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Equilibrium States for Pre-image Pressure

Kesong Yan [1,2*], Fanping Zeng [2,3] and Gengrong Zhang [3]

[1] *Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, 230026, P. R. China,*
[2] *Department of Mathematics and Computer Science, Liuzhou Teachers College, Liuzhou, Guangxi, 545004, P.R China,*
[3] *Institute of Mathematics, Guangxi University, Nanning, Guangxi, 530004, P.R China*

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Abstract: In this paper equilibrium states for pre-image pressure are considered. We study the ergodic decomposition of Cheng–Newhouse metric pre-image entropy. Moreover, for a topological dynamical system (X, T) with finite topological pre-image entropy and upper semi-continuous metric pre-image entropy function $h_{\{pre,\bullet\}}(T)$, we obtain a way to describe a kind of continuous dependence of equilibrium states, and show that all functions with unique equilibrium state is dense in $C(X)$. Last, we also discuss the uniformity of equilibrium states for pre-image pressure.

Keywords: *pre-image pressure, equilibrium states, metric pre-image entropy.*

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1 Introduction

Entropies are fundamental to our current understanding of dynamical systems, and topological pressure is a generalization to topological entropy for a dynamical system (see [1] and [2]). Recently, the pre-image structure of maps has become deeply characterized via entropies and pressures, and several important pre-image entropy and pressure invariants have been introduced (see [3, 4, 5, 6, 7]).

In [3], F. Zeng, K. Yan and G. Zhang studied the topological pre-image pressure of topological dynamical systems, and proved a variational principle for it. They considered a compact metric space X and a continuous map $T : X \to X$. The pre-image pressure is defined as a real-valued continuous convex function $P_{pre}(T, \bullet)$ on $C(X)$, where $C(X)$

* Corresponding author: `mailto:ykshhz@163.com`

denotes the Banach space of all real-valued continuous functions on X with the supremum norm. They showed that $P_{pre}(T,f) = \sup_{\mu\in\mathcal{M}(X,T)}(h_{pre,\mu}(T) + \mu(f))$, where $\mathcal{M}(X,T)$ denotes the collection of all T-invariant probability measures on X, $\mu(f) = \int_X f d\mu$ and $h_{pre,\mu}(T)$ the pre-image entropy of μ with respect to T (see [3, 4] for definition). An $\mu \in \mathcal{M}(X,T)$ such that $h_{pre,\mu}(T) + \mu(f)$ attains its supremum is called equilibrium state. For each $f \in C(X)$, there exist tangent functionals to $P_{pre}(T,\bullet)$ at f, whereas there may be no equilibrium states for f. If $\mathcal{T}_f(X,T)$ denotes the set of tangent functionals to $P_{pre}(T,\bullet)$ at f and $\mathcal{M}_f(X,T)$ the set of equilibrium states for f then one has $\mathcal{M}_f(X,T) \subset \mathcal{T}_f(X,T) \subset \mathcal{M}(X,T)$ and $\mathcal{T}_f(X,T) = \mathcal{M}_f(X,T)$ if and only if the pre-image entropy function $h_{\{pre,\cdot\}}(T)$ is upper semi-continuous at the members of $\mathcal{T}_f(X,T)$(see § 2 for definitions and [3] for some results).

The purpose of this note is to consider equilibrium states for pre-image pressure of the topological dynamical system (X,T) with finite pre-image entropy. In Section 2, we concentrate on the ergodic decomposition of measure pre-image entropy, and review some definitions and some basic properties.

In Section 3, we consider a kind of continuous dependence of the equilibrium states $\mathcal{M}_f(X,T)$ on the function f.

In Section 4, we discuss uniqueness and uniformity of equilibrium states for pre-image pressure. We obtained the collection of continuous functions which has unique equilibrium state relative to pre-image pressure and is a dense G_δ-set of $C(X)$. We also show that for any finite collection of ergodic measures, we can find some continuous function such that they contain its equilibrium states set.

2 Preliminaries

In this section, we will recall some definitions and give some useful lemmas.

