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Painlevé Test to a Reduced System of Six Coupled Nonlinear ODEs

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Abstract: In this paper we investigate the complete integrability of the system of six coupled nonlinear ODEs, which arises in the ODE reduction of rotating stratified Boussinesq equations. We use Painlevé test to investigate the complete integrability of the system. And we conclude that the system is completely integrable only if the Rayleigh number $Ra = 0$. The singular solution of the system admits the movable pole type singularity in complex domain.

Keywords: *Painlevé test; rotating stratified Boussinesq equations; integrable system.*

Mathematics Subject Classification (2000): Primary 37K10, Secondary 76B70.

1 Introduction

We undertake the Painlevé analysis of the system of six coupled nonlinear ODEs arising as a reduction of rotating stratified Boussinesq equations. The rotating stratified Boussinesq equations form a system of partial differential equations modelling the movement of planetary atmosphere. In their study of instability in stratified fluids at large *Richardson number*, Majda and Shefter [1] analyzed certain system of ODE reduction of stratified Boussinesq equations. Srinivasan et al [2] gave the complete analysis of reduced system of ODEs and discussed the stability of degenerate critical point. In their paper Desale and Srinivasan [3] examine the same system in the light of the ARS (Ablowitz, Ramani and Segur [4]) conjecture. Ablowitz, Ramani and Segur have conjectured that a system of PDEs is completely integrable if all its ODE reductions are of Painlevé type. The conjecture has been tested on large class of differential equations and has since been

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employed as a popular test of integrability. Whereas in the basin scale dynamics Maas [5], has considered the flow of fluid contained in rectangular basin of dimension $L \times L \times H$, which is temperature stratified with the fixed zeroth order moments of mass and heat. The container is assumed to be steady, uniformly rotating on an f-plane. With this assumption Maas [5] reduces the rotating stratified Boussinesq equations to an interesting six coupled system of ODEs. Further, Desale [6] has given the complete analysis of the system and also tested the system for complete integrability by determining the four first integrals and uses the Jacobi's theorem. In their recent paper Desale and Sharma [7] have reduced the rotating stratified Boussinesq equations into the system of six coupled ODEs that are also in similar nature with the system which we are looking in this paper.

In this paper we have tested the system of six coupled nonlinear ODEs for its complete integrability via Painlevé analysis. Here we state that our analysis follows similar kind of techniques as used by Desale and Srinivasan in their paper [3]. But our system includes additional terms due to the effects of rotation so that in calculations we are far apart from Desale and Srinivasan [3].

This paper is organized as follows. Section 2 gives the ODE reduction of rotating stratified Boussinesq equations. We implement the Painlevé test to determine the singular solution of the system in Section 3. In Section 4, we illustrate two systems that also exhibit the similar kind of solutions. Finally, we conclude the results in Section 5.

2 Reduced System of Nonlinear ODEs

We now begin by describing the rotating stratified Boussinesq equations (see Majda [8], p. 1)

$$\begin{aligned} \frac{D\mathbf{v}}{Dt} + f(\hat{\mathbf{e_3}} \times \mathbf{v}) &= -\nabla p + \nu(\Delta \mathbf{v}) - \frac{g\tilde{\rho}}{\rho_b}\hat{\mathbf{e_3}}, \\ \operatorname{div} \mathbf{v} &= 0, \\ \frac{D\tilde{\rho}}{Dt} &= \kappa \Delta \tilde{\rho}, \end{aligned} \tag{1}$$

where $\mathbf{v}$ denotes the velocity field, ρ is the density of fluid which is the sum of constant reference density ρ_b and perturb density $\tilde{\rho}$, p is the pressure, g is the acceleration due to gravity that points in $-\hat{\mathbf{e_3}}$ direction, f is the rotation frequency of earth, ν is the coefficient of viscosity, κ the coefficient of heat conduction and $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$ is a convective derivative. For more about rotating stratified Boussinesq equations one may consult with Majda [8].

In the frame of reference of an uniformly stratified fluid contained in rotating rectangular box of dimension $L \times L \times H$, which is temperature stratified with fixed zeroth order moments of mass and heat (so that there is no net evaporation or precipitation, nor any net river input or output, and neither a heating nor cooling). The container is assumed to be in steady uniform rotation on an f-plane. Maas [5] reduces the system of equations (1) into the following system of six coupled ODEs:

$$\begin{aligned} Pr^{-1}\frac{d\mathbf{w}}{dt} + f'\hat{\mathbf{e_3}} \times \mathbf{w} &= \hat{\mathbf{e_3}} \times \mathbf{b} - (w_1,\, w_2,\, rw_3) + \hat{T}\mathbf{T}, \\ \frac{d\mathbf{b}}{dt} + \mathbf{b} \times \mathbf{w} &= -(b_1,\, b_2,\, \mu b_3) + Ra\mathbf{F}. \end{aligned} \tag{2}$$

In these equations, $\mathbf{b} = (b_1,\, b_2,\, b_3)$ is the center of mass, $\mathbf{w} = (w_1,\, w_2,\, w_3)$ is the basin's averaged angular momentum vector, $\mathbf{T}$ is the differential momentum, $\mathbf{F}$ are buoyancy fluxes, $f' = f/2r_h$ is the earth's rotation, $r = r_v/r_h$ is the friction ($r_{v,h}$ are the Rayleigh damping coefficients), Ra is the Rayleigh number, Pr is the Prandtl number, μ the diffusion coefficient and $\hat{T}$ is the magnitude of the wind stress torque.

