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# On Stability of Hopfield Neural Network on Time Scales

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**Abstract:** In the paper uniform asymptotic, exponential and uniform exponential sufficient stability conditions for the neural systems on time scales are obtained. The sufficient conditions of regressivity of system's function are given.

**Keywords:** *neural network; time scale; uniform stability; asymptotic stability; exponential stability; Lyapunov function.*

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## 1 Introduction

The theory of time scale was introduced by Stefan Hilger [10] in order to unify continuous and discrete cases and was intensively developed in many papers (see [1, 4] and references therein). Recently the theory of dynamic systems on time scale have received special attention from many authors, some of them focused their interest on the stability theory for such systems [2, 3, 13].

Proposed in [11] the Hopfield-type neural networks and their generalizations [7, 8] is a special but important case of general differential systems. It derives from biological models in practical investigations and has extensive applications in many different fields such as parallel computation, signal processing, pattern recognition, optimization and associative memories (see [5, 8, 14]).

However, as the theory of dynamic systems on time scale is widely studied the corresponding theory of neural systems is still at an initial stage of its development. In [6], the authors got some stability results for delayed bidirectional associative memory neural networks on time scales. Also in [12], some criteria of stability and existence

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of periodic solutions for delayed bidirectional associative memory neural networks with impulses on time scales were obtained.

Motivated by the above we consider a neural network on time scale the dynamics of which is described by the equation of the type

$$x^{\Delta}(t) = -Bx(t) + Ts(x(t)) + u, \quad t \in \mathbb{T}_\tau, \tag{1}$$

whose solution $x(t; t_0, x_0)$ for $t = t_0$ takes the value $x_0$, i.e.

$$x(t_0; t_0, x_0) = x_0, \quad t_0 \in \mathbb{T}_\tau, \quad x_0 \in \mathbb{R}^n, \tag{2}$$

where $\mathbb{T}$ is an arbitrary time scale, $\sup \mathbb{T} = +\infty$, $\mathbb{T}_\tau = \{t \in \mathbb{T} \colon t \geq \tau\}$, $\tau \in \mathbb{T}$. In system (1) $x^{\Delta}(t)$ is a $\Delta$-derivative on time scale $\mathbb{T}$, $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $x_i$ is the activation of the $i$-th neuron, $T = \{t_{ij}\} \in \mathbb{R}^{n \times n}$, the components $t_{ij}$ describe the interaction between the $i$-th and $j$-th neurons, $s \colon \mathbb{R}^n \to \mathbb{R}^n$, $s(x) = (s_1(x_1), s_2(x_2), \dots, s_n(x_n))^{\mathrm{T}}$, the activation function $s_i$ describes response of the $i$-th neuron, $B \in \mathbb{R}^{n \times n}$, $B = \operatorname{diag}\{b_i\}$, $b_i > 0$ represents the rate with which the $i$-th neuron shell resets its potential to the resting state in isolation when it is disconnected from the network and the external inputs, $i = 1, 2, \dots, n$, n corresponds to the number of neurons in layers, $u \in \mathbb{R}^n$ is a constant external input vector. All needed notations on time scales according to [4] will be given in Section 2.

System (1) is general and unifies two well known neural models. If $\mathbb{T} = \mathbb{R}$ then $x^{\Delta} = d/dt$ and the initial problem (1), (2) is equivalent to the initial problem for a continuous Hopfield type neural network [11]

$$\begin{gathered} \frac{dx(t)}{dt} = -Bx(t) + Ts(x(t)) + u, \quad t \geq \tau, \\ x(t_0; t_0, x_0) = x_0, \quad t_0 \geq \tau, \quad x_0 \in \mathbb{R}^n. \end{gathered} \tag{3}$$

If $\mathbb{T} = \mathbb{N}$ then $x^{\Delta}(k) = x(k+1) - x(k) = \Delta x(k)$, $\mathbb{T}_\tau = \{\tau, \tau+1, \tau+2, \dots\}$ and the initial problem (1)–(2) is equivalent to the initial problem for a discrete Hopfield type neural network [9]

$$\begin{gathered} \Delta x(k) = -Bx(k) + Ts(x(k)) + u, \quad k \in \mathbb{T}_\tau, \\ x(k_0; k_0, x_0) = x_0, \quad k_0 \in \mathbb{T}_\tau, \quad x_0 \in \mathbb{R}^n. \end{gathered} \tag{4}$$

Dynamics of continuous system (3) and discrete systems (4) and their generalizations are widely studied by many authors [7, 8, 9, 11, 15, 17], but there are no stability results for system (1) on time scales. Our purpose in the paper is by using the direct Lyapunov method to study the stability of equilibrium of (1).

The outline of the paper is as follows. In Section 2 we shall give some notations and basic definitions concerning the calculus on time scale and some required assertions. In Section 3 we shall present some new sufficient conditions ensuring the asymptotic and exponential stability of the equilibrium of system (1). Also we shall offer the criteria of regressivity of function $f(x) = -Bx + Ts(x) + u$. In Section 4 we shall give one example to illustrate our results obtained in the previous sections.

## 2 Notations and Preliminaries

In this section all facts concerning time scale calculus are given according to book [4].

