Nonlinear Dynamics and Systems Theory, **10** (2) (2010) 161–174

Practical Stability and Controllability for a Class of Nonlinear Discrete Systems with Time Delay ⋆

Zhan Su [1,2], Qingling Zhang [2] and Wanquan Liu [3]∗

[1] *College of Information Science and Engineering, Northeastern University, Shenyang, Liaoning province, 110004, PR China*
[2] *Institute of Systems Science, Northeastern University, Shenyang, Liaoning province, 110004, PR China*
[3] *Department of Computing, Curtin University of Technology, Perth WA 6102, Australia*

Received: October 27, 2009; Revised: March 26, 2010

Abstract: In this paper we first study the problem of practical asymptotic stability for a class of discrete-time time-delay systems via Razumikhin-type Theorems. Further the estimations of solution boundary and arrival time of the solution into a region are also investigated based on the practical stability results. Finally, the result on practical asymptotic stability is used to analyze the practical controllability of a general class of nonlinear discrete systems with input time delay. Some explicit criteria for the uniform practical asymptotic stability are derived via Lyapunov function and Razumikhin technique. For illustration, a numerical example is given to show the effectiveness of the proposed results.

Keywords: *practical stability; practical controllability; Razumikhin techniques; discrete systems; time delay.*

Mathematics Subject Classification (2000): 70K20, 93C55, 39A10, 03C45, 93D20, 93B05, 37B25, 39A22.

⋆ This work was supported by a grant from National Natural Science Foundation of China with grant number (60574011).

∗ Corresponding author: mailto:W.Liu@curtin.edu.au

1 Introduction

Since Lasalle first introduced the concept of practical stability in [1], it attracts much attention in control community. Many works on practical stability have been published with broad applications in different areas. Being much different from stability in terms of Lyapunov functions, practical stability, which stabilizes a system into a region of phase space, is a significant performance specification from engineering point of view, and this idea is quite satisfactory in many applications for quality analysis. In practice, a system is actually unstable, just because the stable domain or the domain of the desired attractor is not large enough; or sometimes, the desired state of a system may be mathematically unstable, yet the system may oscillate sufficiently near to a state, in which the performance is still acceptable, i.e., it is stable in practice. For example, in practical communication or digital control systems, the signals of controller states, measurement outputs, and control inputs are quantized and then encoded for transmission. A feedback law, which global asymptotically stabilizes a given system without quantization, will in general fail to guarantee global asymptotic stability of the closed-loop system, which arises in the presence of a quantizer with a finite number of values. Instead of using the global asymptotic stability, the practice stability can be used to analyze such systems, where there is a region of attraction in the state and the steady state converges to a small limit cycle [2]–[6]. On the other hand, it is well known that for more than one hundred years, Lyapunov's direct method has been the primary technique for dealing with stability problems in difference equations. However, the construction of Lyapunov's function is much more difficult for time-delay systems than for non-delay systems. Such difficulties can be overcome via using Lyapunov functions and Razumikhin techniques. It should be pointed out that the Razumikhin-type method can deal with the time-delay system effectively and is easier to apply in general, therefore such a method has been a main technique for analyzing the stability for time-delay systems [7]–[10].

Though there are several results on the practical stability for hybrid and descriptor systems [11]–[17], to the best of our knowledge, the Razumikhin-type method on practical stability for discrete time-delay systems has not been investigated. Motivated by results in [9], we will study the Razumikhin-type theorem on practical asymptotic stability for a class of discrete time-delay system in this paper. Also estimations of the solution boundary and arrival time of the solution are discussed. Consequently, the proposed theorems are used to study the practical controllability of a general class of nonlinear discrete systems with input time delay. Some explicit criteria for the uniform practical asymptotic stability are obtained via using the Lyapunov function and Razumikhin technique.

This paper is organized as follows. In Section 2, some definitions and preliminaries are introduced. In Section 3, some criteria for uniform practical asymptotic stability of discrete-time systems with finite delay are derived via using the Lyapunov functions and Razumikhin-technique. In Section 4, estimation of the solution boundary and arrival time of the solution are investigated in terms of practical stability. In Section 5, the proposed theorems are used to analyze the practical controllability for a general class of nonlinear discrete systems with input time delay. In Section 6, a numerical example is given to illustrate the effectiveness of main results obtained in Section 5. The last section gives some conclusions.

2 Preliminaries

To describe the main result of this paper, we include some preliminary knowledge on the practical stability for the following general class of nonlinear discrete systems with finite time delay:

$$x(k+1) = F(k, x_d(k)), \quad k \in \mathbb{Z}^+, \tag{1}$$

where $\mathbb{Z}^+$ is the set of nonnegative integers, $d \geq 0$ is an integer, $x(k) \in \mathbb{R}^n$, $x_d(k) = (x^T(k), x^T(k-1), \ldots, x^T(k-d))^T$, $\mathbb{R}^n$ is the n-dimensional Euclidean space. Denote

$$I_d = \{-d, -d+1, \ldots, -1, 0\}, \;\; I_d^1 = I_d \cup \{1\},$$

$$\Xi(I_d, \mathbb{R}^n) = \{\xi_d = (\xi^T(0), \xi^T(-1), \ldots, \xi^T(-d))^T \,|\, \xi : I_d \to \mathbb{R}^n\},$$

$$\Xi_B(I_d, \mathbb{R}^n) = \Xi(I_d, \mathbb{R}^n) \cap \{\xi_d : \xi(s) \in B, \, s \in I_d\},$$

where B is an open ball. Assume $F : \mathbb{Z}^+ \times \Xi_B(I_d, \mathbb{R}^n) \to \mathbb{R}^n$ with $F(k, 0) = 0$ for $k \in \mathbb{Z}^+$, and satisfies certain conditions to guarantee the global existence and uniqueness of solutions. Thus system (1) has zero solution $x(\cdot) \equiv 0$. For any $k_0 \in \mathbb{Z}^+$ and any given initial function $\phi \in \Xi_B(I_d, \mathbb{R}^n)$, the solution of the systems (1) denoted by $x(k; k_0, \phi)$ satisfies (1) for all integers $k \geq k_0$, and $x(k_0 + s; k_0, \phi) = \phi(s)$ for all $s \in I_d$.

