u - v\|.$$

(H3) $t_0 < t_1 < t_2 < \cdots < t_n = T, (n \in \mathbb{N})$ *and* $g : I_T^n \times E^n \to E$ *and* $\exists\, G$, *a constant such that*

$$\begin{aligned}\|g(t_1, \cdots, t_n, u(t_1), \cdots, u(t_n)) - g(t_1, \cdots, t_n, v(t_1), \cdots, v(t_n))\| \\ \le G\|u - v\|_{C(I_T,E)}.\end{aligned}$$

(H4) *The real valued map* k *is in* $L^q(0, T)$, *where* $1 < q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

(H5) *The constants* M *and* M_0 *are defined as:*

$$\begin{aligned} M &= \max_{\tau \in I_T} \|R(\tau)\|, \\ M_0 &= M[L_1 + \|k\|_{L^q(I_T)}\|L_2\|_{L^p(I_T)}] \end{aligned} \tag{3.1}$$

and satisfy the following inequality

$$MG + (T - t_0)M_0 < 1. \tag{3.2}$$

Then for every $u_0 \in E$ *the nonlocal semilinear problem* (1.1) *has a unique mild solution* $u \in C(I_T, E)$, *Moreover, the mild solution depends continuously on initial data on* I_T.

Proof We fixed $u_0 \in E$. Define a map $X : C(I_T, E) \to C(I_T, E)$ as:

$$\begin{aligned}(Xu)(t) &= R(t-t_0)[u_0 - g(t_1, t_2, \cdots, t_n, u(t_1), u(t_2), \cdots, u(t_n))] \\ &+ \int_{t_0}^{t} R(t-s)[f(s, u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau, u(\tau))d\tau]ds. \end{aligned} \tag{3.3}$$

Now, for $u, v \in C(I_T, E)$, we have

$$\begin{aligned}\|(Xu)(t) - (Xv)(t)\| &\leq \|R(t-t_0)[g(t_1, t_2, \cdots, t_n, v(t_1), v(t_2), \cdots, v(t_n)) \\ &\quad -g(t_1, t_2, \cdots, t_n, u(t_1), u(t_2), \cdots, u(t_n))]\| \\ &\quad +\| \int_{t_0}^{t} R(t-s)[\{f(s, u(s)) - f(s, v(s))\} \\ &\quad + \int_{t_0}^{s} k(s-\tau)\{h(\tau, u(\tau)) - h(\tau, v(\tau))\}d\tau]ds\| \\ &\leq \|R(t-t_0)\|\|g(t_1, t_2, \cdots, t_n, v(t_1), v(t_2), \cdots, v(t_n)) \\ &\quad -g(t_1, t_2, \cdots, t_n, u(t_1), u(t_2), \cdots, u(t_n))\| \\ &\quad + \int_{t_0}^{t} \|R(t-s)\|[\|f(s, u(s)) - f(s, v(s))\| \\ &\quad +\| \int_{t_0}^{s} k(s-\tau)\{h(\tau, u(\tau)) - h(\tau, v(\tau))\}d\tau\|]ds \end{aligned}$$

$$\|(Xu)(t) - (Xv)(t)\|_E \leq MG + M_0(T-t_0)\|u-v\|_{C(I_T,E)}. \tag{3.4}$$

By (3.2) and the well known extension of the Banach contraction principle X has a unique fixed point $u \in C(I_T, E)$. This u satisfies (2.2) and hence it is a unique mild solution to (1.1) on I_T.

To show the continuous dependence of a mild solution u to (1.1) on the initial data, we will show the Lipschitz continuity of the map $u_0 \to u$. The arguments for this are as follows: Let v be a mild solution of (1.1) on I_T with the initial value $v(t_0) = v_0$, then

$$\|u(t) - v(t)\|_E \leq M(\|u_0 - v_0\|_E - G\|u(t) - v(t)\|_E) + M_0 \int_{t_0}^{t} \|u-v\|_{C(I_s,E)}ds. \tag{3.5}$$

Thus for $\eta \in I_t$, we have

$$\|u(\eta) - v(\eta)\|_E \leq \tilde{M}(\|u_0 - v_0\|_E) + \tilde{M}_0 \int_{t_0}^{\eta} \|u-v\|_{C(I_s,E)}ds \tag{3.6}$$

with $\tilde{M} = \frac{M}{1+MG}$, $\tilde{M}_0 = \frac{M_0}{1+MG}$. Thus taking the supremum over I_t, we have

$$\|u-v\|_{C(I_T,E)} \leq \tilde{M}(\|u_0 - v_0\|_E) + \tilde{M}_0 \int_{t_0}^{t} \|u-v\|_{C(I_s,E)}ds. \tag{3.7}$$

Applying Gronwall's inequality and taking the supremum over I_T, we get

$$\|u-v\|_{C(I_T,E)} \leq \tilde{M} \exp\{\tilde{M}_0 T\}\|u_0 - v_0\|_E. \tag{3.8}$$

The inequality (3.8) proves the uniqueness and continuous dependence of a mild solution to (1.1) on the initial data on I_T. Thus, proof of Theorem (3.1) is complete.

The proof of Theorem 3.1 can be modified to get the following result.

Corollary 3.1 *Let A, f, h and k be as in Theorem* 3.1. *Let $r \in C(I_{\tilde{T}}, E)$. Then the integral equation*

$$w(t) = r(t) + \int_{t_0}^{t} R(t-s)[f(s, w(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau, w(\tau))d\tau]ds, \ t \in I_{\tilde{T}}$$

has a unique solution in $C(I_{\tilde{T}}, E)$.

Now we show that, if we assume the conditions of differentiability on the nonlinear maps f, h, we have the regularity result, which proves the existence and uniqueness of classical solution to (1.1), given as follows.