**Definition 2.1** An arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$ is referred to as a *time scale* and denoted by $\mathbb{T}$.

**Definition 2.2**

- The *forward* and *backward jump operators* $\sigma : \mathbb{T} \to \mathbb{T}$ and $\rho : \mathbb{T} \to \mathbb{T}$ are respectively defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.
- If $\sigma(t) = t,\ \rho(t) = t,\ \sigma(t) > t,$ and $\rho(t) < t,$ then the element $t \in \mathbb{T}$ is called *right-dense, left-dense, right-scattered,* and *left-scattered,* respectively. Here it is assumed that $\inf \varnothing = \sup \mathbb{T}$ (i.e. $\sigma(t) = t$, if $\mathbb{T}$ contains the maximal elements $t$) and $\sup \varnothing = \inf \mathbb{T}$ (i.e. $\rho(t) = t$, if $\mathbb{T}$ contains the minimal elements $t$).
- In addition to the set $\mathbb{T}$, the set $\mathbb{T}^k$ is defined as follows

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

- The distance from an arbitrary element $t \in \mathbb{T}$ to its follower is called the *graininess* of the time scale $\mathbb{T}$ and is given by the formula

$$\mu(t) = \sigma(t) - t.$$

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and $\mu(t) = 0$, if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1,\ \rho(t) = t-1$ and $\mu(t) = 1$.

**Definition 2.3**

- The function $f : \mathbb{T} \to \mathbb{R}$ is called $\Delta$-differentiable at a point $t \in \mathbb{T}^k$ if there exists $\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a $W$-neighborhood of $t$ satisfying

$$|\,[f(\sigma(t)) - f(s)] - \gamma[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|$$

for all $s \in W$. In this case we shall write $f^{\Delta}(t) = \gamma$.

- if the function $f : \mathbb{T} \to \mathbb{R}$ is $\Delta$-differentiable for any $t \in \mathbb{T}^k$, then $f$ is called $\Delta$-differentiable on $\mathbb{T}^k$.

**Theorem 2.1** *Assume that the functions $f, g : \mathbb{T} \to \mathbb{R}$ are $\Delta$-differentiable at $t \in \mathbb{T}^k$. Then the following assertions are valid:*

(1) *the sum $f + g$ is $\Delta$-differentiable at $t$ and $(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$;*

(2) *for any $\alpha \in \mathbb{R}$, the function $\alpha f(t)$ is $\Delta$-differentiable at $t$ and $\alpha f^{\Delta}(t) = \alpha f^{\Delta}(t)$;*

(3) *the product $fg$ is $\Delta$-differentiable at $t$ and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t));$$

(4) $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$.

Note that, if $\mathbb{T} = \mathbb{R}$, then $f^{\Delta} = f'$, which is the Euler derivative of $f$, and if $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = \Delta f(t) = f(t + 1) - f(t)$, which is the forward difference of $f(t)$.

**Definition 2.4**

- A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dence points in $\mathbb{T}$ and its left-sided limit exists (finite) at left-dence points in $\mathbb{T}$. The set of all $rd$-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

- A function $f : \mathbb{T} \to \mathbb{R}$ is called *regressive,* if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^k$ and *positive regressive,* if $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}^k$.

- A function $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is called *regressive,* if the mapping $I + \mu(t)f(t, \cdot)$ is invertible at each $t \in \mathbb{T}^k$. Here $I : \mathbb{R}^n \to \mathbb{R}^n$ is identity mapping.

- The set of all regressive and $rd$-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$.

We define the function

$$\beta_k(t) = \begin{cases} \mu^{-1}(t) \log |1 + \mu(t)k(t)|, & \text{if} \quad \mu(t) > 0, \\ k(t), & \text{if} \quad \mu(t) = 0, \end{cases}$$

where $k \in \mathcal{R}$, $t \in [t_0, +\infty)_{\mathbb{T}}$. Here and bellow $[a, +\infty)_{\mathbb{T}} = \{t \in \mathbb{T} : \ a \leq t < +\infty\}$, $a \in \mathbb{T}$.

**Definition 2.5** We recall that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $K$, if it is continuous, strictly increasing on $\mathbb{R}_+$ and $\psi(0) = 0$.

**Definition 2.6** We recall that the matrix $A \in \mathbb{R}^{n \times n}$ is called *$M$-matrix* if its all non-diagonal elements are non-positive and all principle minors are positive.

**Definition 2.7** We recall that the mapping $H : \mathbb{R}^n \to \mathbb{R}^n$ is called a *homeomorphism* of $\mathbb{R}^n$ onto itself, if $H$ is continuous, bijective, $H$ is onto itself and the inverse mapping $H^{-1}$ is also continuous.

For convenience, we introduce some notations. We denote by $\|x\|$ a vector norm of vector $x \in \mathbb{R}^n$ defined by $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$, $\|A\|$ denotes a matrix norm of matrix $A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$ defined by $\|A\| = (\lambda_M(A^{\mathrm{T}}A))^{1/2}$, $\lambda_m(A)$, $\lambda_M(A)$ are minimal and maximal eigenvalues of matrix $A$ respectively. In addition $A^{-1}$ denotes the inverse of A, $|A|$ denotes absolute-value matrix given by $|A| = \{|a_{ij}|\}$.