For all $\xi_d \in \Xi(I_d, \mathbb{R}^n)$, define the norm of ξ_d as $\|\xi_d\| = \max\limits_{s \in I_d} |\xi(s)|$, where $|\cdot|$ stands for any norm in $\mathbb{R}^n$. We further assume that there exists a constant $L > 0$ such that for all $\xi_d \in \Xi_B(I_d, \mathbb{R}^n)$,

$$|F(k, \xi_d)| \leq L\|\xi_d\|, \quad \forall k \in \mathbb{Z}^+. \tag{2}$$

Now we introduce the following definitions.

Definition 2.1 [9] A wedge function is a continuous strictly increasing function $W : \mathbb{R}^+ \to \mathbb{R}^+$ with $W(0) = 0$.

Definition 2.2 System (1) is said to be:

Practically Stable (P.S.): For given (α, β) with $0 < \alpha < \beta$ and some $k_0 \in \mathbb{Z}^+$, if $\|\phi\| < \alpha$ then $|x(k; k_0, \phi)| < \beta$, $k \geq k_0$;

Uniformly Practically Stable (U.P.S.): If P.S. holds for all $k_0 \in \mathbb{Z}^+$;

Practically Asymptotically Stable (P.A.S.): If P.S. holds, and for each $\varepsilon \in (0, \beta)$, there exists a positive number $K = K(k_0, \alpha, \varepsilon)$ such that $\|\phi\| < \alpha$ implies $|x(k; k_0, \phi)| < \varepsilon$, $k \geq k_0 + K$;

Uniformly Practically Asymptotically Stable (U.P.A.S.): If P.A.S. holds for all $k_0 \in \mathbb{Z}^+$.

Definition 2.3 For a function $V : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, define:

$$\Delta V(k, x(k)) \triangleq V(k+1, x(k+1)) - V(k, x(k)).$$

3 Razumikhin-type Theorems

In this section we will prove some Razumikhin-type theorems with the aim of analyzing the uniform practical asymptotical stability (U.P.A.S.) for a general class of nonlinear discrete systems with finite time delay. We first denote the balls B_0, B_1 and B_2 as the following forms, which will be used in main theorems:

$$\begin{aligned} B_0 &= \{x(k):\ V(k,x(k)) < W_2(\alpha)\}; \\ B_1 &= \{x(k):\ V(k,x(k)) < W_1(\beta)\}; \\ B_2 &= \{x(k):\ V(k,x(k)) < W_1(\varepsilon)\}. \end{aligned}$$

Theorem 3.1 *Given positive scalars α and β. Assume that scalars ϖ_1, ϖ_2, ϖ_3 with $0 < \varpi_1 \le \varpi_2$, $\varpi_3 > 0$ are all arbitrary. If there exist a scalar $\eta > 0$, a Lyapunov function $V : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, and wedge functions $W_i(\cdot)(i = 1, 2, 3)$, such that*

$$\begin{aligned} &(i) \quad W_1(|x(k)|) \le V(k,x(k)) \le W_2(|x(k)|); \\ &(ii) \quad \Delta V(k,x(k)) \le -W_3(|x(k+1)|) + \varpi_3 \ \textit{for}\ \varepsilon_0 \le \|x_d\| \le \rho_0, \end{aligned}$$

provided $\varepsilon_0 \le \rho_0$, $V(k+s,x(k+s)) \le \min\{\varpi_2, V(k+1,x(k+1)) + \eta\}$ for $s \in I_d^1$, and $\varpi_1 \le V(k+1,x(k+1))$. Here $\varepsilon_0 = L^{-1}\alpha$, $\rho_0 = \max\{\beta, W_1^{-1}(W_2(\alpha))\}$, L is defined by (2). Then, we have (1) B_0 is an invariable set; (2) If $W_2(\alpha) < W_1(\beta)$, then B_1 is an invariable set and there exists a positive number $K = K(\alpha,\beta)$ such that for any $k_0 \in \mathbb{Z}$, $\phi \in \Xi_{B_1}(I_d,\mathbb{R}^n)$ implies $\forall\ k \ge k_0 + K$, $x(k;k_0,\phi) \in B_0$.

Proof (1) For each $\phi \in \Xi_{B_0}(I_d,\mathbb{R}^n)$, we have $x(k;k_0,\phi) \in B_0$ for $k_0 - d \le k \le k_0$. Now we claim that for all $k \ge k_0$, $x = x(k;k_0,\phi) \in B_0$.

Suppose this is not true. Then there exist some $k^1 \ge k_0$ such that $x \in B_0$ for all $k_0 - d \le k \le k^1$, and

$$V(k^1+1, x(k^1+1)) \ge W_2(\alpha), \tag{3}$$

and consequently,

$$\Delta V(k^1, x(k^1)) = V(k^1+1, x(k^1+1)) - V(k^1, x(k^1)) > 0.$$

On the other hand, by condition (i), we have $W_1(|x(k)|) < W_2(\alpha)$ for $k_0 - d \le k \le k^1$, which implies $\|x_d(k)\| \le \rho_0$ for $k_0 \le k \le k^1$. It follows from (2), (3) and condition (i) that $\alpha \le |x(k^1+1)| \le L\|x_d(k^1)\| \le L\rho_0$, which implies $\varepsilon_0 \le \|x_d(k^1)\| \le \rho_0$, $\varepsilon_0 \le \rho_0$. Let $0 < \varpi_1 \le W_2(\alpha) \le W_2(L\rho_0) \le \varpi_2$, and $0 < \varpi_3 < W_3(\alpha)$. Then, it follows from (3) that $\varpi_1 \le V(k^1+1, x(k^1+1))$, and for $\eta > 0$, $\forall\ s \in I_d^1$,

$$\begin{aligned} &\begin{cases} V(k^1+s, x(k^1+s)) < \varpi_2 \\ V(k^1+s, x(k^1+s)) < V(k^1+1, x(k^1+1)) + \eta \end{cases} \\ \Longrightarrow \quad & V(k^1+s, x(k^1+s)) \le \min\{\varpi_2, V(k^1+1, x(k^1+1)) + \eta\}. \end{aligned}$$

