NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys

Volume 10   Number 4   2010

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NONLINEAR DYNAMICS & SYSTEMS THEORY

Volume 10, No. 4, 2010

Nonlinear Dynamics and Systems Theory

An International Journal of Research and Surveys

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CONTENTS

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ABSTRACTING AND INDEXING SERVICES

Functional Differential Equations with Nonlocal Conditions in Banach Spaces

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Abstract: In this paper we consider a nonlocal initial boundary value problem for a parabolic integro-differential equation. We reformulate this problem as an abstract functional differential equation in a Banach space with a nonlocal history condition. We establish the existence, uniqueness and continuation of mild, strong and classical solutions of the abstract functional differential equation under different conditions.

Keywords: *functional differential equation; mild solution; classical solution; continuation of solution; semigroup of operator; nonlocal condition.*

Mathematics Subject Classification (2000): 34G20, 47D06, 47H10, 34K06.

1 Introduction

Consider the following parabolic integro-differential equation in a bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$:

$$\left.\begin{array}{l} \partial_t w(t,x) + \sum_{|\alpha|\leq 2m} a_\alpha(x) D^\alpha w(t,x) = f_1(t,x) \\ + \left(\int_\Omega f_2(w(t,x))dx\right) \int_{t-\tau}^t k(t-s) f_3(s, w(s,x))ds, \ 0 < t \leq T, \ \tau > 0, \ x \in \Omega, \\ D^\alpha w(t,x) = 0, \quad t \geq 0, \ x \in \partial\Omega, \ |\alpha| \leq m-1, \\ g(w_0)(x) = \phi(x), \quad x \in \Omega, \end{array}\right\} \quad (1.1)$$

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where the sought-for real-valued function w is defined on $[-\tau, T] \times \Omega$, w_0 is the restriction of w on $[-\tau, 0] \times \Omega$, for all multi-indices α, with $|\alpha| \leq 2m$, the functions $a_\alpha(x)$ are sufficiently smooth and are such that the corresponding partial differential operator is strongly elliptic in Ω, f_i, $i = 1, 2, 3$, are smooth real-valued functions defined on $[0, T] \times \Omega$, $\mathbb{R}$, $[-\tau, T] \times \mathbb{R}$, respectively, for $t \in [0, T]$, $k \in L^p(0, \tau)$, $1 < p < \infty$, g is a map from $C([-\tau, 0]; L^p(\Omega))$ into $L^p(\Omega)$ and $\phi \in L^p(\Omega)$.

A few choices of the function g, for instance, are the following:

$$g(\psi)(x) = \int_{-\tau}^0 k_1(-s)\psi(s)(x)ds, \quad x \in \Omega, \ \psi \in C([-\tau, 0]; L^p(\Omega)),$$

where $k_1 \in L^1(0, \tau)$ with $\int_0^\tau k_1(s)ds \neq 0$;

$$g(\psi)(x) = \sum_{i=1}^r c_i \psi(t_i)(x), \quad x \in \Omega, \ \psi \in C([-\tau, 0]; L^p(\Omega)),$$

where $-\tau \leq t_1 < t_2 < \cdots < t_r \leq 0$, $C := \sum_{i=1}^r c_i \neq 0$; and

$$g(\psi)(x) = \sum_{i=1}^r c_i \int_{t_i - \epsilon_i}^{t_i} \psi(s)(x)ds, \quad x \in \Omega, \ \psi \in C([-\tau, 0]; L^p(\Omega)),$$

where r and c_i are as above and $\epsilon_i > 0$, $i = 1, 2, \ldots, r$.

Let $X := L^p(\Omega)$, $1 < p < \infty$. Let the linear operator $A : D(A) \subset X \to X$ be defined by

$$D(A) = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega), \quad Au = \sum_{|\alpha| \leq 2m} a_\alpha(x)D^\alpha u, \quad u \in D(A).$$

Then $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$, of bounded linear operators in X (cf. Theorem 7.3.5 in [14]).

For $t \geq 0$, let $\mathcal{C}_t := C([-\tau, t]; X)$ be the Banach space of all continuous functions from $[-\tau, t]$ into X endowed with the supremum norm

$$\|\psi\|_t := \sup_{-\tau \leq \eta \leq t} \|\psi(\eta)\|_X, \quad u \in \mathcal{C}_t,$$

where $\|.\|_X$ is the norm in X. Define the nonlinear map $F : [0, T] \times X \times \mathcal{C}_0 \to X$ by

$$F(t, u, \psi)(x) = f_1(t, x)$$
$$+ \left(\int_\Omega f_2(u(x))dx\right) \int_{-\tau}^0 k(-\theta)f_3(t + \theta, \psi(\theta))d\theta, \ t \in [0, T], \ u \in X, \ \psi \in \mathcal{C}_0. \quad (1.2)$$