For a given topological dynamical system (X,T) (where X is a compact metric space and T is a continuous map from X to itself), denote by $\mathcal{B}(X)$ the collection of all Borel subsets. A *partition* of X is a finite disjoint collection of Borel subsets of X whose union is X. For finite partitions α,β, we set $\alpha\vee\beta = \{A\cap B : A\in\alpha, B\in\beta\}$ and $T^{-1}\alpha = \{T^{-1}(A) : A\in\alpha\}$. If $0\le j\le n$ are positive integers, we let $\alpha_j^n = \bigvee_{i=j}^n T^{-i}\alpha$ and $\alpha^n = \alpha_0^{n-1}$. Set $\mathcal{B}^- = \bigcap_{n=0}^\infty T^{-n}\mathcal{B}(X)$, then $\mathcal{B}^-$ is a T-invariant sub-σ algebra. We call $\mathcal{B}^-$ the *infinite past σ-algebra related to* $\mathcal{B}(X)$.

Denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X, $\mathcal{M}(X,T)\subset\mathcal{M}(X)$ is the set of T-invariant measures, and $\mathcal{M}^e(X,T)\subset\mathcal{M}(X,T)$ is the set of ergodic measures. Then both $\mathcal{M}(X)$ and $\mathcal{M}(X,T)$ are convex, compact metric spaces endowed with the weak*-topology (see Chapter 6 in [1]).

Given partitions α,β of X, $\mu\in\mathcal{M}(X)$ and a σ-algebra $\mathcal{A}\subset\mathcal{B}(X)$, define

$$H_\mu(\alpha|\mathcal{A}) := \sum_{A\in\alpha}\int_X -\mathbb{E}(1_A|\mathcal{A})\log\mathbb{E}(1_A|\mathcal{A})d\mu,$$

$$H_\mu(\alpha|\beta\vee\mathcal{A}) := H_\mu(\alpha\vee\beta|\mathcal{A}) - H_\mu(\beta|\mathcal{A}),$$

where $\mathbb{E}(1_A|\mathcal{A})$ is the expectation of 1_A with respect to $\mathcal{A}$. It is well-known that $H_\mu(\alpha|\mathcal{A})$ increases with respect to α and decreases with respect to $\mathcal{A}$.

When $\mu\in\mathcal{M}(X,T)$ and $\mathcal{A}$ is a T-invariant measurable sub-σ-algebra of X, it is not hard to see that $a_n = H_\mu(\alpha^n|\mathcal{A})$ is a non-negative sub-additive sequence for a given partition α, i.e. $a_{n+m} \leq a_n + a_m$ for all positive integers n and m. It is well known that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}.$$

The *conditional entropy of* α *with respect to* $\mathcal{A}$ is then defined by

$$h_\mu(T, \alpha|\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha^n|\mathcal{A}) = \inf_{n\geq 1} \frac{1}{n} H_\mu(\alpha^n|\mathcal{A}).$$

Moreover, the *metric conditional entropy of* (X, T) *with respect to* $\mathcal{A}$ is defined by

$$h_\mu(T, X|\mathcal{A}) = \sup_\alpha h_\mu(T, \alpha|\mathcal{A}).$$

Note that if $\mathcal{N}$ is a trivial σ-algebra, we recover the metric entropy, and we write $h_\mu(T, \alpha|\mathcal{N})$ and $h_\mu(T, X|\mathcal{N})$ simple as $h_\mu(T, \alpha)$ and $h_\mu(T)$.

Particularly, if $\mathcal{A}$ is the infinite past σ-algebra $\mathcal{B}^-$, we define the *measure-theoretic (or metric) pre-image entropy of* α *with respect to* (X, T) by

$$h_{pre,\mu}(T, \alpha) := h_\mu(T, \alpha|\mathcal{B}^-) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha^n|\mathcal{B}^-).$$

Moreover, we define the *metric pre-image entropy of* (X, T) by

$$h_{pre,\mu}(T) := \sup_\alpha h_{pre,\mu}(T, \alpha).$$

In [4], Cheng-Newhouse have shown that the quantity $h_{pre,\mu}(T)$ satisfied power and product rules analogous to the standard metric entropy, that the map $\mu \to h_{pre,\mu}(T)$ was affine, and that there was an analog of the Shannon-Breiman-McMillan theorem for the metric pre-image entropy. In [5], Wen-Chiao Cheng obtained a method for calculating the metric pre-image entropy, which is similar to the Kolmogorov-Sinai theorem for the metric entropy.

Now we discuss the ergodic decomposition of metric pre-image entropy. Given a partition α of X, put $\alpha^- = \bigvee_{n=1}^{\infty} T^{-n}\alpha$ and $\alpha^T = \bigvee_{n=-\infty}^{+\infty} T^{-n}\alpha$. The following lemma is a classical result in ergodic theory (see for example [8]).

