Nonlinear Dynamics and Systems Theory, **10** (3) (2010) 283–294

On Nonlinear Abstract Neutral Differential Equations with Deviated Argument

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Received: June 22, 2009; Revised: July 31, 2010

Abstract: In this paper we are concerned with a neutral differential equation with a deviated argument in an arbitrary Banach space X. To study the existence and uniqueness of a solution of the problem considered, we use the theory of the analytic semigroups and the fixed point arguments. Finally, we give an example to demonstrate an application of the abstract results.

Keywords: *neutral differential equation with a deviated argument; Banach fixed point theorem; analytic semigroup.*

Mathematics Subject Classification (2000): 34K30, 34G20, 47H06.

1 Introduction

In this study we are concerned with the following neutral differential equation with a deviated argument considered in a Banach space X:

$$\left\{\begin{array}{c} \frac{d}{dt}[u(t)+g(t,u(a(t)))]+Au(t)=f(t,u(t),u[h(u(t),t)]), \quad 0<t\leq T<\infty, \\ u(0)=u_0, \end{array}\right. \tag{1.1}$$

where $-A$ is the infinitesimal generator of an analytic semigroup. f, g, h and a are suitably defined functions satisfying certain conditions to be stated later.

Initial results related to the differential equations with the deviated arguments can be found in some research papers of the last decade but still a complete theory seems to be missing. For the initial works on the existence, uniqueness and stability of various types of solutions of different kinds of differential equations, we refer to [1]-[14] and the references cited in these papers.

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Hernandez and Henriquez [9, 10] established some results concerning the existence, uniqueness and qualitative properties of a solution operator of the following general partial neutral functional differential equation with the infinite delay:

$$\begin{aligned}\frac{d}{dt}(u(t)-g(t,u_t)) &= Au(t)+f(t,u_t), \quad t\geq 0,\\ u_0 &= \varphi\in C_0,\end{aligned}$$

where A generates an analytic semigroup on a Banach space B, g and f are continuous functions from $[0,\infty)\times C_0$ into B and for each $u:(-\infty,b]\to B,\ b>0$ and $t\in[0,b]$, u_t represents, as usual, the mapping defined from $(-\infty,0]$ into B by

$$u_t(\theta)=u(t+\theta)\quad \text{for}\quad \theta\in(-\infty,0].$$

Adimy *et al* [1] have studies the existence and stability of a solution of the following general class of nonlinear partial neutral functional differential equations:

$$\begin{aligned}\frac{d}{dt}(u(t)-g(t,u_t)) &= A(u(t)-g(t,u_t))+f(t,u_t), \quad t\geq 0,\\ u_0 &= \varphi\in C_0,\end{aligned}\tag{1.2}$$

where the operator A is the Hille-Yosida operator not necessarily densely defined on the Banach space B. The functions g and f are continuous from $[0,\infty)\times C_0$ into B.

In this paper, we use the Banach fixed point theorem and the analytic semigroup theory to prove the existence and uniqueness of different kinds of solutions to the problem (1.1). The plan of the paper is as follows. In Section 3, we prove the existence and uniqueness of a local solution and in Section 4, the existence of a global solution for the problem (1.1) is given. In the last section, we give an example.

The results presented in this paper can be applied easily to the problem (1.1) with a nonlocal condition under some modified assumptions on the function f and the operator A.

2 Preliminaries and Assumptions

As pointed out earlier, we note that if $-A$ is the infinitesimal generator of an analytic semigroup then for $c>0$ large enough, $-(A+cI)$ is invertible and generates a bounded analytic semigroup. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence, without loss of generality, we suppose that

$$\|S(t)\|\leq M\quad \text{for}\quad t\geq 0\quad \text{and}\quad 0\in\rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0\leq\alpha\leq 1$, A^α can be defined as a closed linear invertible operator with domain $D(A^\alpha)$ and being dense in E. We have $E_\kappa\hookrightarrow E_\alpha$, for $0<\alpha<\kappa$ and the embedding is continuous. For more details on the fractional powers of the closed linear operators, we refer to Pazy [15].

It can be proved easily that $E_\alpha:=D(A^\alpha)$ is a Banach space with norm $\|x\|_\alpha=\|A^\alpha x\|$ and it is equivalent to the graph norm of A^α. Also, for each $\alpha>0$, we define $E_{-\alpha}=(E_\alpha)^*$, the dual space of E_α is a Banach space endowed with the norm $\|x\|_{-\alpha}=\|A^{-\alpha}x\|$.

It can be seen easily that $\mathcal{C}_t^\alpha = C([0,t];E_\alpha)$, for all $t \in [0,T]$, is a Banach space endowed with the supremum norm,

$$\|\psi\|_{t,\alpha} := \sup_{0\le\eta\le t} \|\psi(\eta)\|_\alpha, \quad \psi \in \mathcal{C}_t^\alpha.$$

We set $\mathcal{C}_T^{\alpha-1} = C([0,T];E_{\alpha-1}) = \{y \in \mathcal{C}_T^\alpha : \|y(t)-y(s)\|_{\alpha-1} \le L|t-s|, \forall\ t,s \in [0,T]\}$, where L is a suitable positive constant to be specified later and $0 \le \alpha < 1$.