We assume on system (1) as follows.

$\mathrm{S}_1$. The vector-function $f(x) = -Bx + Ts(x) + u$ is regressive.

$\mathrm{S}_2$. There exist positive constants $M_i > 0$, $i = 1, 2, \dots, n$, such that $|s_i(r)| \leq M_i$ for all $r \in \mathbb{R}$.

$\mathrm{S}_3$. There exist positive constants $l_i > 0$, $i = 1, 2, \dots, n$, such that $|s_i(r) - s_i(v)| \leq l_i|r - v|$ for all $r, v \in \mathbb{R}$.

$\mathrm{S}_4$. $0 < \mu(t) \in \mathcal{M}$ for all $t \in \mathbb{T}_\tau$, where $\mathcal{M} \subset \mathbb{R}$ is a compact set.

Note that under conditions $S_1$–$S_3$ there exists a unique solution of problem (1), (2) on $[t_0, +\infty)_{\mathbb{T}}$ for all initial data $(t_0, x_0) \in \mathbb{T}_\tau \times \mathbb{R}^n$ [4].

We denote by $r_0 = \left(\sum_{i=1}^n (\sum_{j=1}^n M_j |T_{ij}| + |u_i|)^2 / b_i^2\right)^{1/2}$ and $\Lambda = \operatorname{diag}\{l_i\} \in \mathbb{R}^{n\times n}$. Similar to Theorem 3.1 from [16] and Theorem 1 from [17] we can easily obtain the following assertion.

**Theorem 2.2** *If for system* (1) *conditions* $S_1$–$S_3$ *are satisfied then there exists an equilibrium state* $x = x^*$ *of system* (1) *and moreover,* $\|x^*\| \le r_0$*. Besides, if the matrix* $B\Lambda^{-1} - |T|$ *is an* $M$*-matrix, this equilibrium state is unique.*

**Definition 2.8** The equilibrium state $x = x^*$ of the system (1) is:

(1) *uniformly stable* if for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that $\|x_0 - x^*\| < \delta$ implies $\|x(t; t_0, x_0) - x^*\| < \varepsilon$ for all $t \in [t_0, +\infty)_{\mathbb{T}}$, $t_0 \in \mathbb{T}_\tau$;

(2) *uniformly asymptotically stable* if it is uniformly stable and there exists $\Delta > 0$ such that $\|x_0 - x^*\| < \Delta$ implies $\lim_{t\to+\infty} \|x(t; t_0, x_0) - x^*\| = 0$ for all $t_0 \in \mathbb{T}_\tau$;

(3) *exponentially stable* if there exist $\beta > 0$ and $\lambda > 0$ such that for all $t_0 \in \mathbb{T}_\tau$ there exists $N = N(t_0) > 0$ such that $\|x_0 - x^*\| < \beta$ implies $\|x(t; t_0, x_0) - x^*\| \le Ne^{-\lambda(t-t_0)}\|x_0 - x^*\|$ for all $t \in \mathbb{T}_\tau$;

(4) *uniformly exponentially stable* if it is exponentially stable and $N$ does not depend on $t_0$.

Let $x^*$ be the equilibrium state of system (1). We perform the change of variables $y(t) = x(t) - x^*$ and rewrite the initial problem (1), (2) as

$$y^{\triangle}(t) = -By(t) + Tg(y(t)), \quad t \in \mathbb{T}_\tau, \tag{5}$$

$$y(t_0; t_0, y_0) = y_0, \quad t_0 \in \mathbb{T}_\tau, \quad y_0 \in \mathbb{R}^n, \tag{6}$$

where $y \in \mathbb{R}^n$, $g\colon \mathbb{R}^n \to \mathbb{R}^n$, $g(y) = (g_1(y_1), g_2(y_2), \dots, g_n(y_n))^{\mathrm{T}}$, $g(y) = s(y + x^*) - s(x^*)$.

If for system (1) assumptions $S_1$–$S_3$ are valid, then for system (5) the following assertions hold true.

$G_1$. The vector-function $\tilde{g}_1(y) = -By + Tg(y)$ is regressive.

$G_2$. For all $r \in \mathbb{R}$ $|g_i(r)| \le 2M_i$, $i = 1, 2, \dots, n$.

$G_3$. For all $r, v \in \mathbb{R}$ $|g_i(r) - g_i(v)| \le l_i|r - v|$, $i = 1, 2, \dots, n$.

Note that under conditions $G_1$–$G_3$ there exists a unique solution of problem (5), (6) on $[t_0, +\infty)_{\mathbb{T}}$ for all initial data $(t_0, x_0) \in \mathbb{T}_\tau \times \mathbb{R}^n$ [4].

Futher we shall need the following result.

