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Stabilization of Controllable Linear Systems

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Abstract: The work consists of two parts. The first part is devoted to linear continuous-time systems and the second one to linear discrete-time systems. In the first part stationary and nonstationary stabilization of linear time-invariant continuous-time systems is considered. A simple and direct proof of Zubov's and Wonham's Theorem on pole assignment in controllable linear systems by means of a suitable static time-invariant output feedback of the state is given. Brockett's problem of stabilization by means of a static time-varying output feedback of linear system is considered. To solve this problem two approaches are considered. Sufficient conditions of low- and high-frequency stabilization of controllable linear systems are given. Also examples of possibility of nonstationary low-frequency and high-frequency stabilization of two-dimensional and three-dimensional linear systems are given. In the second part the discrete-time version of Brockett's problem for linear control systems is considered. It is shown that under mild conditions stabilization for linear time-invariant discrete-time systems is possible by means of piecewise-constant periodic with a sufficiently large period static output feedback control. Sufficient conditions of low-frequency stabilization are given. For second-order systems a necessary and sufficient condition of stabilizability by periodic output feedback is given. Also pole assignment problem for linear time-invariant discrete-time systems by static periodic output feedback is considered.

Keywords: *stabilization, time-invariant system, Brockett problem, pole assignment, time-invariant/time-varying static output feedback.*

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1 Introduction

Within the last 130 years the methods of stabilization of control systems have been constructed, developed, and improved: from creating Vyshnegradsky's cataract to the analysis and synthesis of systems of rocket stabilization and distributed systems of clock-signal generators in multiprocessor clusters. At present the theory and practice of stabilization are the subject of many books and surveys. The various methods of stabilization have entered into textbooks on the control theory and became classical ones. But in the last thirty years there has been a rapid growth of publications, devoted to the methods of stabilization of linear control systems, and the above-mentioned books and surveys have already not reflected them completely.

The increasing interest to stabilization problems is motivated by the needs of the practice of control formulated in the open problems by many famous scholars such as V.I. Zubov, W.M. Wonham, D.S. Bernstein, R. Brockett, J. Rosenthal and J.C. Willems. For solving these problems the new methods of analysis and synthesis of linear control systems were developed.

In this survey the effort is made to describe new methods and results. The authors believe that the acquaintance with these methods and results will be useful for many specialists and will give an impetus to the further development of this interesting and substantial direction: the theory of stabilization of linear control systems.

A more detailed consideration of current methods of stabilization will be in our book [1].

2 Stabilizability and Pole Assignment in Linear Systems by Static Time-Invariant State Feedback

Here we consider the stabilization and pole assignment problems for linear time-invariant continuous-time systems.

Consider a linear time-invariant continuous-time system

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x = x(t) \in \mathbb{R}^n$ is the state vector, $u = u(t) \in \mathbb{R}^m$ is the control input vector, and A, B are real constant matrices of dimension $n \times n$ and $n \times m$, respectively. (The point over the symbol x denotes the differentiation in t).

In the following all matrices have real-valued elements.

We consider for system (1) the classical feedback stabilization problem:

Under the assumption that the uncontrolled system is unstable, find an appropriate stabilizing feedback law.

It is well-known that this problem can be solved by means of a time-invariant static full state feedback $u = Sx$. This result follows from the following theorem on pole assignment.

Zubov's and Wonham's Theorem (on pole assignment) [2, 3]. *The system* (1) *is completely controllable if and only if for every choice of the self-conjugate set* $M = \{\mu_j\}_{j=1}^n$ *of complex numbers* μ_j *there exists* $(m \times n)$*-matrix* S *such that* $A + BS$ *has* M *for its set of eigenvalues.*

According to [4], this theorem was first obtained for the single-input case ($m = 1$) by Bertram in 1959 using locus method. In 1961, Bass independently formulated and proved the same result (but did not publish it) in the context of linear algebra. The single-input case was also considered by Rissanen [5] and Rosenbrock [6]. The above Theorem in the multi-input case for complex matrices A, B, S and arbitrary set M of complex numbers was proved by Popov [7] and by Langenhop [8]. Other contributions concerning pole assignment in multi-input systems by state feedback are due to Simon and Mitter [9], and Brunovsky [10]. In [9] the ability to relocate arbitrarily eigenvalues by state feedback was called *modal controllability*.

Zubov [2] and Wonham [3] were the first to prove the Theorem on pole assignment in the multi-input systems of the type (1) for *real* matrices A, B, S and self-conjugate set of complex numbers.

It should be noted that the proof of this Theorem in complex case (A, B, S and M are complex) is far simpler than real one.

Since then, when Zubov' and Wonham's works appeared, a great number (literally hundreds) of works, concerning pole assignment and its applications has been written. The primary impetus of most of the works mentioned concerns the stabilization problem for system (1).

The proof of Zubov's and Wonham's Theorem in multi-inpute case is rather tedious. Therefore after publication of works [2, 3] there were offered alternative proofs of this theorem (see, e.g., [11]-[13]; and also [14]-[19]). The goal of these papers was to give a simple proof of Zubov's and Wonham's Theorem.

Below we present another, different from the above-mentioned ones simple and direct new proof of Zubov's and Wonham's Theorem [20].

In the following instead of "complete controllability" of system (1) we will simply say about "controllability" of the pair (A, B).

2.1 Elementary Proof of Zubov's and Wonham's Theorem

A. ***Proof*** of Sufficiency.

Suppose that the pair (A, B) is not controllable. Then in system (1) we can separate from (1) a subsystem which contains no input variables. More precisely, there exists a nondegenerate linear transformation of coordinates $x \to Qx$ ($\det Q \neq 0$) such that the system (1) in new coordinates takes a form of the type (1) with the matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \begin{matrix} \}n_1 \\ \}n_2 \end{matrix},$$

$$A_{21} = 0, \quad B_2 = 0 \quad \text{or} \quad A_{12} = 0, \quad B_1 = 0.$$

Then it is clear that whatever $(m \times n)$-matrix

$$S = (\underbrace{S_1}_{n_1} \; \underbrace{S_2}_{n_2}) \}m \quad (n_1 + n_2 = n)$$

we take the spectrum of the closed-loop system matrix $A + BS$ in the form

$$\sigma(A + BS) = \sigma(A_{11} + B_1 S_1) \cup \sigma(A_{22})$$

or

$$\sigma(A + BS) = \sigma(A_{11}) \cup \sigma(A_{22} + B_2 S_2).$$

We see that one of two parts of the spectrum of the matrix $A + BS$ is independent of the choice of matrix $A + BS$. Therefore, the matrix cannot have arbitrarily preassigned eigenvalues. The Sufficiency is proved. □

The proof of Necessity leans on a number of simple propositions.

Lemma 2.1 *Let*

$$\Gamma = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \quad (\beta \neq 0) \tag{2}$$

be a real (2×2)-matrix. Let B be a real (2×2)- or (2×1)-matrix and $B \neq 0$. Then there exists a real matrix R such that the eigenvalues of the matrix $\Gamma + BR$ are real.

The Proof of Lemma 2.1 is straightforward.

Using Lemma 2.1 we can easily prove the following proposition.

Lemma 2.2 *Let Λ and B be real $(n \times n)$- and $(n \times m)$-matrices, respectively. Suppose the pair (Λ, B) is controllable and all eigenvalues of the matrix Λ are nonreal. Then there exists a real $(m \times n)$-matrix R such that all eigenvalues of the matrix $\Lambda + BR$ are real.*

Proof Let $\lambda_1, \bar{\lambda}_1, \dots, \lambda_\ell, \bar{\lambda}_\ell$ $(\lambda_j, \bar{\lambda}_j = \alpha_j \pm i\beta_j,\ \beta_j \neq 0,\ j = 1, \dots, \ell;\ n = 2\ell)$ be the eigenvalues of the matrix Λ, listed according to their multiplicity.

By a similarity matrix Q $(\det Q \neq 0)$ transforms the matrix Λ to the real lower Jordan canonical form

$$\tilde{\Lambda} = Q^{-1}\Lambda Q = \operatorname{diag}\{J_1(\lambda_1), \dots, J_q(\lambda_q)\},\ q \leq \ell.$$

Here $J_k(\lambda_k)$ $(k = 1, \dots, q)$ is a lower λ_k – Jordan block of dimension $2\nu_k \times 2\nu_k$ $\left(\sum_{k=1}^{q} \nu_k = \ell\right)$. That is, the block $J_k(\lambda_k)$ has (2×2)-matrices Γ_j $(j = 1, \dots, \ell)$ of the type (2) on the diagonal, the identity (2×2)-matrices lower the diagonal, and zero – matrices elsewhere.

Let $\tilde{B} := Q^{-1}B$. Find $(m \times n)$-matrix such that the matrix $\tilde{\Lambda} + \tilde{B}\tilde{R}_1$ has two real (may be equal) and $n-2$ nonreal eigenvalues. Then it will be the same for the matrix $\Lambda_1 := \Lambda + BR_1$, where $R_1 = \tilde{R}Q^{-1}$. In this case the pair (Λ_1, B) will be controllable since (Λ, B) is controllable by assumption.

We seek $\tilde{R}_1$ in the form of a block matrix $\tilde{R}_1 = [\tilde{R}_{pq}]$ containing four blocks $\tilde{R}_{pq}$ $(p, q = 1, 2)$ such that $\tilde{R}_{12} = 0, \tilde{R}_{21} = 0, \tilde{R}_{22} = 0$ and (2×2)-block matrix $\tilde{R}_{11}$ is to be determined. (In the case $m = 1$ $\tilde{R}_{11}$ is a row matrix of size 1×2.)

Divide the matrices $\tilde{\Lambda}$ and $\tilde{B}$ into four blocks

$$\tilde{\Lambda} = [\tilde{\Lambda}_{pq}], \quad \tilde{B} = [\tilde{B}_{pq}] \quad (p, q = 1, 2)$$

in such a way that $\tilde{\Lambda}_{11}$ and $\tilde{B}_{11}$ are (2×2)-matrices. (In the case $m = 1$ $\tilde{B}_{11}$ is a column matrix of dimension 2×1.) It is clear that $\tilde{\Lambda}_{11} = \Gamma_1$, $\sigma(\tilde{\Lambda}_{22}) = \{\lambda_2, \bar{\lambda}_2, \dots, \lambda_\ell, \bar{\lambda}_\ell\}$.

We have

$$\tilde{\Lambda}_1 := \tilde{\Lambda} + \tilde{B}\tilde{R}_1 \quad (p, q = 1, 2), \tag{3}$$

where $\tilde{M}_{12} = 0, \tilde{M}_{22} = \tilde{\Lambda}_{22}$ and

$$\tilde{M}_{11} = \Gamma_1 + \tilde{B}_{11}\tilde{R}_{11}. \tag{4}$$

The pair $(\tilde{\Lambda}, \tilde{B})$ is controllable, since the pair (Λ, B) is the same by assumption. Therefore it must be $(\tilde{B}_{11}, \tilde{B}_{12}) \neq 0$. Otherwise the pair $(\tilde{\Lambda}, \tilde{B})$ will not be controllable. Without loss of generality we assume that $\tilde{B}_{11} \neq 0$. Then by virtue of Lemma 2.1 there exists a matrix $\tilde{R}_{11}$ such that (2×2)-matrix (4) has real eigenvalues r_1 and r_2. Whence taking into account the structure of matrix (3) it follows that

$$\sigma(\tilde{\Lambda} + \tilde{B}\tilde{R}_1) = \{r_1, r_2\} \cup \{\lambda_2, \bar{\lambda}_2, \dots, \lambda_\ell, \bar{\lambda}_\ell\}.$$

Rearrange the matrices $\tilde{M}_{11}, \Gamma_j (j = \overline{1,\ell})$ in the diagonal array of matrix (3) in such a way that Γ_2 appears in the top left-hand corner of matrix (3).

