Nonlinear Dynamics and Systems Theory, **10** (4) (2010) 363–374

Design of Decoupling Nonlinear Controllers for Fuzzy Systems

J.-Y. Dieulot [1]* and N. Elfelly [2]

[1] *LAGIS UMR CNRS 8146, Polytech-Lille/University of Lille, Cité Scientifique 59650 Villeneuve d'Ascq, France*
[2] *LAGIS, Ecole Centrale de Lille, France*

Received: March 1, 2010; Revised: October 18, 2010

Abstract: The use of linear matrix inequalities and Lyapunov functions is a powerful and commonplace tool for Takagi–Sugeno fuzzy controlled system analysis and synthesis. This paper shows how to split and handle the coupling terms arising from the existence of different input matrices in the subsystems. Then, a method is proposed which allows to synthesize, for a sufficient number of subsystems, the local gains of a nonlinear parallel distributed controller. It is shown that the controller gains depend on the values of the input matrices and of the membership functions, and are thus able to relax classical stability conditions by embedding information on the fuzzy premises.

Keywords: *fuzzy control; stability; nonlinear control; linear matrix inequalities.*

Mathematics Subject Classification (2000): 93D42, 93D15, 93D21.

1 Introduction

The Takagi–Sugeno fuzzy state-space model allows to describe a nonlinear system using a set of fuzzy rules for which the consequents are a set of linear models, which are smoothly connected by fuzzy membership functions [1]. An intuitive approach to the control of T-S fuzzy systems consists of designing a fuzzy controller which shares the same fuzzy sets with the fuzzy model in the premise parts. In this parallel distributed compensation method (PDC), each control rule is distributively designed for the corresponding rule of a T-S fuzzy model [2].

* Corresponding author: mailto:jean-yves.dieulot@polytech-lille.fr

Most works considering the design of controlled Takagi–Sugeno fuzzy systems lead to express stability conditions and gain synthesis as a set of linear matrix inequalities (LMIs) which can be solved via efficient semi-definite programming optimization software [3]. These works can be extended for very complex systems such as time-delay nonlinear systems modelling and control [6]. However, very few works will consider the relevance of a nonlinear PDC controller. Original results dealing with the search for a common quadratic Lyapunov functions (CQLF) are known to be quite conservative, and, as a result, a number of methods have been proposed to relax standard stability conditions [4, 7, 8, 9], and new tools such as piecewise quadratic Lyapunov functions or fuzzy Lyapunov functions have been introduced (e.g. [10]). Extended results have allowed to consider bounds and/or shapes of the premises' membership functions considering PDC [11, 12] or non PDC [13, 14, 15] controllers (see [24] for a summary of conservativeness issues). An extension of these results to fuzzy nonlinear systems can be done using vector norm approaches [5] with the drawback of adding more conservatism.

A main difficulty to the synthesis of fuzzy controlled systems lies in the combination of closed-loop subsystems which does not result into a parallel distribution of the individual closed-loop subsystems, because of additional coupling terms. These coupling terms result from the linkage of the local subsystems to the other subsystems' local controllers, in particular when input matrices are not identical. Some works, e.g. [22, 23] allow to handle subsystems with different matrices, and a descriptor formulation along with a non quadratic Lyapunov function has been proposed [21] to decouple input and gain matrices. The whole coupling term has also been represented explicitly by a product of matrices involving a single uncertain matrix with a norm smaller than one, leading to a global Riccati equation (e.g. [2]). Finding a global bounding matrix for the coupling term is often not easy to work out, because these terms depend on the membership functions and on the control gains themselves, which prevent the use of the method for control synthesis. The exact cancelation of coupling terms has been tackled explicitly only for large-scale systems [17].

In this paper, it is shown that the closed-loop T-S fuzzy system under PDC control is the sum of distributed closed-loop fuzzy systems and of a coupling term. This coupling term is rewritten as a sum of pairwise products involving input matrices and control gains. A first method is proposed to design fuzzy control gains which attenuate the coupling effect for any of the closed-loop subsystems, considering a common CQLF. This is done by considering bounds on the coupling term, and, when a priori limitations are given for control gains, the stability conditions are resumed to a set of independent Lyapunov equations. As this method still presents high degrees of conservatism, it is shown that when the number of subsystems is large enough, the coupling terms can be canceled by proposing nonlinear control gains for the PDC control structure.