To proceed further, we need to assume the following assumptions on operator A and function f,g,h,a:

(**A1**): $0 \in \rho(-A)$ and $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t) : t \ge 0\}$.

(**A2**): Let $U_1 \subset \mathrm{Dom(f)}$ be an open subset of $\mathbb{R}_+ \times E_\alpha \times E_{\alpha-1}$ and for each $(t,u,v) \in U_1$ there is a neighborhood $V_1 \subset U_1$ of (t,u,v). The nonlinear map $f : \mathbb{R}_+ \times E_\alpha \times E_{\alpha-1} \to E$ satisfies the following condition,

$$\|f(t,x_1,y_1) - f(s,x_2,y_2)\| \le L_f[|t-s|^{\theta_1} + \|x_1-x_2\|_\alpha + \|y_1-y_2\|_{\alpha-1}],$$

where $0 < \theta_1 \le 1$, $0 \le \alpha < 1$, $L_f > 0$ is a constant, $(t,x_1,y_1) \in V_1$, and $(s,x_1,y_2) \in V_2$.

(**A3**): Let $U_2 \subset \mathrm{Dom(h)}$ be an open subset of $E_\alpha \times \mathbb{R}_+$ and for each $(x,t) \in U_2$ there is a neighborhood $V_2 \subset U_2$ of (x,t). The map $h : E_\alpha \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following condition

$$|h(x,t) - h(y,s)| \le L_h[\|x-y\|_\alpha + |t-s|^{\theta_2}],$$

where $0 < \theta_2 \le 1$, $0 \le \alpha < 1$, $L_h > 0$ is a constant, $(x,t),\ (y,s) \in V_2$ and $h(.,0) = 0$.

(**A4**): Let $U_3 \subset \mathrm{Dom(g)}$ be an open subset of $[0,T] \times E_{\alpha-1}$ and for each $(t,x) \in U_3$, there is a neighborhood $V_3 \subset U_3$ of (x,t). The function $g : [0,T] \times E_{\alpha-1} \to E_\beta$ is continuous for $(t,u) \in [0,T_0] \times E_{\alpha-1}$ such that

$$\begin{aligned} \|A^\beta g(t,x) - A^\beta g(s,y)\| &\le L_g\{|t-s| + \|x-y\|_{\alpha-1}\}, \text{ and} \\ L_g\|A^{\alpha-\beta-1}\| &< 1, \end{aligned}$$

where $0 \le \alpha < 1$, $L_g > 0$ is a positive constant $(x,t),(y,s) \in V_3$.

(**A5**): The function $a : [0,T] \to [0,T]$ satisfies the following two conditions:

(i) a satisfies the delay property $a(t) \le t$, for all $t \in [0,T]$;

(ii) The function a is Lipschitz continuous; that is, there exists a positive constant L_a such that

$$|a(t) - a(s)| \le L_a|t-s|, \text{ for all } t,s \in [0,T] \quad \text{and} \quad 1 > \|A^{-1}\|L_a.$$

Definition 2.1 A continuous function $u \in C_T^{\alpha-1} \cap C_T^\alpha$ is said to be a mild solution of equation (1.1) if u is the solution of the following integral equation

$$\begin{aligned} u(t) &= S(t)[u(0) + g(0,u_0)] - g(t,u(a(t))) + \int_0^t AS(t-s)g(s,u(a(s)))ds \\ &+ \int_0^t S(t-s)f(s,u(s),u[h(u(s),s)])ds,\ t \in [0,T] \end{aligned} \tag{2.3}$$

and satisfies the initial condition $u(0) = u_0$.

Definition 2.2 A function $u : [0, T] \to E$ is called a solution of 1.1 if u satisfies the following conditions,

(i) $u(.) + g(., u(a(.))) \in C_T^{\alpha-1} \cap C^1((0, T), E) \cap C([0, T], E)$,

(ii) $u(t) \in D(A)$, and $(t, u(t), u[h(u(t), t)]) \in U_1$,

(iii) $\frac{d}{dt}[u(t) + g(t, u(a(t)))] + A[u(t)] = f(t, u(t), u[h(u(t), t)])$ for all $t \in (0, T]$,

(iv) $u(0) = u_0$.

3 Existence of Local Solutions

In this section, we provide an existence and uniqueness theorem for a mild solution of (1.1). We set

$$\mathcal{W} = \{u \in C_{T_0}^{\alpha} \cap C_{T_0}^{\alpha-1} : u(0) = u_0, \ \ \|u - u_0\|_{T_0,\alpha} \le \delta\}.$$

Clearly, $\mathcal{W}$ is a closed and bounded subset of $C_T^{\alpha-1}$.