We apply to matrix Γ_2 the same procedure as above for Γ_1. Thus we change the matrix Γ_2 by matrix of the type (4) having real eigenvalues. Therefore we obtain a matrix Λ_2 having four (among them may be equal ones) real eigenvalues and $n-4$ remaining nonreal ones $\lambda_3, \bar{\lambda}_3, \dots, \lambda_\ell, \bar{\lambda}_\ell$.

Repeating this process after ℓ steps as a result we obtain a matrix Λ_ℓ having only real eigenvalues. The Lemma 2.2 is proved. □

From Lemma 2.2 immediately it follows

Lemma 2.3 *Let A and B be arbitrary real $(n \times n)$- and $(n \times m)$-matrices, respectively. Let the pair (A, B) be controllable. Then there exists a real $(m \times n)$-matrix R such that all the eigenvalues of the matrix $A + BR$ are real.*

The following lemma solves the pole assignment problem in the field of real numbers $\mathbb{R}$.

Lemma 2.4 (Lemma on pole assignment in $\mathbb{R}$) *Let A and B be arbitrary real $(n \times n)$- and $(n \times m)$-matrices, respectively. Suppose the pair (A, B) is controllable. Let $\{\mu_1, \dots, \mu_n\}$ be an arbitrary set of real numbers. Then there exists a real $(m \times n)$-matrix such that*

$$\sigma(A + BS) = \{\mu_1, \dots, \mu_n\}. \tag{5}$$

Proof By virtue of Lemma 2.3 there exists a real $(m \times n)$-matrix R_0 such that all the eigenvalues of the matrix $A_0 := A + BR_0$ are real. We denote these ones by $\lambda_1, \dots, \lambda_n$, listed according to multiplicity. That is,

$$\sigma(A_0) = \{\lambda_1, \dots, \lambda_n\} \quad (\lambda_j \in \mathbb{R},\ j = 1, \dots, n). \tag{6}$$

The pair (A_0, B) is controllable since the pair (A, B) is the same by assumption.

Let $\mu_1, \dots, \mu_n$ be arbitrary n real numbers (among them may be repeating ones).

The proof of Lemma 2.4 is exactly analogous to that of Lemma 2.2 and consists of solution n intermediate tasks.

1) $\{A_0, B; \lambda_1|\mu_1\}$ – task: *Find real $(m \times n)$-matrix such that*

$$\sigma(A_0 + BS_1) = \{\mu_1; \lambda_2, \dots, \lambda_n\}. \tag{7}$$

As above by a similarity matrix Q_0 we transform the matrix A_0 to the real lower Jordan form: $\tilde{A}_0 := Q_0^{-1} A_0 Q_0$. Let $\tilde{B} = Q_0^{-1} B$.

We first solve $\{\tilde{A}_0, \tilde{B}; \lambda_1|\mu_1\}$ – task. For this purpose as above we seek a corresponding matrix $\tilde{S}_1$ in the form of a block matrix $\tilde{S}_1 = [\tilde{S}_{pq}]$ $(p, q = 1, 2)$ such that $\tilde{S}_{12} = 0$, $\tilde{S}_{21} = 0$, $\tilde{S}_{22} = 0$, and $\tilde{S}_{11} =: \tilde{s}_{11}$ is a real number to be determined.

As above divide the matrices $\tilde{A}_0$ and $\tilde{B}$ into four blocks

$$\tilde{A}_0 = [\tilde{A}_{pq}] \quad \tilde{B} = [\tilde{B}_{pq}] \ (p, q = 1, 2).$$

Here $\tilde{A}_{11} =: \tilde{a}_{11}$ and $\tilde{B}_{11} =: \tilde{b}_{11}$ are real numbers. Clearly, $\tilde{a}_{11} = \lambda_1$, $\sigma(\tilde{A}_{22}) = \{\lambda_2, \dots, \lambda_n\}$. Then we have $\tilde{A}_1 := \tilde{A}_0 + \tilde{B}\tilde{S}_1 = [\tilde{M}_{pq}]$, where $\tilde{M}_{12} = 0$, $\tilde{M}_{22} = \tilde{A}_{22}$ and

$$\tilde{M}_{11} =: \tilde{m}_{11} = \lambda_1 + \tilde{B}_{11}\tilde{S}_{11}. \tag{8}$$

Since the pair (A_0, B) is controllable as above in the proof of Lemma 2.2 one must have $(\tilde{B}_{11}, \tilde{B}_{12}) \neq 0$. Without loss of generality we assume that $\tilde{B}_{11} = \tilde{b}_{11} \neq 0$.

We claim that in (8) $\tilde{m}_{11} = \mu_1$. From here and (8) we determine $\tilde{s}_{11} = (\mu_1 - \lambda_1)/\tilde{b}_{11}$. Therefore we have

$$\sigma(\tilde{A}_0 + \tilde{B}\tilde{S}_1) = \{\mu_1; \lambda_2, \dots, \lambda_n\}.$$

That is, the matrix $\tilde{S}_1$ is solution of the $\{\tilde{A}_0, \tilde{B}; \lambda_1|\mu_1\}$-task. Then the matrix $S_1 = \tilde{S}_1 Q_0^{-1}$ will be solution of the task (7), since the matrix $A_0 + BS_1$ is similar to the matrix $\tilde{A}_0 + \tilde{B}\tilde{S}_1$.

Denote $A_1 := A_0 + BS_1$. Then $A_1 = A + B(R_0 + S_1)$.

We solve the next

2) $\{A_1, B; \lambda_2|\mu_2\}$-task: *Find* $(m \times n)$*-matrix such that*

$$\sigma(A_1 + BS_2) = \{\mu_1, \mu_2; \lambda_3, \dots, \lambda_n\}. \tag{9}$$

We will exactly solve task (9) analogously to task (7). At first we rearrange the diagonal elements $\mu_1, \lambda_2, \dots, \lambda_n$ of the matrix $\tilde{A}_1 = \tilde{A}_0 + \tilde{B}\tilde{S}_1$ in such a way that λ_2 appears in the top left-hand corner of matrix $\tilde{A}_1$.

Apply to matrices A_1, B and numbers λ_2, μ_2 the same procedure of "the replacement λ_1 by μ_1" that we have made in the previous task. In the same way we determine a matrix S_2 and corresponding matrix $A_2 = A + B(R_0 + S_1 + S_2)$ such that

$$\sigma(A_2) = \{\mu_1, \mu_2; \lambda_3, \dots, \lambda_n\}.$$

In this case $S_2 = \tilde{S}_2 Q_0^{-1} Q_1^{-1}$, where Q_1 is a similarity matrix, and $\tilde{S}_2$ is determined analogously to $\tilde{S}_1$.

Repeating this process of solving corresponding $\{A_{j-1}, B; \lambda_j|\mu_j\}$-tasks we sequentially replace each eigenvalue λ_j $(j = 1, \dots, n)$ of the matrix A_0 from (6) by corresponding number μ_j from the given set $\{\mu_j\}_{j=1}^n$. As a result we obtain successively the matrices $S_1, \dots, S_n$ such that the matrix

$$A_n = A + B(R_0 + S_1 + \cdots + S_n)$$

has $\{\mu_j\}_{j=1}^n$ for its desired set of eigenvalues. Hence, the matrix $S := R_0 + S_1 + \cdots + S_n$ has the required property (5).

Lemma 2.4 is proved. $\square$

Immediately from Lemma 2.4 it follows

Lemma 2.5 (Lemma on stabilization of the pair $(\mathbf{A}, \mathbf{B})$) *Let A and B be real $(n \times n)$- and $(n \times m)$-matrices, respectively. Let the pair (A, B) be controllable. Then there exists a real $(m \times n)$-matrix S such that the matrix $A + BS$ is stable, i.e. the pair (A, B) is stabilizable.*

Remark 2.1 A stabilization matrix S in Lemma 2.5 can be constructed by the algorithm described in the proof of Lemma 2.4.

Remark 2.2 In [21] another variant of elementary proof of the Lemma on Stabilization of the system (1) is proposed.

B. ***Proof*** of Necessity of Zubov's and Wonham's theorem.

Without loss of generality we may assume that $\operatorname{rank} B = m$.

If $m = n$, then the solution of pole assignment problem is given by the formula

$$S = B^{-1}(M - A),$$

where M is an arbitrary $(n \times n)$-matrix having the set $\{\mu_j\}_{j=1}^n$ as its spectrum. It remains to consider the case $1 \leq m < n$.

Let $\{\mu_j\}_{j=1}^n$ be an arbitrary set of n complex numbers closed under complex conjugation. We will prove that there exists a $(m \times n)$-matrix S such that

$$\sigma(A + BS) = \{\mu_j\}_{j=1}^n.$$

Assume among the numbers μ_j $(j = 1, \ldots, n)$ we have k real and ℓ complex-conjugate ones. Let $\mu_1, \ldots, \mu_k$ be real numbers and the rest of 2ℓ numbers $\mu_{k+1}, \bar{\mu}_{k+1}, \ldots, \mu_{k+1}, \bar{\mu}_{k+1}$ be complex conjugate ones. Let $\mu_{k+j}, \bar{\mu}_{k+j} = \sigma_{k+j} \pm i\omega_{k+j}$, $\omega_{k+j} \neq 0$ $(j = 1, \ldots .\ell; k + 2\ell = n)$.

Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be an arbitrary set of pairwise distinct real numbers λ_j $(j = 1, \ldots, n)$. By virtue of Lemma 2.4 there exists a real $(m \times n)$-matrix S_0 such that

$$\sigma(A + BS_0) = \{\lambda_1, \ldots, \lambda_n\} \quad (\lambda_p \neq \lambda_q,\ p \neq q,\ p, q = 1, \ldots, n).$$

Denote $A_0 := A + BS_0$.

1. Applying sequentially the algorithm of solving of $\{A_{q-1}, B; \lambda_q|\mu_q\}$ – tasks described in the proof of Lemma 2.4 we construct matrices $S_1, \ldots, S_k$ and the matrix

$$A_k = A_0 + B(S_1 + \cdots + S_k)$$

such that $\sigma(A_k) = \{\mu_1, \ldots, \mu_k; \lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n\}$.