2 Analysis of Fuzzy Systems Under PDC Control

2.1 Closed-loop T-S fuzzy systems decomposition

The fuzzy model proposed by Takagi and Sugeno consists of a set of r fuzzy IF...THEN rules for which the consequents are linear state-space models:

Plant Rule R_i: IF z_1 IS M_{i1} AND $\cdots$ AND z_g IS M_{ig} THEN $\dot{x} = A_i x + B_i u$;
where $x(t)$, $u(t)$ are respectively the state and input vectors, $z_i(t)$, M_{ij} are the premise variables and the corresponding fuzzy models.

The final output of the fuzzy system is inferred as follows:

$$\dot{x} = \sum_{i=1}^{r} \mu_i \big(A_i x + B_i u \big), \tag{1}$$

where $\mu_i = \frac{\omega_i}{\sum_{i=1}^{r} \omega_i}$ and ω_i is the grade of membership function of the rule R_i.

For every subsystem S_i, a local controller can be defined as $u = K_i x$, where K_i is a control gain. The rules which describe the fuzzy controller share the same premises as the fuzzy models, hence distributing the local controllers into the global controllers according to their systems' weights. In general, the controllers are supposed to be linear, but, in this study, it will be shown that nonlinear consequents might be preferred.

Controller C_i: IF z_1 IS M_{i1} AND $\cdots$ AND z_g IS M_{ig} THEN $u = K_i x$, yielding:

$$u = \sum_{i=1}^{r} \mu_i K_i x. \tag{2}$$

Lemma 2.1 *Let the system* $\dot{x} = \sum_{i=1}^{r} \mu_i (A_i x + B_i u)$ *with PDC control* $u = \sum_{i=1}^{r} \mu_i K_i x$ *such that* $A_i + B_i K_i = G_i$ *and* $\sum_{i=1}^{r} \mu_i \leq 1$, $\mu_i \geq 0$. *The closed-loop system is:*

$$\dot{x} = \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j G_i + \sum_{i=1}^{r} \mu_i A_i \Big(1 - \sum_{j=1}^{r} \mu_j \Big) + \sum_{i,j=1, j \neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x. \tag{3}$$

Proof One has

$$\dot{x} = \sum_{i=1}^{r} \mu_i \Big(A_i x + B_i \sum_{j=1}^{r} \mu_j K_j x \Big) = \sum_{i=1}^{r} \mu_i \Big(A_i + \mu_i B_i K_i + B_i \sum_{j=1, j \neq i}^{r} \mu_j K_j \Big) x.$$

$$\dot{x} = \sum_{i=1}^{r} \left(\mu_i^2 G_i + \mu_i A_i (1 - \mu_i) + \mu_i B_i \sum_{j=1, j \neq i}^{r} \mu_j K_j \right) x.$$

Moreover,

$$\sum_{i=1}^{r} \mu_i B_i \sum_{j=1, j \neq i}^{r} \mu_j K_j = \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} \mu_i \mu_j (G_i - A_i) + \sum_{i=1}^{r} \mu_i B_i \sum_{j=1, j \neq i}^{r} \mu_j K_j - \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} \mu_i \mu_j B_i K_i.$$

In this equation, one can rearrange the two last sums into a sum of pairwise terms:

$$\sum_{i,j=1, j \neq i}^{r} \mu_j B_j \mu_i K_i + \mu_i B_i \mu_j K_j - \mu_i \mu_j B_i K_i - \mu_j \mu_i B_j K_j = \sum_{i,j=1, j \neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i).$$

Hence,

$$\sum_{i=1}^{r}\mu_i B_i \sum_{j=1,j\neq i}^{r}\mu_j K_j=\sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\mu_i\mu_j(G_i-A_i)+\sum_{i,j=1,j\neq i}^{r}\mu_i\mu_j(B_i-B_j)(K_j-K_i).$$

One has now:

$$\dot{x}=\left(\sum_{i=1}^{r}\Big(\mu_i^2 G_i+\mu_i A_i(1-\mu_i)\Big)+\sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\mu_i\mu_j(G_i-A_i)\right.$$
$$\left.+\sum_{i,j=1,j\neq i}^{r}\mu_i\mu_j(B_i-B_j)(K_j-K_i)\right)x.$$