**Lemma 2.1** *Assume that* $g_i \in C^2(\mathbb{R})$, $g_i(0) = 0$, $i = 1, 2, \dots, n$, *and constants* $K_i > 0$, $i = 1, 2, \dots, n$, *exist so that* $|g_i''(u)| \le K_i$ *for all* $u \in \mathbb{R}$*. Then the vector-function* $g(y)$ *can be represented as* $g(y) = Hy + \tilde{g}_2(y)$*, where* $H = \operatorname{diag}\{g_i'(0)\} \in \mathbb{R}^{n\times n}$, $\tilde{g}_2\colon \mathbb{R}^n \to \mathbb{R}^n$ *and the estimate*

$$\|\tilde{g}_2(y)\| \le K\|y\|^2, \tag{7}$$

*holds true, where* $K = \max_i\{K_i\}/2$.

***Proof*** Decomposing the functions $g_i(y_i)$ by the Maclaurin formula we easily prove the Lemma.

## 3 Main Results

In this section we consider stability of a *neural network on time scale*. Let $x^*$ be the equilibrium state of system (1). Designate by $\underline{b} = \min\{b_i\},\ \overline{b} = \max\{b_i\},\ L = \max\{l_i\}$.

**Theorem 3.1** *For system* (1) *assume that assumptions* $\mathrm{S}_1$–$\mathrm{S}_4$ *are valid and there exists a constant* $\mu^* \in \mathcal{M}$ *such that* $\mu(t) \le \mu^*$ *for all* $t \in \mathbb{T}_\tau$. *If the inequality*

$$2\underline{b} - 2L\|T\| - \mu^*(\overline{b} + L\|T\|)^2 > 0,$$

*is satisfied, the equilibrium state* $x = x^*$ *of system* (1) *is uniformly asymptotically stable.*

***Proof*** It is clear that the behavior of solution $x(t)$ of system (1) in the neighborhood of the equilibrium state $x^*$ is equivalent to the behavior of solution $y(t)$ of system (5) in the neighborhood of zero. For the proof we shall apply the Lyapunov function $V(y) = y^{\mathrm{T}}y$. If $y(t)$ is $\Delta$-differentiable in the point $t \in \mathbb{T}^k$, for the derivative of function $V(y(t))$ we have the expression

$$\begin{aligned} V^\Delta(y(t)) &= (y^{\mathrm{T}}(t)\,y(t))^\Delta = y^{\mathrm{T}}(t)\,y^\Delta(t) + [y^{\mathrm{T}}(t)]^\Delta y(\sigma(t)) \\ &= y^{\mathrm{T}}(t)\,y^\Delta(t) + [y^{\mathrm{T}}(t)]^\Delta [y(t) + \mu(t) y^\Delta(t)]. \end{aligned}$$

For the derivative of function $V$ along solutions of system (5) we get

$$\begin{aligned} V^\Delta(y(t))|_{(5)} &= 2y^{\mathrm{T}}(t)\,y^\Delta(t) + \mu(t)[y^\Delta(t)]^{\mathrm{T}} y^\Delta(t) \\ &= 2y^{\mathrm{T}}(t)[-By(t) + Tg(y(t))] + \mu(t)\| - By(t) + Tg(y(t))\|^2 \\ &\le -2\lambda_m(B)\|y(t)\|^2 + 2\|y(t)\|\,\|T\|\,\|g(y(t))\| + \mu^*(\|B\|\,\|y(t)\| + \|T\|\,\|g(y(t))\|)^2 \\ &= -2\underline{b}\,\|y(t)\|^2 + 2\|T\|\,\|y(t)\|\,\|y(t)\| + \mu^*(\overline{b}\,\|y(t)\| + \|T\|\,\|g(y(t))\|)^2. \end{aligned}$$

Using obvious estimation $\|g(y(t))\| \le L\,\|y(t)\|$ as a result we have

$$\begin{aligned} V^\Delta(y(t))|_{(f-11)} &\le -2\,\underline{b}\,\|y(t)\|^2 + 2L\|T\|\,\|y(t)\|^2 + \mu^*\left(\overline{b}\,\|y(t)\| + L\|T\|\,\|y(t)\|\right)^2 \\ &= -\left(2\underline{b} - 2L\|T\| - \mu^*(\overline{b} + L\|T\|)^2\right)\|y(t)\|^2. \end{aligned}$$

Hence it follows that all conditions of Corollary 4.2 from the paper [3] are satisfied. Therefore, the equilibrium state $y = 0$ of system (5) is uniformly asymptotically stable. This is equivalent to the uniform asymptotic stability of the equilibrium state $x = x^*$ of system (1).

**Theorem 3.2** *Let the following conditions be satisfied:*

(1) *for system* (1) *on time scale* $\mathbb{T}$ *assumptions* $\mathrm{S}_1$–$\mathrm{S}_4$ *are valid;*

(2) *functions* $s_i \in C^2(\mathbb{R})$ *and there exist constants* $K_i > 0$ *such that* $|s_i''(r)| \le K_i$ *for all* $r \in \mathbb{R},\ i = 1, 2, \dots, n;$

(3) *there exists a constant* $\mu^* \in \mathcal{M}$ *such that* $\mu(t) \le \mu^*$ *for all* $t \in \mathbb{T}_\tau;$

(4) *there exists a positive definite symmetric matrix* $P \in \mathbb{R}^{n\times n}$ *such that the inequality* $\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu^*\|P\|\|B_1\|^2 < 0$ *holds true, where* $B_1 = -B + TH,\ H = \mathrm{diag}\{s_i'(0)\} \in \mathbb{R}^{n\times n}$.