2. We now solve the $\{A_k, B; \lambda_{k+1}, \lambda_{k+2}|\mu_{k+1}, \bar{\mu}_{k+1}\}$-task: *Find* $(m \times n)$*-matrix* S_{k+1} *such that*

$$\sigma(A_k + BS_{k+1}) = \{\mu_1, \ldots, \mu_k; \mu_{k+1}, \bar{\mu}_{k+1}; \lambda_{k+3}, \ldots, \lambda_n\}. \tag{10}$$

Since $\lambda_{k+1} \neq \lambda_{k+2}$ by a similarity matrix P_0 one can reduce the matrix A_k to the form of the four block matrix

$$\tilde{A}_k := P_0^{-1} A_k P{+}_0 = [\tilde{\Lambda}_{pq}] \quad (p, q = 1, 2),$$

where $\tilde{\Lambda}_{11} = (\lambda_{k+1}, \lambda_{k+2})$, $\tilde{\Lambda}_{12} = 0$. It is clear that

$$\sigma(\tilde{\Lambda}_{22}) = \{\mu_1, \ldots, \mu_k; \lambda_{k+3}, \lambda_{k+4}, \ldots, \lambda_n\}.$$

Divide the matrix $\tilde{B} := P_0^{-1}B$ into four blocks such that the matrix $\tilde{B}_{11}$ has the dimension 2×2 (or 2×1 in the case $m = 1$): $\tilde{B} = [\tilde{B}_{pq}]$ $(p, q = 1, 2)$. Let

$$\tilde{B}_{11} = (\tilde{b}_{rt})_{r,t=1}^2.$$

(In the case $m = 1$ $\tilde{B}_{11} = \mathrm{column}(\tilde{b}_{11}, \tilde{b}_{21}.)$

Since the pair (A, B) is controllable the pair (A_k, B), and therefore the pair $(\tilde{A}_k, \tilde{B})$ is controllable. Hence, as we have noted above in the proofs of Lemmas 2.2 and 2.4 one may assume that

$$\tilde{b}_{11} \neq 0 \quad \text{and} \quad \tilde{b}_{22} \neq 0. \tag{11}$$

To establish (10) we first solve $\{\tilde{A}_k, \tilde{B}; \lambda_{k+1}, \lambda_{k+2}|\mu_{k+1}, \bar{\mu}_{k+1}\}$-task. For this purpose as above we seek a matrix $\tilde{S}_{k+1}$ in the form of a block matrix $\tilde{S}_{k+1} = [\tilde{S}_{pq}]$ $(p, q = 1, 2)$, where $\tilde{S}_{12} = 0$, $\tilde{S}_{21} = 0$, $\tilde{S}_{22} = 0$ and (2×2)-matrix $\tilde{S}_{11}$ is to be determined (in the case $m = 1$ $\tilde{S}_{11}$ is a row (1×2)-matrix).

We have

$$\tilde{A}_{k+1} := \tilde{A}_k + \tilde{B}\tilde{S}_{k+1} = [\tilde{M}_{pq}], \tag{12}$$

where $\tilde{M}_{11} = \Lambda_{11} + \tilde{B}_{11}\tilde{S}_{11}$, $\tilde{M}_{12} = 0$, $\tilde{M}_{22} = \tilde{\Lambda}_{22}$. Two cases are possible: $a)$ $\det \tilde{B}_{11} \neq 0$ and $b)$ $\det \tilde{B}_{11} = 0$.

Case a). Here we claim that

$$\tilde{\Lambda}_{11} + \tilde{B}_{11}\tilde{S}_{11} = \Sigma_1, \tag{13}$$

where

$$\Sigma_1 = \begin{pmatrix} \sigma_{k+1} & -\omega_{k+1} \\ \omega_{k+1} & \sigma_{k+1} \end{pmatrix}.$$

From (13) we at once determine the matrix $\tilde{S}_{11} = (\tilde{B}_{11})^{-1}(\Sigma_1 - \tilde{\Lambda}_{11})$.

Case b). In this case we determine the matrix $\tilde{S}_{11}$ from the condition of equality of the characteristic polynomials of matrices in the right-hand and left-hand sides of (13):

$$\det(pI_2 - \tilde{\Lambda}_{11} - \tilde{B}_{11}\tilde{S}_{11}) = \det(pI_2 - \Sigma_1). \tag{14}$$

Here I_2 is the identity (2×2)-matrix.

Let $\tilde{S}_{11} = (c_{rt})^2_{r,t=j}$. (In the case $m = 1$ $\tilde{S}_{11} = (c_{11}, c_{12})$.) Taking into account inequalities (2.11) and equality $\tilde{b}_{11}\tilde{b}_{22} - \tilde{b}_{21}\tilde{b}_{12} = 0$, from (14) we can determine one of possible values of the entries c_{rt} of the matrix $\tilde{S}_{11}$:

$$\tilde{c}_{11} = d_1/\tilde{b}_{11} \quad c_{21} := 0; \quad c_{22} = d_2/\tilde{b}_{22} \quad (\tilde{b}_{22} \neq 0), \tag{15}$$

where

$$d_1 = (\sigma^2_{k+1} + \omega^2_{k+1} + \lambda^2_{k+1} - 2\lambda_{k+1}\sigma_{k+1})/(\lambda_{k+2} - \lambda_{k+1}),$$
$$d_2 = (\sigma^2_{k+1} + \omega^2_{k+1} + \lambda^2_{k+2} - 2\lambda_{k+2}\sigma_{k+1})/(\lambda_{k+1} - \lambda_{k+2}).$$

Since $\lambda_{k+1} \neq \lambda_{k+2}$ by choice of the set Λ the last expressions have a meaning. (In the case $m = 1$ $c_{11} = d_1/\tilde{b}_{11}$, $c_{12} = d_2/\tilde{b}_{21}$).

From (12) and (14) it follows that

$$\det(pI_n - \tilde{A}_{k+1}) = \det(pI_2 - \Sigma_1)\det(pI_{n-2} - M_{22}). \tag{16}$$

Here I_2, I_{n-2}, I_n are the identity matrices of respective dimensions. The equality (16) implies that for matrix (12) corresponding to the matrix $\tilde{S}_{k+1}$ with found above entries (15) the relation

$$\sigma(\tilde{A}_{k+1}) = \{\mu_1, \ldots, \mu_k; \mu_{k+1}, \bar{\mu}_{k+1}; \lambda_{k+3}, \ldots, \lambda_n\} \tag{17}$$

holds.

Set $S_{k+1} := \tilde{S}_{k+1}P_0^{-1}$. Since the matrix $\tilde{A}_{k+1}$ is similar to the matrix $A_{k+1} := A_k + BS_{k+1}$, from (17) it follows that relation (10) is valid for the matrix S_{k+1}.

Further we solve the $\{A_{k+1}, B; \lambda_{k+3}, \lambda_{k+4} | \mu_{k+2}, \tilde{\mu}_{k+2}\}$ – task exactly analogously to the preceding one. As a result we find a matrix S_{k+2} and a corresponding matrix $A_{k+2} := A_{k+1} + BS_{k+2}$ such that

$$\sigma(A_{k+2}) = \{\mu_1, \ldots, \mu_k; \mu_{k+1}, \bar{\mu}_{k+1}; \mu_{k+2}\bar{\mu}_{k+2}; \lambda_{k+5}, \ldots, \lambda_n\}.$$

Repeating this process as above after ℓ steps we find matrices $S_{k+1}, \dots, S_{k+\ell}$ and the matrix $A_{k+\ell} = A_k + B(S_{k+1} + \dots + S_{k+\ell})$ such that

$$\sigma(A_{k+\ell}) = \{\mu_1, \dots, \mu_k; \mu_{k+1}, \bar{\mu}_{k+1}, \dots, \mu_{k+\ell}\bar{\mu}_{k+\ell}\}. \tag{18}$$

Since

$$A_{k+\ell} = A + B(S_0 + \sum_{q=1}^{k} S_q + \sum_{j=1}^{\ell} S_{k+j}),$$

from (18) it follows that the $(m \times n)$-matrix

$$S = S_0 + \sum_{q=1}^{k} S_q + \sum_{j=1}^{\ell} S_{k+j}$$

has the required property.

Zubov's and Wonham's Theorem is completely proved. $\square$

Remark 2.3 In just proposed proof of Zubov's and Wonham's theorem we only used the fact of possibility of matrices reduction to Jordan canonical form. But there is also an elementary proof of the theorem on reduction of a matrix to Jordan form proposed by A. F. Filippov [22]. Together with this Filippov's theorem our above proof of Zubov's and Wonham's theorem is completely elementary.

Remark 2.4 As is seen from the proofs of Lemmas 2.2, 2.4 and the proof of the sufficiency of Zubov's and Wonham's theorem there is no necessity to reduce matrices to Jordan form. It is sufficient only to reduce them to the following forms. In the proof of Lemma 2.2 in the top left-hand corner of the considered matrices we must have a (2×2)-matrix Γ of the type (2) and the elements of the first two rows except for the entries of matrix Γ must be equal to zero.

Also, in the proof of Lemma 2.4 in the top left-hand corner we must have a number λ_j and the elements of the first row except, may be, for λ_j must be equal to zero. This observation also applies to the proof of sufficiency of Zubov's and Wonham's Theorem. As a result the finding of the required matrix S becomes more "economical" for computations: much less number of operations must be done.

3 Pole Assignment in Linear Systems with Output Feedback

In the preceding section we have considered linear systems with full state feedback.

We now turn our attention to pole assignment for linear systems by output feedback.

Consider a linear time-invariant continuous-time system described by

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{19}$$

where $x \in \mathbb{R}^n$ is a state vector, $u \in \mathbb{R}^m$ is an input vector, $y \in \mathbb{R}^\ell$ is an output vector, and A, B, C are real constant matrices of sizes $n \times n$, $n \times m$, $\ell \times n$, respectively $(\ell \le n)$.

Assume that the linear system (19) is controlled by a linear static output feedback

$$u = Sy \tag{20}$$

with a real constant $m \times \ell$-matrix S. Then the resulting closed-loop system (19), (20) is described by

$$\dot{x} = (A + BSC)x.$$

The poles of this system are the eigenvalues of the matrix $A + BSC$.

The problem of pole assignment arises in a natural way for closed-loop system using static output feedback.

Recall that this problem for system (19), (20) or simply triple matrices A, B, C is formulated as follows:

Given a triple real matrices (A, B, C) and an arbitrary set $\{\mu_j\}_{j=1}^{n}$ of the complex numbers μ_j closed under complex conjugation, find a real matrix S such that the spectrum of the matrix $A + BSC$ coincides with the set $\{\mu_j\}_{j=1}^{n}$, i.e.

$$\sigma(A + BSC) = \{\mu_j\}_{j=1}^{n}. \tag{21}$$

As we remarked above in the previous section this problem for two matrices A and B was first stated and solved by Zubov [2] and Wonham [3].

The problem of pole assignment by time-invariant static output feedback (20) has received much attention of researchers. Many works are devoted to solution of this problem and its various modifications (see surveys [23, 24]). Sufficient conditions have been obtained under which the pole assignment problem (21) can be resolved.

We note that for system (19) as for the system (1) property of controllability of the pair (A, B) is a necessary condition for the solvability of the pole assignment problem (21).

One of the pioneer works devoted to solving this problem was Davison's work [25]. In this work Davison proved the following assertion.

Theorem 3.1 (Davison [25]) *If the matrix A is cyclic (i.e. in its Jordan form to the distinct boxes correspond the distinct eigenvalues), the pair (A, B) is controllable and $\operatorname{rank} B = m, \operatorname{rank} C = \ell$, then there exists a matrix S such that the eigenvalues of the matrix $A + BSC$ of closed-loop system* (19), (20) *are arbitrary close to ℓ preassigned arbitrary numbers on the complex plane placed symmetrically with respect to the real axis.*

In the work [26] it was shown that if system (19) is controllable and observable, then there exists a matrix S such that the matrix $A + BSC$ is cyclic. Taking into account this result, in the paper [27] a theorem was proved which strengthen the Davison's Theorem. Namely, the following result is valid

Theorem 3.2 (Davison, Chatterjee [27]) *If (A, B) is controllable, (A, C) is observable, and $\operatorname{rank} B = m, \operatorname{rank} C = \ell$, then there exists a matrix S such that the* $\max\{\ell, m\}$ *eigenvalues of the matrix $A + BSC$ are arbitrary close to the* $\max\{\ell, m\}$ *preassigned arbitrary complex numbers closed under complex conjugation.*

In [28] an algorithm based on this theorem is given which allows pole assignment to be carried out on large linear systems (19) with output feedback (20).

In the case when A is a cyclic matrix an alternative proof of Davison's and Chatterjee's Theorem based on another approach was suggested in Sridhar's and Lindorff's work [29].