By condition (ii), we have $\Delta V(k^1, x(k^1)) \le -W_3(|x(k^1+1)|) + \varpi_3 < 0$. This is a contradiction. Thus for all $k \ge k_0$, $x(k) \in B_0$, i.e., B_0 is an invariable set.
(2) If $W_2(\alpha) < W_1(\beta)$, we first prove that B_1 is an invariable set. In fact, $\rho_0 = \beta$,

and $\varepsilon_0 = L^{-1}\alpha < L^{-1}W_2^{-1}(W_1(\beta))$. Similar to the proof of (1), one can derive that, $\phi \in \Xi_{B_1}(I_d, \mathbb{R}^n)$ implies $x(k) \in B_1$ for all $k \geq k_0$.

Next, we will find an integer $K = K(\alpha, \beta) > 0$ such that for all $k_0 \in \mathbb{Z}^+$, $\phi \in \Xi_{B_1}(I_d, \mathbb{R}^n)$ implies $x(k; k_0, \phi) \in B_0$ for all $k \geq k_0 + K$.

Assume that $0 < \varpi_1 \leq W_2(\alpha) < W_1(\beta) \leq \varpi_2$, $0 < \varpi_3 < (1/2)W_3(\alpha)$. Let N_η be the first positive integer satisfying

$$W_1(\beta) < W_2(\alpha) + \eta N_\eta. \tag{4}$$

For each $i \in \{0, 1, \dots, N_\eta\}$, let $k_i = k_0 + i(d + \left[\frac{W_1(\beta)}{\varpi_3}\right])$, where $[\cdot]$ denotes the greatest integer function, η is dependent on ϖ_1 and ϖ_3. We show that for all $i \in \{0, 1, \dots, N_\eta\}$,

$$V(k, x(k)) < W_2(\alpha) + \eta(N_\eta - i), \quad \forall\, k \geq k_i. \tag{5}$$

Obviously, it follows (4) that (5) holds for $i = 0$ since $x(k) \in B_1$ for all $k \geq k_0$. Suppose (5) holds for some $i \in \{0, 1, \dots, N_\eta - 1\}$, we aim to show that (5) also holds for $i + 1$, i.e.,

$$V(k, x(k)) < W_2(\alpha) + \eta(N_\eta - i - 1), \quad \forall\, k \geq k_{i+1}.$$

Next we present proof in two steps for clarity.
Step 1. We show that there does exist some $k' \in [k_i + d, k_{i+1}]$ such that

$$V(k', x(k')) < W_2(\alpha) + \eta(N_\eta - i - 1). \tag{6}$$

Suppose this is not true, for all $k \in [k_i + d, k_{i+1}]$, we would have

$$V(k, x(k)) \geq W_2(\alpha) + \eta(N_\eta - i - 1). \tag{7}$$

Noting the assumption that (5) holds for some $i \in \{0, 1, \dots, N_\eta - 1\}$, then, for all $k \in [k_i + d, k_{i+1} - 1]$, $s \in I_d^1$, from (7) we have

$$V(k + s, x(k + s)) < W_2(\alpha) + \eta(N_\eta - i) \leq V(k + 1, x(k + 1)) + \eta.$$

On the other hand, for all $k \in [k_i + d, k_{i+1} - 1]$, it follows from condition (i), (2) and (7) that $W_2(\alpha) \leq V(k+1, x(k+1)) \leq W_2(|x(k+1)|)$, which implies that $\alpha \leq |x(k+1)| \leq L\rho_0$, $\varepsilon_0 \leq \|x_d(k)\| \leq \rho_0$, $\varepsilon_0 \leq \rho_0$. Then, for all $k \in [k_i + d, k_{i+1} - 1]$, $V(k + s, x(k + s)) \leq \varpi_2$, $s \in I_d^1$, and it follows from (7) that $V(k + 1, x(k + 1)) \geq \varpi_1$. By condition (ii), for all $k \in [k_i + d, k_{i+1} - 1]$,

$$\Delta V(k, x(k)) \leq -W_3(|x(k + 1)|) + \varpi_3 < -\varpi_3.$$

Hence, we have

$$\begin{aligned} V(k_{i+1}, x(k_{i+1})) &\leq V(k_i + d, x(k_i + d)) - \varpi_3(k_{i+1} - k_i - d) \\ &< W_1(\beta) - \varpi_3\left[\frac{W_1(\beta)}{\varpi_3}\right] < 0. \end{aligned}$$

This is a contradiction to the definition of Lyapunov function $V(k, x(k))$. Thus, there does exist some $k' \in [k_i + d, k_{i+1}]$ such that (6) holds.
Step 2. We need to show that

$$V(k, x(k)) < W_2(\alpha) + \eta(N_\eta - i - 1), \quad \forall\, k \geq k'. \tag{8}$$

In fact, suppose this is not true, there must be some $k_1' \geq k'$ such that

$$\begin{aligned} V(k_1', x(k_1')) &< W_2(\alpha) + \eta(N_\eta - i - 1), \\ V(k_1' + 1, x(k_1' + 1)) &\geq W_2(\alpha) + \eta(N_\eta - i - 1). \end{aligned} \tag{9}$$