Neglecting diffusive and viscous terms, Maas [5] consider the dynamics of an ideal rotating, uniformly stratified fluid in response to forcing. He assumes this to be due solely to differential heating in the meridional (y) direction $\mathbf{F} = (0, 1, 0)$; the wind effect is neglected i.e. $\mathbf{T} = 0$. For Prandtl number, Pr, equal to one the system of equations (2) reduces to the following ideal rotating, uniformly stratified system of six coupled ODEs

$$\begin{aligned} \frac{d\mathbf{w}}{dt} &= -f'\hat{\mathbf{e_3}} \times \mathbf{w} + \hat{\mathbf{e_3}} \times \mathbf{b}, \\ \frac{d\mathbf{b}}{dt} &= -\mathbf{b} \times \mathbf{w} + Ra\mathbf{F}. \end{aligned} \tag{3}$$

In his paper Desale [6] has demonstrated the complete integrability of the system (3) for $Ra = 0$. Our approach to discuss the integrability of above system is quite different than Desale has used in his paper [6]. In the following section we deploy the Painlevé test for complete integrability of the system (3).

3 Singular Solution of the System

We can write the system of six coupled ODEs (3) component-wise as:

$$\begin{aligned} &\dot{w_1} = f'w_2 - b_2, \quad \dot{w_2} = -f'w_1 + b_1, \quad \dot{w_3} = 0, \\ &\dot{b_1} = w_2 b_3 - w_3 b_2, \quad \dot{b_2} = w_3 b_1 - w_1 b_3 + Ra, \quad \dot{b_3} = w_1 b_2 - w_2 b_1. \end{aligned} \tag{4}$$

Since $\dot{w_3} = 0$, hence we get $w_3 = \text{constant} = k_1$ say and consequently we have the system of five ODEs

$$\begin{aligned} &\dot{w_1} = f'w_2 - b_2, \quad \dot{w_2} = -f'w_1 + b_1, \\ &\dot{b_1} = w_2 b_3 - k_1 b_2, \quad \dot{b_2} = k_1 b_1 - w_1 b_3 + Ra, \quad \dot{b_3} = w_1 b_2 - w_2 b_1. \end{aligned} \tag{5}$$

We are looking for the solution of system (5) in the form of power series as given below

$$\begin{aligned} &w_1(t) = \sum_{j=0}^{\infty} w_{1j}\tau^{j+m_1}, \quad w_2(t) = \sum_{j=0}^{\infty} w_{2j}\tau^{j+m_2}, \\ &b_1(t) = \sum_{j=0}^{\infty} b_{1j}\tau^{j+n_1}, \quad b_2(t) = \sum_{j=0}^{\infty} b_{2j}\tau^{j+n_2}, \quad b_3(t) = \sum_{j=0}^{\infty} b_{3j}\tau^{j+n_3}, \end{aligned} \tag{6}$$

where $\tau = t - t_0$ and t_0 is the arbitrary position of singularity. As per the Painlevé algorithm there are three main steps in determination of singular solution. These steps are:

1. Determination of dominant behavior.
2. Determination of resonances.
3. Examining the compatibility conditions at the resonances.

It is natural that the algorithm may stop at the first step, second step or third step. For more details about this algorithm one may consult with Ablowitz et al [4]. The convergence of the series solution by use of this algorithm is guaranteed by Kichenassamy and Littman [9, 10].

Now we proceed for implementation of algorithm so in the first step we determine dominant behavior of the system (5). There are the several possible cases for dominant balance but the system of ODEs (5) admits the singular solution only in the following case of principle dominant balance

$$\dot{w_1} = -b_2, \quad \dot{w_2} = b_1, \quad \dot{b_1} = w_2 b_3, \quad \dot{b_2} = -w_1 b_3, \quad \dot{b_3} = w_1 b_2 - w_2 b_1. \tag{7}$$

In the following subsection we determine exponents and leading order coefficients.

3.1 Determination of exponents

To determine the singular exponents m_1, m_2, n_1, $n_2 \,\&\, n_3$, which appear in (6), it is sufficient to truncate the expansions up to the leading order and then substituting these truncated expansions into (7) we obtain the following system of equations

$$\begin{aligned} & m_1 w_{10} \tau^{m_1-1} = -b_{20} \tau^{n_2}, \quad m_2 w_{20} \tau^{m_2-1} = b_{10} \tau^{n_1}, \\ & n_1 b_{10} \tau^{n_1-1} = w_{20} b_{30} \tau^{m_2+n_3}, \quad n_2 b_{20} \tau^{n_2-1} = -w_{10} b_{30} \tau^{m_1+n_3}, \\ & n_3 b_{30} \tau^{n_3-1} = \left(w_{10} b_{20} \tau^{m_1+n_2} - w_{20} b_{10} \tau^{m_2+n_1} \right). \end{aligned} \tag{8}$$