An analogous result under some other conditions is established in Jameson's work [30] for the systems with scalar input ($m = 1$). In this work for the case ($m = 1$) it is also proved that if the pair (A, B) is not controllable or the pair (A, C) is not observable and the eigenvalues λ_j $(j = 1, \dots, n)$ of the matrix A are distinct, then there is not any feedback matrix S such that the eigenvalues λ_{j_k} $(k = 1, \dots, r; r \leq n)$ which correspond to either the uncontrolled or unobserved variables, can be changed. Later, an alternative, more simple, proof of the second part of Jameson's assertion, extending his result to the systems with the vector input $(m > 1)$ was suggested in Nandi's and Herzog's note ([31]).

In later Davison's and Wang's [32] and Kimura's [33, 34] works it was established that under the same as above conditions on the matrices A, B and C for almost all A, B and C the $\min(n, m+\ell-1)$ eigenvalues of the matrix $A+BSC$ can be made arbitrarily close to the $\min(n, m+\ell-1)$ preassigned arbitrary complex numbers closed under complex conjugation.

This implies that if

$$m + \ell \geq n + 1,$$

then the pole assignment problem (21) is solvable for almost all matrices A, B and C.

Thus, the last inequality is a sufficient condition of solvability of the problem (21) in the typical case.

In Brockett's, Byrnes's [35] and Shumacher's [36] works there was given another sufficient condition of solvability of the problem (21) in the typical case. Namely, they show that

if $m\ell = n$ and the number

$$d(m, \ell) = \frac{1!2! \dots (\ell-1)!(m\ell)!}{m!(m+1)! \dots (m+\ell-1)!}$$

is odd, then the problem (21) *is solvable in the typical case.*

A sufficient condition in the case when the number $d(m, \ell)$ is even was obtained by Wang [37]:

if $m\ell > n$ and the number $d(m, \ell)$ is even, then the problem (21) *is solvable in the typical case.*

The distinct elementary proofs of this assertion were given in the works [38]-[41].

Another sufficient conditions of solvability of the problem (21) (and "near" problems) in the typical case were obtained in the works of many authors.

In Hermann's and Martin's [42] and Willems's and Hesselink's [43] papers it was established a general necessary condition

$$m\ell \geq n$$

of solvability of the problem (21) in the typical case. Later, this condition was strengthened in the work [44].

In [43] it is shown that, generally speaking, the inequality $m\ell \geq n$ is not a sufficient condition of solvability of the problem (21) in the typical case. Namely,

if $m = \ell = 2$ and $n = 4$, then the problem (21) *in the typical case is unsolvable.*

Note that in many works (see, for example, [45]-[51]) a more general than (21) eigenstructure assignment problem was considered. In this case the eigenvalues $\mu_1, \dots, \mu_r$ of the matrix of closed-loop system together with the corresponding to them eigenvectors $\xi_1, \dots, \xi_r$ are arbitrarily given or the elementary divisors, corresponding to these eigenvalues, are given. The problem is to find a matrix S such that either the spectrum of the matrix $A + BSC$ contains the set $\{\mu_j\}_{j=1}^r$ as a subset and the corresponding to the numbers μ_j eigenvectors of the matrix $A + BSC$ are equal to ξ_j (or are arbitrarily close to ξ_j) or the characteristic polynomial of the matrix $A+BSC$ has the preassigned polynomials $\psi_1,\dots,\psi_r(p)$ as its invariant factors (or elementary divisors).

One of the first works devoted to the eigenstructure assignment problem were the works of Rosenbrock [52], Kalman [53], Moore [54], and Srinathkumar [55]. The following result is valid.

Theorem 3.3 (Rosenbrock and Kalman [18, 52, 53]) *Suppose the pair (A,B) is controllable with the indices of controllability $k_1 \ge k_2 \ge \dots \ge k_m$. Let $\{\psi_i(p)\}_{i=1}^{q}$, $q \le m$ be a set of polynomials the leading coefficients of which are equal to 1. Assume that each polynomial ψ_i $(i=1,\dots,q-1)$ is divided by the successive one ψ_{i+1} without residue and $\sum\limits_{i=1}^{n} deg\,\psi_i = n$.*

Then for the existence of the matrix S such that the given polynomials ψ_i are the nontrivial (not equal identically to the unity) invariant factors of the characteristic polynomial $pI-A-BS$ it is necessary and sufficient that the following inequalities hold

$$\sum_{i=1}^{r} deg\psi_{q+1-i} \le \sum_{i=1}^{r} k_{q+1-i},\ \ r=1,2,\dots,q.$$

In this case the equality occurs for $r=q=m$.

(Here "deg" denotes a "degree of polynomial".)

In the papers [56, 57] Rosenbrock's and Kalman's theorem (and results of other authors) are generalized.

In the work [54] it was described the class of all sets of the eigenvectors of the matrix $A+BS$ of closed-loop system with state feedback, which can correspond to the preassigned arbitrarily distinct eigenvalues of this matrix. In the same work in the case of distinct eigenvalues there was given the solution of the problem of simultaneous assignment of the eigenvalues and the corresponding eigenvectors of the matrix of closed-loop system.

In the paper [55] a tool developed in [33, 58] was used for study of the eigenstructure assignment problem for systems with state feedback. In [55] Srinathkumar has proved, in particular, the following assertion.

If the pair (A,B) is controllable, the pair (A,C) is observable and rank (B) = m, rank $(C) = \ell$, then there exists a matrix S such that the eigenvalues of the matrix $A+BSC$ are equal to the $\max(m,\ell)$ preassigned numbers with the corresponding $\max(m,\ell)$ eigenvectors with $\max(m,\ell)$ preassigned arbitrary components.

We also note Van der Woude's paper where a general theorem is proved giving a necessary and sufficient condition (in geometric terms) of solvability of pole assignment problem (21) by output feedback (20) for single-input system (19) $(m=1)$.

Theorem 3.4 (Van der Woude [59]) *Suppose the system* (19) *is controllable and $f(p)$ is an arbitrary real polynomial with leading coefficient 1 of degree n.*

Then for the existence of a real $(\ell \times 1)$-matrix S such that

$$\det(pI-(A+BSC)) = f(p)$$

it is necessary and sufficient that

$$f(A)Ker(C) \subset Lin(B,AB,\dots,A^{n-2}B).$$

Lately Van der Woude's theorem was essentially used by Aeyels and Willems [60, 61] for pole assignment in linear time-invariant discrete-time systems by periodic static output feedback. In the end we note that at the present time the pole assignment problem and the related with it adjoining questions are in the focus of attention of many scholars and the flow of literature in this direction does not weaken.

Remark 3.1 Some above-mentioned result can be regarded as results for output stabilization problem, since the latter is a special case of pole assignment problem. These results are formulated in terms of matrices whereas in the well-known Nyquist criterion the necessary and sufficient condition of stabilization of the system (19) is formulated in terms of behavior of hodograph of the frequency response of this system.

4 Nonstationary Stabilization. The Brockett Problem

In 1999, R. Brockett in the book [62] formulated the problem on stabilizability of a linear time-invariant system by means of a static time-varying output feedback.

To solve this problem two approaches are developed. The first of them is developed for constructing a low-frequency time-varying feedback, and the second approach for constructing a high-frequency one.

The Brockett problem is formulated as follows.

Problem 4.1 (Brockett Problem) Given a linear time-invariant continuous-time system (19), find a static time-varying output feedback

$$u = S(t)y, \tag{22}$$

such that the resulting closed-loop system

$$\dot{x} = (A + BS(t)C)x \tag{23}$$

is asymptotically stable.

In the previous section some aspects of the problem of stabilization of system (19) by output feedback (22) with a constant matrix $S(t) \equiv S=$ const are considered. In the Brockett problem it is required to find a variable stabilizing matrix $S = S(t)$ with the property mentioned above. In this case the Brockett problem can be reformulated in the following way.

Does the introduction of the time-dependent matrices $S(t)$ in feedback gain extend the possibility of stationary stabilization?

In the works [63]-[66] for some important cases the solution of the Brockett problem of nonstationary linear stabilization for system (19) in the class of piecewise-constant periodic with a sufficiently large period stabilizing functions $S(t)$ is given(a low-frequency stabilization).

In the works [67]-[70] for single-input single-output system (19) the Brockett problem is solved in the other class of the stabilizing functions. Namely, this is solved in the class of continuous periodic with a sufficiently small period functions $S(t)$ (a high-frequency stabilization). Below we consider these two types of nonstationary stabilization.

4.1 Nonstationary low-frequency stabilization

Basic hypotheses. Suppose that there exist real constant $(m \times \ell)$-matrices S_1 and S_2 such that the linear systems

$$\dot{x} = (A + BS_jC)x \quad (x \in \mathbb{R}^n) \quad (j = 1, 2) \tag{24}$$

possess stable invariant linear manifolds L_j and invariant linear manifolds M_j.

Suppose

$$M_j \cap L_j = \{0\}, \quad \dim M_j + \dim L_j = n.$$

We assume also that for solutions $x_j(t; x_0)$ $(x_j(0; x_0) = x_0)$ of systems (24) the following inequalities

$$|x_j(t; x_0)| \le \alpha_j |x_0| e^{-\lambda_j t} \quad \forall\, x_0 \in L_j, \tag{25}$$

$$|x_j(t; x_0)| \le \beta_j |x_0| e^{-\kappa_j t} \quad \forall\, x_0 \in M_j, \tag{26}$$

are satisfied for positive numbers $\lambda_j, \kappa_j, \alpha_j, \beta_j$.

Suppose that there exist a continuous $(m \times \ell)$-matrix $\Sigma(t)$ and a number $r > 0$ such that during the time from $t = 0$ to $t = r$ the phase flow $\{\theta_{t_0}^r\}$ of the system

$$\dot{x} = (A + B\Sigma(t)C)x \quad (x \in \mathbb{R}^n) \tag{27}$$

takes the manifold M_1 to a manifold lying in L_2:

$$\theta_0^r M_1 \subset L_2. \tag{28}$$

Under these assumptions the following theorem holds.

Theorem 4.1 (The fundamental theorem) *Suppose the following inequality holds*

$$\lambda_1 \lambda_2 > \kappa_1 \kappa_2.$$

Then there exists a periodic $(m\times\ell)$-matrix $S(t)$ such that the system (23) is asymptotically stable. In this case stabilizing matrix $S(t)$ in (22) has the form

$$S(t) = \begin{cases} S_1 \quad \textit{for} \quad t \in [0, t_1), \\ \Sigma(t - t_1) \quad \textit{for} \quad t \in [t_1, t_1 + \tau), \quad S(t + T) = S(t), \\ S_2 \quad \textit{for} \quad t \in [t_1 + \tau, t_1 + t_2 + \tau), \end{cases} \tag{29}$$

where $T := t_1 + t_2 + \tau$ and positive numbers t_1 and t_2 are determined from conditions

$$\begin{cases} -\lambda_1 t_1 + \kappa_2 t_2 < -\tilde{T}, \\ -\lambda_2 t_2 + \kappa_1 t_1 < -\tilde{T}. \end{cases}$$

Here $\tilde{T}$ is a sufficiently large number.

Consider separately an important case of single-input single-output system (19). Let in (24)-(28)

$$S_1 = S_2 = S_0, \quad \Sigma(t) \equiv \Sigma_0, \quad S_0, \Sigma_0 \in \mathbb{R}, \tag{30}$$

$$S_0 \Sigma_0 < 0, \quad \lambda_1 = \lambda_2 = \lambda, \quad \kappa_1 = \kappa_2 = \kappa. \tag{31}$$

Suppose that all the eigenvalues λ_k of the matrix $A+B\Sigma_0 C$ have nonpositive real parts, in this case the eigenvalues with zero real parts have the prime divisors only.