For $u \in \mathcal{C}_T$, let $u_t \in \mathcal{C}_0$ be defined by $u_t(\theta) = u(t + \theta)$, $\theta \in [-\tau, 0]$. Then (1.1) can be reformulated as the following functional differential equation with a nonlocal history condition in the Banach space $X = L^p(\Omega)$:

$$\left.\begin{array}{rcl} u'(t) + Au(t) & = & F(t, u(t), u_t), \quad 0 < t \leq T, \\ g(u_0) & = & \phi. \end{array}\right\} \quad (1.3)$$

If we define $\Phi \in \mathcal{C}_0$ given by $\Phi(\theta) \equiv \phi$ for all $\theta \in [-\tau, 0]$ and $H : \mathcal{C}_0 \to \mathcal{C}_0$ given by $H(\chi)(\theta) \equiv g(\chi)$ for all $\theta \in [-\tau, 0]$ and all $\chi \in \mathcal{C}_0$, then the condition $g(\chi) = \phi$ is

equivalent to the condition $H(\chi) = \Phi$. Thus we may consider the following functional differential equation with a more general nonlocal history condition:

$$\left. \begin{array}{rcl} u'(t) + Au(t) & = & F(t, u(t), u_t), \quad 0 < t \leq T, \\ H(u_0) & = & \Phi, \end{array} \right\} \tag{1.4}$$

which also includes the functional differential equation:

$$\left. \begin{array}{rcl} u'(t) + Au(t) & = & F(t, u(t), u_t), \quad 0 < t \leq T, \\ u_0 & = & \Phi, \end{array} \right\} \tag{1.5}$$

as a particular case.

The functional differential equation (1.5) has been extensively studied in literature. We refer to Kartsatos [10, 11], Kartsatos and Liu [9], Kartsatos and Parrott [12, 13].

Amraoui and Rhali [3] have used integrated semigroups to study the existence and uniqueness of integral solutions and other forms of solutions of the abstract Cauchy problem $u'(t) = Bu(t) + Lu_t$, $t > 0$, where B is a nondensely defined linear operator in a Banach space X and L is a bounded linear operator on X.

Recently, Bahuguna [4], Bahuguna, Dabas and Shukla [5], Bahuguna and Dabas [6], Bahuguna and Muslim [7, 8], Agarwal and Bahuguna [1, 2] have linear as well as nonlinear nonlocal history-valued evolution equations using the theory of semigroups and the theory of accretive operators.

Let $\psi \in \mathcal{C}_0$ such that $H(\psi) = \Phi$. The function $u \in \mathcal{C}_{\tilde{T}}$, $0 < \tilde{T} \leq T$, such that

$$u(t) = \left\{ \begin{array}{ll} \psi(t), & t \in [-\tau, 0], \\ S(t)\psi(0) + \int_0^t S(t - s)F(s, u(s), u_s)ds, & t \in [0, \tilde{T}], \end{array} \right. \tag{1.6}$$

is called a *mild solution* of (1.4) on $[-\tau, \tilde{T}]$. If a mild solution u of (1.4) on $[-\tau, \tilde{T}]$ is such that $u(t) \in D(A)$ for a.e. $t \in [0, \tilde{T}]$, u is differentiable a.e. on $[0, \tilde{T}]$ and

$$u'(t) + Au(t) = F(t, u(t), u_t), \quad \text{a.e. on } [0, \tilde{T}],$$

it is called a *strong solution* of (1.4) on $[-\tau, \tilde{T}]$. If a mild solution u of (1.4) on $[-\tau, \tilde{T}]$ is such that $u \in C^1((0, \tilde{T}]; X)$, $u(t) \in D(A)$ for $t \in (0, \tilde{T}]$ and satisfies

$$u'(t) + Au(t) = F(t, u(t), u_t), \quad t \in (0, \tilde{T}],$$

then it is called a *classical solution* of (1.4) on $[-\tau, \tilde{T}]$.

We first establish the existence of a *mild solution* $u \in \mathcal{C}_{\tilde{T}}$ of (1.4) for some $0 < \tilde{T} \leq T$ and its continuation to either on the whole of $[-\tau, T]$ or show that there exists the maximal interval $[-\tau, t_{max})$, $0 < t_{max} \leq T$ such that u is a mild solution of (1.4) on every subinterval $[-\tau, \tilde{T}]$, $0 < \tilde{T} < t_{max}$, under the assumptions that there exists a $\psi \in \mathcal{C}_0$ such that $H(\psi) = \Phi$ and $-A$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$, of bounded linear operators in X. In the later case, since $t_{max} \leq T < \infty$, we obtain that

$$\lim_{t \to t_{max}-} \|u(t)\|_X = \infty.$$

Under the additional assumption of Lipschitz continuity on ψ on $[-\tau, 0]$, we show that the mild solution u is a strong solution of (1.4) on the interval of existence and it is Lipschitz continuous. Under further additional assumption that $S(t)$ is analytic, we show that u is a classical solution of (1.4) on the interval of existence. We also show that u is unique if and only if ψ satisfying $H(\psi) = \Phi$ is unique.