Under the assumptions **(A2)-(A3)**, $0 \le \alpha < 1$ and $u \in \mathcal{C}_{T_0}^{\alpha}$ imply that $f(s, u(s), u[h(u(s), s)])$ is continuous on $[0, T_0]$. Therefore, we can show that there exists a positive constant N such that

$$\|f(s, u(s), u[h(u(s), s)])\| \le N = L_f[{T_0}^{\theta_1} + \delta(1 + LL_h) + LL_h T_0^{\theta_2}] + N_0,$$

where $N_0 = \|f(0, u_0, u_0)\|$. Similarly, with the help of the assumptions **(A4)-(A5)**, we can easily show that $\|A^{\beta} g(t, u(a(t)))\| \le L_g[T_0 + \delta] + \|g(0, u_0)\|_{\alpha} = N_1$. Also, we denote $\|A^{-1}\| = M_2$ and $\|A^{-\alpha}\| = M_3$.

Theorem 3.1 *Let us assume that the assumptions (A1)-(A5) are satisfied and $u_0 \in D(A^{\alpha})$, for $0 \le \alpha < 1$. Then, the differential equation (1.1) has a unique local mild solution $u(t)$, for $t \in (0, T_0)$, where $T_0 = T_0(\alpha, \beta, u_0) > 0$ is sufficiently small.*

Proof For a fixed $\delta > 0$, we choose $0 < T_0 = T_0(\alpha, \beta, u_0) \le T$ such that

$$C_{\alpha+1-\beta} L_g \frac{T_0^{\beta-\alpha}}{\beta - \alpha} + C_{\alpha} L_f[2 + LL_h] \frac{T_0^{1-\alpha}}{(1 - \alpha)} \le 1 - \eta, \tag{3.4}$$

where $\eta = L_g\|A^{\alpha-\beta-1}\| < 1$ and satisfying the following

$$\|(S(t) - I)A^{\alpha}[u_0 + g(0, u_0)]\| + \|A^{\alpha-\beta}\| L_g[T_0 + \delta] \le \frac{\delta}{2} \tag{3.5}$$

for all $t \in [0, T_0]$ and

$$C_{\alpha+1-\beta} N_1 \frac{T_0^{\beta-\alpha}}{\beta - \alpha} + C_{\alpha} N \frac{T_0^{1-\alpha}}{1 - \alpha} \le \frac{\delta}{2}. \tag{3.6}$$

For more details of choosing such a T_0, we refer to Theorem 2.2 of [8].

We define a map $\mathcal{F} : C_{T_0}^{\alpha} \cap C_{T_0}^{\alpha-1} \to C_{T_0}^{\alpha} \cap C_{T_0}^{\alpha-1}$ as

$$\begin{aligned} (\mathcal{F}u)(t) \quad &= \quad S(t)[u_0 + g(0, u_0)] - g(t, u(a(t))) + \int_0^t AS(t - s) g(s, u(a(s)))ds \\ &+ \quad \int_0^t S(t - s) f(s, u(s), u[h(u(s), s)])ds, \quad t \in [0, T]. \end{aligned} \tag{3.7}$$

In order to prove this theorem, we need to show that $\mathcal{F}u \in \mathcal{C}_{T_0}^{\alpha-1}$, for any $u \in \mathcal{C}_{T_0}^{\alpha-1}$. Clearly, $\mathcal{F} : \mathcal{C}_T^{\alpha} \to \mathcal{C}_T^{\alpha}$.

If $u \in \mathcal{C}_{T_0}^{\alpha-1}$, $\quad T > t_2 > t_1 > 0$, and $0 \le \alpha < 1$, then we get

$$\begin{aligned}
\|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\|_{\alpha-1} &\le \|(S(t_2) - S(t_1))(u_0 + g(0, u_0))\|_{\alpha-1} \\
&+ \|A^{\alpha-\beta-1}\| \|A^{\beta} g(t_2, u(a(t_2))) - A^{\beta} g(t_1, u(a(t_1)))\| \\
&+ \int_0^{t_1} \|(S(t_2 - s) - S(t_1 - s))A^{\alpha-\beta}\| \|A^{\beta} g(s, u(a(s)))\| ds \\
&+ \int_{t_1}^{t_2} \|S(t_2 - s)A^{\alpha-\beta}\| \|A^{\beta} g(s, u(a(s)))\| ds. \\
&+ \int_0^{t_1} \|(S(t_2 - s) - S(t_1 - s))A^{\alpha-1}\| \\
&\qquad \times \|f(s, u(s), u[h(u(s), s)])\| ds \\
&+ \int_{t_1}^{t_2} \|S(t_2 - s)A^{\alpha-1}\| \|f(s, u(s), u[h(u(s), s)])\| ds. \qquad (3.8)
\end{aligned}$$

For the first part of the right hand side of (3.8), we have

$$\begin{aligned}
\|(S(t_2) - S(t_1))(u_0 + g(0, u_0))\|_{\alpha-1} &\le \int_{t_1}^{t_2} \|A^{\alpha-1} S'(s)(u_0 + g(0, u_0))\| ds \\
&= \int_{t_1}^{t_2} \|A^{\alpha} S(s)(u_0 + g(0, u_0))\| ds \\
&\le \int_{t_1}^{t_2} \|S(s)\| [\|u_0\|_{\alpha} + \|A^{\alpha-\beta}\| \|g(0, u_0)\|_{\beta}] ds \\
&\le C_1(t_2 - t_1), \qquad (3.9)
\end{aligned}$$

where $C_1 = [\|u_0\|_{\alpha} + \|A^{\alpha-\beta}\| \|g(0, u_0)\|_{\beta}]M$.