Theorem 3.2 *Let* (D1),(D2),(H1)–(H2),(H5) *be satisfied. If $f, h : I_T \times E \to E$ are continuously differentiable from their domain into E, $g : I_T^n \times E^n \to E$ satisfies the condition* (H3) *and k is continuous on I_T satisfying* (H4)*, then the mild solution u to* (1.1) *obtained in Theorem* 3.1*, with $u_0 \in D(A)$ is a unique classical solution to* (1.1) *on I_T.*

Proof If f, h are continuously differentiable from $I_T \times E$ into E then for any compact subinterval $I_{\tilde{T}}$ of I_T, f, h are continuous in t on $I_{\tilde{T}}$ and satisfy (H1)-(H2). Therefore, (1.1) has a unique mild solution u on $I_{\tilde{T}}$ such that $u(t_0) = u_0 - g(t_1, t_2, \cdots, t_n, u(t_1), u(t_2), \cdots, u(t_n))$. To show that it is also a classical solution of (1.1), we have to show that u is continuously differentiable on $I_{\tilde{T}}$.
Let

$$B_1(t) = \frac{\partial}{\partial u} f(t, u), \tag{3.9}$$

$$B_2(t) = \frac{\partial}{\partial u} h(t, u), \tag{3.10}$$

$$\begin{aligned} r(t) &= A[R(t-t_0)u(t_0) + \int_{t_0}^{t} F(t-s)R(s)u(t_0)ds] \\ &+ R(t-t_0)f(t_0, u(t_0)) + \int_{t_0}^{t} R(t-s)k(s-t_0)h(t_0, u(t_0)) \\ &+ \int_{t_0}^{t} R(t-s)[\frac{\partial}{\partial s} f(s, u(s)) + \int_{t_0}^{s} k(s-\tau)\frac{\partial}{\partial \tau} h(\tau, u(\tau))d\tau]ds. \end{aligned} \tag{3.11}$$

Consider the integral equation

$$w(t) = r(t) + \int_{t_0}^{t} R(t-s)[B_1(s)w(s) + \int_{t_0}^{s} k(s-\tau)B_2(\tau)w(\tau)d\tau]ds. \tag{3.12}$$

Conditions assumed on f, h imply that r is continuous on $I_{\tilde{T}}$ and $B_i(t)u$ are continuous in t from $I_{\tilde{T}}$ into E and uniformly Lipschitz continuous in u. From Corollary 3.1, it follows that (3.12) has a unique mild solution w on $I_{\tilde{T}}$. Now from the assumption on f and h we have

$$f(s, u(s+\triangle)) - f(s, u(s)) = B_1(s)[u(s+\triangle) - u(s)] + \omega_1(s, \triangle), \tag{3.13}$$

$$h(s, u(s+\triangle)) - h(s, u(s)) = B_2(s)[u(s+\triangle) - u(s)] + \omega_2(s, \triangle), \tag{3.14}$$

$$f(s+\triangle, u(s+\triangle)) - f(s, u(s+\triangle)) = \frac{\partial}{\partial s} f(s, u(s+\triangle)) \triangle + \omega_3(s, \triangle), \tag{3.15}$$

$$h(s+\triangle, u(s+\triangle)) - h(s, u(s+\triangle)) = \frac{\partial}{\partial s} h(s, u(s+\triangle)) \triangle + \omega_4(s, \triangle), \tag{3.16}$$

where $\triangle^{-1}\ \|\omega_i(s,\triangle)\|_E \to 0$ as $\triangle\to 0$ uniformly on $I_{\tilde{T}}$, for $i=1,2,3,4$.
Let

$$w_\triangle(t) = \frac{u(t+\triangle)-u(t)}{\triangle} - w(t),\ t\in I_{\tilde{T}}. \tag{3.17}$$

Then

$$\begin{aligned}
w_\triangle(t) &= [\frac{1}{\triangle}(R(t+\triangle-t_0)u(t_0) - R(t-t_0)u(t_0)) \\
&\quad +A\{R(t-t_0)u(t_0) + \int_{t_0}^{t} F(t-s)R(s)u(t_0)ds\}] \\
&+ [\frac{1}{\triangle}\int_{t_0}^{t_0+\triangle} R(t+\triangle-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds \\
&- R(t-t_0)f(t_0,u(t_0)) - \int_{t_0}^{t} R(t-s)k(s-t_0)h(t_0,u(t_0))ds] \\
&+ \frac{1}{\triangle}[\int_{t_0+\triangle}^{t+\triangle} R(t+\triangle-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds \\
&- \int_{t_0}^{t} R(t-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds] \\
&- \int_{t_0}^{t} R(t-s)[\frac{\partial}{\partial s}f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)\frac{\partial}{\partial\tau}h(\tau,u(\tau))d\tau]ds \\
&- \int_{t_0}^{t} R(t-s)[B_1(s)w(s)) + \int_{t_0}^{s} k(s-\tau)B_2(\tau)w(\tau))d\tau]ds.
\end{aligned} \tag{3.18}$$

Consider

$$\int_{t_0+\triangle}^{t+\triangle} R(t+\triangle-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds. \tag{3.19}$$

Putting $s=\eta+\triangle$ in (3.19), and then replacing η by s, we have

$$\begin{aligned}
&= \int_{t_0}^{t} R(t-\eta)[f(\eta+\triangle,u(\eta+\triangle)) + \int_{t_0}^{\eta+\triangle} k(\eta+\triangle-\tau)h(\tau,u(\tau))d\tau]d\eta. \\
&= \int_{t_0}^{t} R(t-s)[f(s+\triangle,u(s+\triangle)) \\
&\quad + \int_{t_0}^{s+\triangle} k(s+\triangle-\tau)h(\tau,u(\tau))d\tau]ds.
\end{aligned} \tag{3.20}$$

Again, in the inner integral on the right of (3.20), putting $\tau=\gamma+\triangle$ and then replacing γ by τ, we get

$$\begin{aligned}
&\int_{t_0}^{t} R(t-s)[f(s+\triangle,u(s+\triangle)) + \int_{t_0}^{s+\triangle} k(s+\triangle-\tau)h(\tau,u(\tau))d\tau]d\eta \\
&= \int_{t_0}^{t} R(t-s)[f(s+\triangle,u(s+\triangle)) + \int_{t_0-\triangle}^{s} k(s-\tau)h(\tau+\triangle,u(\tau+\triangle))d\tau]ds.
\end{aligned}$$

The last term can be rewritten as

$$\begin{aligned} &= \int_{t_0}^{t} R(t-s)[f(s+\triangle, u(s+\triangle)) + \int_{t_0}^{s} k(s-\tau)h(\tau+\triangle, u(\tau+\triangle))d\tau]ds \\ &+ \int_{t_0}^{t}\int_{t_0-\triangle}^{t_0} R(t-s)k(s-\tau)h(\tau+\triangle, u(\tau+\triangle))d\tau ds. \end{aligned} \tag{3.21}$$