Lemma 2.1 *(Pinsker formula) Let* α, β *be two partitions of* X. *Then*

$$h_\mu(T, \alpha \vee \beta) = h_\mu(T, \beta) + H_\mu(\alpha|\beta^T \vee \alpha^-).$$

Lemma 2.2 *(Ergodic decomposition of metric entropy, [1, Theorem 8.4]) Let* (X, T) *be a topological dynamical system and* α *be a partition of* X. *If* $\mu \in \mathcal{M}(X, T)$ *and* $\mu = \int_{\mathcal{M}^e(X,T)} m d\tau(m)$ *is the ergodic decomposition of* μ, *then we have:*

$$h_\mu(T, \alpha) = \int_{\mathcal{M}^e(X,T)} h_m(T, \alpha) d\tau(m).$$

Lemma 2.3 *([5, Lemma 4.13]) Let* (X, T) *be a topological dynamical system,* $\mu \in \mathcal{M}(X, T)$ *and* α *be a partition of* X. *Then*

$$h_{pre,\mu}(T, \alpha) = H_\mu(\alpha|\alpha^- \vee \mathcal{B}^-).$$

Theorem 2.1 *(Ergodic decomposition of metric pre-image entropy). Let* (X, T) *be a topological dynamical system,* $\mu \in \mathcal{M}(X, T)$ *and* α *be a partition of* X*. If* $\mu = \int_{\mathcal{M}^e(X,T)} m d\tau(m)$ *is the ergodic decomposition of* μ*, then*

$$h_{pre,\mu}(T, \alpha) = \int_{\mathcal{M}^e(X,T)} h_{pre,m}(T, \alpha) d\tau(m),$$

and

$$h_{pre,\mu}(T) = \int_{\mathcal{M}^e(X,T)} h_{pre,m}(T) d\tau(m).$$

Proof Take an increasing sequence of finite Borel partitions β_j of X with $diam(\beta_j) \to 0$. Then using the Pinsker formula, the ergodic decomposition of metric entropy, Lemma 2.3 and dominated convergence theorem, we have

$$\begin{aligned}
h_{pre,\mu}(T, \alpha) = H_\mu(\alpha|\alpha^- \vee \mathcal{B}^-) &= \lim_{k\to\infty} H_\mu(\alpha|\alpha^- \vee T^{-k}\mathcal{B}(X)) \\
&= \lim_{k\to\infty} \lim_{j\to\infty} H_\mu(\alpha|\alpha^- \vee (T^{-k}\beta_j)^T) \\
&= \lim_{k\to\infty} \lim_{j\to\infty} [h_\mu(T, \alpha \vee T^{-k}\beta_j) - h_\mu(T, T^{-k}\beta_j)] \\
&= \lim_{k\to\infty} \lim_{j\to\infty} \int_{\mathcal{M}^e(X,T)} [h_m(T, \alpha \vee T^{-k}\beta_j) - h_m(T, T^{-k}\beta_j)] d\tau(m) \\
&= \lim_{k\to\infty} \lim_{j\to\infty} \int_{\mathcal{M}^e(X,T)} H_m(\alpha|\alpha^- \vee (T^{-k}\beta_j)^T) d\tau(m) \\
&= \int_{\mathcal{M}^e(X,T)} \lim_{k\to\infty} \lim_{j\to\infty} H_m(\alpha|\alpha^- \vee (T^{-k}\beta_j)^T) d\tau(m) \\
&= \int_{\mathcal{M}^e(X,T)} h_{pre,m}(T, \alpha) d\tau(m).
\end{aligned}$$

Moreover, we can get

$$\begin{aligned}
h_{pre,\mu}(T) = \lim_{j\to\infty} h_{pre,\mu}(T, \beta_j) &= \lim_{j\to\infty} \int_{\mathcal{M}^e(X,T)} h_{pre,m}(T, \beta_j) d\tau(m) \\
&= \int_{\mathcal{M}^e(X,T)} \lim_{j\to\infty} h_{pre,m}(T, \beta_j) d\tau(m) \\
&= \int_{\mathcal{M}^e(X,T)} h_{pre,m}(T) d\tau(m).
\end{aligned}$$

Theorem 2.1 is proved. □

Following the idea of topological pressure (see [1]), F.Zeng etc. defined a new notion of pre-image pressure, which extends Cheng-Newhouse pre-image entropy [4]. For a given topological dynamical system (X, T), the pre-image pressure of T is a map $P_{pre}(T, \bullet) : C(X) \to \mathbb{R}$ which is convex, Lipschitz continuous, increasing, with $P_{pre}(T, 0) = h_{pre}(T)$ (see [3] for definition).