For the second part of the right hand side of (3.8), we can see that

$$\begin{aligned}
&\|A^{\alpha-\beta-1}\| \|A^{\beta} g(t_2, u(a(t_2))) - A^{\beta} g(t_1, u(a(t_1)))\| \\
&\qquad \le \|A^{\alpha-\beta-1}\| L_g [|(t_2 - t_1)| + \|u(a(t_2)) - u(a(t_1))\|_{\alpha-1}] \\
&\qquad \le \|A^{\alpha-\beta-1}\| [L_g + L L_a] |(t_2 - t_1)| \\
&\qquad \le C_2 |(t_2 - t_1)|. \qquad (3.10)
\end{aligned}$$

where $C_2 = \|A^{\alpha-\beta-1}\| [L_g + L L_a]$.

To handle the third and fifth parts of the right hand side of (3.8), we observe that

$$\begin{aligned}
\|(S(t_2 - s) - S(t_1 - s))\|_{\alpha-1} &\le \int_0^{t_2 - t_1} \|A^{\alpha-1} S'(l) S(t_1 - s)\| dl \\
&\le \int_0^{t_2 - t_1} \|S(l) A^{\alpha} S(t_1 - s)\| dl \\
&\le M C_{\alpha} (t_2 - t_1)(t_1 - s)^{-\alpha}. \qquad (3.11)
\end{aligned}$$

Now we use the inequality (3.11) to get the bound for the third part we have

$$\int_0^{t_1} \|(S(t_2 - s) - S(t_1 - s))A^{\alpha-\beta}\| \times \|A^{\beta} g(s, u(a(s))])\| ds \le C_4(t_2 - t_1), \qquad (3.12)$$

where $C_4 = N_1 M C_{\alpha-\beta+1} \frac{T_0^{1-(\alpha-\beta)}}{1-(\alpha-\beta)}$. Similarly, bound for the fifth part is given as

$$\int_0^{t_1} \|(S(t_2-s)-S(t_1-s))A^{\alpha-1}\| \times \|f(s,u(s),u[h(u(s),s)])\|ds \leq C_3(t_2-t_1), \quad (3.13)$$

where $C_3 = NMC_\alpha \frac{T_0^{1-\alpha}}{1-\alpha}$. For the bound for the sixth part, we have

$$\int_{t_1}^{t_2} \|S(t_2-s)A^{\alpha-1}\|\|f(s,u(s),u[h(u(s),s)])\|ds \leq C_5(t_2-t_1), \quad (3.14)$$

where $C_5 = \|A^{\alpha-1}\|MN$. Finally, for the fourth part we have the following

$$\int_{t_1}^{t_2} \|S(t_2-s)A^{\alpha-\beta}\|\|A^\beta g(s,u(a(s))\|ds \leq C_6(t_2-t_1), \quad (3.15)$$

where $C_6 = \|A^{\alpha-\beta}\|MN_1$.

We use the inequalities (3.9), (3.10), (3.13)-(3.15) in inequality (3.8) to get the following inequality

$$\|(\mathcal{F}u)(t_2)-(\mathcal{F}u)(t_1)\|_{\alpha-1} \leq \tilde{L}|t_2-t_1|, \quad (3.16)$$

where, $\tilde{L} = \max\{C_i,\ i=1,2,\cdots 6\}$. Hence, $\mathcal{F}: \mathcal{C}_{T_0}^{\alpha-1} \to \mathcal{C}_{T_0}^{\alpha-1}$ follows.

Our next task is to show that $\mathcal{F}: \mathcal{W} \to \mathcal{W}$. Now, for $t \in (0,T_0]$ and $u \in \mathcal{W}$, we have

$$\begin{aligned}
\|(\mathcal{F}u)(t)-u_0\|_\alpha &\leq \|(S(t)-I)A^\alpha[u_0+g(0,u_0)]\| \\
&\quad + \|A^{\alpha-\beta}\|\|A^\beta g(s,u(a(s)))-A^\beta g(0,u(a(0)))\| \\
&\quad + \int_0^t \|S(t-s)A^{1+\alpha-\beta}\|\|A^\beta g(s,u(a(s)))\|ds \\
&\quad + \int_0^t \|S(t-s)A^\alpha\|\|f(s,u(s),u[h(u(s),s)])\|ds \\
&\leq \|(S(t)-I)A^\alpha[u_0+g(0,u_0)]\| + \|A^{\alpha-\beta}\|L_g[T_0+\delta] \\
&\qquad + C_\alpha N \frac{T_0^{1-\alpha}}{1-\alpha} + C_{1+\alpha-\beta} N_1 \frac{T_0^{\beta-\alpha}}{\beta-\alpha}.
\end{aligned}$$

Hence, from inequalities (3.5) and (3.6), we get $\|\mathcal{F}u-u_0\|_{T_0,\alpha} \leq \delta$. Therefore, $\mathcal{F}: \mathcal{W} \to \mathcal{W}$.