*Then the equilibrium state $x = x^*$ of system* (1) *is uniformly asymptotically stable.*

***Proof*** We apply the function $V(y) = y^{\mathrm{T}} P y$. For the derivative of function $V$ along solutions of system (5) we have

$$\begin{aligned}
V^{\Delta}(y(t))|_{(5)} &= y^{\mathrm{T}}(t)Py^{\Delta}(t) + [y^{\mathrm{T}}(t)]^{\Delta}Py(\sigma(t)) = y^{\mathrm{T}}(t)Py^{\Delta}(t) + [y^{\mathrm{T}}(t)]^{\Delta}Py(t) \\
&+ \mu(t)[y^{\Delta}(t)]^{\mathrm{T}}Py^{\Delta}(t) = y^{\mathrm{T}}(t)P\big[B_1 y(t) + T\tilde{g}_2(y(t))\big] + \big[B_1 y(t) + T\tilde{g}_2(y(t))\big]^{\mathrm{T}}Py(t) \\
&+ \mu(t)\big[B_1 y(t) + T\tilde{g}_2(y(t))\big]^{\mathrm{T}}P\big[B_1 y(t) + T\tilde{g}_2(y(t))\big] \le y^{\mathrm{T}}(t)\big[PB_1 + B_1^{\mathrm{T}}P\big]y(t) \\
&+ 2y^{\mathrm{T}}(t)PT\tilde{g}_2(y(t)) + \mu(t)\|P\|\,\|B_1 y(t) + T\tilde{g}_2(y(t))\|^2 \le \big(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) \\
&+ \mu(t)\|P\|\,\|B_1\|^2\big)\|y(t)\|^2 + 2\|P\|\,\|T\|\,\|\tilde{g}_2(y(t))\|\,\|y(t)\| + \mu(t)\|P\|\,\|\tilde{g}_2(y(t))\|^2\|T\|^2 \\
&+ 2\mu(t)\|P\|\,\|B_1\|\,\|T\|\,\|\tilde{g}_2(y(t))\|\,\|y(t)\|.
\end{aligned}$$

Using inequality (7) and condition (3) of Theorem 3.2 we get

$$\begin{aligned}
V^{\Delta}(y(t))|_{(5)} &\le \big(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu^*\|P\|\,\|B_1\|^2\big)\|y(t)\|^2 + 2K\|P\|\,\|T\|\,\|y(t)\|^3 \\
&\quad + 2\mu^* K\|P\|\,\|B_1\|\,\|T\|\,\|y(t)\|^3 + \mu^* K^2\|P\|\,\|T\|^2\|y(t)\|^4.
\end{aligned}$$

Designate

$$\begin{aligned}
&\psi(\|y\|) = a\|y\|^2, \\
&a = -\big(\lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu^*\|B_1\|\|P\|^2\big) > 0, \\
&m(\psi) = 2a^{-\frac{1}{3}}K\|P\|\|T\|\,(1 + \mu^*\|B_1\|)\,\psi^{\frac{1}{3}} + \mu^* a^{-2}K^2\|P\|\|T\|^2\psi.
\end{aligned}$$

For the derivative of function $V$ along solutions of system (5) we obtain the inequality

$$V^{\Delta}(y(t))|_{(5)} \le -\psi(\|y\|) + m(\psi(\|y\|)).$$

Since the function $\psi \in K$-class, $\lim\limits_{\psi \to 0} m(\psi) = 0$, all conditions of Corollary 4.2 from [3] are satisfied and therefore, the equilibrium state $y = 0$ of system (5) is uniformly asymptotically stable. This is equivalent to the uniform asymptotic stability of the equilibrium state $x = x^*$ of system (1).

**Theorem 3.3** *Let the following conditions be satisfied*

(1) *for system* (1) *assumptions* $\mathrm{S}_1$-$\mathrm{S}_3$ *hold true.*

(2) *functions* $s_i \in C^2(\mathbb{R})$ *and there exist constants* $K_i > 0$ *such that* $|s_i''(r)| \le K_i$ *for all* $r \in \mathbb{R}$, $i = 1, 2, \dots, n$.

(3) *there exist a positive definite symmetric matrix* $P \in \mathbb{R}^{n\times n}$ *and a constant* $M > 0$ *such that* $|1 + \mu(t)A(t)| \ge M$ *for all* $t \in \mathbb{T}_\tau$, *where* $B_1 = -B + TH$, $H = \mathrm{diag}\{s_i'(0)\} \in \mathbb{R}^{n\times n}$, $A(t) = \lambda_M(PB_1 + B_1^{\mathrm{T}}P) + \mu(t)\|P\|\,\|B_1\|^2$.