Hence we have $\Delta V(k_1', x(k_1')) > 0$. On the other hand, $\varpi_1 \leq W_2(\alpha) \leq V(k_1' + 1, x(k_1' + 1))$, $V(k_1' + s, x(k_1' + s)) \leq \varpi_2$. Noting the assumption that (5) holds for some $i \in \{0, 1, \ldots, N_\eta - 1\}$, then, we have for $s \in I_d^1$,

$$V(k_1' + s, x(k_1' + s)) < W_2(\alpha) + \eta(N_\eta - i) \leq V(k_1' + 1, x(k_1' + 1)) + \eta.$$

From condition (i), (2) and (9), we have $W_2(\alpha) \leq V(k_1' + 1, x(k_1' + 1)) \leq W_2(|x(k_1' + 1)|)$, and hence, $\alpha \leq |x(k_1' + 1)| \leq L\rho_0$, $\varepsilon_0 \leq \|x_d(k_1')\| \leq \rho_0$, $\varepsilon_0 \leq \rho_0$. With condition (ii), one can derive that

$$\Delta V(k_1', x(k_1')) \leq -W_3(|x(k_1' + 1)|) + \varpi_3 \leq -\varpi_3 < 0.$$

This is a contradiction again to the definition of Lyapunov function $V(k, x(k))$. Thus (8) holds, and consequently, (5) holds for all $i \in \{0, 1, \ldots, N_\eta\}$. Therefore, we obtain that $x(k) \in B_0$ for all $k \geq k_{N_\eta} = k_0 + K$, where $K = N_\eta(d + \left[\frac{W_1(\beta)}{\varpi_3}\right])$ is independent of k_0 and ϕ. □

Corollary 3.1 *Given positive scalars α and β and assume that $P_V(s) \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $P_V(s) > s$ for $s > 0$. If there exist a Lyapunov function $V : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, and wedge functions $W_i(\cdot)(i = 1, 2, 3)$, satisfying the conditions (i) in Theorem 3.1 and the following condition :*

$$\text{(ii)}' \quad \Delta V(k, x(k)) \leq -W_3(|x(k + 1)|) \text{ for } \varepsilon_0 \leq \|x_d(k)\| \leq \rho_0,$$

provided $\varepsilon_0 \leq \rho_0$, $V(k+s, x(k+s)) < P_V(V(k+1, x(k+1)))$ for $s \in I_d^1$, where $\varepsilon_0 = L^{-1}\alpha$, $\rho_0 = \max\{\beta, W_1^{-1}(W_2(\alpha))\}$, L is defined by (2). Then, the conclusion of Theorem 3.1 still holds.

Proof For any $0 < \varpi_1 \leq \varpi_2$, and any $\varpi_3 > 0$, choose $\eta \in (0, \inf\{P_V(s) - s : \varpi_1 \leq s \leq \varpi_2\})$. Then, if $V(k + s, x(k + s)) \leq \min\{\varpi_2, V(k + 1, x(k + 1) + \eta)\}$ for $s \in I_d^1$, and $\varpi_1 \leq V(k + 1, x(k + 1))$, we have

$$V(k + s, x(k + s)) \leq V(k + 1, x(k + 1)) + \eta < P_V(V(k + 1, x(k + 1))),$$

for $s \in I_d^1$. Hence, by condition (ii)$'$, we have

$$\Delta V(k, x(k)) \leq -W_3(|x(k + 1)|) \leq -W_3(|x(k + 1)|) + \varpi_3.$$

Then, the conditions (i) and (ii) in Theorem 3.1 are both satisfied. Therefore, the result follows. □

By using Theorem 3.1 and Corollary 3.1, we obtain the following Razumikhin-type theorem for the U.P.A.S. with regard to the zero solution of systems (1).

Theorem 3.2 *For given scalar pair (α, β) with $0 < \alpha < \beta$, $\varepsilon \in (0, \beta)$ is arbitrary. Assume that scalars ϖ_1, ϖ_2, ϖ_3 with $0 < \varpi_1 \leq \varpi_2$, $\varpi_3 > 0$ are all arbitrary, $P_V(s) \in$*

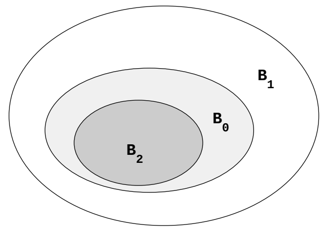

Figure 1: The relationship of the balls B_0, B_1 and B_2.

$C(\mathbb{R}^+, \mathbb{R}^+)$ with $P_V(s) > s$ for $s > 0$. If there exist a Lyapunov function $V : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, wedge functions $W_i(\cdot)(i = 1, 2, 3)$, satisfying

$$\begin{aligned} &\textit{(i)} \quad W_2(\alpha) \le W_1(\beta);\\ &\textit{(ii)} \quad W_1(|x(k)|) \le V(k, x(k)) \le W_2(|x(k)|);\end{aligned}$$

and either of the following conditions $(iii)_a$ or $(iii)_b$ for $\varepsilon_0 \le \|x_d(k)\| \le \rho_0$, $\varepsilon_0 \le \rho_0$:

$$\begin{aligned} \textit{(iii)}_a \quad & \Delta V(k, x(k)) \le -W_3(|x(k+1)|) + \varpi_3, \text{ provided}\\ & V(k+s, x(k+s)) \le \min\{\varpi_2, V(k+1, x(k+1)) + \eta\}\\ & \text{for } s \in I_d^1, \text{ and } \varpi_1 \le V(k+1, x(k+1));\\ \textit{(iii)}_b \quad & \Delta V(k, x(k)) \le -W_3(|x(k+1)|), \text{ provided for } s \in I_d^1,\\ & V(k+s, x(k+s)) < P_V(V(k+1, x(k+1))),\end{aligned}$$

where $\varepsilon_0 = L^{-1}W_2^{-1}(W_1(\varepsilon))$, $\rho_0 = \beta$, L is defined by (2). Then the zero solution of systems (1) is U.P.A.S.