As $\sum_{i=1}^{r}\mu_i^2 G_i+\sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\mu_i\mu_j G_i=\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j G_i$, and

$$\sum_{i=1}^{r}\mu_i A_i(1-\mu_i)-\sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\mu_i\mu_j A_i=\sum_{i=1}^{r}\mu_i A_i\Big(1-\sum_{j=1}^{r}\mu_j\Big),$$

we demonstrate the final result:

$$\dot{x}=\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j G_i+\sum_{i=1}^{r}\mu_i A_i\Big(1-\sum_{j=1}^{r}\mu_j\Big)+\sum_{i,j=1,j\neq i}^{r}\mu_i\mu_j(B_i-B_j)(K_j-K_i)\right)x.$$

2.2 Specific cases

Note that in the Lemma, the formula could also be valid for $\sum_{i=1}^{r}\mu_i\leq 1$. One can derive more specific cases.

Polytopic systems: When $\sum_{i=1}^{r}\mu_i=1$, formula (3) is reduced to:

$$\dot{x}=\left(\sum_{i=1}^{r}\mu_i G_i+\sum_{i,j=1,j\neq i}^{r}\mu_i\mu_j(B_i-B_j)(K_j-K_i)\right)x.$$

Two-subsystems: The coupling term is now $\mu_1\mu_2(B_1-B_2)(K_2-K_1)$. In this case, the deviation from the polytopic closed-loop system only depends on the difference between gains K_2 and K_1, and this only degree of freedom is a limitation to the cancellation of the coupling term and of the choice of the local controllers.

Common input matrix: Suppose that $\forall i, B_i=B$, and $\sum_{i=1}^{r}\mu_i=1$, then formula (3) is reduced to:

$$\dot{x}=\left(\sum_{i=1}^{r}\mu_i G_i\right)x.$$

As a remark, one can say that, when the system exhibits a common input matrix, the closed-loop system behavior is a polytope of closed-loop local systems, and, thus, the coupling terms vanishes. The analysis of the whole closed-loop system can be handled easily.

Proportional input matrices: Suppose that $\forall i, B_i = \alpha_i B$, where $\alpha_i \in \mathbb{R}$, then the closed-loop subsystem is

$$\dot{x} = \left(\sum_{i=1}^{r} \mu_i G_i + B \sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (\alpha_i - \alpha_j)(K_j - K_i) \right) x.$$

This case arises often in Takagi-Sugeno modelling, and it can be seen that the coupling term is strongly dependent of the membership functions and the gains amplitude.

2.3 Global stability verification

Theorem 2.1 [2] *The system $\dot{x} = \sum_{i=1}^{r} \mu_i (A_i x + B_i u)$, under PDC control $u = \sum_{i=1}^{r} \mu_i K_i x$, such that $A_i + B_i K_i = G_i$ and $A_i + B_i K_j = G_{ij}$, is stable if there exists a common positive definite matrix P such that:*

$$\begin{aligned} &\forall\, i = 1, \cdots, r, PG_i + G_i^T P \prec 0, \\ &\forall\, i < j, P\left(G_{ij} + G_{ji}\right) + \left(G_{ij} + G_{ji}\right)^T P \prec 0. \end{aligned} \tag{4}$$

Remark 2.1 Theorem (2.1) allows the determination of both the Lyapunov matrix and the controller gain, using a change of variable $N_i = K_i P^{-1}$, when being replaced in the stability conditions, leads to a set of LMIs in N_i and in P, the PDC controller being provided by $K_i = N_i P$. The existence of a common quadratic Lyapunov function is only a sufficient stability condition, and, moreover, the conditions of Theorem (2.1) are independent of the membership functions, leading to conservative results. Coupling terms are not accounted for, since any of local subsystems i under any local controller $u = K_j x$, where $j \neq i$, should be performing, whereas it cannot be expected that a system with a controller designed for another plant has necessarily a "good" behavior. Hence, the PDC controller is designed according to the "worst" case among the pairs $\{$Plant i, Controller $j\}$.