Now, if $t \in (0,T_0]$ and $u,v \in \mathcal{W}$, then

$$\begin{aligned}
\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\|_\alpha &\leq \|A^{\alpha-\beta}\|\|A^\beta g(t,u(a(s)))-A^\beta g(t,v(a(s)))\| \\
&\quad + \int_0^t \|S(t-s)A^{1+\alpha-\beta}\|\|A^\beta g(s,u(a(s)))-A^\beta g(s,v(a(s)))\|ds. \\
&\quad + \int_0^t \|S(t-s)A^\alpha\| \\
&\quad \times\|f(s,u(s),u[h(u(s),s)])-f(s,v(s),v[h(u(s),s)])\|ds. \qquad (3.17)
\end{aligned}$$

We have the following inequalities

$$\|A^\beta g(t,u(a(s)))-A^\beta g(t,v(a(t)))\| \quad \leq \quad L_g\|A^{-1}\|\|u-v\|_{T_0,\alpha}, \quad (3.18)$$

$$\begin{aligned}&\|f(s,u(s),u[h(u(s),s)])-f(s,v(s),v[h(v(s),s)])\|\\&\quad\le L_f[2+LL_h]\|u-v\|_{T_0,\alpha}.\end{aligned}\tag{3.19}$$

We use the inequalities (3.18) and (3.19) in the inequality (3.17) and get

$$\begin{aligned}\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\|_\alpha \quad\le\quad & [L_g(\|A^{\alpha-\beta-1}\|+C_{1+\alpha-\beta}\frac{T_0^{\beta-\alpha}}{\beta-\alpha})\\ &+C_\alpha L_f[2+LL_h]\frac{T_0^{1-\alpha}}{(1-\alpha)}]\|u-v\|_{T_0,\alpha}.\end{aligned}\tag{3.20}$$

Hence, from inequality (3.4), we get the following inequality given below

$$\|\mathcal{F}u-\mathcal{F}v\|_{T_0,\alpha}<\|u-v\|_{T_0,\alpha}.$$

Therefore, the map $\mathcal{F}$ has a unique fixed point $u\in\mathcal{W}$ which is given by

$$\begin{aligned}u(t)\quad=\quad & S(t)[u_0+g(0,u_0)]-g(t,u(a(t)))+\int_0^t AS(t-s)g(s,u(a(s)))ds\\ +\quad & \int_0^t S(t-s)f(s,u(s),u[h(u(s),s)])ds \ \ t\in[0,T_0].\end{aligned}\tag{3.21}$$

Hence, the mild solution u of equation (1.1) is given by the equation (3.21) and belongs to $C_{T_0}^{\alpha}\cap C_{T_0}^{\alpha-1}$. Also, on the similar lines of the proof of Theorem 6.3.1, we can easily check that

$$\|u(t+h)-u(t)\|\le L'|h|^\gamma$$

for some $0<\gamma<1-\alpha$. Furthermore, the inequality of (A2), implies the local Hölder continuity of the function f for $t,s\in[t_0,T], 0<t_0<T$. Precisely for $u\in C_{T_0}^{\alpha-1}$ and moreover, $u\in C^\gamma((0,T],E_\alpha)$ for $0<\gamma<1-\alpha$:

$$\begin{aligned}&\|f(t,u(t),u[h(u(t),t)])-f(s,u(s),u[h(u(s),s)])\|\\&\le L_f\{|t-s|^{\theta_1}+\|u(t)-u(s)\|_\alpha+L|h(u(t),t)-h(u(s),s)|\}\\&\le L_f\{|t-s|^{\theta_1}+\|u(t)-u(s)\|_\alpha+LL_h[|t-s|^{\theta_2}+\|u(t)-u(s)\|_\alpha]\}\\&\le L_f\{|t-s|^{\theta_1}+L'|t-s|^\gamma+LL_h[|t-s|^{\theta_2}+L'|t-s|^\gamma]\}.\end{aligned}\tag{3.22}$$

Hence, the map $t\mapsto f(t,u(t),u[h(u(t),t)])$ is locally Hölder continuous. Therefore,

$$f(t,u(t),u[h(u(t),t)])\in C([0,T],E)\cap C^{\beta'}((0,T],E),$$

where $0<\beta'<\min\{\theta_1,\ \gamma,\ \theta_2\}$. Similarly, we can prove that $u(.)+g(.,u(a(.)))$ is also Hölder continuous on $(0,T_0]$. Therefore, from Theorem 3.1 pp. 110 and Corollary 3.3, pp. 113, Pazy [15], the function $u(.)+g(.,u(a(.)))\in C_{T_0}^{\alpha-1}\cap C^1((0,T_0),E)\cap C([0,T_0],E)$ and $u(.)$ is the unique solution of the problem (1.1) in the sense of definition (3.2) of Pazy [15]. This completes the proof of the Theorem. □

4 Existence of Global Solutions

In order to establish the global existence of a mild solution to (1.1), we need the following lemma.