Proof By condition (i), $B_0 \subseteq B_1$, as shown in Fig 1. Since

$$\varepsilon_0 = L^{-1}W_2^{-1}(W_1(\varepsilon)) < L^{-1}W_2^{-1}(W_1(\beta)),$$

Then, by Theorem 3.1 and Corollary 3.1, we can assert that, both B_1 and B_2 are invariant sets, and there exists a positive number $K = K(\alpha, \varepsilon)$ such that for any $k_0 \in \mathbb{Z}$, $\phi \in \Xi_{B_0}(I_d, \mathbb{R}^n)$ implies $\forall\ k \ge k_0 + K$, $x(k; k_0, \phi) \in B_2$. By condition (ii), $|x(k)| < \alpha$ implies $x(k) \in B_0$; $x(k) \in B_1$ implies $|x(k)| < \beta$; $x(k) \in B_2$ implies $|x(k)| < \varepsilon$. Then, for any $k_0 \in \mathbb{Z}$, $\|\phi\| < \alpha$ implies $\forall\ k \ge k_0 + K$, $|x(k; k_0, \phi)| < \varepsilon$, i.e., the zero solution of the systems (1) is U.P.A.S. □

Remark 3.1 In Theorem 3.1 and Corollary 3.1, whenever $\varepsilon_0 \le L^{-1}W_2^{-1}(W_1(\beta))$, $\rho_0 \ge \beta$ and $\Delta V(k, x(k)) \le 0$ in the conditions (ii) and (ii)$'$, one can obtain the result that B_1 is an invariable set. Here, the conditions of Theorem 3.1 and Corollary 3.1 are corresponding to the case that u, v, w are wedge functions in the conditions of Theorem 1 and Corollary 1 in [9]. Moreover, it is more convenient to apply Theorem 3.1 and Corollary 3.1 in this paper to estimate relations between balls B_0 and B_1 in the light of information on $\varepsilon_0 \le \|x_d(k)\| \le \rho_0$, which are not mentioned in Theorem 1, Corollary 1 and Corollary 2 in [9].

4 Estimation of the Solution Boundary and Arrival Time

Now let us consider Theorem 3.2, Corollary 3.1 and Theorem 3.2 from previous section without the condition $W_2(\alpha) < W_1(\beta)$. If $\varepsilon_0 = L^{-1}W_2^{-1}(W_1(\beta))$, $\rho_0 = \max\{\beta, W_1^{-1}(W_2(\alpha))\}$, then we can assert that B_1 is an invariant set. In addition, the trajectory of the solution of system (1) starting from B_0, will fall into B_1 in finite time when $B_0 \supset B_1$, or stay in the region of B_1 when $B_0 \subseteq B_1$. On the other hand, with the assumption of $W_2(\alpha) < W_1(\beta)$, all trajectories which exit from the ball B_0, will take the ball B_1 to be their boundary and can not get out of the region of B_1. Thus, by the proposed theorems and Remark 3.1, as long as $\varepsilon_0 \le L^{-1}W_2^{-1}(W_1(\beta))$, and $\Delta V(k, x(k)) \le 0$ in conditions (iii)$_a$~(iii)$_b$, the system is U.P.S.. Following the above analysis, one can observe that it is more convenient to apply Theorem 3.1 and Corollary 3.1 in this paper to estimate relations between the balls B_0 and B_1 by using the information on $\varepsilon_0 \le \|x_d(k)\| \le \rho_0$, which are not discussed in Theorem 1, Corollary 1 and Corollary 2 in [9]. We give the following theorem to estimate both the boundary of the solution of system (1) and arrival time K.

Theorem 4.1 *Given scalars α, ε with $0 < \varepsilon < \alpha$, $\sigma_1 > 1$. If there exist a Lyapunov function $V : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^+$, wedge functions $W_i(\cdot)(i = 1, 2, 3)$, satisfying*

(i) $W_1(|x(k)|) \le V(k, x(k)) \le W_2(|x(k)|)$;

(ii) $\Delta V(k, x(k)) \le -W_3(|x(k+1)|)\, for \|x_d(k)\| \le \rho_0$, *provided* $V(k+s, x(k+s)) < \sigma_1(V(k+1, x(k+1)))\, for\ s \in I_d^1$,

then
(1) $\beta_\alpha = W_1^{-1}(W_2(\alpha))$;
(2) $K = k_0 + m_1(d + m_2)$,
where

$$m_1 = \begin{cases} \frac{W_2(\alpha)+(\sigma_1-2)W_1(\varepsilon)}{(\sigma_1-1)W_1(\varepsilon)}, & \frac{W_2(\alpha)-W_1(\varepsilon)}{(\sigma_1-1)W_1(\varepsilon)} \text{ is integer;} \\ \left[\frac{W_2(\alpha)-W_1(\varepsilon)}{(\sigma_1-1)W_1(\varepsilon)}\right], & \text{otherwise,} \end{cases}$$

$$m_2 = \begin{cases} \frac{2W_2(\alpha)}{W_3(W_2^{-1}(W_1(\varepsilon)))} + 1, & \frac{2W_2(\alpha)}{W_3(W_2^{-1}(W_1(\varepsilon)))} \text{ is integer;} \\ \left[\frac{2W_2(\alpha)}{W_3(W_2^{-1}(W_1(\varepsilon)))}\right], & \text{otherwise,} \end{cases}$$

$[\cdot]$ *denotes the greatest integer function, $\rho_0 = W_1^{-1}(W_2(\alpha))$, β_α is the estimation of the solution boundary of system (1), and K is the time that the solution exists from the given ball $\{\phi : \|\phi\| < \alpha\}$ and falls into the region $\{x(k) : \|x(k)\| < \varepsilon\}$.*

Proof (1) In Theorem 3.1, let $W_1(\beta_\alpha) = W_2(\alpha)$. Then, $\varepsilon_0 = L^{-1}\alpha$, $\rho_0 = \beta_\alpha$, and $B_1 = B_2 = \{x(k) : V < W_1(\beta_\alpha)\}$. It follows from Theorem 3.1 that the solution starting from B_2 can not exits from B_1, which implies that the solution starting from set $\{\phi : \|\phi\| < \alpha\}$ will have a boundary $\beta_\alpha = W_1^{-1}(W_2(\alpha))$.