*Then, if*

(a) $\limsup\limits_{t\to\infty} \beta_A(t) = q < 0$, *the equilibrium state* $x = x^*$ *of system* (1) *is exponentially stable;*

(b) $\sup\{\beta_A(t):\ t\in\mathbb{T}_\tau\}=\overline{q}<0$, *the equilibrium state* $x=x^*$ *of system* (1) *is uniformly exponentially stable.*

***Proof*** We shall apply function $V(y)=y^{\mathrm{T}}Py$ and for the derivative of function $V$ along solutions of system (5) we shall use the expression obtained in the previous theorem

$$\begin{aligned}V^{\Delta}(y(t))|_{(5)} &\le \big(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\,\|B_1\|^2\big)\|y(t)\|^2\\ &+2\|P\|\,\|T\|\,\|\tilde g_2(y(t))\|\,\|y(t)\|+2\mu(t)\|P\|\,\|B_1\|\,\|T\|\,\|\tilde g_2(y(t))\|\,\|y(t)\|\\ &+\mu(t)\|P\|\,\|\tilde g_2(y(t))\|^2\|T\|^2\le \big(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\,\|B_1\|^2\big)\|y(t)\|^2\\ &+\big(2K\|P\|\,\|T\|\,\|y(t)\|+2\mu(t)K\|P\|\,\|B_1\|\,\|T\|\,\|y(t)\|\\ &+\mu(t)K^2\|P\|\,\|T\|^2|y(t)\|^2\big)\|y(t)\|^2=A(t)\|y(t)\|^2+\Phi(t,V(y)),\end{aligned}$$

where $\Phi(t,V)=\big[2K\|P\|\,\|T\|(1+\mu(t)\|B_1\|)\sqrt{V}+\mu(t)K^2\|P\|\,\|T\|^2V\big]V$.

Consider the set $\mathcal{T}=\{t\in\mathbb{T}_\tau:\ \mu(t)\ne 0\}$. If there exists $\sup\mathcal{T}<+\infty$ then there exists $t_1\in\mathbb{T}_\tau$ such that $\mu(t)=0$ for all $t\in[t_1,+\infty)_{\mathbb{T}}$. If the set $\mathcal{T}$ is not bounded, the condition $\limsup\limits_{t\to\infty}\beta_A(t)=q<0$ implies that there exists a sufficiently large $t_2\in\mathbb{T}_\tau\cap\mathcal{T}$ such that for all $t\in[t_2,+\infty)_{\mathbb{T}}\cap\mathcal{T}$ inequality $\beta_A(t)<0$ holds true. This yields that for all $t\in[t_2,+\infty)_{\mathbb{T}}\cap\mathcal{T}$ the inequality

$$\log\big|1+\mu(t)(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\|B_1\|^2)\big|<0$$

is true. Then

$$\begin{aligned}\mu(t)(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\,\|B_1\|^2)-1<1,\\ \|P\|\,\|B_1\|^2\mu^2(t)+\lambda_M(PB_1+B_1^{\mathrm{T}}P)\mu(t)-2\le 0.\end{aligned}$$

Since $D=\lambda_M(PB_1+B_1^{\mathrm{T}}P)^2+8\|P\|\|B_1\|^2\ge 0$, we obtain the estimate $\mu(t)\le\mu_1$ for all $t\in[t_2,+\infty)\cap\mathcal{T}$, where $\mu_1=(-\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\sqrt{D})/2\|P\|\|B_1\|^2\ge 0$. Hence, one can conclude that $\mu(t)\le\mu_1$ for all $t\in[t_3,+\infty)_{\mathbb{T}}$, $t_3=\max\{t_1,t_2\}$. If $t\in[\tau,\rho(t_3)]\cap\mathbb{T}$ then $\mu(t)\le t_3$. This implies the estimate $\mu(t)\le\mu^*=\max\{\mu_1,t_3\}$ for all $t\in\mathbb{T}_\tau$. Since

$$\begin{aligned}\frac{\Phi(t,V)}{V}&=2K\|P\|\,\|T\|(1+\mu(t)\|B_1\|)\sqrt{V}+\mu(t)K^2\|P\|\,\|T\|^2V\\ &\le 2K\|P\|\,\|T\|(1+\mu^*\|B_1\|)\sqrt{V}+\mu^*K^2\|P\|\,\|T\|^2V,\end{aligned}$$

we get $\Phi(t,V)/V\to 0$ for $V\to 0$ uniformly in $t$. According to Theorem 2 from the paper [13] we conclude that the equilibrium state $y=0$ of system (5) is exponentially stable. This is equivalent to the exponential stability of the equilibrium state $x=x^*$ of system (1).

Now we shall prove the second part of the theorem. Condition $\sup\{\beta_A:\ t\in\mathbb{T}_\tau\}=\overline{q}<0$ for $t\in\mathcal{T}$ implies

$$\log|1+\mu(t)(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\|B_1\|^2)|\le\mu(t)\overline{q}<0$$

for all $t\in\mathcal{T}$. Hence, we get

$$\mu(t)\le\frac{-\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\sqrt{D}}{2\|P\|\|B_1\|^2}=\mu^*,\quad \mu^*\ge 0,\quad t\in\mathcal{T}.$$

That is that $\mu(t) \le \mu^*$ for all $t \in \mathbb{T}_\tau$. Then, similar to the above, we have $\Phi(t,V)/V \to 0$ for $V \to 0$ uniformly in $t$.