Lemma 4.1 *Let $u_0(t,s) \geq 0$ be continuous on $0 \leq s \leq t \leq T < \infty$. If there are positive constants A, E and α such that*

$$u_0(t,s) \leq A + B\int_s^t (t-\sigma)^{\alpha-1} u_0(\sigma,s)d\sigma, \tag{4.1}$$

for $0 \leq s < t \leq T$, then there is a constant C such that $u_0(t,s) \leq C$.

Proof For $0 \leq s < t \leq T$, we have

$$\int_s^t (t-\tau)^{(\alpha-1)}(\tau-s)^{(\beta-1)}d\tau = (t-s)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{4.2}$$

which holds for every $\alpha, \beta > 0$. Integrating (4.1) $n-1$ times and using (4.2) and replacing $t-s$ by T, we get

$$u_0(t,s) \leq A\sum_{j=0}^{n-1}\left(\frac{BT^\alpha}{\alpha}\right)^j + \frac{(B\Gamma(\alpha))^n}{\Gamma(n\alpha)}\int_s^t (t-\sigma)^{n\alpha-1}u_0(\sigma,s)d\sigma. \tag{4.3}$$

Let n be large enough so that $n\alpha > 1$. We majorize $(t-\sigma)^{n\alpha-1}$ by $T^{n\alpha-1}$ to obtain

$$u_0(t,s) \leq c_1 + c_2\int_s^t u_0(\sigma,s)d\sigma. \tag{4.4}$$

Application of Gronwall's inequality leads to

$$u_0(t,s) \leq c_1 e^{c_2(t-s)} \leq c_1 e^{c_2 T} \leq C. \tag{4.5}$$

This completes the proof of the lemma.

Theorem 4.1 *Suppose that $0 \in \rho(-A)$ and the operator $-A$ generates the analytic semigroup $S(t)$ with $\|S(t)\| \leq M$, for $t \geq 0$, the conditions **(A1)–(A5)** are satisfied and $u_0 \in D(A^\alpha)$. If there are continuous nondecreasing real valued functions $k_1(t)$, $k_2(t)$ and $k_3(t)$ such that*

$$\begin{aligned} \|f(t,x,y)\| &\leq k_1(t)(1+\|x\|_\alpha + \|y\|_{\alpha-1}), &(4.6)\\ |h(x,t)| &\leq k_2(t)(1+\|z\|_\alpha), &(4.7)\\ \|g(t,y)\|_\beta &\leq k_3(t)(1+\|v\|_{\alpha-1}), &(4.8)\end{aligned}$$

for $t \geq 0$, $x \in E_\alpha$ and $y \in E_{\alpha-1}$, then the initial value problem (1.1) has a unique solution which exists for all $t \in [0,T]$.

Proof Let T_0 be sufficiently small as defined in the proof of Theorem 3.1 and let $u(t), t \in (0,T_0)$, be the local mild solution of (1.1). To prove the global existence of $u(t)$, we need to show that we can continue the solution of equation (1.1) as long as $\|u(t)\|_\alpha$ stays bounded. It is therefore sufficient to show that if u exists on $[0,T)$, then $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$.

We have the following inequality

$$\begin{aligned} \|u[h(u(s),s)]\|_{\alpha-1} &\leq \|u[h(u(s),s)] - u(0)\|_{\alpha-1} + \|u_0\|_{\alpha-1}\\ &\leq L|h(u(s),s)| + \|u_0\|_{\alpha-1}\\ &\leq Lk_2(T) + Lk_2(T)\|u\|_{s,\alpha} + \|u_0\|_{\alpha-1}. &(4.9)\end{aligned}$$