Now, using (3.21) in (3.18) we have

$$\begin{aligned} w_\triangle(t) &= [\frac{1}{\triangle}(R(t+\triangle-t_0))u(t_0) - R(t-t_0)u(t_0) \\ &\quad +A\{R(t-t_0)u(t_0) + \int_{t_0}^{t} F(t-s)R(s)u(t_0)ds\}] \\ &+ [\frac{1}{\triangle}\int_{t_0}^{t_0+\triangle} R(t+\triangle-s)[f(s,u(s) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds \\ &- R(t-t_0)f(t_0,u(t_0)) - \int_{t_0}^{t} R(t-s)k(s-t_0)h(t_0,u(t_0))ds] \\ &+ \frac{1}{\triangle}\{[\int_{t_0}^{t} R(t-s)[f(s+\triangle, u(s+\triangle)) \qquad (3.22) \\ &\quad +\int_{t_0}^{s} k(s-\tau)h(\tau+\triangle, u(\tau+\triangle))d\tau]ds \\ &- \int_{t_0}^{t} R(t-s)[f(s,u(s+\triangle) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau+\triangle))d\tau]ds] \\ &+ [\int_{t_0}^{t} R(t-s)[f(s,u(s+\triangle) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau+\triangle)d\tau]ds \\ &- \int_{t_0}^{t} R(t-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds] \\ &+ \int_{t_0}^{t}\int_{t_0}^{t_0-\triangle} R(t-s)k(s-\tau)h(\tau+\triangle, u(\tau+\triangle))d\tau ds\} \\ &- \int_{t_0}^{t} R(t-s)[\frac{\partial}{\partial s}f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)\frac{\partial}{\partial \tau}h(\tau,u(\tau))d\tau]ds \\ &- \int_{t_0}^{t} R(t-s)[B_1(s)w(s)) + \int_{t_0}^{s} k(s-\tau)B_2(\tau)w(\tau))d\tau]ds. \end{aligned} \tag{3.23}$$

Now, using (3.13)-(3.16) in (3.23) and readjusting the terms, we have

$$\begin{aligned} w_\triangle(t) &= [\frac{1}{\triangle}(R(t+\triangle-t_0))u(t_0) - R(t-t_0)u(t_0) \\ &\quad +A\{R(t-t_0)u(t_0) + \int_{t_0}^{t} F(t-s)R(s)u(t_0)ds\}] \\ &\quad +[\frac{1}{\triangle}\int_{t_0}^{t_0+\triangle} R(t+\triangle-s)[f(s,u(s+\triangle)) \end{aligned}$$

$$
\begin{aligned}
&+ \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds - R(t-t_0)f(t_0,u(t_0))] \\
&+\frac{1}{\triangle}\int_{t_0}^{t} R(t-s)[\omega_1(s,\triangle)+\omega_3(s,\triangle) \\
&+\int_{t_0}^{s} k(s-\tau)\{\omega_2(s,\triangle)+\omega_4(s,\triangle)\}d\tau]ds \\
&+\int_{t_0}^{t} R(t-s)[\{\frac{\partial}{\partial s}f(s,u(s+\triangle))-\frac{\partial}{\partial s}f(s,u(s))\} \qquad (3.24)\\
&+\int_{t_0}^{s} k(s-\tau)\{\frac{\partial}{\partial \tau}h(\tau,u(\tau+\triangle))-\frac{\partial}{\partial \tau}h(\tau,u(\tau))\}d\tau]ds \\
&-\int_{t_0}^{t} R(t-s)[\frac{1}{\triangle}\int_{t_0-\triangle}^{t_0} k(s-\tau)h(\tau+\triangle,u(\tau+\triangle))d\tau + k(s-t_0)h(t_0,u(t_0))]ds \\
&+\int_{t_0}^{t} R(t-s)[B_1(s)w_\triangle(s)+\int_{t_0}^{s} k(s-\tau)B_2(\tau)w_\triangle(\tau)d\tau]ds. \qquad (3.25)
\end{aligned}
$$

Since the norms in E of all but the term in the last line of (3.25) tend to zero as $\triangle\to 0$, we have

$$\|w_\triangle\|_{C(I_t,E)} \leq \epsilon(\triangle) + D(\tilde{T})\int_{t_0}^{t}\|w_\triangle\|_{C(I_s,E)}ds, \qquad (3.26)$$

where $\epsilon(\triangle)\to 0$ as $\triangle\to 0$ and

$$D(\tilde{T}) = \max\{\|R(t-s)\|_{B(E)}[\|B_1(s)\|_{B(E)} + \|k\|_{L^p(I_T)}\|B_2(s)\|_{B(E)}] : s\in I_{\tilde{T}}\}.$$

Applying Gronwall's inequality in (3.26), we obtain

$$\|w_\triangle\|_{C(I_t,E)} \leq \epsilon(\triangle)\exp\{D(\tilde{T})\tilde{T}\}. \qquad (3.27)$$

Therefore $\|w_\triangle(t)\|_{E)} \to 0$ as $\triangle\to 0$. Hence u is differentiable on $I_{\tilde{T}}$ and its derivative is w on $I_{\tilde{T}}$. Since $w\in C(I_{\tilde{T}},E),\ u\in C^1(I_{\tilde{T}},E)$. Finally, to show that u is the required classical solution of problem (1.1), assumptions on f,h and $u\in C^1(I_{\tilde{T}},E)$ imply that the maps $s\to f(s,u(s))$ and $s\to h(s,u(s))$ are continuously differentiable on $I_{\tilde{T}}$.

$$
\begin{aligned}
v(t) &= R(t-t_0)[u_0 - g(t_1,t_2,\cdots,t_n,u(t_1),u(t_2),\cdots,u(t_n))] \\
&+ \int_{t_0}^{t} R(t-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds, \\
v(t_0) &= u_0 - g(t_1,t_2,\cdots,t_n,u(t_1),u(t_2),\cdots,u(t_n)) \qquad (3.28)
\end{aligned}
$$

is a unique solution to

$$\frac{dv}{dt} = Av(t) + f(t,u(t)) + \int_{t_0}^{t} k(t-s)h(s,u(s))ds, t\in I_{\tilde{T}}. \qquad (3.29)$$

By definition, u is a mild solution to (3.29) on $I_{\tilde{T}}$. By uniqueness of a mild solution to (3.29), we have $u=v$ on $I_{\tilde{T}}$. Thus u satisfies (3.29) and therefore u is a unique classical solution (1.1) on $I_{\tilde{T}}$. Since $\tilde{T},\ t_0<\tilde{T}<T$, arbitrary, u is a classical solution to (1.1) on I_T. This completes the proof.