2   Local Existence of Mild Solutions

We first prove the following result establishing the local existence and uniqueness of a mild solution of (1.4).

Theorem 2.1 *Suppose that $-A$ is the infinitesimal generator of a C_0-semigroup $S(t)$, $t \geq 0$ of bounded linear operators in X. Let $H : C_0 \to C_0$ be such that there exists a function $\psi \in C_0$ such that $H(\psi) = \Phi$. Let $F : [0,T] \times X \times C_0 \to X$ satisfy a Lipschitz-like condition*

$$\|F(t_1, u_1, \phi_1) - F(t_2, u_2, \phi_2)\|_X \leq L_F(r)[|t_1 - t_2| + \|u_1 - u_2\|_X + \|\phi_1 - \phi_2\|_{C_0}],$$

for all $t_i \in [0,T]$, $u_i \in B_r(X, \psi(0))$, $\phi_i \in B_r(C_0, \psi)$ $i = 1, 2$, where $L_F : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function. Then there exists a mild solution u of (1.4) on $[-\tau, T_0]$ for some $0 < T_0 \leq T$. Here $B_r(Z, z_0) := \{z \in Z : \|z - z_0\|_Z \leq r\}$ for any Banach space $(Z, \|.\|_Z)$, $z_0 \in Z$ and $r > 0$. Moreover, the mild solution u is unique if and only if ψ is unique.

Proof Let $R > 0$ be fixed. Let $M \geq 1$ and $\omega \geq 0$ be such that $\|S(t)\|_{B(X)} \leq Me^{\omega t}$ for $t \geq 0$. Here $B(X)$ is the space of all bounded linear operators on X. Choose $0 < T_0 \leq T$ be such that

$$
\begin{aligned}
T_0 L_F(R) &\leq 3/8, \\
\sup_{0 \leq t \leq T_0} \|(S(t) - I)\psi(0)\|_X &\leq R/2, \\
T_0 M_0 &\leq R/2,
\end{aligned}
$$

where
$$M_0 := T + 2\|\psi\|_0 + 2MRL_F(R)e^{\omega T} + \|F(0, \psi(0), \psi)\|_X.$$

Define a map $\mathcal{F} : C_{T_0} \to C_{T_0}$ by

$$\mathcal{F}w(t) = \begin{cases} \psi(t), & t \in [-\tau, 0], \\ S(t)\psi(0) + \int_0^t S(t-s)F(s, w(s), w_s)ds, & t \in [0, T_0], \end{cases} \quad w \in C_{T_0}. \quad (2.7)$$

Let $\tilde{\psi} \in C_T$ be defined by

$$\begin{cases} \tilde{\psi}(t) = \psi(t), & t \in [-\tau, 0], \\ \psi(0), & t \in [0, T]. \end{cases}$$

Then from the choice of T_0 it follows that $\mathcal{F}$ maps $B_R(C_{T_0}, \tilde{\psi})$ into itself. Here and subsequently, any function in C_T is also in $C_{\tilde{T}}$, $0 \leq \tilde{T} \leq T$, as its restriction on the subinterval. Also, for $w_i \in B_R(C_{T_0}, \tilde{\psi})$, $i = 1, 2$, we have

$$\|\mathcal{F}w_1(t) - \mathcal{F}w_2(t)\|_X \leq 2T_0 L_F(R)\|w_1 - w_2\|_{T_0}.$$

Since $T_0 L_F(R) \leq 3/8$, $\mathcal{F}$ is a strict contraction on $B_R(C_{T_0}, \tilde{\psi})$ and hence has a unique fixed point $u \in B_R(C_{T_0}, \tilde{\psi})$. Clearly u is a mild solution of (1.4) on $[-\tau, T_0]$. It can be shown that if $\psi \in C_0$ satisfying $H(\psi) = \Phi$ is unique then the mild solution $u \in C_{T_0}$ is unique. If there are two different ψ_1 and ψ_2 in C_0 satisfying $H(\psi_1) = H(\psi_2) = \Phi$, then the corresponding mild solutions differ on $[-\tau, 0]$. This completes the proof of Theorem 2.1. $\square$