Equating the powers of τ so that equations (8) get satisfied we have the following linear equations

$$\begin{aligned} & m_1 - 1 = n_2, \quad m_2 - 1 = n_1, \quad n_1 - 1 = m_2 + n_3, \\ & n_2 - 1 = m_1 + n_3, \quad n_3 - 1 = m_1 + n_2 = m_2 + n_1. \end{aligned} \tag{9}$$

From equations (9) the exponents can be uniquely determined as given below.

$$m_1 = m_2 = -1, \quad n_1 = n_2 = n_3 = -2. \tag{10}$$

Substituting the values of m_1, m_2, n_1, $n_2 \,\&\, n_3$ into equations (8) and then equating the coefficients of like powers of τ on both sides of each equation, we get the following system of equations to determine the leading order coefficients

$$\begin{aligned} & w_{10} = b_{20}, \quad w_{20} = -b_{10}, \\ & b_{10} = -\tfrac{1}{2} w_{20} b_{30}, \quad b_{20} = \tfrac{1}{2} w_{10} b_{30}, \\ & b_{30} = -\tfrac{1}{2} (w_{10} b_{20} - w_{20} b_{10}). \end{aligned} \tag{11}$$

Solving these equations we find that there are two possible branches of leading order involving one leading order coefficient to be an arbitrary constant. Suppose that $w_{20} = k_2$ is an arbitrary constant. The possible branches of leading order are as given below

$$w_{10} = \pm\sqrt{-4 - k_2^2},\, w_{20} = k_2,\, b_{10} = -k_2,\, b_{20} = \pm\sqrt{-4 - k_2^2},\, b_{30} = 2. \tag{12}$$

Here we notice that there are two possible branches of leading order. Hence, we will get two different singular solutions in complex domain. The next step of Painlevé algorithm is to determine the resonances. In the following section we proceed to determine the resonances.

3.2 Determination of resonances

As per the Painlevé algorithm this is the second step. Here we determine the resonances. So we rewrite the equations (6) by substituting the values of exponents

$$\begin{aligned}
&w_1(t) = w_{10}\tau^{-1} + \sum_{j=1}^{\infty} w_{1j}\tau^{j-1}, \quad w_2(t) = w_{20}\tau^{-1} + \sum_{j=1}^{\infty} w_{2j}\tau^{j-1},\\
&b_1(t) = b_{10}\tau^{-2} + \sum_{j=1}^{\infty} b_{1j}\tau^{j-2}, \quad b_2(t) = b_{20}\tau^{-2} + \sum_{j=1}^{\infty} b_{2j}\tau^{j-2},\\
&b_3(t) = b_{30}\tau^{-2} + \sum_{j=1}^{\infty} b_{3j}\tau^{j-2}.
\end{aligned} \tag{13}$$

Substituting the above equations into the system (5) we obtained the following recursion relations for determining the coefficients of different powers of τ in the equations (13), which are valid for $j \geq 2$,

$$\begin{pmatrix} j-1 & 0 & 0 & 1 & 0\\ 0 & j-1 & -1 & 0 & 0\\ 0 & -b_{30} & j-2 & 0 & -w_{20}\\ b_{30} & 0 & 0 & j-2 & w_{10}\\ -b_{20} & b_{10} & w_{20} & -w_{10} & j-2 \end{pmatrix} \begin{pmatrix} w_{1j}\\ w_{2j}\\ b_{1j}\\ b_{2j}\\ b_{3j} \end{pmatrix} = \begin{pmatrix} A_j\\ B_j\\ C_j\\ D_j\\ E_j \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned}
&A_j = f' w_{2(j-1)}, \quad B_j = -f' w_{1(j-1)}, \quad C_j = -k_1 b_{2(j-1)} + \sum_{k=1}^{j-1} w_{2k} b_{3(j-k)},\\
&D_j = k_1 b_{1(j-1)} - \sum_{k=1}^{j-1} w_{1k} b_{3(j-k)}, \quad E_j = \sum_{k=1}^{j-1} w_{1k} b_{2(j-k)} - \sum_{k=1}^{j-1} w_{2k} b_{1(j-k)}.
\end{aligned} \tag{15}$$

Now we denote by $M(j)$ the matrix

$$M(j) = \begin{pmatrix} j-1 & 0 & 0 & 1 & 0\\ 0 & j-1 & -1 & 0 & 0\\ 0 & -b_{30} & j-2 & 0 & -w_{20}\\ b_{30} & 0 & 0 & j-2 & w_{10}\\ -b_{20} & b_{10} & w_{20} & -w_{10} & j-2 \end{pmatrix}. \tag{16}$$

The above recursion relations (14) determine the unknown expansion coefficients uniquely unless the determinant of matrix $M(j)$ is zero. Those values of j at which the determinant $det(M(j))$ vanishes are called *resonances*. Here we see that for both possible branches of leading orders given in equations (12) the determinant of matrix $M(j)$ is

$$det(M(j)) = (j+1)j(j-2)(j-3)(j-4). \tag{17}$$

Hence, the resonances are

$$j = -1,\ 0,\ 2,\ 3,\ 4. \tag{18}$$

Here $j = -1$ is a usual resonance and $j = 0$ is corresponding to the arbitrariness of w_{20} in leading order behavior.

For the next step in the algorithm we check the compatibility conditions at non negative resonances given in equation (18).