4 Modified Results

In this section, we study the special case when $\|R(t)\|_{B(E)} \leq Me^{-\alpha t}, 0 \leq t \leq T$ for some constant $\alpha \geq 0$ and when the nonlocal condition (1.2) is given by (1.7). Since we are going to assume a weaker condition and so we hope to get improved conditions in assumption (3.2) in Theorem 3.1. To find those improved conditions we move as follows: We first prove the existence and uniqueness of mild solution $u(.,v)$ of Cauchy problem

$$u'(t) = A[u(t) + \int_o^t F(t-s)u(s)ds] + f(t,u(t)) + \int_0^t k(t-s)h(s,u(s))ds,$$
$$u(0) = v, \quad 0 \leq t \leq T.$$

for any $v \in E$, and then we define an operator along the curve of $u(.,v)$ and show that the operator is a contraction, and finally conclude that operator gives rise to a mild solution of (1.1)-(1.2) by finding its fixed point.

To prove the desired result we need to assume the following (see [16]):

(H6) For some constant $\alpha > 0$, the resolvent operator of (1.1) with $f \equiv 0$ satisfies

$$\|R(t)\|_{B(E)} \leq Me^{-\alpha t}, \quad 0 \leq t \leq T. \tag{4.1}$$

(H7) Nonlocal condition (1.2) is given by (1.7) and

$$\beta \equiv \alpha - M_0 > 0, \quad M\sum_{i=1}^{p}|c_i|e^{-\beta(t_i-t_0)} < 1. \tag{4.2}$$

(M_0 from 3.1, α, M from 4.1.)

Remark 4.1 Note that conditions given in (H7) are better than conditions given in (H5) in some situations.

We need the following inequality to find our results.

Lemma 4.1 [11] *Let $u(t)$ and $b(t)$ be non negative continuous functions for $t \geq \alpha$, and let*

$$u(t) \leq ae^{-\gamma(t-\alpha)} + \int_\alpha^t e^{-\gamma(t-s)}b(s)u(s)ds, t \geq \alpha, \tag{4.3}$$

where $\alpha \geq 0$ and γ are constants. Then

$$u(t) \leq ae^{(-\gamma(t-\alpha)+\int_\alpha^t b(s)ds)}, \quad t \geq \alpha, \tag{4.4}$$

We will use Lemma 4.1 to prove the uniqueness of a mild solution of (1.1)-(1.2). The result is as follows:

Theorem 4.1 *Let assumptions* (H1)–(H4), (H6) *and* (H7) *be satisfied. Then for every $u_0 \in E$* (1.1)–(1.2) *has a unique mild solution.*

Proof Let $u_0 \in E$ be fixed. Then for any $v \in E$, define an operator $X : C([0,T],E) \to C([0,T],E)$ by

$$(Xu)(t) = R(t-t_0)v + \int_{t_0}^{t} R(t-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)h(\tau,u(\tau))d\tau]ds. \quad (4.5)$$

Then for $u, w \in C([0,T],E), t \in [0,T]$, we have

$$\begin{aligned}
\|(Xu)(t) - (Xw)(t)\| &\leq \| \int_{t_0}^{t} R(t-s)[\{f(s,u(s)) - f(s,w(s))\} \\
&\quad + \int_{t_0}^{s} k(s-\tau)\{h(\tau,u(\tau)) - h(\tau,w(\tau))\}d\tau]ds\| \\
&\leq \int_{t_0}^{t} \|R(t-s)\|[\|f(s,u(s)) - f(s,w(s))\| \\
&\quad + \| \int_{t_0}^{s} k(s-\tau)\{h(\tau,u(\tau)) - h(\tau,w(\tau))\}d\tau\|]ds
\end{aligned}$$

and hence

$$\|(Xu)(t) - (Xw)(t)\|_E \leq (M_0(t-t_0))\|u-w\|_{C(I_T,E)}. \quad (4.6)$$

with M_0, defined in (3.1). Using (4.5) and repeated application of the inequality (4.6), we have

$$\|(X^n u)(t) - (X^n w)(t)\|_E \leq \frac{[M_0(t-t_0)]^n}{n!}\|u-w\|_{C(I_T,E)}. \quad (4.7)$$

Therefore, we have

$$\|(X^n u) - (X^n w)\|_{C(I_T,E))} \leq \frac{[M_0 T - t_0)]^n}{n!}\|u-w\|_{C(I_T,E)}. \quad (4.8)$$

For n large enough $\frac{[M_0(T-t_0)]^n}{n!} < 1$ and by the well known extension of the Banach contraction principle X has a unique fixed point $u(.,v)$. This u satisfies (2.2) and hence it is a unique mild solution to (1.1) on I_T with $u(t_0) = v$.

Next, define an operator $X_1 : E \to E$ by

$$X_1 v = u_0 - \sum_{i=1}^{p} c_i u(t_i), \quad (4.9)$$

where $u(.) = u(.,v)$ is the unique fixed point of (4.5). Let $u_i(.) = u_i(.,v_i),\ i = 1,2,$ be the unique fixed point of (4.5) with $u_i(t_0) = v_i$. Now, we have:

$$\|X_1 v_1 - X_1 v_2\|_E \leq \sum_{i=1}^{p} |c_i|(\|u_1(t_i) - u_2(t_i)\|_E. \quad (4.10)$$