For $t \in [0, T)$, we have

$$\begin{aligned}
\|u(t)\|_\alpha \quad &\leq \quad \|S(t)A^\alpha[u_0 + g(0, u_0)]\| + \|A^{\alpha-\beta}\|\|g(t, u(a(t)))\|_\beta \\
&+ \quad \int_0^t \|A^{\alpha+1-\beta}S(t-s)\|\|A^\beta g(s, u(s)))\|ds \\
&+ \quad \int_0^t \|A^\alpha S(t-s)\|\|f(s, u(s), u[h(u(s), s)])\|ds \\
&\leq \quad M[\|u_0\|_\alpha + k_3(T)\|A^{\alpha-\beta}\|\{1 + \|A^{-1}\|\|u_0\|_\alpha\}] \\
&+ \quad k_3(T)\|A^{\alpha-\beta}\|[1 + \|A^{-1}\|\|u\|_{t,\alpha}] \\
&+ \quad C_{\alpha+1-\beta}\int_0^t (t-s)^{-1+\beta-\alpha}k_3(T)[1 + \|A^{-1}\|\|u\|_{s,\alpha}]ds, \\
&+ \quad C_\alpha \int_0^t (t-s)^{-\alpha}k_1[1 + \|u\|_{s,\alpha} + \|u[h(u(s), s)]\|_{\alpha-1}]ds \\
&\leq \quad M[\|u_0\|_\alpha + k_3(T)\|A^{\alpha-\beta}\|\{1 + \|A^{-1}\|\|u_0\|_\alpha\}] \\
&+ \quad k_3(T)\|A^{\alpha-\beta}\|[1 + \|A^{-1}\|\|u\|_{t,\alpha}] \\
&+ \quad k_3(T)C_{\alpha+1-\beta}\int_0^t (t-s)^{-(1+\alpha-\beta)}ds \\
&+ \quad \|A^{-1}\|k_3(T)C_{\alpha+1-\beta}\int_0^t (t-s)^{-(1+\alpha-\beta)}\|u\|_{s,\alpha}ds \\
&+ \quad k_1(T)C_\alpha \int_0^t (t-s)^{-\alpha}ds + k_1(T)C_\alpha \int_0^t (t-s)^{-\alpha}\|u\|_{s,\alpha}ds \\
&+ \quad (Lk_2(T) + \|u_0\|_{\alpha-1})k_1(T)C_\alpha \int_0^t (t-s)^{-\alpha}ds \\
&+ \quad Lk_2(T)k_1(T)C_\alpha \int_0^t (t-s)^{-\alpha}\|u\|_{s,\alpha}ds.
\end{aligned}$$

Hence,

$$\|u\|_{t,\alpha} \quad \leq \quad C_1 + \int_0^t (C_2(t-s)^{-\alpha} + C_3(t-s)^{\beta-\alpha-1})\|u\|_{s,\alpha}ds, \tag{4.10}$$

where

$$\begin{aligned}
C_1 \quad &= \quad \frac{M[\|u_0\|_\alpha + k_3(T)\|A^{\alpha-\beta}\|\{1 + \|A^{-1}\|\|u_0\|_\alpha\}] + k_3(T)\|A^{\alpha-\beta}\|}{(1 - k_3(T)\|A^{\alpha-\beta-1}\|)} \\
&+ \frac{k_1(T)C_\alpha T^{1-\alpha}}{(1 - k_3(T)\|A^{\alpha-\beta-1}\|)(1-\alpha)} \\
&+ \frac{(Lk_2(T) + \|u_0\|_{\alpha-1})k_1(T)C_\alpha T^{1-\alpha}}{(1 - k_3(T)\|A^{\alpha-\beta-1}\|)(1-\alpha)} \\
&+ \frac{k_3(T)C_{\alpha+1-\beta}T^{\alpha-\beta}}{(1 - k_3(T)\|A^{\alpha-\beta-1}\|)(\alpha-\beta)}, \\
C_2 \quad &= \quad \frac{k_1(T)C_\alpha[1 + Lk_2(T)]}{(1 - k_3(T)\|A^{\alpha-\beta-1}\|)}, \\
C_3 \quad &= \quad \frac{\|A^{-1}\|k_3(T)C_{\alpha+1-\beta}}{(1 - k_3(T)\|A^{\alpha-\beta-1}\|)}.
\end{aligned}$$

Now, we rewrite (4.10) as follows

$$\|u\|_{t,\alpha} \quad \leq \quad C_1 + \int_0^t \widetilde{C}_{2,3}(t-s)^{-\widetilde{\gamma}} \|u\|_{s,\alpha} ds, \tag{4.11}$$

where

$$\widetilde{C}_{2,3}(t-s)^{-\widetilde{\gamma}} = 2 \times \max[C_2(t-s)^{-\alpha}, C_3(t-s)^{\beta-\alpha-1}]. \tag{4.12}$$

Hence, by applying Lemma 4.1 to the above inequality (4.11), we get the required results. This completes the proof of the theorem. $\Box$

5 Example

Let $E = L^2(0,1)$. We consider the following partial differential equations with a deviated argument,

$$\begin{cases} \partial_t[w(t,x) + \partial_x f_1(t, w(a(t), x))] - \partial_x^2[w(t,x)] \\ \quad = f_2(x, w(t,x)), + f_3(t, x, w(t,x)), \quad x \in (0,1),\ t > 0, \\ \quad w(t,0) = w(t,1) = 0,\ t \in [0,T],\ 0 < T < \infty, \\ \quad w(0,x) = u_0,\ x \in (0,1), \end{cases} \tag{5.1}$$

where

$$f_2(x, w(t,x)) = \int_0^x K(x,s) w(s, h(t)(a_1|w(s,t)| + b_1|w_s(s,t)|)) ds.$$

The function $f_3 : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u and uniformly continuous in x. Further, we assume that $a_1, b_1 \geq 0, (a_1, b_1) \neq (0,0)$, $h : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in t with $h(0) = 0$ and $K : [0,1] \times [0,1] \to \mathbb{R}$.