(2) In Theorem 3.1, let $B_1 = \{x(k) : V(k, x(k)) < W_2(\alpha)\}$, and $B_2 = \{x(k) : V(k, x(k)) < W_1(\varepsilon)\}$. Notice that $\varepsilon_0 = L^{-1}W_2^{-1}(W_1(\varepsilon))$ and $\rho_0 = W_1^{-1}(W_2(\alpha))$, let $P_V(s) = \sigma_1 s$, then $P_V(s)$ has the required property in Corollary 3.1 and there exist two scalars $\delta_1 > 0$ and $\delta_2 \in (0, 1/2)$, such that $\frac{W_2(\alpha)-W_1(\varepsilon)}{(\sigma_1-1)W_1(\varepsilon)} < \frac{W_2(\alpha)-W_1(\varepsilon)}{(\sigma_1-1)W_1(\varepsilon)-\delta_1} < m_1$, and $\frac{2W_2(\alpha)}{W_3(W_2^{-1}(W_1(\varepsilon)))} < \frac{W_2(\alpha)}{\delta_2 W_3(W_2^{-1}(W_1(\varepsilon)))} < m_2$. With the similar analysis process in the proofs of Theorem 3.1 and Corollary 3.1, one can derive the conclusion of (2) for $\varepsilon \le \|x(k)\| < \alpha$ with $\eta = (\sigma_1 - 1)W_1(\varepsilon) - \delta_1 \in (0, \inf(P_V(V) - V))$ and $\varpi_3 = \delta_2 W_3(W_2^{-1}(W_1(\varepsilon)))$. □

5 Practical Controllability

In this section we will use the results from previous sections to study the practical controllability for a general class of nonlinear discrete systems with input time delay. Consider the following system:

$$x(k+1) = f(k, x(k)) + \sum_{i=0}^{d} B(k-i)u(k-i), \tag{10}$$

where $f : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $B : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $i = 1, \ldots, d$, $u(k) \in \mathbb{R}^m$ is input, and is supposed to guarantee the existence and uniqueness of the solution. This type of model is generally studied in networked control systems (NCSs). We first introduce the following definitions:

Definition 5.1 System (10) is called to be:

Uniformly Practically Controllable (U.P.C.) with respect to (α, β), $0 < \alpha < \beta$, if there exist finite time K and a control $u(\cdot)$ defined on $[k_0, K]$ such that all the solutions $x(k) = x(k; k_0, \phi, u)$ that exit from $\{\phi \in \mathbb{R}^n : \|\phi\| < \alpha\}$ will enter into a bounded region $\{x(k) \in \mathbb{R}^n : \|x_d(k)\| < \beta\}$ at time K instant for all $k_0 \in \mathbb{Z}^+$;

Uniformly Practically Asymptotically Controllable (U.P.A.C.) with respect to (α, β), $0 < \alpha < \beta$, if U.P.C. holds, and for each $\varepsilon \in (0, \beta)$, there exists a positive number $K = K(k_0, \alpha, \varepsilon)$ such that $\|\phi\| < \alpha$ implies $|x(k; k_0, \phi, u)| < \varepsilon$ for all $k \geq k_0 + K$.

Theorem 5.1 *Assume that there exists a control law $u(k)$ such that system (10) can be expressed by the form of (1), and the conditions of Theorem 3.2 are satisfied. Then, system (10) is U.P.A.C.with respect to (α, β).*

For system (10), adopt the feedback control law $u(k) = F(k, x(k))x(k)$. Assume $f_u(k, x(k)) = f(k, x(k)) + B(k)u(k)$ and

$$\|f_u(k, x(k))\| \leq \|\Psi_0(k)\|\|x(k)\|.$$

Let $\Psi_i(k) = B(k-i)F(k-i, x(k-i))$, where $F(k, x(k))$ is the control gain matrix, $\Psi_0(k)$ and $\Psi_i(k)$ are of compatible dimensions. Consequently, the closed-loop system of (10) has the following form:

$$x(k+1) = f_u(k, x(k)) + \sum_{i=1}^{d} \Psi_i(k)x(k-i). \tag{11}$$

Let $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ be the maximum eigenvalue and the minimum eigenvalue of a real symmetric matrix, respectively. $\|\cdot\|_2$ stands for the Euclidean vector norm or the 2-norm of a matrix. Then, we have the following corollary.

Corollary 5.1 *If there exists $F(k)$ such that*

$$\sup_{k\in\mathbb{Z}} \sum_{i=0}^{d} \|\Psi_i(k)\|_2^2 < 1 - (\frac{\alpha}{\beta})^2 \tag{12}$$

then, the closed-loop system (11) is U.P.A.S., and system (10) is U.P.A.C. with respect to (α, β) with $0 < \alpha < \beta$.

Proof In fact, by (12), noting that $0 < \alpha < \beta$, then, $\forall \epsilon \in (0, \alpha^2/\beta^2)$, there exist scalars $\delta_1 \in [\alpha^2/\beta^2, 1]$ and $\delta_2 > 1$ such that

$$\sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2 + \delta_2 \sup_{k\in\mathbb{Z}} \sum_{i=1}^{d} \|\Psi_i(k)\|_2^2 < \delta_1 - (\frac{\alpha}{\beta})^2 + \epsilon < \delta_1.$$