Let $w(.) \equiv u_1(.) - u_2(.)$, then we can rewrite (4.5) as

$$
\begin{aligned}
\|w(t)\|_E &\leq \|R(t-t_0)\|_{B(E)}\|v_1 - v_2\|_E \\
&\quad + \int_{t_0}^{t} \|R(t-s)\|[\|f(s,u_1(s)) - f(s,u_2(s))\| \\
&\quad + \|\int_{t_0}^{s} k(s-\tau)\{h(\tau,u_1(\tau)) - h(\tau,u_2(\tau))\}d\tau\|]ds \\
&\leq M\|v_1 - v_2\|_E e^{-\alpha(t-t_0)} + \int_{t_0}^{t} M_0 e^{-\alpha(t-s)}\|u_1(s) - u_2(s)\|_E ds \\
&\leq M\|v_1 - v_2\|_E e^{-\alpha(t-t_0)} + \int_{t_0}^{t} M_0 e^{-\alpha(t-s)}\|w(s)\|_E ds, \quad t \in [t_0, T].
\end{aligned}
$$

Thus by the lemma (4.1),

$$
\begin{aligned}
\|w(t)\|_E &\leq M\|v_1 - v_2\|_E e^{-(\alpha - M_0)(t-t_0)} && (4.11) \\
&= M\|v_1 - v_2\|_E e^{-\beta(t-t_0)}, \quad t \in [t_0, T]. && (4.12)
\end{aligned}
$$

By use of (4.12), we can rewrite (4.10) as:

$$
\|X_1 v_1 - X_1 v_2\|_E \leq (M \sum_{i=1}^{p} |c_i| e^{-\beta(t_i - t_0)} \|v_1 - v_2\|_E. \tag{4.13}
$$

By $(H7)$, X_1 is a contraction operator on E and so X_1 has a unique fixed point $v_0 \in E$.
So, for the unique fixed $u(., v_0)$ of (4.5) with to $u(t_0) = v_0$, we obtain

$$
u(t_0, v_0) = v_0 = u_0 - \sum_{i=1}^{p} c_i u(t_i, v_0). \tag{4.14}
$$

This implies that

$$
u(t, v_0) = R(t-t_0)[u_0 - \sum_{i=1}^{p} c_i u(t_i, v_0)] + \int_{t_0}^{t} R(t-s) f(s, , u(s, v_0)) ds, \quad t \in [t_0, T],
$$

and hence, $u(., v_0)$ is a mild solution of (1.1-1.2). At last, we show that mild solutions of (1.1)-(1.2) are unique. Since if $u(.)$ is a mild solution of (1.1)-(1.2) with (1.2) given by (1.7), then

$$
u(t_0) = u_0 - \sum_{i=1}^{p} c_i u(t_i)
$$

and $u(.)$ is also the mild solution of (1.1) with $v = u(t_0)$.

However, X_1 is the contraction map operator and so (4.7) implies that $u(t_0)$ is uniquely determined by X_1. X is also a contraction operator and fixed point of (4.5) is uniquely determined by $v = u(t_0)$. Therefore, it is clear that mild solutions of (1.1) with (1.7) are unique. This completes the proof.

Similar to Theorem 3.1, we have the following result for the classical solution provided that $f, h : I_T \times E \to E$ are continuously differentiable.

Theorem 4.2 *Let the assumptions* (H1)-(H4), (H6) *and* (H7) *be satisfied and let* $u(.)$ *be the unique mild solution of* (1.1) *and* (1.2) *guaranteed by Theorem* 3.1 *with* (1.2) *being given by* (1.7). *Assume further that*

$$u_0 \in D(A), \quad \sum_{i=1}^{p} c_i u(t_i) \in D(A), \quad f, h \in C^1([t_0, T] \times E, E). \tag{4.15}$$

Then $u(.)$ *gives rise to a unique classical solution of* (1.1) *with* (1.7).

5 Application

1. The case in which $k, F, g \equiv 0$ was considered by I. Segal [4]. Different forms of solutions in this particular case have been considered by Pazy [18] and R. H. Martin [3]. The case when $k, F \equiv 0$ was considered by Ludwik Byszewski [5]. The case when $F, g \equiv 0$ has been considered in D. Bahuguna [2]. Also in the case when $F \equiv 0$, existence and uniqueness results for a solution to 1.1, have been analyzed in [1]. Therefore, our results presented here for the problem (1.1) generalizes the results given in [1]–[5] and [18].

2. Consider the following integrodifferential equation termed as classical heat equation for a material with a memory. Let u be the internal energy and

$$f(t, u(t, x)) + \int_{t_0}^{t} k(t-s)h(s, u(s, x))ds$$

be the external heat with

$$\begin{cases} \alpha(t,x) = -u_x(t,x) - \int_{t_0}^{t} b(t-s)u_x(s,x)ds, \quad \text{Heat flux} \\ u_t(t,x) = \frac{\partial}{\partial x}\alpha(t,x) + f(t,u(t,x)) \\ \qquad\qquad + \int_{t_0}^{t} k(t-s)h(s,u(s,x))ds, \quad \text{Balance equation} \\ u(t_0,x) + \sum_{i=1}^{n} u(t_i,x) = u_0(x). \end{cases} \tag{5.1}$$

We can rewrite (5.1) as

$$\begin{aligned} u_t(t,x) &= \frac{\partial^2}{\partial x^2}[u(t,x) + \int_{t_0}^{t} b(t-s)u(s,x)ds] + f(t,u(t,x)) \\ &\quad + \int_{t_0}^{t} k(t-s)h(s,u(s,x))ds, \quad (t,x) \in [t_0,T] \times [0,1], \qquad (5.2) \\ u(t_0,x) &+ \sum_{i=1}^{n} u(t_i,x) = u_0(x). \end{aligned}$$

This is of the type of (1.1) with $A = \frac{\partial^2}{\partial x^2}$ on $H^2[0,1] \cap H_0^1[0,1]$ which generates the strongly continuous semigroup on $L^2[0,1]$ and $b(t)$ is a continuous function. It can be verified that the conditions of Theorem 3.1 are satisfied and thus our analysis ensures existence and uniqueness of a solution to (5.1).

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A Short Note on Semilinear Elliptic Equations in Unbounded Domain

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Abstract: Let $\Omega \subset \mathbb{R}^n$ be a domain (not necessarily bounded) with smooth boundary $\partial\Omega$. Let $1 \le n \le 6$ and $f \in C^{0,\alpha}(\overline{\Omega}) \cap L^2(\Omega)$ be a given function with $f < 0$. In the present study, we prove that the following BVP

$$-\Delta u = u^2 + f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

has a solution $u \in H_0^1(\Omega)$ and satisfies $u \le 0$ in Ω.