Corollary 2.1 *Suppose that $\forall i, j, B_i \neq B_j$ iff $\mu_i \mu_j \equiv 0$, then the closed-loop system in Theorem (2.1) is stable if there exists a common positive definite matrix P such that:*

$$\forall i = 1, \cdots, r, PG_i + G_i^T P \prec 0.$$

This corollary shows that, when there exists a common input matrix, the closed-loop systems are uncoupled. What is more interesting is that, within the coupling term, the contributions involving different input matrices can be canceled when their corresponding membership functions do not overlap, i.e. their product is identically zero.

3 Coupling Terms Attenuation

Theorem 3.1 [16] *First, we consider the linear uncertain system for which $\dot{x} = A + \sum_{i=1}^{r} D_i \delta_i E_i, \|\delta_i\| \leq 1$, and the elements of the time-varying matrices δ_i are Lebesgue measurable. Then the positive-definite matrix P is a common Lyapunov matrix for this system if there exists r positive scalars η_i such that:*

$$PA + A^T P + \sum_{i=1}^{r} \eta_i P D_i D_i^T P + \eta_i^{-1} E_i^T E_i \prec 0,$$

or, as a specific case:

$$PA + A^T P + \sum_{i=1}^{r} PD_i D_i^T P + E_i^T E_i \prec 0. \tag{5}$$

Remark 3.1 This Theorem was applied first by Tanaka et al. [2] and then by numerous authors to the whole coupling term. Note that some authors [19, 18] introduce a DδE component within the consequent part. Whereas this method provides for a rather non-conservative solution, it is clear that finding individual uncertain matrices might be a tedious task, because the rate of variation and thus the bounds of the uncertain matrix depend on the control gains themselves. It can thus be applied to analyze an existing solution (when the gains are fixed a priori) but not for gain synthesis considering models/controllers coupling. The following theorem proposes a different application of this method to every individual component of the coupling term.

Theorem 3.2 *Consider the system $\dot{x} = \sum_{i=1}^{r} \mu_i(A_i x + B_i u)$, under PDC controller:*

$$\dot{x} = \left(\sum_{i=1}^{r} \mu_i G_i + \sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x.$$

Let us suppose that: $\forall\, i$, there exists b_i such that:

$$\sum_{B_i \neq B_j} \mu_j (B_j - B_i) = b_i \delta_i, \text{ where } \|\delta_i\| \leq 1.$$

The matrices δ_i thus depend on membership functions μ_i and other input matrices μ_j and B_j; as μ_j may vary with time, δ_i is a matrix which may vary with time or with the state space x.

The closed-loop system is quadratically stable if:

$$\forall\, i = 1, \cdots, r, PG_i + G_i^T P + P b_i b_i^T P + K_i^T K_i \prec 0. \tag{6}$$

This can be turned into:

$$\forall\, i = 1, \cdots, r, \begin{pmatrix} PG_i + G_i^T P & Pb_i & K_i^T \\ b_i^T P & -I & 0 \\ K_i & 0 & -I \end{pmatrix} \prec 0. \tag{7}$$

Proof

$$\begin{aligned} \dot{x} &= \left(\sum_{i=1}^{r} \mu_i G_i + \sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x \\ &= \left(\sum_{i=1}^{r} \mu_i \sum_{j=1, j\neq i}^{r} (G_i + \mu_j (B_i - B_j) K_i) \right) x. \end{aligned}$$

One has now: $\sum_{j=1, j\neq i}^{r} (G_i + \mu_j(B_i - B_j)K_i) = G_i + b_i \delta_i K_i$, and one can apply the Theorem 3.1.

Remark 3.2 Results involving a CQLF are known to be conservative. However, other Lyapunov functions can be searched for the TS system represented with an explicit coupling term, e.g. piecewise or fuzzy Lyapunov functions. However, this theorem wants to show that, taking explicitly the coupling term into account, one may relax standard or existing conditions for a given method.

Uncertain matrices δ_i do not depend anymore on the control gains but only on input matrices and membership functions which are supposed to be known as a part of the fuzzy model representation. Their determination is thus quite easy and the membership functions are indeed embedded in the control synthesis. Of course, it is assumed that such matrices exist. Note also that the corresponding i Riccati equations in (6) are decoupled, i.e. the i^{th} equation only depends on the i^{th} control gain, the influence of the other subsystems are merged into the matrix $b_i\delta_i$. The following corollary is a simplified condition of equation (7).