Suppose there exists a sequence $\{\tau_j\} \to +\infty$ such that

$$\theta^{\tau_j} M_1 \subset L_2. \tag{32}$$

Here $\theta^t = e^{(A+\Sigma_0 BC)t}$ is the phase flow of system (27), where $\Sigma(t) \equiv \Sigma_0$. Then the following result is valid.

Theorem 4.2 *Suppose for system* (19) *the hypotheses* (30)-(32) *are satisfied. Suppose the inequality*

$$\lambda > \kappa$$

is valid. Then there exists T-periodic function with zero mean on the period such that the system (23) *is asymptotically stable. In this case the stabilizing function has the from*

$$S(t) = \begin{cases} S_0 & \textit{for} \quad t \in [0, t^0), \\ \Sigma_0 & \textit{for} \quad t \in [t^0, t^0 + \tau), \\ S_0 & \textit{for} \quad t \in [t^0 + \tau_j, 2t^0 + \tau_j), \end{cases} \qquad S(t+T) = S(t),$$

Here $T = \tau_j(1 - \Sigma_0/S_0)$ is a period of the function $S(t)$, $t^0 = |\tau_j \Sigma_0 / 2S_0|$ and τ_j is a sufficiently large number satisfying condition (32).

We remark that there are propositions which provide effective test of the "condition of manifolds embedding" (28) [63]-[66].

Applying Theorem 4.1 to two-dimensional case of system (19) ($n = 2$) one can prove the following assertion.

Theorem 4.3 *Suppose there exist $(m \times \ell)$-matrices S_0 and Σ_0 satisfying the following hypotheses:*

1) $\det(BS_0C) \neq 0$, $Tr\,(BS_0C) \neq 0$*; if* $\det(BS_0C) = 0$*, then, at least one of inequalities* $\det A \neq 0$ *or* $\det(a_1, r_2) + \det(r_1, a_2) \neq 0$*, is valid. Here a_1, a_2 and r_1, r_2 are the first and the second columns of the matrices A and BS_0C, respectively.*

2) *The matrix $A + B\Sigma_0 C$ has complex-conjugate eigenvalues.*

Then there exists a periodic matrix S(t) such that the system (23) *is asymptotically stable.*

4.2 Stabilization of linear system in the scalar case

Consider the system (19) with scalar input u and scalar output y ($m = \ell = 1$).

In the sequel we shall assume that the transfer function $W(p) = C(A - pI)^{-1}B$ of system (19) is nondegenerate. This is equivalent to the fact that the pair (A, B) is controllable and the pair (A, C) is observable.

By applying Theorem 4.1 one can prove a number of assertions.

A. The case of codimension 1 of the stable manifold.

In this case the following theorems hold.

Theorem 4.4 *Suppose the systems* (24) *have a stable invariant manifold L_2 of dimension $n-1$ and an one-dimensional invariant manifold M_1, satisfying basic conditions* (25)-(28).

Suppose also that S_1, S_2 *and* Σ_0 *are numbers such that* $\Sigma_0 \neq S_j\,(j = 1,2)$ *and the matrix* $Q = A + \Sigma_0 BC$ *has the complex-conjugate eigenvalues* $\alpha \pm i\beta$ *of multiplicity 1 and the rest of its eigenvalues* λ_k *satisfy the condition* $Re\lambda_k < \alpha$ $(k = 1, \ldots, n-2)$.

Then there exists a periodic function $S(t)$ *of the type* (29) *with* $\Sigma(t) = \Sigma_0$ *such that the system* (23) *is asymptotically stable.*

Theorem 4.5 *Let the system* (24) $(j = 1,2)$ *have a stable invariant manifold* L_2 *of dimension* $n-1$ *and an one-dimensional invariant manifold* M_1 *satisfying basic conditions* (25)-(28). *Suppose* $CB = 0$. *Then there exists a feedback* (22)*, where* $S(t)$ *is a piecewise-constant periodic function of the type* (29)*, such that the system* (23) *is asymptotically stable.*

Theorem 4.6 *Let in system* (19) $CB \neq 0$. *Suppose the matrix* A *has the eigenvalue* $\kappa > 0$ *and* $n-1$ *eigenvalues with the real part smaller than* $-\lambda$*, where* $\lambda > \kappa$. *Suppose that the inequality*

$$\frac{CB}{\lim\limits_{p \to \kappa} (\kappa - p) W(p)} < 1$$

is satisfied. Here $W(p)$ *is the transfer function of system* (19). *Then there exists a periodic function* $S(t)$ *of the type* (29) *such that the system* (23) *is asymptotically stable.*

Theorem 4.7 *Let* $CB \neq 0$. *Suppose that there exist numbers* $S_1 \neq S_2$ *such that:*

1) *the matrix* $A + S_1 BC$ *has the positive eigenvalue* κ_1.

2) *the matrix* $A + S_2 BC$ *has the one positive eigenvalue* κ_2 *and* $n-1$ *eigenvalues with negative real parts;*

3) *the inequality*

$$(CB)\frac{S_1 - S_2}{\kappa_2 - \kappa_1} < 1$$

holds. Suppose the condition $\lambda_1 \lambda_2 > \kappa_1 \kappa_2$ *of Fundamental Theorem is satisfied.*

Then there exists a periodic function $S(t)$ *of the type* (29) *such that the system* (23) *is asymptotically stable.*

B. The case of codimension 2 of the stable manifold

In this case the following result is valid.

Theorem 4.8 *Suppose the systems* (24) *have a* $n-2$*-dimensional stable invariant manifold* L_2 *and an one-dimensional invariant manifold* M_1 *satisfying basic conditions* (25)-(28). *Suppose that for a certain number* $\Sigma_0 \neq S_j$ $(j = 1,2)$ *the matrix* $A + \Sigma_0 BC$ *has two complex-conjugate eigenvalues* $\alpha \pm i\beta$ *of multiplicity 1 and the rest of its eigenvalues* λ_j *satisfy the condition* $Re\lambda_j < \alpha$. *Then there exists a periodic function* $S(t)$ *of the type* (29)*, where* $\Sigma(t) \equiv \Sigma_0$, $S_1, S_2, \Sigma_0 \in \mathbb{R}$*, such that the system* (23) *is asymptotically stable.*

4.3 Necessary conditions of stabilization

Above we derived some sufficient conditions of stabilizability of the system (19). Here we give necessary conditions of stabilizability of the system (19) with a scalar input u and a scalar output y.

A simple and general necessary condition of the impossibility of stabilization of system (19) is given by the following

Proposition 4.1 *If the inequality* $Tr(A + S(t)BC) \geq \alpha > 0$ *is satisfied for all* $t \in \mathbb{R}$ *and some positive number* α*, then the system* (23) *is not asymptotically stable.*

Here Tr denotes the trace of a matrix.

The statement of this proposition follows from the well-known Liouville formula.

Suppose now that the transfer function of system (19) is nondegenerate. Then it can be represented as the quotient

$$W(p) = \frac{\nu(p)}{\Delta(p)}$$

of the two polynomials

$$\begin{aligned} \nu(p) &= c_n p^{n-1} + c_{n-1}p^{n-2} + \cdots + c_1, \;\; c_k \in \mathbb{R}, \\ \Delta(p) &= p^n + a_n p^{n-1} + \cdots + a_1, \;\; a_k \in \mathbb{R} \;\; (k = 1, \ldots, n), \end{aligned}$$

with no common zeros. Here $\Delta(p)$ is the characteristic polynomial of the matrix A.

Assume that $c_n \neq 0$. In this case, without loss of generality, we set $c_n = 1$.

The following theorem gives sufficient conditions of the impossibility of stabilization of system (19).

Theorem 4.9 *Suppose for system* (19) *the following conditions are valid:*
1) *for* $n > 2$ $c_1 \leq 0, \ldots, c_{n-1} \leq 0$ *(for* $n = 2$ $c_1 \leq 0$*),*

$$\begin{aligned} 2) \quad & c_1(a_n - c_{n-1}) > a_1, \\ & c_1 + c_2(a_n - c_{n-1}) > a_2 \\ & \ldots\ldots\ldots\ldots\ldots \\ & c_{n-2} + c_{n-1}(a_n - c_{n-1}) > a_{n-1}. \end{aligned}$$

Then there does not exist a function $S(t)$ *such that the system* (23) *is asymptotically stable.*

Thus, a necessary condition of stabilization of the system (19) is the violation of at least one of hypotheses either 1) or 2) of Theorem 4.9 or the violation of inequality in the above Proposition.

4.4 Low-frequency stabilization of two-dimensional and three-dimensional systems

Now we apply the above results to the two-dimensional and three-dimensional systems.

A. Two-dimensional systems. Consider a system with a scalar input $u(t)$ and a scalar output $y(t)$, the transfer function of which is equal to the following

$$W(p) = \frac{c_2 p + c_1}{p^2 + a_2 p + a_1}. \tag{33}$$

Here $a_1, a_2; c_1, c_2$ are real numbers.

Let $c_2 \neq 0$. Then without loss of generality we can assume that $c_2 = 1$. Suppose also that the function $W(p)$ is nondegenerate, i.e.

$$c_1^2 - a_2 c_1 + a_1 \neq 0. \tag{34}$$

Then the system with transfer function (33) can be realized in the phase space as a system of the type (19)

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -a_1 x_1 - a_2 x_2 - u. \quad y = c_1 x_1 + x_2. \end{cases} \tag{35}$$

From the Routh-Hurwitz conditions it follows that by the feedback $u = S_0 y$, $S_0 =$ const $\neq 0$ the stationary stabilization of system (35) is possible if and only if either the inequality $c_1 > 0$ or the relations $c_1 \leq 0, a_2 c_1 < a_1$, are valid.

Consider the case when the stationary stabilization is impossible: $c_1 \leq 0, \quad a_2 c_1 \geq a_1$.

Applying Theorem 4.3 or Theorem 4.6 we can obtain the following sufficient condition of nonstationary stabilization of system (35) $c_1^2 - a_2 c_1 + a_1 > 0$. If the inequality $c_1^2 - a_2 c_1 + a_1 < 0$. holds, then the hypotheses of Theorem 4.9 are satisfied. Therefore, system (35) cannot be stabilizable by any feedback $u = S(t)y$.

Thus, we have the following

Theorem 4.10 *Suppose that the transfer function $W(p)$ of system (35) is non-degenerate, i.e. inequality (34) is valid. Then a necessary and sufficient condition of stabilizability of system (35) is that at least one of the conditions holds:*

$$1)\, c_1 > 0 \quad or \quad 2)\, c_1 \leq 0,\, c_1^2 - a_2 c_1 + a_1 > 0, \tag{36}$$

In this case for the stabilizing control $u = S(t)y$ the function $S(t)$ can be chosen as the piecewise-constant periodic one with sufficiently large period (a low-frequency stabilization).

Remark 4.1 Theorem 4.10 very well illustrates the fact that the introduction of a function $S(t) \neq S_0,\ S_0 =$ const, in the feedback $u = S(t)y$ (a nonstationary stabilization) extends the possibility of stationary stabilization ($S(t) \equiv S_0$).. Namely, in the space of parameters $\{(a_1, a_2; c_1)\}$ of system (35) conditions (36) select a more wide domain than the domain $\{c_1 > 0\} \cup \{c_1 < 0, a_2 c_1 < a_1\}$, defined by the Routh-Hurwitz conditions for stationary stabilization.