Given $f \in C(X)$. A member $\mu \in \mathcal{M}(X,T)$ is called an *equilibrium state* for f if $P_{pre}(T,f) = h_{pre,\mu}(T) + \mu(f)$. By the variational principle (Theorem 3.1 in [3])this is equivalent to requiring

$$h_{pre,\mu}(T) + \mu(f) = \sup\{h_{pre,m}(T) + m(f) : m \in \mathcal{M}(X,T)\}.$$

Let $\mathcal{M}_f(X,T)$ denote the collection of all equilibrium states for f. Note that this set could be empty (see Example 5.1 in [3]).

A *tangent functional* to $P_{pre}(T,\bullet)$ at f is a finite signed Borel measure μ on X such that

$$P_{pre}(T,f+g) - P_{pre}(T,f) \geq \mu(g),\ \forall\ g \in C(X).$$

Let $\mathcal{T}_f(X,T)$ denote the collection of all tangent functionals to $P_{pre}(T,\bullet)$ at f. An application of the Hahn-Banach theorem gives $\mathcal{T}_f(X,T) \neq \emptyset$. It is easy to see that $\mu \in \mathcal{T}_f(X,T)$ if and only if

$$P_{pre}(T,f) - \mu(f) = \inf\{P_{pre}(T,h) - \mu(f) : h \in C(X)\}.$$

Also we have $\mathcal{T}_f(X,T) \subset \mathcal{M}(X,T)$ (see [3] for details).

Proposition 2.1 *The following holds.*

(1) $\mathcal{M}_f(X,T)$ *is convex;*

(2) if the pre-image entropy map $h_{pre,\bullet}(T)$ *is upper semi-continuous then* $\mathcal{M}_f(X,T)$ *is compact and non-empty;*

(3) the extreme points of $\mathcal{M}_f(X,T)$ *are precisely the ergodic members of* $\mathcal{M}_f(X,T)$*;*

(4) If $\mu \in \mathcal{M}_f(X,T)$ *and* $\mu = \int_{\mathcal{M}^e(X,T)} m d\tau(m)$ *is the ergodic decomposition of* μ*, then for* τ*-a.e.* $m \in \mathcal{M}^e(X,T)$, $m \in \mathcal{M}_f(X,T)$.

Proof (1)-(3) can see Theorem 5.1 in [3].

(4) This follows from the following two facts: (i) $h_{pre,m}(T) + m(f) \leq P_{pre}(T,f)$ for each $m \in \mathcal{M}^e(X,T)$; (ii) $\int_{\mathcal{M}^e(X,T)}[h_{pre,m}(T) + m(f)]d\tau(m) = h_{pre,\mu}(T) + \mu(f) = P_{pre}(T,f)$ by Theorem 2.1. □

Proposition 2.2 *Let* (X,T) *be a topological dynamical system with* $h_{pre}(T) < \infty$ *and* $f \in C(X)$*. Then the following holds.*

(1) $\mathcal{M}_f(X,T) \subset \mathcal{T}_f(X,T) \subset \mathcal{M}(X,T)$*;*

(2) $\mathcal{T}_f(X,T) = \bigcap_{n=1}^{\infty} \overline{\{\mu \in \mathcal{M}(X,T) : h_{pre,\mu}(T) + \mu(f) > P_{pre}(T,f) - 1/n\}}$*;*

(3) $\mathcal{M}_f(X,T) = \mathcal{T}_f(X,T)$ *if and only if* $h_{pre,\bullet}(T)$ *is upper semi-continuous at the members of* $\mathcal{T}_f(X,T)$.

Proof Theorem 5.2 in [3]. □

3 Continuous Dependence of Equilibrium State

Let (X,T) be a topological dynamical system. Throughout the following sections, we assume the topological pre-image entropy $h_{pre}(T) < \infty$, and the metric pre-image entropy function $h_{\{pre,\bullet\}}(T) : \mathcal{M}(X,T) \to \mathbb{R}$ is upper semi-continuous.

In this section, we prove a theorem to describe a kind of continuous dependence of the set $\mathcal{M}_f(X,T)$ on the function $f \in C(X)$.