Thus, there exists a positive definite matrix P such that $\lambda_{\min}(P) = \delta_1\lambda_{\max}(P)$. Choose $V(k, x(k)) = x^T(k)Px(k)$, $W_1(|x(k)|) = \lambda_{\min}(P)x^T(k)x(k)$, and $W_2(|x(k)|) = \lambda_{\max}(P)x^T(k)x(k)$. It is obvious that

$$W_1(|x(k)|) \leq V(k, x(k)) \leq W_2(|x(k)|).$$

Let $P_V(s) = \delta_2 s$ for $s \geq 0$. Then $P_V(s) > s$ for $s \geq 0$. For all $i \in \{1, \ldots, d\}$, if $V(k-i, x(k-i)) < P_V(V(k+1, x(k+1)))$, then, $\|x(k-i)\|_2^2 < \|x(k+1)\|_2^2\delta_2/\delta_1$, and it follows (11) that

$$\begin{aligned}\|x(k+1)\|_2^2 &\leq \sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2\|x(k)\|_2^2 \\ &\quad + \sup_{k\in\mathbb{Z}} \sum_{i=1}^{d} \|\Psi_i(k)\|_2^2\|x(k-i)\|_2^2 \\ &\leq \sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2\|x(k)\|_2^2 \\ &\quad + \frac{\delta_2 \sup_{k\in\mathbb{Z}} \sum_{i=1}^{d} \|\Psi_i(k)\|_2^2}{\delta_1}\|x(k+1)\|_2^2.\end{aligned}$$

Consequently,

$$-\|x(k)\|_2^2 \leq \frac{\delta_2 \sup_{k\in\mathbb{Z}} \sum_{i=1}^{d} \|\Psi_i(k)\|_2^2 - \delta_1}{\delta_1 \sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2}\|x(k+1)\|_2^2.$$

Let $\tilde{e} = \frac{\delta_1 - \sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2 - \delta_2 \sup_{k\in\mathbb{Z}} \sum_{i=1}^{d} \|\Psi_i(k)\|_2^2}{\sup_{k\in\mathbb{Z}} \|\Psi_0(k)\|_2^2}$. Since scalar $\epsilon \in (0, \alpha^2/\beta^2)$ is arbitrary, thus, $\tilde{e} > \frac{\alpha^2}{\beta^2-\alpha^2} > 0$, and

$$\begin{aligned}\Delta V(k, x(k)) &= x^T(k+1)Px(k+1) - x^T(k)Px(k) \\ &\leq -\lambda_{\max}(P)\frac{\alpha^2}{\beta^2-\alpha^2}\|x(k+1)\|_2^2.\end{aligned}$$

Then, conditions (i), (ii) and (iii)$_b$ of Theorem 3.2 are all satisfied, and hence, the conclusion follows. □

Remark 5.1 In Theorem 3.1, Corollary 3.1, Theorem 3.2 and Theorem 4.1, there is a relation between $V(k+s, x(k+s))(s \in I_d^1)$ and $V(k+1, x(k+1))$, namely, "provided $\mathcal{R}(V(k+s, x(k+s)), V(k+1, x(k+1)))$", where $\mathcal{R}(\cdot,\cdot)$ defines a relation. We call this relation as the $\mathcal{R}$-relation. The conditions (ii), (ii)$'$, (iii)$_a$ and (iii)$_b$ describe the constraint on $\Delta V(k, x(k))$ under the $\mathcal{R}$-relation, but no constrain on $\Delta V(k, x(k))$ without

$\mathcal{R}$-relation. Thus, the condition that the constraint on $\Delta V(k, x(k))$ holds not only with but also without the $\mathcal{R}$-relation, is more restrictive than the condition that the constraint on $\Delta V(k, x(k))$ holds only with the $\mathcal{R}$-relation. Therefore, we can obtain a class of particular cases of Theorem 3.2 with conditions (i), (ii), either $(\text{iii})_a$ or $(\text{iii})_b$, which in fact are corresponding to the well-known Lyapunov-like theorems.

6 Illustrative Numerical Example

To illustrate the effectiveness of the results obtained in previous sections, we consider the following nonlinear discrete system with input time delay:

$$x(k+1) \quad = \quad 1.44x(k) - x^3(k) + 0.069u(k) + 0.031u(k-1), \quad x(k) \in [-1.2, 1.2]. \tag{13}$$

Assume that $\alpha = 0.45$ and $\beta = 0.60$. To obtain the zero solution $x(k) = 0$ in U.P.A.S with (α, β), adopt the following fuzzy control law:

$$\begin{aligned} R_1: \quad & \text{IF } x \text{ is about } \pm 1.2, \text{ THEN} \\ & u = F_1 x(k), \\ R_2: \quad & \text{IF } x \text{ is about } 0, \text{ THEN} \\ & u = F_2 x(k). \end{aligned}$$

The references on fuzzy control can be found in [18, 19]. Then, the overall control law is

$$u(k) = \sum_{i=1}^{2} \mu_i F_i x(k), \tag{14}$$

where $\mu_1 = \frac{x^2}{1.44}$ and $\mu_2 = 1 - \mu_1$ are both membership functions, as shown in Figure 2. The control gain matrices are designed to be $F_1 = -0.0694$ and $F_2 = -18.9114$. Then, the closed-loop system can be expressed as follows:

$$x(k+1) \quad = \quad \big(1.44 - x^2(k) + 0.069 \sum_{i=1}^{2} \mu_i F_i\big) x(k) + 0.031 \sum_{i=1}^{2} \mu_i F_i x(k-1).$$

Denote discriminant function by

$$g(x(k)) = \big(1.44 - x^2(k) + 0.069 \sum_{i=1}^{2} \mu_i F_i\big)^2 + (0.031 \sum_{i=1}^{2} \mu_i F_i)^2.$$

The profile of $g(x)$ is illustrated in Figure 3. We can calculate that $g(x) \le 0.3619 < 1 - \alpha^2/\beta^2 = 0.4375$ for $x \in [-1.2, 1.2]$. By (12) and Corollary 5.1, system (13) is U.P.A.C. with respect to (α, β). The state curve with initial values $x(-1) = 0.3$, $x(0) = 0.4$ of system (13) with and without fuzzy controller (14) are shown in Figure 4. Without fuzzy controller, i.e., $u(k) = 0$, the zero solution is unstable, and the nonlinear discrete system converges to $x(k) \approx 0.6633 > \beta$; whereas, with fuzzy controller (14), the closed-loop system is U.P.A.S. with (α, β).