Therefore, all conditions of Theorem 2 from the paper [13] are satisfied and the equilibrium state $y = 0$ of system (5) is uniformly exponentially stable. This is equivalent to the uniform exponential stability of the equilibrium state $x = x^*$ of system (1).

**Remark 3.1** Consider the scale $\mathbb{T} = \mathbb{N}$ $(\mu(t) \equiv 1)$. In this case system of equations (1) is equivalent to system (4) and the condition of uniform asymptotic stability of the equilibrium state of system (1) established in Theorem 3.1 for $\mu^* = 1$ becomes

$$2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2 > 0.$$

This result coincides completely with the below result for discrete system (4).

**Theorem 3.4** *For neural discrete system* (4) *let assumptions* $\mathrm{S}_2$, $\mathrm{S}_3$ *be satisfied. Then the equilibrium state* $x = x^*$ *of system* (4) *is uniformly asymptotically stable, provided that*

$$2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2 > 0.$$

***Proof*** Consider function $y(k) = x(k) - x^*$ and rewrite equations (4) as

$$y(k+1) = (-B + I)y(k) + Tg(x(k)), \quad k \in \mathbb{T}_\tau, \tag{8}$$

where $I$ is an identity $n \times n$-matrix and for the first difference of function $V(y) = y^{\mathrm{T}}y$ we get the estimate

$$\begin{aligned}
\Delta V(y(k))|_{(8)} &= y^{\mathrm{T}}(k+1)y(k+1) - y^{\mathrm{T}}(k)y(k) \\
&= [(-B+I)y(k) + Tg(y(k))]^{\mathrm{T}}[(-B+I)y(k) + Tg(y(k))] - y^{\mathrm{T}}(k)y(k) \\
&= y^{\mathrm{T}}(k)B^{\mathrm{T}}By(k) - 2y^{\mathrm{T}}(k)B^{\mathrm{T}}y(k) - 2y(k)^{\mathrm{T}}BTg(y(k)) \\
&\quad + 2y^{\mathrm{T}}(k)Tg(y(k)) + G^{\mathrm{T}}(y(k))T^{\mathrm{T}}Tg(y(k)) \\
&\le \|B\|^2\|y(k)\|^2 - 2\lambda_m(B)\|y(k)\|^2 + 2L\|B\|\,\|T\|\,\|y(k)\|^2 \\
&\quad + 2L\|T\|\,\|y(k)\|^2 + \|T\|^2\|g(y(k))\|^2 \\
&\le \left[\overline{b}^2 - 2\underline{b} + 2L\overline{b}\|T\| + 2L\|T\| + \|T\|^2L^2\right]\|(y(k))\|^2 \\
&= -\left[2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2\right]\|(y(k))\|^2.
\end{aligned}$$

This yields the assertion of the theorem.

The regressivity of function $f(x) = -Bx + Ts(x) + u$ is one of conditions for existence of solution of problem (1), (2). Here we give some sufficient regressivity conditions for the function $f(x)$.

**Theorem 3.5** *Let assumption* $\mathrm{S}_3$ *be fulfilled. If for every fixed* $t \in \mathbb{T}$ *the matrix* $(I - \mu(t)B)\Lambda^{-1} - \mu(t)|T|$ *is an* $M$*-matrix, the function* $f(x) = -Bx + Ts(x) + u$ *is regressive.*

***Proof*** We fix $t \in \mathbb{T}$ and consider the mapping $R\colon \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$R(x) = x + \mu(t)f(t,x) = (I - \mu(t)B)x + \mu(t)Ts(x) + \mu(t)u.$$

Designate by $\widetilde{B} = (I - \mu(t)B)$, $\widetilde{T} = \mu(t)T$ and $\widetilde{u} = \mu(t)u$. Then we get

$$R(x) = \widetilde{B}x + \widetilde{T}s(x) + \widetilde{u}.$$

Since the matrix $\widetilde{B}\Lambda^{-1} - |\widetilde{T}|$ is an $M$-matrix, the mapping $R\colon \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism [17]. Hence follows the reversibility of the mapping $R(x)$ which is equivalent to the reversibility of the operator $I + \mu(t)f(t,\cdot)\colon \mathbb{R}^n \to \mathbb{R}^n$.

## 4 Example

On the time scale $\mathbb{P}_{1,\gamma} = \bigcup_{j=0}^{\infty}[j(1+\gamma), j(1+\gamma)+1]$, $\quad \gamma > 0$, we consider a neural network

$$\begin{aligned} x_1^{\Delta}(t) &= -b_1x_1(t) + t_{11}s(x_2(t)) + t_{12}s(x_2(t)) + u_1, \\ x_2^{\Delta}(t) &= -b_2x_1(t) + t_{21}s(x_1(t)) + t_{22}s(x_2(t)) + u_2, \end{aligned} \tag{9}$$

where $x_1, x_2 \in \mathbb{R}$, $u_1, u_2 \in \mathbb{R}$, $b_1 = b_2 = 1$, $T = \left(\begin{smallmatrix} 0.1 & -0.5 \\ 0.5 & 0.1 \end{smallmatrix}\right)$, $s(u) = \tanh u$.