Keywords: *monotone iteration method; maximum principle; unbounded domain.*

Mathematics Subject Classification (2000): 35J60, 35J25.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain (i.e open and connected) with smooth boundary $\partial\Omega$. Let $1 \le n \le 6$ and $f \in C^{0,\alpha}(\overline{\Omega}) \cap L^2(\Omega)$ be a nonzero given function. We consider the BVP

$$\begin{aligned} -\Delta u &= u^2 + f \quad \text{in } \Omega, & (1.1)\\ u &= 0 \quad \text{on } \partial\Omega. & (1.2) \end{aligned}$$

The variational or the weak formulation of (1.1) and (1.2) is to find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u.\nabla v = \int_\Omega u^2 v + \int_\Omega f v, \quad \text{for all } v \in H_0^1(\Omega). \tag{1.3}$$

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For a bounded domain Ω and $f < 0$, monotone iteration technique has been used to prove the existence of a solution of (1.1) and (1.2). For details of the proof, we refer to the book by Kesavan, S. [3, p.227]. The Rellich–Kondrasov theorem has been used. The present study deals with the existence of a solution of the BVP when Ω is an unbounded domain with smooth boundary.

We assume $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, where Ω_i is a bounded domain with smooth boundary $\partial\Omega_i$, for $i = 1, 2, 3....$ with $\Omega_i \subseteq \Omega_{i+1}$. Before we proceed, as a consequence of embedding theorem, we note that $u^2 \in L^{3/2}(\Omega)$ if $u \in H_0^1(\Omega)$ for $n \le 6$, and so

$$\Big| \int_\Omega u^2 v \Big| \ \le\ |u|_{0,3,\Omega}^2 |v|_{0,3,\Omega} \ \le\ c\ \|u\|_{1,\Omega}^2 \|v\|_{1,\Omega},$$

which shows that $u \in H^{-1}(\Omega)$. Here c is a generic constant, $\|.\|_{1,\Omega}$ denotes the norm in $H_0^1(\Omega)$ and $|.|_{0,\Omega}$ denotes the norm in $L^2(\Omega)$. Hence, the integrals on the right side of (1.3) exist.

2 The Main Results

Let G be a bounded domain in $\mathbb{R}^n$, $n \le 6$, with smooth boundary ∂G, $f \in C^{0,\alpha}(\overline{G}) \cap L^2(\Omega)$ and $f < 0$. Let $w \in H_0^1(G)$ be the smooth solution of

$$-\Delta u = f \ \text{ in } G, \quad u = 0 \ \text{ on } \partial G. \tag{2.1}$$

The following result is proved in the book [3, p. 227].

Lemma 2.1 *Let G be a bounded domain in $\mathbb{R}^n$, $n \le 6$ with smooth boundary ∂G . Let $f \in C^{0,\alpha}(\overline{G}) \cap L^2(\Omega)$ with $f < 0$. Then, there exists, $u \in H_0^1(G)$ satisfying*

$$\int_G \nabla u . \nabla v = \int_G u^2 v + \int_G f v, \ \textit{for every } v \in H_0^1(G)$$

such that $w \le u \le 0$ in G and

$$\|u\|_{1,G} \le c\ (|f|_{0,G} + |w|_{0,G}). \tag{2.2}$$

Remark 2.1 By Lemma 9.17 [1, p. 242], we obtain $|w|_{0,G} \le c\ |f|_{0,G}$, where c depends only on n and G and (2.2) reduces to

$$\|u\|_{1,G} \le c|f|_{0,G}, \tag{2.3}$$

where $c > 0$ depends on n and G only. Let $n \le 6$ and $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, where Ω_i is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega_i$, for each $i \ge 1$. Let $f \in C^{0,\alpha}(\overline{\Omega}) \cap L^2(\Omega)$. By Lemma 2.1, for $i \ge 1$ there exists a sequence u_i such that

$$\int_{\Omega_i} \nabla u_i . \nabla v_i = \int_{\Omega_i} u_i^2 v_i + \int_{\Omega_i} f_i v_i, \ \text{for all } v_i \in H_0^1(\Omega_i), \tag{2.4}$$

$$\|u_i\|_{1,\Omega_i} \le c|f|_{0,\Omega}, \tag{2.5}$$

where c depends on Ω_i and n. Here $f_i = f|_{\Omega_i}$ is the restriction of f on $\Omega_i, i \ge 1$.

With these preliminaries, we have the main result stated below.

Theorem 2.1 *Let* $\Omega = \bigcup_{i=1}^{\infty} \Omega_i,\ \Omega_i \subseteq \Omega_{i+1}$ *be open bounded domains in* Ω*. We suppose that* $f \in L^2(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$*. Then* (1.1) *and* (1.2) *have a solution* $u \in H_0^1(\Omega)$ *with* $u \leq 0$ *a.e in* Ω*.*

Proof A part of the following proof is similar to the one found in [2]. Let M be any fixed (but arbitrary) bounded domain such that $\overline{M} \subseteq \Omega$. Then there exists an integer i such that $\overline{M} \subseteq \Omega_j$ for $j \geq i$. Let $\tilde{u}_j$ (for $j \geq i$) denote the extension of u_j by zero outside Ω_j. We continue to denote $\tilde{u}_j$ by u_j. By a lemma [4, p.124] and (2.4),(2.5) of Remark 2.1, there exists a positive constant $k > 0$ depending on α, n and M such that

$$\|u_j\|_{H_0^1(M)} \leq k, \text{ for all } j \geq i,$$

where k is independent of $j \geq i$. Since $\{u_j\}$ is a bounded sequence in $H_0^1(M)$, $\{u_j\}$ has a weakly convergent subsequence in $H_0^1(M)$, which we still denote by $\{u_j\}$. A little computation shows that $u_j^2 \rightharpoonup u^2$ weakly in $H_0^1(M)$. Since M is an arbitrary bounded domain in Ω, we have

$$\int_{\Omega} \nabla u.\nabla v = \int_{\Omega} u^2 v + \int_{\Omega} f v, \text{ for every } v \in H_0^1(\Omega)$$

which completes the proof of the theorem. □

Remark 2.2 When $f > 0$, (1.1) and (1.2) may not admit a solution even in bounded domain. We refer to example 5.4.1 in the book [3, p. 230].