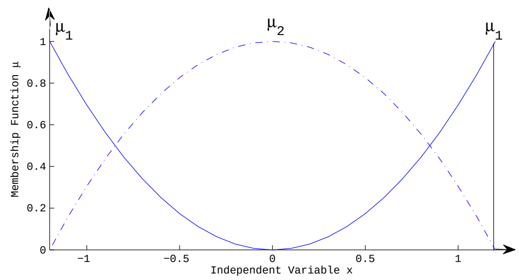

Figure 2: The membership functions of μ_1 and μ_2

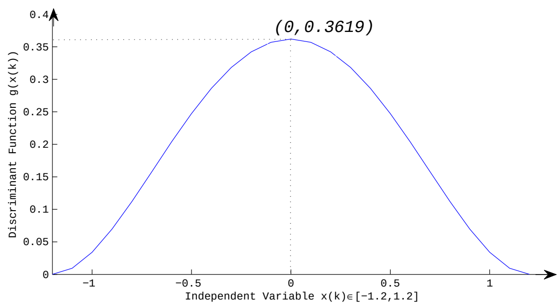

Figure 3: The profile of $g(x(k))$

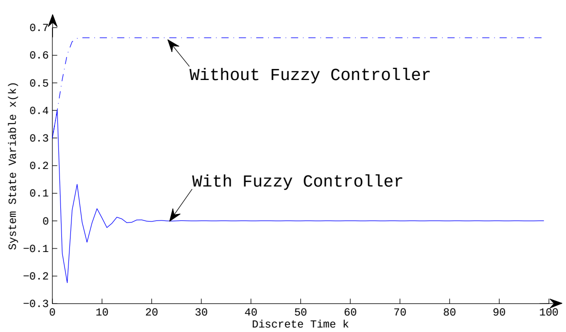

Figure 4: The state curve of system (13) with and without fuzzy controller (14)

7 Conclusions

Motivated by the idea in [9], practical asymptotic stability and controllability are studied for a class of nonlinear discrete systems with time delay. Some explicit criteria for the uniform practical asymptotic stability are established by means of Lyapunov function and Razumikhin technique. Estimations of the solution boundary and arrival time of the solution are also investigated. In addition, the proposed theorems are used to study the practical controllability for a general class of nonlinear discrete systems with input time delay. Finally, a numerical example is presented to illustrate the effectiveness of the proposed results. We believe the results in this paper are useful for the study networked control systems.

References

[1] Lasalle, J. and Lefshetz, S. *Stability by Lyapunov direct method and application*. Academic Press, New York, 1961.

[2] Lakshmikantham, V., Leela, S. and Martynyuk, A. A. *Practical Stability of Nonlinear Systems*. World Scientific, Singapore, 1990.

[3] Delchamps, D. F. Stabilizing a linear system with quantized state feedback. *IEEE Transactions on Automatic Control* **35** (8) (1990) 916–924.

[4] Chou, J. H., Chen, S. H. and Horng, I. R. Robust stability bound on linear time-varying uncertainties for linear digital control systems under finite wordlength effects. *JSMS International Journal: Series C* **39** (4) (1996) 767–771.

[5] Elia, N. and Mitter, S. K. Stabilization of linear systsems with limited information. *IEEE Transactions on Automatic Control* **46** (9) (2001) 1384–1400.

[6] Fagnani, F. and Zampieri, S. Stability analysis and synthesis for scalar linear systems with a quantized feedback. *IEEE Transactions on Automatic Control* **48** (9) (2003) 1569–1584.

[7] Hou, C. and Qian, J. Decay estimates for applictions of Razumikhin-type theorems. *Automatica* **34** (7) (1998) 921–924.

[8] Blanchini, F. and Ryan, E. P. A Razumikhin-type lemma for functional differential equations with application to adaptive control. *Automatica* **35** (1999) 809–818.

[9] Zhang, S. and Chen, M. P. A new Razumikhin theorem for delay difference equations. *Computers and Mathematics with Applications* **36** (1998) 405–412.

[10] Zhang, S. A new technique in stability of infinite delay differential equations. *Computers and Mathematics with Applications* **44** (2002) 1275–1287.

[11] Zhai, G. and Michel, A.N. On practical stability of switched systems. In: *Proceeding of the 41st IEEE Conference on Decision and Control*. Las Vegas, Nevada USA, December 2002, 3488–3493.

[12] Xu, X. and Zhai, G. Practical stability and stabilization of hybrid and switched systems. *IEEE Transactions on automatic control* **50** (11) (2005) 1897–1903.

[13] Yang, C., Zhang, Q. and Zhou, L. Practical stabilization and controllability of descriptor systems. *International Journal of Information and Systems Sciences* (1) (3–4) (2005) 455–465.

[14] Yang, C., Zhang, Q., Lin, Y. and Zhou, L. Practical stability of closed-loop descriptor systems. *International Journal of Systems Science* **37** (14) (2006) 1059–1067.

[15] Peng, S. and Chen, C. Estimation of the practical stability region of a class of robust controllers with input constraint. *Journal of Franklin Institute* **335B** (7) (1998) 1271–1281.

[16] Kapitaniak, T. and Brindley, J. Practical stability of chaotic attractors. *Chaos, Solitons & Fractals* **9** (7) (1998) 43–50.

[17] Anabtawi, M.J. and Sathananthan, S. Quantitative analysis of hybrid parabolic systems with Markovian regime switching via practical stability. *Nonlinear Analysis: Hybrid Systems* **2** (2008) 980–992.

[18] Takagi, T. and Sugeno, M. Fuzzy identification of system and its applictions to modeling and control. *IEEE Transactions on Systems, Man, and Cybernetics* **15** (1) (1985) 116–132.

[19] Babuska, R. *Fuzzy Modeling For Control.* Kluwer Academic Publishers, 1998.