We define an operator A, as follows,

$$Au = -u'' \quad \text{with} \quad u \in D(A) = \{u \in H_0^1(0,1) \cap H^2(0,1) :\ u'' \in E\}. \tag{5.2}$$

Here, clearly the operator A is self-adjoint with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$. Now we take $\alpha = 1/2$, $D(A^{1/2}) = H_0^1(0,1)$ is the Banach space endowed with the norm,

$$\|x\|_{1/2} := \|A^{1/2}x\|, \quad x \in D(A^{1/2})$$

and we denote this space by $E_{1/2}$. Also, for $t \in [0,T]$, we denote

$$C_t^{1/2} = C([0,t]; D(A^{1/2})),$$

endowed with the sup norm

$$\|\psi\|_{t,1/2} := \sup_{0 \leq \eta \leq t} \|\psi(\eta)\|_\alpha, \quad \psi \in \mathcal{C}_t^{1/2}.$$

We observe some properties of the operators A and $A^{1/2}$ defined by (5.2). For $u \in D(A)$ and $\lambda \in \mathbb{R}$, with $Au = -u'' = \lambda u$, we have $\langle Au, u \rangle = \langle \lambda u, u \rangle$; that is,

$$\langle -u'', u \rangle = |u'|_{L^2}^2 = \lambda |u|_{L^2}^2,$$

so $\lambda > 0$. A solution u of $Au = \lambda u$ is of the form

$$u(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

and the conditions $u(0) = u(1) = 0$ imply that $C = 0$ and $\lambda = \lambda_n = n^2\pi^2$, $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, the corresponding solution is given by

$$u_n(x) = D\sin(\sqrt{\lambda_n}x).$$

We have $\langle u_n, u_m\rangle = 0$ for $n \neq m$ and $\langle u_n, u_n\rangle = 1$ and hence $D = \sqrt{2}$. For $u \in D(A)$, there exists a sequence of real numbers $\{\alpha_n\}$ such that

$$u(x) = \sum_{n\in\mathbb{N}} \alpha_n u_n(x), \quad \sum_{n\in\mathbb{N}} (\alpha_n)^2 < +\infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} (\lambda_n)^2(\alpha_n)^2 < +\infty.$$

We have

$$A^{1/2}u(x) = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}\,\alpha_n\, u_n(x)$$

with $u \in D(A^{1/2})$; that is, $\sum_{n\in\mathbb{N}} \lambda_n(\alpha_n)^2 < +\infty$. $E_{-\frac{1}{2}} = H^1(0,1)$ is a Sobolev space of negative index with the equivalent norm $\|.\|_{-\frac{1}{2}} = \sum_{n=1}^{\infty} |\langle ., u_n\rangle|^2$. For more details on the Sobolev space of negative index, we refer to Gal [8].

The equation (5.1) can be reformulated as the following abstract equation in $E = L^2(0,1)$:

$$\begin{aligned} \frac{d}{dt}[u(t) + g(t, u(a(t)))] + A[u(t)] &= f(t, u(t), u[h(u(t), t)]) \quad t > 0,\\ u(0) &= u_0, \end{aligned} \tag{5.3}$$

where $u(t) = w(t, .)$ that is, $u(t)(x) = w(t, x)$, $x \in (0,1)$. The function $g : \mathbb{R}_+ \times E_{1/2} \to E$, such that $g(t, u(a(t)))(x) = \partial_x f_1(t, w(a(t), x))$ and the operator A is same as in equation (5.2).

The function $f : \mathbb{R}_+ \times E_{1/2} \times E_{-1/2} \to E$, is given by

$$f(t, \psi, \xi)(x) = f_2(x, \xi) + f_3(t, x, \psi), \tag{5.4}$$

where $f_2 : [0,1] \times E \to H_0^1(0,1)$ is given by

$$f_2(t, \xi) = \int_0^x K(x, y)\xi(y)dy, \tag{5.5}$$

and $f_3 : \mathbb{R} \times [0,1] \times H^2(0,1) \to H_0^1(0,1)$, satisfies the following

$$\|f_3(t, x, \psi)\| \leq Q(x, t)(1 + \|\psi\|_{H^2(0,1)}) \tag{5.6}$$

with $Q(., t) \in E$ and Q is continuous in its second argument. We can easily verify that the function f satisfies the assumptions (A1)-(A4). For more details see [8].

For the function a we can take

(i) $a(t) = kt$, where $t \in [0, T]$ and $0 < k \leq 1$;

(ii) $a(t) = kt^n$ for $t \in I = [0,1]$ $k \in (0,1]$ and $n \in \mathbb{N}$;

(iii) $a(t) = k\sin t$ for $t \in I = [0, \frac{\pi}{2}]$, and $k \in (0,1]$.

Acknowledgments

The authors would like to thank the referees for the valuable suggestions. The second and third authors would like to acknowledge the financial help provided by the Department of Science & Technology (DST), New Delhi, under its research project SR/S4/MS:581/09.

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