3.3 Compatibility conditions

In this section we check whether the compatibility conditions hold at positive resonances which are determined in previous section. The recursion relations (14) will be valid if and only if the vector appearing on the right hand side of (14) must be annihilated by every left null vector of $M(j)$ (when j is a resonance) resulting in a set of compatibility conditions to be satisfied by the previously determined coefficients. When these conditions hold, the j-th coefficient vector enters as an arbitrary coefficient vector in the expansion (13). On the other hand if the compatibility condition fails at a resonant level, logarithms need to be introduced in the expansion (see [9, 10] for details). We investigate this in each case of possible branches of leading order coefficients given by (12) and we determine the expansion coefficients in each case up to the last resonant level.

• **Case 1:** Consider the leading order coefficients

$$\begin{aligned} &w_{10} = \sqrt{-4-k_1^2}, \quad w_{20} = k_2 \,(\text{arbitrary constant}),\\ &b_{10} = -k_2, \quad b_{20} = \sqrt{-4-k_1^2}, \quad b_{30} = 2. \end{aligned} \tag{19}$$

• **Compatibility condition at** $j = 1$**.** Since the recursion relations (14) come into force when $j \geq 2$, hence, we have directly substituted equations (19) into (13) and then into the equations (5). After simplifying we equate the like powers of τ on both sides of the resulting expansion thereby obtaining the following system of linear equations for w_{11}, w_{21}, b_{11}, b_{21} and b_{31}

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 & -k_2 \\ 2 & 0 & 0 & -1 & \sqrt{-4-k_2^2} \\ -\sqrt{-4-k_2^2} & -k_2 & k_2 & -\sqrt{-4-k_2^2} & -1 \end{pmatrix} \begin{pmatrix} w_{11} \\ w_{21} \\ b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} f'k_2 \\ -f'\sqrt{-4-k_2^2} \\ -k_1\sqrt{-4-k_2^2} \\ -k_1k_2 \\ 0 \end{pmatrix}. \tag{20}$$

The system of linear equations (20) has a unique solution, hence w_{11}, w_{21}, b_{11}, b_{21} and b_{31} are uniquely determined and these are given below

$$\begin{aligned} &w_{11} = \tfrac{1}{2}(f'k_2 - k_1k_2), \quad w_{21} = \tfrac{1}{2}(-f' + k_1)\sqrt{-4-k_2^2},\\ &b_{11} = f'\sqrt{-4-k_2^2}, \quad b_{21} = f'k_2, \quad b_{31} = 0. \end{aligned} \tag{21}$$

• **Compatibility condition at the resonance** $j = 2$. Now substituting the values of w_{ij} and b_{ij} for $i = 1, 2, 3$ and $j = 0, 1$ into the recursion relations (14) for $j = 2$, we get the following set of linear equations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 & -k_2 \\ 2 & 0 & 0 & 0 & \sqrt{-4-k_2^2} \\ -\sqrt{-4-k_2^2} & -k_2 & k_2 & -\sqrt{-4-k_2^2} & 0 \end{pmatrix} \begin{pmatrix} w_{12} \\ w_{22} \\ b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \\ E_2 \end{pmatrix}, \tag{22}$$

where

$$\begin{aligned} &A_2 = \tfrac{f'}{2}(k_1 - f')\sqrt{-4-k_2^2}, \quad B_2 = -\tfrac{f'k_2}{2}(f' - k_1), \quad C_2 = -f'k_1k_2,\\ &D_2 = k_1 f'\sqrt{-4-k_2^2}, \quad E_2 = \tfrac{(f'-k_1)}{2}(f'k_2^2 - k_2^2 - 4). \end{aligned} \tag{23}$$

Since $j = 2$ is a resonance, the coefficient matrix to the left hand side of equation (22) vanishes. Hence, we have infinitely many solutions to above system of linear equations with one arbitrary constant say $b_{32} = k_3$. Solving the system (22) with the help of (23) we get the following set of values of w_{12}, w_{22}, b_{12}, b_{22} and b_{32}.

$$\begin{aligned}&w_{12} = \tfrac{1}{2}(f'k_1 - k_3)\sqrt{-4 - k_2^2}, \quad w_{22} = \tfrac{k_2}{2}(f'k_1 - k_3),\\ &b_{12} = \tfrac{k_2}{2}\left[(f')^2 - k_3\right], \quad b_{22} = \tfrac{1}{2}\left[k_3 - (f')^2\right]\sqrt{-4 - k_2^2}, \quad b_{32} = k_3.\end{aligned} \tag{24}$$

• **Compatibility condition at the resonance** $j = 3$. Now we check the compatibility condition at the resonant level $j = 3$. At this resonance level we observe that recurrence relations fail to collect the additional term Ra, which is one of the terms involved in the equations (3) due to the effects of rotation. So we substitute the equations (13) into the system of differential equations (5), then equating the like powers of τ with $j = 3$ we get the following system of nonhomogeneous linear equations

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & -2 & 1 & 0 & -k_2 \\ 2 & 0 & 0 & 1 & \sqrt{-4-k_2^2} \\ -\sqrt{-4-k_2^2} & -k_2 & k_2 & -\sqrt{-4-k_2^2} & 1 \end{pmatrix} \begin{pmatrix} w_{13} \\ w_{23} \\ b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \\ E_3 \end{pmatrix}, \tag{25}$$

where

$$\begin{aligned}&A_3 = f'w_{22}, \quad B_2 = -f'w_{12}, \quad C_2 = w_{21}b_{32} - k_1 b_{22},\\ &D_2 = k_1 b_{12} - w_{11}b_{32} + Ra, \quad E_2 = w_{11}b_{22} + w_{12}b_{21} - w_{21}b_{12} - w_{22}b_{11}.\end{aligned} \tag{26}$$