Corollary 3.1 *Let us suppose that:*

$$\forall\, i = 1, \cdots, r,\ \exists\, Q_i \succ 0,\ K_i^T K_i - Q_i \prec 0.$$

Then, condition (7) can be expressed as:

$$\exists\, P \prec 0, \forall\, i = 1, \cdots, r, PG_i + G_i^T P + Q_i' \prec 0, \tag{8}$$

where $Q_i' = Pb_i b_i^T P + Q_i$*, with* $K_i^T K_i - Q_i \prec 0$*, which can be turned into:*

$$\forall\, i = 1, \cdots, r, \begin{cases} \begin{pmatrix} PG_i + G_i^T P + Q_i' & Pb_i \\ b_i^T P & -I \end{pmatrix} \prec 0, \\ K_i^T K_i - Q_i \prec 0. \end{cases}$$

The Corollary simply reduces the search for a common Lyapunov matrix to a series of r Lyapunov equations and thus r LMIs. This is really an improvement to other methods because, now, control gains can nearly be selected independently without the need of taking care of coupling terms, at the expense of gain limitation. The synthesis gains are now completely uncoupled, the interdependence being lumped into the matrices b_i; in general, matrices b_i can be obtained from simple membership functions analysis. The following corollary focuses on the specific (and commonly encountered) case for which input matrices are proportional, and shows that the computation of matrices b_i is quite direct.

Corollary 3.2 *Suppose that the input matrices are proportional, i.e.* $\forall i, B_i = \alpha_i B$*, where* $\alpha_i \in \mathbb{R}$*, then the bounding matrices in Theorem 3.2 are given by:*

$$b_i = B \max \left(\sum_{j=1, j\neq i}^{r} \mu_i \mu_j (\alpha_j - \alpha_i) \right).$$

4 Coupling Terms Exact Compensation

In the previous section, a method has been proposed to choose control gains by balancing the effect of coupling terms resulting from other subcontrollers. The problem is that a

CQLF is still needed and that the coupling still exixsts, still yielding conservative solutions. Of course, a high number of subsystems increases the size of the set of Lyapunov equations but offers more degrees of freedom. It will be shown that, when these degrees of freedom are numerous enough, they can be used to cancel explicitly the coupling terms.

Proposition 4.1 *Let the system:*

$$\dot{x} = \left(\sum_{i=1}^{r} \mu_i G_i + \sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x,$$

and let $n = dim(x)$. Let us suppose also that $rank[B_1 \cdots B_n] = n$ and $\mu_i\mu_j \not\equiv 0$. There exists a nonlinear PDC controller $K(\mu_i)$, such that $\sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) = 0$ and $\exists i, j, K_i \neq K_j$, only if $r > n + 1$.

Proof There exists of course a trivial solution $K_i = K, \forall i$. The system has a solution different from this trivial solution, i.e. a true nonlinear PDC iff the system

$$\sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) = 0$$

is compatible. The weight corresponding to control gain K_i is:

$$w_i = \sum_{j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j).$$

One can notice that $\sum_{i=1}^{r} w_i = 0$. Hence, there is a solution $K_i \not\equiv 0$ only if $r > n + 1$.

In this case, the nonlinear PDC gain is membership-function dependent and non linear; one has to check that all the subsystems share a CQLF – or some other common Lyapunov function – which can however be more complicated. The workout will be shown in the example section.

5 Examples

5.1 Example 1

Let us take the following 3 systems:

$$A_1 = \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix}, B_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$A_3 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, B_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

along with local gains: $K_1 = \begin{pmatrix} 2 & 1 \end{pmatrix}, K_2 = \begin{pmatrix} -2 & 1 \end{pmatrix}, K_3 = \begin{pmatrix} 2 & 0 \end{pmatrix}$.

In these examples, gains were fixed a priori. The grades of membership corresponding to systems 1, 2 and 3 are: $\omega_1 = z$, $\omega_2 = 1 - z$ and $\omega_3 = z$ where $z \in [-1 \cdots 1]$.
For every subsystem i, it is quite easy to compute the matrices b_i such that

$$\sum_{B_i \neq B_j} \mu_i (B_j - B_i) = b_i \delta_i$$

since the upper bound depends on the fuzzy variable z. One finds: ${b_1}^T = \begin{pmatrix} 1 & 0.25 \end{pmatrix}$, ${b_2}^T = \begin{pmatrix} 0.75 & 0.25 \end{pmatrix}$, ${b_3}^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$.