Theorem 3.1 *Consider $f, g_n \in C(X)$ and $t_n \in (-1,1)$ such that $t_n \to 0$ and $||g_n||_\infty \to 0$. Let $\mu_n \in \mathcal{M}_{(1+t_n)f+g_n}(X,T)), n > 0$. Then the following holds.*

(1) If $\{\mu_n\}_{n\geq 1}$ converges weakly to some $\mu \in \mathcal{M}(X,T)$ (i.e. $\mu_n(h) \to \mu(h)$ for all $h \in C(X)$), then $\mu \in \mathcal{M}_f(X,T)$;

(2) If $\mathcal{M}_f(X,T) = \{\mu\}$, then $\lim_{n\to\infty} \mu_n = \mu$.

Proof (1) Observe that

$$\begin{aligned} & P_{pre}(T,(1+t_n)f+g_n) \\ = & \sup_{\mu\in\mathcal{M}(X,T)} (h_{pre,\mu}(T) + \mu((1+t_n)f+g_n)) \\ = & \sup_{\mu\in\mathcal{M}(X,T)} ((1+t_n)(h_{pre,\mu}(T)+\mu(f)) - t_n h_{pre,\mu}(T) + \mu(g_n)) \qquad (1) \\ \geq & (1+t_n)P_{pre}(T,f) - |t_n| h_{pre}(T) - ||g_n||_\infty \end{aligned}$$

Since the metric pre-image entropy function $h_{pre,\bullet}(T)$ is upper semi-continuous,

$$\begin{aligned} & h_{pre,\mu}(T) + \mu(f) \\ \geq & \limsup_{n\to\infty} h_{pre,\mu_n}(T) + \limsup_{n\to\infty} \mu_n(f) \\ \geq & \limsup_{n\to\infty} (h_{pre,\mu_n}(T) + \mu_n((1+t_n)f+g_n) - \mu_n(t_n f + g_n)) \\ \geq & \limsup_{n\to\infty} (P_{pre}(T,(1+t_n)f+g_n) - |t_n\mu_n(f)| - ||g_n||_\infty) \\ \geq & \limsup_{n\to\infty} ((1+t_n)P_{pre}(T,f) - |t_n|h_{pre}(T) - |t_n\mu_n(f)| - 2||g_n||_\infty) \quad (by\ (1)) \\ \geq & P_{pre}(T,f) - \limsup_{n\to\infty} |t_n\mu_n(f)| \\ \geq & P_{pre}(T,f) - \limsup_{n\to\infty} |t_n|\mu_n(|f|) \\ = & P_{pre}(T,f) \quad (Since \ \limsup_{n\to\infty} \mu_n(|f|) = \mu(|f|) < \infty). \end{aligned}$$

Therefore, $\mu \in \mathcal{M}_f(X,T)$.

(2) If ω is a limit point of $\{\mu_n\}_{n\geq 1}$, then $\omega = \mu$ by (1). It follows that $\mu_n \to \mu$ as $n \to \infty$. □

4 Uniqueness and Uniformity of Equilibrium State

In this section, we study uniqueness and uniformity of equilibrium state for pre-image pressure. First, we have the following lemma.

Lemma 4.1 *For a given topological dynamical system (X,T), there is a dense subset $C(X)$ such that each function in this set has a unique equilibrium state for pre-image pressure.*

Proof It follows directly from (3) in Proposition 2.2 and the fact that a convex continuous function on a separable Banach space has a unique tangent functional at a dense set of points (can see [9, page 450] or [10, Appendix A.3.6]). □

Denote by $2^{\mathcal{M}(X,T)}$ the hyperspace of compact metric space $\mathcal{M}(X,T)$. Define $\Phi : C(X) \to 2^{\mathcal{M}(X,T)}$ by

$$\Phi(f) = \mathcal{M}_f(X,T), \quad \forall\ f \in C(X).$$

Lemma 4.2 Φ *is upper semi-continuous.*

Proof If $f_n \in C(X)$ with $f_n \to f \in C(X)$ and $\mu_n \in \mathcal{M}_{f_n}(X,T)$ with $\mu_n \to \mu$ for some $\mu \in \mathcal{M}(X,T)$, then for each n we have

$$h_{pre,\mu_n}(T) + \mu_n(f_n) = P_{pre}(T, f_n).$$

Letting $n \to \infty$, then by the continuity of pre-image pressure function $P_{pre}(T, \bullet)$ (see [3, Lemma 4.1 (3)]) and the upper semi-continuity of $h_{pre,\bullet}(T)$, we have

$$h_{pre,\mu}(T) + \mu(f) \geq P_{pre}(T, f).$$

Using the variational principle of pre-image pressure, $\mu \in \mathcal{M}_f(X,T)$. $\square$

Theorem 4.1 *Let (X,T) be a topological dynamical system. Then the following holds.*

(1) $f \in C(X)$ has a unique equilibrium state relative to pre-image pressure if and only if Φ is continuous at f;

(2) $\mathcal{C} \subset C(X)$ is a dense G_δ set, where each $f \in \mathcal{C}$ has unique equilibrium state for pre-image pressure.