After substituting the values of w_{ij} and b_{ij} for $i = 1, 2, 3$ and $j = 0, 1, 2$ in above equation and simplifying we see that the rank of coefficient matrix is 4, whereas the rank of augmented matrix is 5. This shows the inconsistency of the system (25). This is because of the term Ra, the Rayleigh number. Hence, we reduce the augmented matrix to its triangular form by use of elementary row transformation, which is given below

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 & \frac{f'k_2(-k_3+f'k_1)}{2} \\ 0 & 1 & 0 & -\frac{1}{2k_2}\sqrt{-4-k_2^2} & \frac{1}{k_1} & \frac{f'}{4}(k_3 - f'k_1)\sqrt{-4-k_2^2} \\ 0 & 0 & 1 & -\frac{1}{k_2}\sqrt{-4-k_2^2} & \frac{2}{k_2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & Ra \end{pmatrix}.$$

From the above triangular matrix we notice that the system (25) is consistent if and only if $Ra = 0$. Hence, the compatibility condition at resonance level $j = 3$ will hold only if $Ra = 0$. Now we assume that $Ra = 0$ (Note that with this assumption we have one more term in equations (3) due to the effect of rotation), so that the linear equations (25) can be solved and we see that there are infinitely many solutions with one independent variable. We found that the variable b_{23} to be independent. We assign the arbitrary value k_4 to b_{23} that is to say $b_{23} = k_4$. The solutions of system (25) are given below

$$\begin{aligned}&w_{13} = -\tfrac{k_4}{2} + \tfrac{f'k_2}{4}(-k_3 + f'k_1), \quad w_{23} = \left(\tfrac{k_4}{2k_2} + \tfrac{f'k_3}{4} - (f')^2 k_1\right)\sqrt{-4 - k_2^2},\\ &b_{13} = \tfrac{k_4}{k_2}\sqrt{-4 - k_2^2}, \quad b_{23} = k_4, \quad b_{33} = 0.\end{aligned} \tag{27}$$

• **Compatibility condition at the resonance** $j = 4$. At the resonant level $j = 3$ we notice that compatibility conditions hold only if $Ra = 0$ and there we assume that

$Ra = 0$. Now we proceed to check the compatibility conditions at the resonance $j = 4$. We substitute the equations (27), (24), (21) and (19) into the recurrence relations given by (14) for $j = 4$; and then equating the like powers of τ with $j = 3$ we get the following system of linear equations

$$\begin{pmatrix} 3 & 0 & 0 & 1 & 0 \\ 0 & 3 & -1 & 0 & 0 \\ 0 & -2 & 2 & 0 & -k_2 \\ 2 & 0 & 0 & 2 & \sqrt{-4-k_2^2} \\ -\sqrt{-4-k_2^2} & -k_2 & k_2 & -\sqrt{-4-k_2^2} & 2 \end{pmatrix} \begin{pmatrix} w_{14} \\ w_{24} \\ b_{14} \\ b_{24} \\ b_{34} \end{pmatrix} = \begin{pmatrix} A_4 \\ B_4 \\ C_4 \\ D_4 \\ E_4 \end{pmatrix}, \tag{28}$$

where

$$\begin{aligned} A_4 &= \tfrac{f'}{4k_2}\Big(2k_4 + f'k_2[k_3 - f'k_1]\Big)\sqrt{-4-k_2^2}, \\ B_4 &= -\tfrac{f'}{4}\big(-2k_4 - f'k_2k_3 + (f')^2k_1k_2\big), \\ C_4 &= -k_1k_4 + \tfrac{k_2k_3}{2}(-k_3 + f'k_1), \\ D_4 &= \tfrac{1}{2k_2}\big(2k_1k_4 + k_2k_3^2 - f'k_1k_2k_3\big)\sqrt{-4-k_2^2}, \\ E_4 &= \tfrac{1}{2}k_4\big(f'k_2 - k_1k_2\big) - \tfrac{1}{4}k_2^2\big((f')^2 - k_3\big)\big(-k_3 + f'k_1\big) \\ &+ \tfrac{f'k_2}{4}\big(-2k_4 - f'k_2k_3 + (f')^2k_1k_2\big) \\ &+ \tfrac{1}{4}(4+k_2^2)\Big[k_4(f' + k_1) - [(f')^2 + k_3](-k_3 + f'k_1) \\ &\quad + \tfrac{f'}{k_2}\big(2k_4 + f'k_2k_3 - [f']^2k_1k_2\big)\Big]. \end{aligned} \tag{29}$$