The application of Theorem 3.2 allows to find a common positive definite matrix $P = \begin{pmatrix} 1.28 & -0.37 \\ -0.37 & 0.87 \end{pmatrix}$ whereas it is impossible to find one by the classical method; it is easy to check that the gain K_2 is unable to stabilize matrix A_1 and the converse for K_1 and A_2. It is quite interesting to note that the result is quite tied to the value of the matrices b_i. When all other variables keep the same values, but $b_2^T = \begin{pmatrix} 1 & 1 \end{pmatrix}$, then Theorem 3.2 is no more applicable because a positive definite CQLF cannot be found. Thus, Theorem 3.2 is able to relax stability conditions, depending strongly on the membership functions and input matrices values. Yet, results may remain conservative with respect to other methods, but, such methods as piecewise Lyapunov or fuzzy functions can also be applied (with further insight) to the TS fuzzy system with coupling terms.

Suppose that, now, we add the following subsystem

$$A_4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, B_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

along with the grade of membership $\omega_4 = (1 - z)/2$. It is possible, in this case, to find a nonlinear PDC controller such that

$$\sum_{i,j=1, j \neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) = 0.$$

Indeed, the solution of this system of equations is:

$$K_1 = \begin{pmatrix} k_{31} + (1 - z)k_{41} & k_{32} + (1 - z)k_{42} \end{pmatrix}^T, K_2 = K_1,$$

$$K_3 = \begin{pmatrix} k_{31} & k_{32} \end{pmatrix}^T, K_4 = \begin{pmatrix} k_{41} & k_{42} \end{pmatrix}^T.$$

In this case, one only has to ensure that the local closed-loop controlled systems share a CQLF. If $A_i + B_i K_i(z) = G_i(z)$, one has to check that there exists a common positive definite matrix P such that $\forall\, i = 1 \cdots r, P G_i(z) + G_i(z)^T P \prec 0$, which is easy to solve since the closed-loop matrices are affine in z.

5.2 Example 2

Consider the model of a stirred tank reactor:

$$\begin{aligned} \dot{C_A} =& \frac{q}{V}(C_{Af} - C_A) - k_0 C_A e^{-\frac{E}{RT}}, \\ \dot{T} =& \frac{q}{V}(T_f - T) - \frac{\Delta H k_0}{\rho_{C_p}} C_A e^{-\frac{E}{RT}} + \frac{\rho_c C_{pc}}{\rho C_p V} q_c (1 - e^{-\frac{h_A}{\rho_c C_{pc} q_c}})(T_{cf} - T), \end{aligned} \tag{9}$$

where q,q_c are the process and coolant flowrates, C_A and C_{Af} are the ouput and feed concentrations, T, T_f, T_{cf} are the reactor, feed and coolant temperatures. V is the reactor volume, h_a a heat transfer coefficient, E/R an energy activation term, ΔH the heat of reaction, ρ_c, ρ the liquid and coolant densities, and C_{pc}, C_p their specific heats. All values can be found in [20]. The coolant flowrate q_c is the control, C_A is the measured variable, and one supposes that $C_A \in [0.06 \cdots 0.13]$, the operating points for $C_A^1 = 0.06, C_A^2 = 0.1, C_A^3 = 0.13$ have the following linear models:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -16.67 & -0.047 \\ 0 & 3133.33 & 7.42 \end{pmatrix}, B_1 = \begin{pmatrix} 0 \\ 0 \\ -0.99 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -10 & -0.047 \\ 0 & 1800 & 7.33 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 0 \\ -0.88 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -7.69 & -0.046 \\ 0 & 1338.46 & 7.19 \end{pmatrix}, B_3 = \begin{pmatrix} 0 \\ 0 \\ -0.82 \end{pmatrix}.$$

$$T^1 = 449.47, q_c^1 = 89.03, T^2 = 438.54, q_c^2 = 103.41, \ T^3 = 432.92, q_c^3 = 110.03.$$