B. Three-Dimensional Systems

1) Suppose that the transfer function of system with a scalar input $u(t)$ and the scalar output $y(t)$ has the form

$$W(p) = \frac{1}{p^3 + \alpha p^2 + \beta p + \gamma}, \tag{37}$$

where α, β, γ are real numbers. Then such a system can be realized in the phase space $\mathbb{R}^3$ as a system of the type (19)

$$\begin{cases} \dot{x} = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -(\alpha x_3 + \beta x_2 + \gamma x_1) - u,\ y = x_1. \end{cases} \tag{38}$$

By the Routh-Hurwitz conditions the stationary stabilization $u = S_0 y$ of system (38) is possible if and only if

$$\alpha > 0 \quad \text{and} \quad \beta > 0.$$

Let $\alpha > 0$, $\beta \le 0$. In this case the stationary stabilization is impossible. Now we make use of Theorem 4.8.

By applying Theorem 4.8 to system (38) one can show that if $\alpha > 0$, $\beta \le 0$, there exists a control $u = S(t)y$, where $S(t)$ is a piecewise-constant periodic function with sufficiently large period, such that the system (38) with $u = S(t)y$ is asymptotically stable.

For system (38) with any feedback $u = S(t)y$ we have

$$\mathrm{Tr}(A + BS(t)C) = -\alpha \quad \forall t \in \mathbb{R}. \tag{39}$$

Then by Proposition from section 4.3 system (38) ($u = S(t)y$) is not asymptotically stable for $\alpha \le 0$.

Thus, we have the following

Theorem 4.11 *The system* (38) *with transfer function* (37) *is stabilized by feedback* (22) *if and only if* $\alpha > 0$. *In this case the function* $S(t)$ *for the stabilizing control can be chosen as the piecewise-constant periodic one with sufficiently large period (a low-frequency stabilization).*

2) Consider a system with a scalar input $u(t)$ and a scalar output $y(t)$ and the transfer function of the form

$$W(p) = \frac{p}{p^3 + \alpha p^2 + \beta p + \gamma}, \tag{40}$$

where α, β and γ are real numbers.

Let $\gamma \ne 0$. This condition is a condition of nondegenaracy of the function (40). Then this system can be realized in the phase space $\mathbb{R}^3$ as a system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -(\alpha x_3 + \beta x_2 + \gamma x_1) - u, \ y = x_2. \end{cases} \tag{41}$$

By the Routh-Hurwitz conditions the stationary stabilization of system (41) is possible if and only if $\alpha > 0, \gamma > 0$. Consider the case $\alpha > 0, \gamma < 0$. Then the stationary stabilization is impossible. We apply Theorem 4.5 with $S_1 = S_2$; $\lambda_1 = \lambda_2 = \lambda$, $\kappa_1 = \kappa_2 = \kappa$. Then we obtain that the conditions $\alpha > 0, \gamma < 0$, are sufficient for nonstationary stabilization of system (41).

Since for system (41) with any feedback $u = S(t)y$ the equality (39) holds, asymptotic stability of the system (41) is impossible for $\alpha \le 0$ by Proposition from section 4.3.

Thus, we have the following

Theorem 4.12 *Let* $\alpha \ne 0, \gamma \ne 0$. *Then for system* (41) *to be stabilized by feedback* (22) *it is necessary and sufficient that* $\alpha > 0$.

3) Consider a system with a scalar input $u(t)$ and a scalar output $y(t)$ and the transfer function of the form

$$W(p) = \frac{p^2}{p^3 + \alpha p^2 + \beta p + \gamma}, \tag{42}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

Suppose that the function (42) is nongenerate, i.e. $\gamma \neq 0$. Then this system can be realized in the phase space $\mathbb{R}^3$ as a system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -(\alpha x_3 + \beta x_2 + \gamma x_1) - u, \ y = x_3. \end{cases} \tag{43}$$

The stationary stabilization $u = S_0 y$ of system (43) is possible if and only if $\beta > 0, \gamma > 0$. In the case $\beta < 0, \gamma < 0$ by Theorem 4.9 the stabilization (a stationary or nonstationary) is impossible.

Consider the case $\beta > 0, \gamma < 0$, when the stationary stabilization is impossible. By applying the fundamental theorem (Theorem 4.1) from section 4.1 and as above, letting $S_1 = S_2;\ \lambda_1 = \lambda_2 = \lambda,\ \kappa_1 = \kappa_2 = \kappa$, one can prove the following assertion.

Theorem 4.13 *Let $\beta \neq 0, \gamma < 0$. Then for system* (43) *to be stabilized by feedback* (22) *it is necessary and sufficient that $\beta > 0$.*

Remark 4.2 As Theorem 4.10 Theorems 4.11–4.13 very well illustrate advantages of nonstationary stabilization in comparison with the stationary one.

4.5 Nonstationary high-frequency stabilization

In the previous section for some important cases the solution of the Brockett problem of nonstationary linear stabilization of system (19) in the class of piecewise-constant periodic stabilizing functions $S(t)$ is given.

In the works [67]-[70] another approach is proposed for solving the Brockett problem. This approach differs from the technique considered in the previous section and is based on the averaging method and uses some ideas and methods from vibrational control theory [71]-[74].

Also in this approach some research methods are used developed for the investigation well-known phenomenon of stabilization of the upper pendulum equilibrium position when the suspension point performs sufficiently fast oscillations in the vertical direction.

In [67]-[70] the Brockett problem is solved in the class of continuous periodic functions with a sufficiently small period (a high-frequency stabilization). There there are considered the functions of the form $S(t) = \alpha + \beta\omega^k \cos(\omega t)$, where $k \in \mathbb{N}$ and ω is a sufficiently large parameter.

We present corresponding results. Consider two cases:

1) $CB \neq 0$ and 2) $CB = CAB = 0$.

A. Stabilization in the case $CB \neq 0$.

In this case the following theorem holds.

Theorem 4.14 ([70]). *Let in system* (19) $CB \neq 0$. *Suppose that there exist real numbers α and $\kappa \geq 0$ such that the matrix*

$$A + \kappa(CB)BCA + (\alpha - \kappa CAB)BC \tag{44}$$

is stable. Then there exists a periodic function

$$S(t) = \alpha + \beta\omega\cos\omega t, \tag{45}$$

where ω is a sufficiently large number and $\beta \in \mathbb{R}$ satisfies the relation

$$\frac{\left(\frac{1}{2\pi}\int\limits_0^{2\pi}\exp(\beta CB\sin t)dt\right)^2 - 1}{(CB)^2} = \kappa, \tag{46}$$

such that the closed-loop system (23) *is exponentially stable uniformly with respect to ω for all sufficiently large ω.*

B. Stabilization in the case $CB = CAB = 0$.

In this case the following result is valid.

Theorem 4.15 ([70]). *Let in system* (19) *$CB = CAB = 0$. Suppose that there exist real numbers α and $\kappa \geq 0$ such that the matrix*

$$A - 3\kappa(CA^2B)BCA + (\alpha + \kappa CA^3B)BC \tag{47}$$

is stable. Then there exists a periodic function

$$S(t) = \alpha + \gamma\omega^2\cos\omega t, \tag{48}$$

where ω is a sufficiently large number and $\gamma \in \mathbb{R}$ satisfies the relation

$$\gamma^2 = 2\kappa. \tag{49}$$

such that the closed-loop system (23) *is exponentially stable uniformly with respect to ω for sufficiently large ω.*

Remark 4.3 In the work [70] the case when in system (19) $CB = CAB = \cdots = CA^{2k-1}B = 0$ is also considered. In the case $k > 1$ ($k \in \mathbb{N}$) the corresponding stabilization theorem is formulated similarly to Theorem 4.15: instead of the stability property of matrix (47) the stability property of the matrix

$$A + (-1)^k(2k+1)\kappa(CA^{2k}B)BCA + [\alpha + (-1)^{k+1}(2k+1)\kappa(CA^{2k+1}B)BC]$$

is required. In this case the stabilizing function has the form $S(t) = \alpha + \beta\omega^{k+1}\cos\omega t$.

4.6 High-frequency stabilization of two-dimensional and three-dimensional systems

Here we consider examples of application of Theorems 4.14 and 4.15 to two-dimensional and three-dimensional systems.

A. Two-dimensional systems. Consider the system (35). Suppose that inequality (34) holds. For system (35) the condition $CB \neq 0$ is valid. Therefore, we can apply Theorem 4.14. In this case the matrix (44) takes the form

$$\begin{pmatrix} 0 & 1 \\ -a_1 - \alpha c_1 - \kappa(c_1^2 - a_2c_1 + a_1) & -a_2 - \alpha \end{pmatrix}. \tag{50}$$

The matrix (50) is stable if and only if there exist the values of parameters $\alpha \in \mathbb{R}$ and $\kappa \in [0, +\infty)$ such that the inequalities

$$a_2 + \alpha > 0, \quad a_1 + \alpha c_1 + \kappa(c_1^2 - a_2c_1 + a_1) > 0 \tag{51}$$

hold. Relations (51) are satisfied if at least one of the inequalities

$$c_1 > 0 \quad \text{or} \quad c_1^2 - a_2 c_1 + a_1 > 0,\ c_1 \le 0 \tag{52}$$

is satisfied.

Thus, by Theorem 4.14 the condition (52) is sufficient for the existence of a control $u = S(t)y$, which stabilizes the system (35). Here $S(t)$ is a function of the type (45). As α, one may take an arbitrary number satisfying inequalities (51) for some $\kappa \ge 0$. As β, one should take a number satisfying the equation

$$\int_0^{2\pi} e^{-\beta \sin t} dt = 2\pi\sqrt{1+\kappa}. \tag{53}$$

It is easy to show that the equation (53) has a solution with respect to β.

By Theorem 4.9 if the inequality $c_1^2 - a_2 c_1 + a_1 < 0$ $(c_1 \le 0)$ is satisfied, then system (35) cannot be stabilized by any feedback of the type $u = S(t)y$.

Thus, we have the following

Theorem 4.16 ([67, 70]) *Suppose the inequality* (34) *holds. Then*

1) *if at least one of inequalities* (52) *is satisfied, then there exists a feedback*

$$u = S(t)y, \quad S(t) = \alpha + \beta\omega\cos\omega t, \tag{54}$$

where α and β are determined from (51) *and* (53)*, respectively, such that the closed-loop system* (35),(54) *is exponentially stable uniformly with respect to ω for sufficiently large ω;*

2) *if condition* (52) *is not satisfied, then for any choice of function $S(t)$ the system* (35)*, where $u = S(t)y$, is not exponentially stable.*

Thus, condition (52) is necessary and sufficient one for the existence of feedback (54) such that it stabilizes uniformly exponentially system (35) in the class of continuous and periodic functions $S(t)$.

The same condition (52), as was shown in section 4.4, is also necessary and sufficient one for stabilization of system (35) in the other class of the piecewise-constant periodic functions $S(t)$.

B. Three-dimensional systems. Consider a system (38), where $\alpha := a_3, \beta := a_2, \gamma := a_1 (a_1, a_2, a_3$ are real numbers).

The stationary stabilization ($S(t) \equiv \text{const}$) is possible if and only if $a_2 > 0, a_3 > 0$.

For system (38) the relations $CB = CAB = 0$ are valid. We apply Theorem 4.15. Here the matrix (47) takes the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 + \alpha + \kappa a_3 & -a_2 - 3\kappa & -a_3 \end{pmatrix}. \tag{55}$$

The matrix (55) is stable if and only if there exist values of the parameters $\alpha \in \mathbb{R}$ and $\kappa \in [0, +\infty)$ such that the following inequalities

$$\left.\begin{aligned} &a_3 > 0,\ a_1 - \alpha - \kappa a_3 > 0. \\ &a_3(a_2 + 3\kappa) - a_1 + \alpha + \kappa a_3 > 0 \end{aligned}\right\} \tag{56}$$

hold. The relations (56) are equivalent to the inequality $a_3 > 0$.