We see that the linear system (28) is consistent and admits infinitely many solutions with one independent variable. Reducing the augmented matrix to its upper triangular form we found the variable b_{24} to be an independent variable. Let $b_{24} = k_5$ be an arbitrary constant. Solving the system (28) with this independent variable we get the following solutions

$$\begin{aligned} w_{14} &= -\tfrac{k_5}{3} - \tfrac{\sqrt{-4-k_2^2}}{12k_2}\big[-(f')^2k_2k_3 - 2f'k_4 + (f')^3k_1k_2\big], \\ w_{24} &= -\frac{k_2k_5}{3\sqrt{-4-k_2^2}} + \frac{(f')^2k_2k_3 + 2f'k_4 - (f')^3k_1k_2}{12}, \\ b_{14} &= \frac{k_2k_5}{4+k_2^2}, \quad b_{24} = k_5, \\ b_{34} &= -\frac{4k_5}{3\sqrt{-4-k_2^2}} - \frac{(f')^2k_2k_3 - 3k_2k_3^2 + 2f'k_4 - (f')^3k_1k_2 + 3f'k_1k_2k_3 - k_1k_4}{6k_1}. \end{aligned} \tag{30}$$

• **Compatibility condition for** $j \geq 5$. From the equation (16) we observe that the matrix $M(j)$ for $j \geq 5$ is nonsingular matrix in this case of leading order coefficients as given by equations (19). So the system (14) with (15) in this case of leading order coefficients possesses unique solution. For the calculations of w_{ij} and b_{ij} for $i = 1, 2, 3$ and $j \geq 5$, we substitute (30), (27), (24), (21) into the recursion relations (14) and (15) for passing successively $j = 5, 6, \ldots$. In this fashion we find all the coefficients are uniquely determined for $j \geq 5$.

As we notice the compatibility conditions hold provided that $Ra = 0$. Hence, the system (4) passes the Painlevé test implying the complete integrability of the system.

So we can write the general solution of the system (4). In that respect we substitute all these coefficients into the Laurent's series expansions as given in equations (13). The general solution of system (4) in this case of leading order coefficients consists of five arbitrary constants k_1, k_2, k_3, k_4, k_5 and an arbitrary position of t_0 singularity and the required solution is as given below

$$
\begin{aligned}
w_1(t) &= \sqrt{-4-k_2^2}\tau^{-1} + \tfrac{1}{2}(f'k_2 - k_1k_2) + \big[\tfrac{1}{2}(f'k_1 - k_3)\sqrt{-4-k_2^2}\big]\tau \\
&\quad + \big[-\tfrac{k_4}{2} + \tfrac{f'k_2}{4}(-k_3 + f'k_1)\big]\tau^2 \\
&\quad + \big[-\tfrac{k_5}{3} - \tfrac{\sqrt{-4-k_2^2}}{12k_2}\big(-(f')^2k_2k_3 - 2f'k_4 + (f')^3k_1k_2\big)\big]\tau^3 \\
&\quad + \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\
w_2(t) &= k_2\tau^{-1} + \big[\tfrac{1}{2}(-f' + k_1)\sqrt{-4-k_2^2}\big] + \big[\tfrac{k_2}{2}(f'k_1 - k_3)\big]\tau \\
&\quad + \Big[\Big(\tfrac{k_4}{2k_2} + \tfrac{f'k_3}{4} - (f')^2k_1\Big)\sqrt{-4-k_2^2}\Big]\tau^2 \\
&\quad + \Big[-\frac{k_2k_5}{3\sqrt{-4-k_2^2}} + \frac{(f')^2k_2k_3 + 2f'k_4 - (f')^3k_1k_2}{12}\Big]\tau^3 \\
&\quad + \sum_{j=5}^{\infty} w_{2j}\tau^{j-1}, \\
w_3(t) &= k_1 \text{ (arbitrary constant)}, \\
b_1(t) &= -k_2\tau^{-2} + \big[f'\sqrt{-4-k_2^2}\big]\tau^{-1} + \tfrac{k_2}{2}\big[(f')^2 - k_3\big] + \big[\tfrac{k_4}{k_2}\sqrt{-4-k_2^2}\big]\tau^1 \\
&\quad + \Big[\frac{k_2k_5}{4+k_2^2}\Big]\tau^2 + \sum_{j=5}^{\infty} b_{1j}\tau^{j-2}, \\
b_2(t) &= \sqrt{-4-k_1^2}\tau^{-2} + f'k_2\tau^{-1} + \big[\tfrac{1}{2}\big((k_3 - (f')^2\big)\sqrt{-4-k_2^2}\big] + k_4\tau \\
&\quad + k_5\tau^2 + \sum_{j=5}^{\infty} b_{2j}\tau^{j-2}, \\
b_3(t) &= 2\tau^{-2} + k_3 + \Big[-\tfrac{4k_5}{3\sqrt{-4-k_2^2}} \\
&\quad - \frac{(f')^2k_2k_3 - 3k_2k_3^2 + 2f'k_4 - (f')^3k_1k_2 + 3f'k_1k_2k_3 - k_1k_4}{6k_1}\Big]\tau^2 \\
&\quad + \sum_{j=1}^{\infty} b_{3j}\tau^{j-2}.
\end{aligned}
\tag{31}
$$

Equations (31) contain five arbitrary constants k_1, k_2, k_3, k_4, k_5 and arbitrary position of t_0; these equations satisfy the system of differential equations (3) for $Ra = 0$. Hence, in the present case of leading order coefficient, equations (31) represent the general solution of (3). The convergence of such series solutions is guaranteed by Kichenassamy and Littman [9, 10]. And it seems that the solution contains the movable pole type singularity. Similar kind of steps are involved for another branch of leading order coefficients. In the following subparagraphs we listed these calculations.