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The Boundedness of Solutions to Nonlinear Third Order Differential Equations

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Abstract: In this paper, we establish some new sufficient conditions under which all solutions of nonlinear third order differential equations of the form

$$x''' + \psi(x, x')x'' + f(x, x') = p(t, x, x', x'')$$

are bounded. For illustrations, an example is also given on the bounded solutions.

Keywords: *nonlinear differential equations; third order; boundedness of solutions; Lyapunov's second method.*

Mathematics Subject Classification (2000): 34D20.

1 Introduction

In a recent paper, Omeike [5] considered the following nonlinear third order differential equation:

$$x''' + \psi(x, x')x'' + f(x, x') = 0. \tag{1.1}$$

He introduced a Lyapunov function and then discussed the global asymptotic stability of zero solution $x(t) = 0$ of this equation. By this work, the author proved under less restrictive conditions the stability result obtained by Qian [6] for equation (1.1). The Lyapunov function introduced in that paper, [5], raised this case. It should be noted that, first in 1970, Barbashin [2] proved some results related to the qualitative behaviors of solution of some systems of third order differential equation. Later, based on the results of Barbashin [2], some results have been improved concerning the qualitative behaviors of

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solutions of (1.1) and various nonlinear third order differential equations in the literature (see Omeike [5], Qian [6], Tunç [7, 8] and the references thereof). At the same time, for some papers published on the qualitative behaviors of solutions of various nonlinear third order differential equations and the stability and boundedness of nonlinear systems, we refer the reader to the papers of Aleksandrov and Platonov [1], Barbashin and Tabueva [3, 4], Tunç [9, 10, 11], Tunç and Ateş [12] and the references thereof. Now, we consider the following nonlinear third order differential equation

$$x''' + \psi(x, x')x'' + f(x, x') = p(t, x, x', x''). \tag{1.2}$$

This equation can be stated as the following equivalent system:

$$x' = y, \quad y' = z,$$

$$z' = -\psi(x, y)z - f(x, y) + p(t, x, y, z), \tag{1.3}$$

where $\psi \in C(\Re \times \Re,\ \Re)$, $f \in C(\Re \times \Re,\ \Re)$ and $p \in C([0, \infty) \times \Re \times \Re \times \Re,\ \Re)$. We also assume that the functions ψ, f and p depend only on the arguments displayed explicitly, and the primes in equation (1.2) denote differentiation with respect to t; the derivatives

$$\frac{\partial \psi(x, x')}{\partial x} \equiv \psi_x(x, x'), \quad \frac{\partial f(x, x')}{\partial x} \equiv f_x(x, x')$$

exist and are also continuous. The motivation for the present paper has been inspired basically by the papers of Barbashin [2], Omeike [5], Qian [6] and Tunç [7, 8] and the papers mentioned above. The principal aim of this paper is to improve the result achieved in Omeike [5] on the boundedness of solutions of nonlinear differential equation (1.2). It should also be noted that we prove our main result here by using the Lyapunov's second method.

2 Boundedness of Solutions

Our main result is the following theorem.

Theorem 2.1 *In addition to the basic assumptions imposed on the functions ψ, f and p appearing in equation* (1.2)*, we assume that there exist positive constants a, b and c such that the following conditions hold:*

(i) $\frac{f(x,0)}{x} \geq c$, *$(x \neq 0)$, $f_y(x, \theta y) \geq b$, $0 \leq \theta \leq 1$, $\psi(x, y) \geq a$ and*

$$a[f(x, y) - f(x, 0) - \int_0^y \psi_x(x, v)v dv]y \geq y \int_0^y f_x(x, v)dv;$$

(ii) $|p(t, x, y, z)| \leq q(t)$, *where $q \in L^1(0, \infty)$, $L^1(0, \infty)$ is a space of integrable Lebesgue functions.*

Then, there exists a finite positive constant K such that every solution $(x(t), y(t), z(t))$ of system (1.3) *satisfies*

$$|x(t)| \leq \sqrt{K}, \quad |y(t)| \leq \sqrt{K}, \quad |z(t)| \leq \sqrt{K}.$$

Proof The proof of this theorem depends on a scalar differentiable Lyapunov's function $V = V(x,y,z)$. This function and its time derivative satisfy some fundamental inequalities. We use here the Lyapunov's function V introduced in [5]:

$$V = \int_0^x f(u,0)du + \int_0^y \psi(x,v)vdv + a^{-1}\int_0^y f(x,v)dv + \frac{1}{2a}z^2 + yz. \tag{2.1}$$

This function, (2.1), can be rearranged as follows:

$$V = (1+a^{-1})\int_0^x f(u,0)du + \int_0^y \psi(x,v)vdv + a^{-1}\int_0^y f_v(x,\theta v)vdv + \frac{1}{2a}z^2 + yz$$

since

$$f_v(x,\theta v) = \frac{f(x,v) - f(x,0)}{v}, \quad (v \neq 0,\ 0 \leq \theta \leq 1),$$

that is

$$f(x,v) = f_v(x,\theta v)v + f(x,0), (v \neq 0,\ 0 \leq \theta \leq 1).$$

This arrangement implies

$$V = (1+a^{-1})\int_0^x \left[u^{-1}f(u,0) - c\right]\ udu + a^{-1}\int_0^y [f_v(x,\theta v) - b]\ vdv$$

$$+ \int_0^y [\psi(x,v) - a]\ vdv + \frac{1}{2a}(z+ay)^2 + \frac{b}{2a}y^2 + \frac{(1+a^{-1})c}{2}x^2. \tag{2.2}$$

Obviously, it follows from (2.2) that there exist some positive constants D_i, $(i = 1,\ 2,\ 3)$, such that

$$\begin{aligned} V &\geq \frac{1}{2a}(z+ay)^2 + \frac{b}{2a}y^2 + \frac{(1+a^{-1})c}{2}x^2 \\ &\geq D_1x^2 + D_2y^2 + D_3z^2 \\ &\geq D_4(x^2+y^2+z^2), \end{aligned}$$

where $D_4 = \min\{D_1, D_2, D_3\}$. Now, let $(x,y,z) = (x(t), y(t), z(t))$ be any solution of system (1.3). Differentiating the function V, (2.1), along system (1.3) with respect to the independent variable t, we have

$$\begin{aligned} \frac{d}{dt}V(x,y,z) &= f(x,0)y + y\int_0^y \psi_x(x,v)vdv + a^{-1}y\int_0^y f_x(x,v)dv + z^2 \\ &\quad -a^{-1}\psi(x,y)z^2 - f(x,y)y + (y + a^{-1}z)p(t,x,y,z) \\ &= -[f(x,y) - f(x,0) - y\int_0^y \psi_x(x,v)vdv] + a^{-1}y\int_0^y f_x(x,v)dv \\ &\quad -[a^{-1}\psi(x,y) - 1]z^2 + (y + a^{-1}z)p(t,x,y,z). \end{aligned} \tag{2.3}$$