Thus, by Theorem 4.15 the condition $a_3 > 0$ is sufficient for the existence of the periodic function $S(t)$ of the type (48) such that the feedback $u = S(t)y$ exponentially stabilizes the system (38). Here one may take as α an arbitrary number satisfying relation (56) for some $\kappa \geq 0$. As γ a number satisfying the equation (49) should be taken.

The relation (39) holds. Therefore, if $a_3 \leq 0$ by Proposition from section 4.3 the system (38) is not asymptotically (and exponentially) stable for any feedback $u = S(t)y$.

Thus, the following result is valid.

Theorem 4.17 ([69, 70]) 1) *If in system* (38) $a_3 > 0$*, then there exists a feedback of the type* (48) *such that the system* (38),(48) *is uniformly with respect to* ω *exponentially stable for sufficiently large values of* ω.

2) *If* $a_3 \leq 0$*, then for no function* $S(t)$ *the exponential stabilization of system* (38) *is possible by means of the feedback* $u = S(t)y$.

Thus, the condition $a_3 > 0$ is necessary and sufficient one for the existence of the feedback of the type (48), which stabilizes uniformly exponentially the system (38). As was shown above in section 4.4 (Theorem 4.11) the same condition $a_3 > 0$ is also necessary and sufficient one for stabilization of system (38) in the class of the piecewise-constant periodic functions $S(t)$ with sufficiently large period (a low-frequency stabilization).

5 Discrete-time systems. Problem statement

In this part the discrete-time version of Brockett stabilization problem and pole assignment in discrete-time systems by periodic output feedback will be considered.

Consider a linear time-invariant discrete-time system

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k \quad (k = 0, 1, 2, \ldots), \tag{57}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input vector, $y_k \in \mathbb{R}^\ell$ is the output vector, A, B and C are real constant matrices of dimension $n \times n, n \times m$ and $\ell \times n$, respectively.

It is well known that if $C = I_n$ (I_n is the identity matrix) and the pair (A, B) is controllable then the poles of the system (57) can be assigned arbitrarily by time-invariant static state feedback [2, 3]. Hence the system (57) under the mentioned assumptions can be stabilizable. When only the output but not the state is available the problem of stabilizability and pole assignability by time-invariant static output feedback has also received much attention.

Necessary and/or sufficient conditions have been obtained under which stabilizability and pole assignability by time-invariant static output feedback are guaranteed. The basic results are available in the literature (see, for example, [18],[19] and surveys [23],[24]).

The question arises as to what extent the stabilization or pole assignment problem can be resolved by introducing time-varying static output feedback. The problem can be formulated as follows.

Problem 5.1 The Stabilization Problem.

Given a triple real constant matrices A, B and C, find a sequence of real $(m \times \ell)$-matrices $\{S_k\}$ $(k = 0, 1, 2, \ldots)$ such that the system (57) with the feedback

$$u_k = S_k y_k \quad (k = 0, 1, 2, \ldots), \tag{58}$$

i.e. the closed-loop system

$$x_{k+1} = (A + BS_kC)x_k \quad (k = 0, 1, 2, \ldots) \tag{59}$$

is asymptotically stable.

The Problem 5.1 is the discrete analog of the Brockett problem of stabilization of a linear continuous-time system by means of time-varying static output feedback.

It is important to notice that the discrete-time and continuous-time versions of Brockett problem are essentially different. This becomes clear, for example, from the fact that several difficulties and obstructions, which arise in solving of the Brockett problem in the continuous-time case, are lacking in the discrete-time case. For the statement the the next problem we assume that the time-dependent feedback (58) is periodic. i.e.

$$S_{k+p} = S_k \quad \forall k \in \{0, 1, 2, \ldots\}, \tag{60}$$

where p is a positive integer.

Then the system (59) is a periodic linear system of period p. This system can be considered as a time-invariant system with time interval equal to the period p:

$$\xi_{r+1} = M_s\xi_r \quad (r = 0, 1, 2, \ldots), \tag{61}$$

where

$$M_s = (A + BS_{p-1}C)(A + BS_{p-2}C)\ldots(A + BS_0C). \tag{62}$$

The eigenvalues of the composite matrix M_s determine the dynamics of the system (61), which in turn determines the dynamics of the periodic system (59),(60). These eigenvalues called multiplicators will be referred to as the poles of the periodic system (59). The matrix M_s is called the monodromy matrix for system (59),(60).

Now the pole assignment problem for the system (57) can be formulated in the following way:

Problem 5.2 The pole assignment problem.

Given a triple real constant matrices A, B and C, find real $(m \times \ell)$-matrices $S_0, S_1, \ldots, S_{p-1}$ such that the eigenvalues of the closed-loop system matrix M_s from (62) are the roots of a polynomial

$$f(z) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_0 \tag{63}$$

with real coefficients $\alpha_i\,(i = 0, 1, \ldots, n-1)$.

Clearly, Problem 5.2 is more general than Problem 5.1. Problem 5.1 has been studied in [75], and Problem 5.2 in [60],[61]. In these works two different approaches were offered in solving these problems. Here we present the corresponding results. We begin with Problem 5.2.

5.1 Pole assignability

Consider the system (57) with scalar input and scalar output ($m = l = 1$). Then B is a column matrix, C is a row matrix, the feedback gains S_k are numbers.

A. Two-Dimensional Case

In this case the following theorem yields a complete solution of Problem 5.2 in the sense of giving conditions for the realizability of pole assignment by static periodic output feedback.

Theorem 5.1 (On pole assignment: $n = 2$ [60]). *Let in system* (57) $n = 2$. *Suppose that the pair* (A, B) *is controllable and the pair* (A, C) *is observable. Then for the problem of pole assignment in system* (57) *by means of periodic feedback* (58),(60) *with period 3 to be solvable it is necessary and sufficient that* 1) $CA^{-1}B \neq 0$ *and* 2) $|CB| + |TrA| \neq 0$.

Remarks to Theorem 5.1

1. In Theorem 5.1 it is assumed that the system (57) is controllable and observable. This assumption is necessary for pole assignability. This follows from the well-known fact that uncontrollable and unobservable modes cannot be moved by static output feedback: neither by time-invariant nor by time-varying feedback [76].

2. In Theorem 5.1 it is assumed implicitly that the system matrix A is non-singular. This assumption entails no restriction as soon as the system (57) is controllable and observable. The nonsingularity of matrix A in controllable and observable system can be realized by a preliminary output feedback. Really, this follows from the well-known formula

$$\det[zI - (A + SBC)] = \Delta(z) + S\nu(z),$$

where the characteristic polynomial $\Delta(z) = \det(zI - A)$ of the matrix A and the polynomial $\nu(z)$ of degree not greater than $n - 1$ have no common roots.

3. From the result stated in [77] it follows that the pole assignability of system (57) is not possible in general by means of a periodic static output feedback of period 2. Therefore, at least periodic static output feedback of period 3, as considered in Theorem 5.1, is necessary to realize pole assignment.

B. Multidimensional Case

Let $W(z)$ denote the transfer function of system (57). Consider its representation in the form of rational function

$$W(z) = C(Iz - A)^{-1}B = \frac{q_{n-1}z^{n-1} + \cdots + q_1 z + q_0}{z^n + p_{n-1}z^{n-1} + \cdots + p_1 z + p_0}, \qquad (64)$$

where $p, q \in \mathbb{R}$ $(i = 0, 1, \ldots, n-1)$. Here in the denominator of (64) we have the characteristic polynomial of the matrix A.

The following theorem gives sufficient conditions under which the poles of system (57) of arbitrary order can be assigned by means of periodic output feedback (58),(60).

Theorem 5.2 (On pole assignment: $n > 2$ [61]) *Suppose that the pair* (A, B) *is controllable and the pair* (A, C) *is observable. Suppose that the coefficients* q_i, p_i $(i = 0, 1, \ldots, n-1)$ *of polynomials in the numerator and denominator of fraction* (64) *are non-zero and all quotients* p_i/q_i $(i = 0, 1, \ldots, n-1)$ *are mutually different. Let* $\alpha_0 \neq 0$ *in* (63). *Then for the problem of pole assignment in system* (57) *by means of periodic feedback* (58),(60) *with period* $p = n + 1$ *to be solvable it is sufficient that*

$$rank\,[B, A\Pi_{s^0}B, \ldots, (A\Pi_{s^0})^{n-1}B] = n,$$

where

$$\Pi_{s^0} = (A + S^0_{n-1}BC)\ldots(A + S^0_0 BC),\;\; s^0 := (S^0_0, S^0_1, \ldots, S^0_{n-1}) = \left(\frac{p_0}{q_0}, \frac{p_1}{q_1}, \cdots, \frac{p_{n-1}}{q_{n-1}}\right).$$

Remarks to Theorem 5.2

1. As in Theorem 5.1 (see Remark 1 to it) the conditions that (A, B) is controllable and (A, C) is observable are necessary for pole assignability. Therefore, without loss of

generality, the system matrix A is assumed to be nonsingular. By analogy with this we may also assume that the coefficients $p_i\,(i = 0, 1, \ldots, n-1)$ of characteristic polynomial of the matrix A are non-zero.

2. The assumption that the coefficients $q_i\,(i = 0, 1, \ldots, n-1)$ of the numerator in (64) are all non-zero is not necessary in general. This assumption is only a consequence of approach offered in [61]. The condition $q_0 \neq 0$ is equivalent to condition $CA^{-1}B \neq 0$, since $q_0 = -(CA^{-1}B)p_0$. This is necessary. Really, otherwise the determinant of the monodromy matrix M_s from (62) would be independent of the numbers $S_0, S_1, \ldots, S_{p-1}$ for any values of p, since

$$\det M_s = (\det A)^p \cdot (1 + S_0 CA^{-1}B)\ldots(1 + S_{p-1}CA^{-1}B).$$

On the other hand the condition $q_{n-1} \neq 0$ or equivalent (since $q_{n-1} = CB$) to it condition $CB \neq 0$ is not necessary in general. Indeed, in two-dimensional case (see Theorem 5.1) the zero value of CB may be allowed, but then the trace $\operatorname{Tr} A$ of the matrix A must be different from zero.

3. The condition that all quotients $p_i/q_i\,(i = 0, 1, \ldots, n-1)$ are mutually different is not necessary in general. It is a consequence of approach offered in [61]. For the second-order systems this condition is not necessary [60].

4. The condition that the period of the output feedback gain S_k is equal to $n+1$ is sufficient only. As remarked above for second order systems periodic feedback of period $p = 2$ cannot solve the pole assignment problem.

5. The condition $\alpha_0 \neq 0$ is equivalent to the condition that the poles of closed-loop system (59) must differ from the origin. This condition is not necessary. In [61] an example of third order system, where $\alpha_0 = 0$, is given but nevertheless the pole assignment is possible.

6. The result of Theorem 5.2 can be generalized for multi-input multi-output systems (see [61]).

5.2 Examples

A. A second order system ([60]).

Consider the system

$$x_{k+1} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} x_k + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_k, \quad y_k = (-1\ 1)x_k. \tag{65}$$

The system (65) is controllable and observable. Here $CB = 1, CA^{-1}B = -1/2$. Therefore by Theorem 5.1 the pole assignment for system (65) is solvable by means of periodic feedback (58),(60) with period $p = 3$.