• **Case 2:** Consider the leading order coefficients

$$\begin{aligned} &w_{10} = -\sqrt{-4-k_2^2}, \quad w_{20} = k_2 \,(\text{arbitrary constant}), \\ &b_{10} = -k_2, \quad b_{20} = -\sqrt{-4-k_2^2}, \quad b_{30} = 2. \end{aligned} \tag{32}$$

Using the same approach as in the previous case we have determined the expansion coefficients of (13) for $j=1, j=2, j=3$, and $j=4$ which are listed below.

• **Leading order coefficients at** $j=1$ **:**
As we notice already $j=1$ is not a resonance and hence, in this branch of leading order coefficients for $j=1$ we can determine w_{ij} and b_{ij} uniquely for $i=1,2,3 \quad j=1$ as given below.

$$\begin{aligned} &w_{11} = \frac{f'k_2 - k_1k_2}{2}, \quad w_{21} = \frac{f'-k_1}{2}\sqrt{-4-k_2^2}, \\ &b_{11} = -f'\sqrt{-4-k_2^2}, \quad b_{21} = f'k_2, \quad b_{31} = 0. \end{aligned} \tag{33}$$

• **At the resonance** $j=2$: At this resonant level $j=2$, we find that one of the coefficients is independent. Let b_{32} be independent. Assign the value to $b_{32}=k_3$ and consequently other expansion coefficients for $j=2$ are given below

$$\begin{aligned} &w_{12} = \frac{k_3 - f'k_1}{2}\sqrt{-4-k_2^2}, \quad w_{22} = \frac{k_2}{2}(k_1f' - k_3), \\ &b_{12} = \frac{k_2}{2}[(f')^2 - k_3], \quad b_{22} = \frac{((f')^2-k_3)}{2}\sqrt{-4-k_2^2}, \quad b_{32} = k_3. \end{aligned} \tag{34}$$

• **At the resonance** $j=3$: As we noticed in previous case at this resonant level $j=3$ is that system of linear equations (25) is inconsistent unless $Ra=0$. Similarly in this case we also notice that a system of linear equations is inconsistent unless $Ra=0$. Again assuming that $Ra=0$, we determine the expansion coefficients with one independent variable. Let b_{23} be independent. Assign $b_{23}=k_4$ and other expansion coefficients for $j=3$ are given below

$$\begin{aligned} &w_{13} = \frac{1}{4}[-2k_4 + f'k_2(f'k_1 - k_3)], \; w_{23} = \frac{\sqrt{-4-k_2^2}}{4}\Big[\frac{-2k_4}{k_2} - f'k_3 + (f')^2k_1\Big], \\ &b_{13} = \frac{-k_4\sqrt{-4-k_2^2}}{k_2}, \quad b_{23} = k_4, \quad b_{33} = 0. \end{aligned} \tag{35}$$

At the resonance $j=4$: Also, at this resonant level $j=4$ we found that one of the expansion coefficients is independent. Let b_{24} be independent and assign the arbitrary value say $b_{24}=k_5$. Other expansion coefficients are as listed below

$$\begin{aligned} &w_{14} = \frac{-k_4}{3} - \frac{\sqrt{-4-k_2^2}}{12k_2}\big[(f')^2k_2k_3 + 2f'k_4 - (f')^3k_1k_2\big], \\ &w_{24} = \frac{k_2k_5}{3\sqrt{-4-k_2^2}} + \frac{f'}{12}[2k_4 + f'k_2k_3 - (f')^2k_1k_2], \\ &b_{14} = \frac{k_2k_5}{\sqrt{-4-k_2^2}}, \\ &b_{24} = k_5, \\ &b_{34} = \frac{4k_5}{3\sqrt{-4-k_2^2}} - \frac{1}{6k_2}\big[k_2k_3\big((f')^2 - 3k_3\big) + 2k_4(f' - 3k_1) + f'k_1k_2(3k_3 - f'^2)\big]. \end{aligned} \tag{36}$$

For $j \geq 5$**:** Plugging the equations (36), (35), (34), (33) and (32) into the recursion relations (14), we can uniquely determine the expansion coefficients w_{ij} and b_{ij} for $j \geq 5$. The general solution of system (4) in this case of leading order is as given below