For Gaussian validity functions, the nominal T-S model is given by:

$$\begin{pmatrix} \dot{\int} C_A(t)dt \\ \dot{C}_A(t) \\ \dot{T}(t) \end{pmatrix} = \sum_{i=1}^{3} \mu_i(C_A) \left[A_i \begin{pmatrix} C_A(t) \\ C_A(t) - C_A^i \\ T(t) - T^i \end{pmatrix} + B_i(q_c(t) - q_c^i) \right],$$

where $\mu_i = \omega_i(C_A)/\sum_{j=1}^{3}(\omega_j(C_A))$, $\omega_i = exp(-\frac{1}{2}(\frac{C_A - CA^i}{\sigma_i})^2)$, and $\sigma_i = 0.01, i = 1, 2, 3$ is a reasonable choice to represent with a good accuracy the nonlinear model (see [20] for full details).

The state space is $x = (\int c_A dt, c_A, T)^T$, and the control gains have been chosen to place the poles at $\lambda = (-3.4205 + 1.8701i, -3.4205 - 1.8701i, -5)^T$.

In this case the products $\mu_1\mu_2$ and $\mu_2\mu_3$ are bounded by 0.25 and $\mu_1\mu_3$ is bounded by 10^{-5}. Thus, it is easy to find bounds for b_1, b_2, b_3. It is impossible to find a common Lyapunov matrix P for the T-S system using Theorem (2.1), but it is possible to find one using Theorem (3.2) with

$$P = 10^5. \begin{pmatrix} 10 & -0.72 & -0.23 \\ -0.72 & 2.71 & 0.014 \\ -0.23 & 0.014 & 0.0001 \end{pmatrix}.$$

The magnitude of elements of P is still important because of the small overlapping between membership functions. Of course, this result only guaranties the convergence of the Takagi–Sugeno fuzzy system and not that of the corresponding nonlinear system, for which uncertainties should be lumped into the T-S fuzzy model as for example in [19].

6 Conclusion

In this paper, the stability of a Takagi–Sugeno fuzzy system under the Parallel Distributed Compensation controller has been studied. This control strategy allocates the same weight to a local controller as the one in the fuzzy combination of local submodels. The influence of the coupling between any local subsystem and any local controller (different from the corresponding local controller designed from the local subsystem considered) in the closed-loop response has been highlighted, and it has been shown to be effective when the input matrices of the subsystems are different. It has been subsequently shown that a controller synthesis based on an analysis of each local subsystem controlled by any local compensator, would lead to conservative results. A new approach has been proposed which, for every local subsystem, takes the coupling term coming from other subsystems into account, and proposes to choose the gain in order to cope with the effect of the coupling terms. This strategy allows to minimize the number of linear matrix inequalities to be solved for controller synthesis and to take into account the shape of the membership functions. Moreover, an exact compensation using a nonlinear PDC controller has been proposed, which is tractable only if the number of subsystems is greater than the model order plus one. Further investigation will be undertaken to generalize the results for Lyapunov functions leading to less conservative results, i.e. piecewise and fuzzy Lyapunov functions.

References

[1] Takagi, T. and Sugeno, M. Fuzzy identification of systems and its applications to modeling and control, *IEEE Transactions on Systems, Man And Cybernetics* **15** (1985) 116–132.

[2] Tanaka, K., Ikeda, T. and Wang, H. O. Robust Stabilization of a Class of Uncertain Nonlinear Systems via Fuzzy Control: Quadratic Stabilizability, H^∞ Control Theory, and Linear Matrix Inequalities. *IEEE Transactions on Fuzzy Systems* **4** (1996) 1–13.

[3] Gahinet, P., Nemirovski, A., Laub, A. and Chilali, M. *The LMI Control Toolbox.* Natick, MA: The Mathworks, Inc., 1995.

[4] Tanaka, K., Ikeda, T. and Wang, H. O. Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs. *IEEE Transactions on Fuzzy Systems* **6** (1998) 250–265.

[5] Benrejeb, M., Gasmi, M. and Borne, P. New stability conditions for TS continuous nonlinear models. *Nonlinear Dynamics and Systems Theory* **5** (4) (2005) 369–379.