This result can be obtained also by Theorem 5.2. Really, the transfer function of system (65) is

$$W(z) = \frac{z-1}{z^2 - z - 2}.$$

We have $p_0 = -2, p_1 = -1, q_0 = -1, q_1 = 1, S_0^0 = 2, S_1^0 = -1$. Also,

$$\Pi_{s^0} = (A + S_0^0 BC)(A + S_1^0 BC) = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \quad \text{rank } (B, A\Pi_{s^0}) = \text{rank} \begin{pmatrix} 0 & 3 \\ 1 & 3 \end{pmatrix} = 2.$$

Therefore, all conditions of Theorem 5.2 are satisfied.

It should be noted that the open-loop system (65) $x_{k+1} = Ax_k$ ($u_k := 0$) is unstable and cannot be stabilized by time-invariant output feedback $u_k = Sy_k$ ($S = \text{const}$).

B. A third order system ([61]).

Consider the discrete-time third-order system of the type (57) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = (q_0 \; q_1 \; q_1).$$

The transfer function is

$$W(z) = \frac{q_2 z^2 + q_1 z + q_0}{z^3 + p_2 z^2 + p_1 z + p_0}.$$

For the values of parameters $p_0 = 1.875, p_1 = 5.75, p_2 = 4.5, q_0 = 2, q_1 = 3, q_2 = 1$, the transfer function $W(z)$ has three poles $z_1 = -0.5, z_2 = -1.5, z_3 = -2.5$ and two real zeros $z_1^0 = -1, z_2^0 = -2$. By the root locus method it can be established that this system cannot be stabilized by constant output feedback.

Let the characteristic polynomial to be realized be $f(z) = z^3$, i.e. the closed-loop system is required to have all poles at the origin by introducing the periodic output feedback of period $p = 4$. A numerical analysis [61] yields the following result for feedback gains $S_0 = 0.9375,\ S_1 = 2.528322,\ S_2 = -8.928145,\ S_3 = 10$.

Note that the condition $\alpha_0 \neq 0$ of Theorem 5.2 is not satisfied. But nevertheless the pole assignment problem for considered system is solvable for the polynomial $f(z) = z^3$. Therefore, as is remarked above (Remark 5 to Theorem 5.2) the condition $\alpha_0 \neq 0$ in Theorem 5.2 is indeed not necessary.

Remark 5.1 Above for the pole assignment in system (57) the periodic memoryless output feedback has been used, i.e. the value of the input at a particular time $t = k$ depends on the output value at the same moment of time k. Contrary to this approach in the works [78]-[80] a memory in the periodic output feedback law is introduced. That is, value of the input at a moment $t = k$ depends on an output value at a time prior to this moment, namely at the beginning of the period. We adduce a result on pole assignment for single-input single-output system (57) by such kind (with memory) of periodic output feedback. In [79] the following theorem is established:

Let the pair (A, B) be controllable. Then the pole assignment problem has a solution if and only if the pair (A^n, C) is observable.

5.3 Stabilizability

We now turn to Problem 5.1 stated above.

5.3.1 Low-frequency stabilization of multi-input multi-output systems

Basic hypotheses. Suppose that there exist real constant $(m\times\ell)$-matrices $S_{(j)} (j = 1, 2)$ such that the systems

$$x_{k+1} = (A + BS_{(j)}C)x_k,\ x_k \in R^n\ (j = 1, 2)(k = 0, 1, 2, \ldots) \tag{66}$$

have stable invariant linear manifolds L_j and invariant linear manifolds M_j. Assume that $\dim M_j + \dim L_j = n$ and $M_j \cap L_j = \{0\}$. Suppose that for solutions $x_k^{(j)}(x_0^{(j)} = x_0)$ of systems (66) the following inequalities

$$\|x_k^{(j)}\| \le \alpha_j \|x_0\| e^{-\lambda_j k} \quad \forall x_0 \in L_j, \tag{67}$$

$$\|x_k^{(j)}\| \le \beta_j \|x_0\| e^{\mu_j k} \quad \forall x_0 \in M_j, \tag{68}$$

are satisfied for positive numbers $\lambda_j, \mu_j, \alpha_j, \beta_j$.

Assume that there exist a sequence of matrices $\{\Sigma_k\}_{k=0}^{\infty}$ and an integer $r \ge 1$ such that for the system

$$x_{k+1} = (A + B\Sigma_k C)x_k$$

the inclusion $\theta_0^r M_1 \subset L_2$ holds, where $\theta_0^r = \prod_{j=0}^{r-1}(A + B\Sigma_j C)$.

Under these assumptions we have the following

Theorem 5.3 (Fundamental Theorem on Stabilization [75]). *Suppose the inequality $\lambda_1\lambda_2 > \mu_1\mu_2$ holds. Then there exists a K-periodic matrix sequence $\{S_k\}(S_{k+K} = S_k, k = 0, 1, 2, \ldots; K \in \mathbb{N})$ such that the system* (59) *is asymptotically stable.*

In this case the stabilizing feedback gain matrix S_k has the form

$$S_k = \begin{cases} S_1 \quad \text{for} \quad k \in [0, k_1), \\ \Sigma_{k-k_1} \quad \text{for} \quad k \in [k_1, k_1 + r), \\ S_2 \quad \text{for} \quad k \in [k_1 + r, k_1 + k_2 + r), \end{cases}$$

where $K := k_1 + k_2 + r$, and positive integers k_1 and k_2 are determined from conditions

$$-\lambda_1 k_1 + \mu_2 k_2 < -T, \quad -\lambda_2 k_2 + \mu_1 k_1 < -T.$$

Here T is a sufficiently large positive number. (The notation $k \in [\alpha, \beta)$ means that k takes only integer values from the interval $[\alpha, \beta)$.)

5.3.2 Stabilization of single-input single-output systems

Consider the system (57) with scalar input u_k and scalar output y_k.

Theorem 5.3 implies the following assertion.

Theorem 5.4 (On Stabilization: $m = l = 1$ [75]). *Suppose the pair (A, B) is controllable and the pair (A, C) is observable. Let*

$$M_1 = M_2, \quad \dim M_1 = 1, \quad \mu_1 = \mu_2 = \mu, \qquad L_1 = L_2, \quad \dim L_1 = n - 1, \quad \lambda_1 = \lambda_2 = \lambda,$$

where $L_j, M_j (j = 1, 2)$ are linear manifolds introduced for systems (66)*, and λ_j, μ_j are numbers from* (67)*,* (68)*. Then if the inequality $\lambda > \mu$ is satisfied, there exists a K-periodic number sequence $\{S_k\}_{k=0}^{\infty}$ such that the system* (59) *is asymptotically stable.*

Using Theorem 5.4 one can prove the following

Theorem 5.5 (On Stabilization: $m = l = 1$ [75]). *Suppose the pair (A, B) is controllable and the pair (A, C) is observable. Suppose that there exists a number S_0 such that the matrix $A + S_0BC$ has $n-1$ eigenvalues $\rho_j (j = 1, \dots, n-1)$ located inside the unit circle and for the eigenvalue ρ_n the inequality*

$$\max_j |\rho_n \cdot \rho_j| < 1 \tag{69}$$

holds. Then there exists a K-periodic number sequence $\{S_k\}_{k=0}^{\infty}$ such that the system (59) *is asymptotically stable.*

Remark 5.2 It is well known that for time-invariant system $x_{k+1} = Dx_k$ to be asymptotically stable it is necessary and sufficient that all eigenvalues of the matrix D should be located inside the unit circle $|z| < 1$. The matrix $A + S_0BC$ is closed-loop system obtained after introducing in system (57) a time-invariant output feedback $u_k = S_0y_k (S_0 \in \mathbb{R})$. As is seen the condition (69) of Theorem 5.5 relaxes the requirement of locating all eigenvalues of the matrix $A+S_0BC$ inside the unit circle. Hence Theorem 5.5 extends the possibility of stationary stabilization (by time-invariant output feedback).

5.3.3 Stabilization of linear second order systems

Consider a linear single-input single-output system with the transfer function

$$W(z) = \frac{c_2z + c_1}{z^2 + a_2z + a_1}, \tag{70}$$

Here $a_1, a_2.c_1, c_2$ are real numbers.

Suppose the function $W(z)$ is nondegenerate, i.e.

$$c_1^2 - a_2c_1c_2 + a_1c_2^2 \neq 0. \tag{71}$$

A state-space realization of the system considered is a system of the type (57) with

$$A = \begin{pmatrix} 0 & 1 \\ -a_1 & -a_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (c_1 \;\; c_2). \tag{72}$$

Relation (71) is a necessary and sufficient condition for controllability of the pair (A, B) and observability of the pair (A, C).

Apply Theorem 5.5. The condition (69) is equivalent to the inequality $|a_1 - S_0c_1| < 1$ which have to be satisfied for some number S_0. This yields the conditions

$$c_1 \neq 0 \quad \text{or} \quad |a_1| < 1, \tag{73}$$

which by Theorem 5.5 are sufficient conditions for stabilizability of the system (57), (72).

One can show that conditions (73) are also necessary for stabilizability of the system considered.

Thus, since the similarity transformation $(A, B, C) \to (T^{-1}AT, T^{-1}B, CT)$ does not change the transfer function $W(z)$ and $\det A$, we have the following

Theorem 5.6 (On Stabilization: $n = 2, m = l = 1$ [75]). *Suppose that inequality* (71) *is satisfied. Then for the system* (57) *with transfer function* (70) *to be stabilizable it is necessary and sufficient that at least one of conditions*

$$W(0) \neq 0 \quad \textit{or} \quad |\det A| < 1 \tag{74}$$

is valid.

We note that the conditions 1) and 2) of Theorem 5.1 are also sufficient but, in general, not necessary conditions for stabilization of system (57) in the two-dimensional case. Since $W(0) = CA^{-1}B$, it is clear that for the stabilization of system (57) conditions (74) are milder than the conditions 1) and 2) of Theorems 5.1.

Remark 5.3 Comparing the conditions (73) of nonstationary stabilization of system defined by matrices (72) for special case $c_2 := 0 (c_1 \neq 0)$ with necessary and sufficient condition $|a_2| < 2$ of stationary stabilization we see the additional possibilities opened up by introducing time-variance in the feedback gain.

6 Conclusion

It is well known that although some interesting results are obtained for arbitrary pole assignment in linear time-invariant systems by means of time-invariant static output feedback, the possibility of this approach is limited. Another approach to the pole assignment stabilization problems is to consider the potential of time-varying static output feedback. This approach was developed for stabilization of continuous-time systems by Brockett [62]. For pole assignment in discrete-time systems this approach was considered by Aeyels and Willems [60, 61].

It is shown that the stabilization by means of periodic output feedback is possible under weak conditions. Necessary and sufficient conditions for nonstationary low- and high-frequency stabilization of two- and three-dimensional systems are derived. It turns out that time-varying feedback control strategy can achieve results that cannot be obtained by time-invariant feedback.

Analogous problems are considered for pole assignment and stabilization of time-invariant discrete-time control systems.

It is shown that under mild conditions stabilization of time-invariant control systems is possible by means of piecewise-constant periodic with a sufficiently large period output feedback (low-frequency stabilization). For second order systems necessary and sufficient conditions of stabilizability are obtained. Also, it is shown that introducing time-variance in the feedback gain opens up additional possibilities of stabilization of time-invariant discrete-time control systems.

Further, the results of works [60, 61] on pole assignment in discrete-time systems by time-varying static output feedback are presented.

Finally, we remark that the problems of stabilization of linear controllable systems are the high-capacity impetus for the development of new mathematical methods, which are presented in the present paper. Here an attempt is made to represent a constantly increasing number of publications, concerning this subject. In these publications not only the classical problems of stabilization are solved but the new notions are introduced and the new problems, arising in different applications, are considered. For the solution of these problems the methods, suggested in the present paper, can be useful.

Some of the methods described here are useful for investigations of nonlinear systems [81].

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