$$
\begin{aligned}
w_1(t) &= -\sqrt{-4-k_2^2}\,\tau^{-1} + \frac{f'k_2 - k_2k_1}{2} + \frac{\sqrt{-4-k_2^2}}{2}(k_3 - f'k_1)\tau \\
&+ \left(-\frac{k_4}{2} + \frac{f'k_2}{4}\big[-k_3 + f'k_1\big]\right)\tau^2 \\
&- \left(\frac{k_5}{3} + \frac{f'\sqrt{-4-k_2^2}}{12k_1}\big[f'k_2k_3 + 2k_4 - (f')^2k_2k_1\big]\right)\tau^3 \\
&+ \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\
w_2(t) &= k_2\tau^{-1} + \left(\frac{\sqrt{-4-k_2^2}}{2}\big[f' - k_1\big]\right) + \frac{-k_2k_3 + f'k_1k_2}{2}\tau \\
&+ \frac{\sqrt{-4-k_2^2}}{4}\left(\frac{-2k_4}{k_2} - k_3f' + (f')^2k_1\right)\tau^2 \\
&+ \left(\frac{k_2k_5}{3\sqrt{-4-k_2^2}} + \frac{f'}{12}\big[f'k_2k_3 + 2k_4 - (f')^2k_2k_1\big]\right)\tau^3 + \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\
w_3(t) &= k_1, \\
b_1(t) &= -k_2\tau^{-2} - \left(f'\sqrt{-4-k_2^2}\right)\tau^{-1} - \frac{k_2k_3 - (f')^2k_2}{2} - \frac{k_4\sqrt{-4-k_2^2}}{k_2}\tau \\
&+ \frac{k_2k_5}{\sqrt{-4-k_2^2}}\tau^2 + \sum_{j=5}^{\infty} b_{1j}\tau^{j-2}, \\
b_2(t) &= -\sqrt{-4-k_2^2}\,\tau^{-2} + f'k_2\tau^{-1} + \frac{\sqrt{-4-k_2^2}}{2}\left(-k_3 + (f')^2\right) + k_4\tau + k_5\tau^2 \\
&+ \sum_{j=5}^{\infty} b_{2j}\tau^{j-2}, \\
b_3(t) &= 2\tau^{-2} + k_3 + \Big[\frac{4k_5}{3\sqrt{-4-k_2^2}} \\
&\quad - \frac{(f')^2k_2k_3 - 3k_2k_3^2 + 2f'k_4 - (f')^3k_1k_2 + 3f'k_1k_2k_3 - k_1k_4}{6k_1}\Big]\tau^2 \\
&\quad + \sum_{j=1}^{\infty} b_{3j}\tau^{j-2}.
\end{aligned} \tag{37}
$$

4 Examples

In this section we present two systems of ODEs that are in similar analog with our system (3) for $Ra = 0$. Hence, these systems will have the singular solutions and these solutions will be in similar nature as we have obtained so far.

Now consider the equations for the motion under gravity of a rigid body about a fixed point

$$\begin{aligned}\frac{d\mathbf{l}}{dt} &= \mathbf{l}\times\omega+\mathbf{c}\times\mathbf{g},\\ \frac{d\mathbf{g}}{dt} &= \mathbf{g}\times\omega;\quad \mathbf{l}=\mathbf{I}\omega.\end{aligned}\tag{38}$$

In the above equations, $\mathbf{l}$ and ω are respectively the angular momentum and angular velocity of the body, $\mathbf{g}$ is the gravitational acceleration with respective the moving frame. The vector $\mathbf{c}$ is the center of mass and inertia tensor $\mathbf{I}$ are both constants. The explicit details about the system (38) have been discussed by Andrew Hone [11]. This system will be as similar to our system (3) for $Ra = 0$ and assigning the value $f' = 0$. So that the singular solutions of a system (38) will be obtained in similar fashion as we discussed above.

In their paper Julien et al [12] employ a multiscale expansion in both time and space. Specifically, they define the Ekman number $E \equiv \nu/2\Omega d^2$, where ν is kinematic viscosity, d is typical length scale, and $\mathbf{\Omega} \equiv \Omega\hat{\mathbf{z}}$ (which is equivalent to $f\hat{\mathbf{e_3}}$ in our equations (1)) is the rotation vector, and treat E as a small parameter. With these assumptions and in the absence of stratification the incompressible Navier-Stokes equations then become

$$\begin{aligned}\frac{D\mathbf{u}}{Dt}+\hat{\mathbf{\Omega}}\times\mathbf{u} &= -\nabla\pi+E\nabla^2\mathbf{u}+\mathbf{f},\\ \nabla\cdot\mathbf{u} &= 0,\end{aligned}\tag{39}$$

where $\mathbf{f}$ is an unspecified body force and π is the pressure. Further Julien et all [12] present their results for the specific case of rotating convection for which they took $\mathbf{f} = (Ra/\sigma)E^2T\hat{\mathbf{z}}$ and (39) were supplemented with the energy equation

$$\sigma\frac{DT}{Dt} = E\nabla^2 T.\tag{40}$$

In equation (40), T is the temperature, Ra is the Rayleigh number, and $\sigma = \nu/\kappa$ is the Prandtl number; κ is the thermal diffusivity.

Here we observe that if we take $E \equiv 0$ and unspecified body forces to be equal to zero, and going through the local analysis as Desale and Sharma [7] deploy it to a similar equations. We can have a system of ODEs which is equivalent to system (3). Hence for $Ra = 0$ the singular solutions in this case will be in similar nature with the solutions which we have investigated in Section 3.

5 Conclusion

Now we conclude that the system of ODE reduction of rotating Stratified Boussinesq Equations (3) is completely integrable (in the light of ARS conjecture). There are several possible cases of principle dominant balance cases among these the system of ODEs (3) admits the singular solution only in the case of (7). There are two possible branches of leading orders and in both cases of leading orders system (3) passes the strong Painlevé test only if the Rayleigh number $Ra = 0$. The general solutions are given by(31) and (37). We found that these solutions are in complex domain and contain the movable pole type singularity at $t = t_0$. In Section 4 we illustrate the systems which also exhibit similar kind of solutions so far we obtained in Section 3.

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