For the time scale $\mathbb{P}_{1,\gamma}$ the granularity function

$$\mu(t) = \begin{cases} 0, & t \in \bigcup_{j=0}^{\infty} [j(1+\gamma), j(1+\gamma)+1)\,, \\ \gamma, & t \in \bigcup_{j=0}^{\infty} \{j(1+\gamma)+1\}\,. \end{cases}$$

We take matrix $P = \operatorname{diag}\{0.5, 0.5\}$ and write out all the functions and constants mentioned in the conditions of Theorem 3.3

$$M_1 = M_2 = L_1 = L_2 = 1, \quad A(t) = -0.9 + 0.53\,\gamma,$$

$$K_1 = K_2 = 8\left|e^{\frac{2+\sqrt{3}}{2}} - e^{-\frac{2+\sqrt{3}}{2}}\right| \Big/ \left(e^{\frac{2+\sqrt{3}}{2}} + e^{-\frac{2+\sqrt{3}}{2}}\right)^3,$$

$$\beta_A(t) = \begin{cases} \gamma^{-1}\log|1 + \gamma(-0.9 + 0.53\,\gamma)|, & t \in \bigcup\limits_{j=0}^{\infty} \{j(1+\gamma)+1\}\,, \\ -0.9 + 0.53\,\gamma, & t \in \bigcup\limits_{j=0}^{\infty} [j(1+\gamma), j(1+\gamma)+1)\,. \end{cases}$$

The regressivity condition has the form of the inequalities

$$\begin{cases} 1 - 1.1\,\gamma > 0, \\ (1 - 1.1\gamma)^2 - 0.25\,\gamma^2 > 0, \end{cases}$$

which yields $\gamma < 0.625$. Since $1 + \gamma(-0.9 + 0.53\,\gamma) \geq 1 + \gamma_0(-0.9 + 0.53\,\gamma_0)$, $\gamma_0 = 0.9/(2 \cdot 0.53)$ for any $\gamma$, we can take for the constant $M$ the following value: $M = 1 + \gamma_0(-0.9 + 0.53\,\gamma_0) = 0.61$.

For $\gamma < 1.69$ the system of inequalities

$$\begin{cases} M \leq |1 + \gamma(-0.9 + 0.53\,\gamma)| < 1, \\ -0.9 + 0.53\,\gamma < 0 \end{cases}$$

is satisfied. This implies that $\sup_t \beta_A(t) = \max\{\gamma^{-1}\log|1 + \gamma(-0.9 + 0.53\,\gamma)|, -0.9 + 0.53\} < 0$. Since the matrix $B\Lambda^{-1} - |T| = \left(\begin{smallmatrix} 0.9 & -0.5 \\ -0.5 & 0.9 \end{smallmatrix}\right)$ is an $M$-matrix, for $0 < \gamma < 0.625$ system (9) possesses a unique equilibrium state for any $u_1, u_2 \in \mathbb{R}$ and this equilibrium state is uniformly exponentially stable.

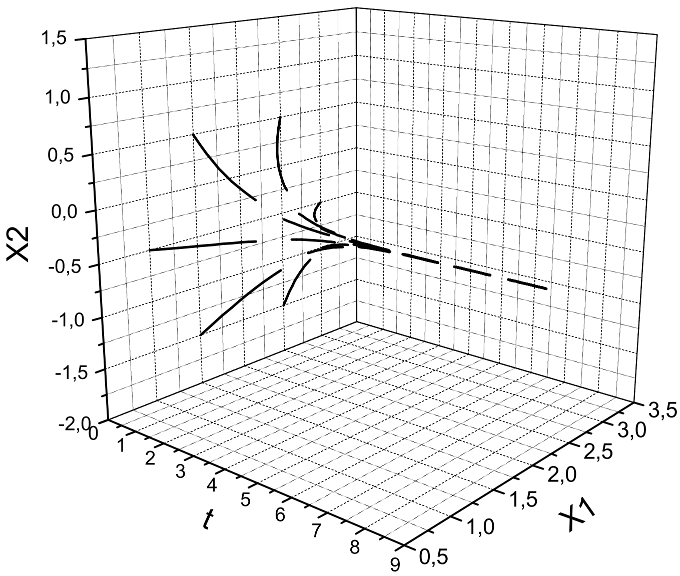


**Figure 1**: Dependence of the function $x(t)$ on time $t$ obtained by numerical solution of system of equations (9).

We shall consider a model example for this problem. We take the following values of the constants: $u_1 = 2$, $u_2 = -1$, $\gamma = 0.5$. The result of numerical solution of system (9) is shown in Figure 1. It is seen from the figure, for arbitrary chosen initial conditions $(1, -0.5)$, $(1.5, -1.5)$, $(2.5, -1.5)$, $(3, -0.5)$, $(2.5, 0.5)$, $(1.5, 0.5)$ the function $x(t)$ approaches asymptotically with time $t$ to the equilibrium state $(x_1^*, x_2^*)^{\mathrm{T}} = (2.35, -0.56)^{\mathrm{T}}$.

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