Proof (1) It follows directly from Lemma 4.2 that Φ is continuous at f whenever $\mathcal{M}_f(X,T)$ has only one element.

Now we let Φ be continuous at f. By Lemma 4.1, there is a sequence $f_n \in C(X)$ such that $f_n \to f$ and each $\mathcal{M}_{f_n}(X,T)$ is a single point set. Since Φ is continuous at f, $\mathcal{M}_f(X,T)$ also has only one element.

(2) It follows directly from Lemma 4.1, Lemma 4.2 and (1) above. $\square$

Now we discuss uniformity of equilibrium states for pre-image pressure. Set

$$\mathcal{M}_{pre}(X,T) = \bigcup_{f \in C(X)} \mathcal{M}_f(X,T),$$

which denote the set of all equilibrium states for pre-image pressure.

Lemma 4.3 *Given $f \in C(X)$. Then for any $\mu \in \mathcal{M}(X,T)$ and $\epsilon > 0$, there is $f' \in C(X)$ and $\mu' \in \mathcal{M}_{f'}(X,T)$ such that*

$$||\mu - \mu'|| = \sup_{g \in C(X), ||g||=1} |\mu(g) - \mu'(g)| \leq \epsilon,$$

and

$$||f - f'|| \leq \frac{1}{\epsilon}[P_{pre}(T,f) - h_{pre,\mu}(T) - \mu(f)].$$

Proof The proof follows the arguments of the proof of [10, Theorem 3.16]. First we have $P_{pre}(T, \bullet) : C(X) \to \mathbb{R}$ is convex and continuous (see [3, Lemma 4.1 (3) and (4)]). Since $\mu(g) \leq P_{pre}(T,g)$ for all $g \in C(X)$, it follows from [10, Appendix A.3.6] that there is $f' \in C(X)$ and $\mu' \in \mathcal{T}_{f'}(X,T) = \mathcal{M}_{f'}(X,T)$ such that $||\mu - \mu'|| \leq \epsilon$, and

$$\begin{aligned} ||f - f'|| &\leq \frac{1}{\epsilon}[P_{pre}(T,f) - \mu(f) - \inf\{P_{pre}(T,g) - \mu(g) : g \in C(X)\}] \\ &= \frac{1}{\epsilon}[P_{pre}(T,f) - \mu(f) - h_{pre,\mu}(T)] \ \ (\textit{By } [3,\ \textit{Theorem } 4.2]). \end{aligned}$$

The lemma is proved. $\square$

Theorem 4.2 *The following holds.*

(1) The set $\mathcal{M}_{pre}(X,T)$ is dense in $\mathcal{M}(X,T)$;

(2) For any finite collection of ergodic measures $\{\mu_1, \mu_2, \cdots, \mu_n\}$, there is a $f \in C(X)$ such that $\{\mu_1, \mu_2, \cdots, \mu_n\} \subset \mathcal{M}_f(X,T)$.

Proof (1) It follows directly from Lemma 4.3.

(2) Use (1), we know that there is $f \in C(X)$ and $\mu \in \mathcal{M}_f(X,T)$ such that

$$||\mu - \frac{1}{n}(\mu_1 + \mu_2 + \cdots + \mu_n)|| < \frac{1}{n}.$$

Let $\mu = \int_{\mathcal{M}^e(X,T)} md\tau(m)$ be the ergodic decomposition of μ. Then we have

$$||\tau - \frac{1}{n}(\delta_{\mu_1} + \delta_{\mu_2} + \cdots + \delta_{\mu_n})|| < \frac{1}{n},$$

(see [10, Appendix A.5.5]), and therefore $\tau(\{\mu_1\}) > 0, \cdots, \tau(\{\mu_n\}) > 0$. Thus $\{\mu_1, \mu_2, \cdots, \mu_n\} \subset \mathcal{M}_f(X,T)$ by (4) in Proposition 2.1. □

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