[6] Karimi, H.R., Moshiri, B. and Lucas, C. Robust Fuzzy Linear Control of a Class of Stochastic Nonlinear Time-Delay Systems. *Nonlinear Dynamics and Systems Theory* **4** (3) (2004) 317-332

[7] Feng, G. A Survey on Analysis and Design of Model-Based Fuzzy Control Systems. *IEEE Transactions on Fuzzy Systems* **14** (2006) 676–697.

[8] Kim, E. and Lee, H. New approaches to relaxed quadratic stability condition of fuzzy control systems. *IEEE Transactions on Fuzzy Systems* **8** (2000) 523–534.

[9] Fang, C.-H., Liu, Y.-S., Kau, S.-W., Hong, L. and Lee, C.-H. A New LMI-Based Approach to Relaxed Quadratic Stabilization of T-S Fuzzy Control Systems. *IEEE Transactions on Fuzzy Systems* **14** (2006) 286–397.

[10] Feng, G., Chen, C. L., Sun, D. and Guan, X. P. H^∞ controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions and bilinear matrix inequalities. *IEEE Transactions on Fuzzy Systems* **13** (2005) 94–103.

[11] Sala, A. and Ariño, C. Relaxed stability and performance conditions for Takagi–Sugeno fuzzy systems with knowledge on membership function overlap. *IEEE Transactions on Systems, Man, and Cybernetics, Part B* **37** (2007) 727–732.

[12] Sala, A. and Ariño, C. Relaxed Stability and Performance LMI Conditions for Takagi-Sugeno Fuzzy Systems With Polynomial Constraints on Membership Function Shapes. *IEEE Transactions on Fuzzy Systems* **16** (2008) 1328–1336.

[13] Lam, H.K. and Leung, F.H.F. Stability analysis of fuzzy control systems subject to uncertain grades of membership. *IEEE Transactions on Systems, Man and Cybernetics, Part B* **35** (2005) 1322–1325.

[14] Lam, H. K. and Leung, F. H. F. LMI-Based Stability and Performance Conditions for Continuous-Time Nonlinear Systems in Takagi–Sugeno's Form, *IEEE Transactions on Systems, Man, and Cybernetics, Part B* **37** (2007) 1396–1406.

[15] Abbaszadeh, M. and Marquez, H. J. LMI optimization approach to robust H^∞ observer design and static output feedback stabilization for discrete-time nonlinear uncertain systems. *Int. J. Robust Nonlinear Control* **19** (2009) 313–340.

[16] Khargonekar, P.P., Petersen, I.R. and Zhou, K. Robust stabilization of uncertain linear systems: Relations between quadratic stabilizability and H_1 control theory. *IEEE Transactions on Automatic Control* **37** (1990) 356–361.

[17] Wang, W.J. and Lin, W.W. PDC Synthesis for T-S Fuzzy Large-scale Systems. *IEEE Transactions on Fuzzy Systems* **12** (2004) 309–315.

[18] Yoneyama, J. Robust H^∞ control analysis and synthesis for Takagi-Sugeno general uncertain fuzzy systems. *Fuzzy Sets and Systems* **157** (16) (2006) 2205–2223.

[19] Lo, J.-C. and Lin, M.-L. Robust H^∞ nonlinear modeling and control via uncertain fuzzy systems. *Fuzzy Sets and Systems* **143** (2004) 189–209.

[20] Toscano, R. Robust synthesis of a PID controller by uncertain multimodel approach. *Information Sciences* **177** (2007) 1441–1451.

[21] Tanaka, K., Ohtake, H. and Wang, H.O. A Descriptor System Approach to Fuzzy Control System Design via Fuzzy Lyapunov Functions. *IEEE Transactions on Fuzzy Systems* **15** (2007) 333–341.

[22] Tuan, H.D., Apkarian, P., Narikiyo, T. and Yamamoto, Y. Parametrized linear matrix inequality techniques in fuzzy control design. *IEEE Transactions on Fuzzy Systems* **9** (2001) 324–332.

[23] Liu, X. and Zhang, Q. New approaches to H^∞ controller design based on fuzzy observers for fuzzy T-S systems via LMI. *Automatica* **39** (2003) 1571–1582.

[24] Sala, A. On the conservativeness of fuzzy and fuzzy-polynomial control of nonlinear systems. *Annual Reviews in Control* **33** (2009) 48–58.