Making use of assumption (i) of the theorem, we obtain

$$\frac{d}{dt}V(x,y,z) \le (y + a^{-1}z)p(t,x,y,z).$$

By using assumption (ii) of the theorem, the inequality $2\,|uv| \le u^2 + v^2$ and the fact

$$y^2 + z^2 \le x^2 + y^2 + z^2 \le D_4^{-1}V(x,y,z), \tag{2.4}$$

one can easily obtain that

$$\begin{aligned}
\frac{d}{dt}V(x,y,z) &\le \left(|y| + a^{-1}\,|z|\right) q(t) \\
&\le D_5\,(|y| + |z|)\, q(t) \\
&\le D_5\left(2 + y^2 + z^2\right)\; q(t). \\
&\le D_5\left(2 + D_4^{-1}V(x,y,z)\right) q(t) \\
&= 2D_5 q(t) + D_5 D_4^{-1}V(x,y,z)q(t),
\end{aligned} \tag{2.5}$$

where $D_5 = \min\{1, a^{-1}\}$. Integrating (2.5) from 0 to t, using the assumption $q \in L^1(0,\infty)$ and the Gronwall–Reid–Bellman inequality, we have

$$\begin{aligned}
V(x,y,z) &\le V(0,0,0) + 2D_5A + D_5D_4^{-1}\int\limits_0^t (V(x(s),y(s),z(s)))q(s)ds \\
&\le (V(0,0,0) + 2D_5A)\exp\left(D_5D_4^{-1}\int\limits_0^t q(s)ds\right) \\
&= (V(0,0,0) + 2D_5A)\exp\left(D_5D_4^{-1}A\right) = K_1 < \infty,
\end{aligned} \tag{2.6}$$

where $K_1 > 0$ is a constant, $K_1 = (V(0,0,0) + 2D_5A)\exp\left(D_5D_4^{-1}A\right)$ and $A = \int\limits_0^\infty q(s)ds$. In view of inequalities (2.4) and (2.6), we get

$$x^2(t) + y^2(t) + z^2(t) \le D_4^{-1}V(x,y,z) \le K,$$

where $K = K_1D_4^{-1}$. Aforementioned inequality implies that

$$|x(t)| \le \sqrt{K}, \quad |y(t)| \le \sqrt{K}, \quad |z(t)| \le \sqrt{K}$$

for all $t \ge t_0 \ge 0$. Hence,

$$|x(t)| \le \sqrt{K}, \quad |x'(t)| \le \sqrt{K}, \quad |x''(t)| \le \sqrt{K}$$

for all $t \ge t_0 \ge 0$. Thus, the proof of theorem is now complete. $\square$

Example 2.1 We consider nonlinear third order scalar differential equation:

$$x''' + (x' \sin x + (x')^2 + 4)x'' + (x')^3 + x' + x + \frac{x}{1+x^2} = \frac{1}{1+t^2+x^2+(x')^2+(x'')^2}. \quad (2.7)$$

Now, it can be seen that differential equation (2.7) has the form of (1.2), and its equivalent system is

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= -\{(\sin x)y + y^2 + 4\}z - y^3 - y - x - \frac{x}{1+x^2} + \frac{1}{1+t^2+x^2+y^2+z^2}. \end{aligned} \quad (2.8)$$

Clearly, by comparing (2.8) with (1.3) and taking into account the assumptions of the theorem, it follows:

$$f(x,y) = x + \frac{x}{1+x^2} + y + y^3,$$

$$\frac{f(x,0)}{x} = 1 + \frac{1}{1+x^2} \geq 1 = c,$$

$$f_x(x,y) = 1 + \frac{1-x^2}{(1+x^2)^2},$$

$$f_y(x,y) = 1 + 3y^2 \geq 1 = b;$$

$$\psi(x,y) = (\sin x)y + y^2 + 4 \geq -|\sin x| \ |y| + y^2 + 4$$

$$\geq - \ |y| + y^2 + 4 = \left(|y| - \frac{1}{2}\right)^2 + \frac{15}{4} > 3 = a,$$

$$\psi_x(x,y) = (\cos x)y;$$

$$a[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v)v\,dv]y = 3[y + y^3 - \int_0^y (\cos x)v^2 dv]\ y$$

$$= 3[y + y^3 - (\cos x)\frac{y^3}{3}]\ y$$

$$= 3[y^2 + y^4 - (\cos x)\frac{y^4}{3}];$$

$$y\int_0^y f_x(x,v)dv = y\int_0^y \left[1 + \frac{1-x^2}{(1+x^2)^2}\right] dv$$

$$= \left[1 + \frac{1-x^2}{(1+x^2)^2}\right] y^2$$

$$= y^2 + \frac{y^2}{(1+x^2)^2} - \frac{x^2y^2}{(1+x^2)^2},$$

Now, we observe

$$3\big[y^2 + y^4 - (\cos x)\frac{y^4}{3}\big] \geq y^2 + \frac{y^2}{(1+x^2)^2} - \frac{x^2y^2}{(1+x^2)^2}.$$

That is,

$$a[f(x,y) - f(x,0) - \int_0^y \psi_x(x,v)vdv]y \geq y \int_0^y f_x(x,v)dv.$$

Finally, we have

$$p(t,x,y,z) = \frac{1}{1+t^2+x^2+y^2+z^2} \leq \frac{1}{1+t^2}$$

and

$$\int_0^\infty q(s)ds = \int_0^\infty \frac{1}{1+s^2}ds = \frac{\pi}{2} < \infty,$$

that is, $q \in L^1(0,\infty)$.

Hence, the above whole discussion shows that all the conditions of the theorem hold. Thus, one can conclude that all solutions of equation (2.7) are bounded.

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