3   Continuation of Solutions

Theorem 3.1 *Assume the hypotheses of Theorem 2.1. Then the local mild solution u of (1.4) on $[-\tau, T_0]$, $0 < T_0 \leq T$, can be continued either on the whole interval $[-\tau, T]$ or on the maximal interval $[-\tau, t_{max})$ of existence and since in the later case $t_{\max} \leq T < \infty$, we have*

$$\lim_{t \to t_{max}-} \|u(t)\|_X = \infty.$$

Proof Assume that $T_0 < T$. Consider the functional differential equation

$$\left. \begin{array}{rcl} v'(t) + Av(t) & = & G(t, v(t), v_t), \quad 0 < t \leq T - T_0, \\ \tilde{H}(v_0) & = & \tilde{\Phi}, \end{array} \right\} \tag{3.8}$$

where $G : [0, T - T_0] \times X \times C([-\tau, 0]; X) \to X$ is defined by $G(t, u, \chi) = F(t + T_0, u, \chi)$, $\tilde{H} : \mathcal{C}_0 \to \mathcal{C}_0$ given by $\tilde{H}\chi = \chi$ for $\chi \in \mathcal{C}_0$ and $\tilde{\Phi}(\theta) = u(T_0 + \theta)$ for $\theta \in [-\tau, 0]$. Since all the hypotheses of Theorem 2.1 are satisfied for problem (3.8), we have the existence of a mild solution $w \in \mathcal{C}_{T_1}$, $0 < T_1 \leq T - T_0$ of (3.8). This mild solution w is unique as $\tilde{H}$ in (3.8) is the identity map on $\mathcal{C}_0$. We define

$$\bar{u}(t) = \left\{ \begin{array}{ll} u(t), & t \in [-\tau, T_0], \\ w(t - T_0), & t \in [T_0, T_0 + T_1]. \end{array} \right. \tag{3.9}$$

Then $\bar{u}$ is a mild solution of (1.4) on $[-\tau, T_0 + T_1]$. Continuing this way, we get the existence of a mild solution u either on the whole interval $[-\tau, T]$ or on the maximal interval $[-\tau, t_{max})$ of existence. In the later case we may use the arguments similar to those in the proof of Theorem 6.2.2 in [14] (pp. 193–194) to conclude that $\lim_{t \to t_{max}-} \|u(t)\|_X = \infty$. This completes the proof of Theorem 3.1. $\square$

4   Regularity of Solutions

Theorem 4.1 *Assume the hypotheses of Theorem 2.1. If, in addition, ψ is Lipschitz continuous on $[-\tau, 0]$ and $\psi(0) \in D(A)$, then u is Lipschitz continuous on every compact subinterval of existence. If, in addition, X is reflexive, then u is a strong solution of (1.4) on the interval of existence and this strong solution is a classical solution of (1.4) provided $S(t)$ is an analytic semigroup.*

Proof We shall prove the result for the first case when the mild solution u exists on the whole interval. The proof can be modified easily for the second case.

We need to show the Lipschitz continuity of u only on $[0, T]$. In what follows, C_i's are positive constants depending only on R, T and $\|\phi\|_0$. Let $t \in [0, T]$ and $h \geq 0$. Then

$$\begin{aligned} \|u(t+h) - u(t)\|_X & \leq \|(S(h) - I)S(t)\psi(0)\|_X \\ & + \int_{-h}^{0} \|S(t-s)f(s+h, u(s+h), u_{s+h})\|_X ds \\ & + \int_{0}^{t} \|s(t-s)[f(s+h, u(s+h), u_{s+h}) - f(s, u(s), u_s)]\|_X ds \\ & \leq C_1 \left[h + \int_{0}^{t} [\|u(s+h) - u(s)\|_X + \|u_{s+h} - u_s\|_{\mathcal{C}_0}] ds \right] \\ & \leq 2C_1 \left[h + \int_{0}^{t} \sup_{-\tau \leq \theta \leq 0} \|u(s+h+\theta) - u(s+\theta)\|_X \right] ds, \quad (4.10) \end{aligned}$$

For the case when $-\tau \le t < 0$ and $0 \le t + h$ (clearly, $t + h \le h$ in this case), we have

$$
\begin{aligned}
\|u(t+h) - u(t)\|_X \quad &\le \quad \|(S(t+h) - I)\psi(0)\|_X + \|\psi(t) - \psi(0)\|_X \\
&\quad + \int_0^h \|S(t+h-s)f(s, u(s), u_s)\|_X \, ds \\
&\le \quad C_2 h. \tag{4.11}
\end{aligned}
$$

Combining inequalities (4.10) and (4.11), we have for $-\tau \le \bar{t} \le t$,

$$
\|u(\bar{t}+h) - u(\bar{t})\|_X \le C_3 \left[h + \int_0^t \sup_{-\tau \le \theta \le 0} \|u(s+h+\theta) - u(s+\theta)\|_X ds \right]. \tag{4.12}
$$

Putting $\bar{t} = t + \bar{\theta}$, $-t - \tau \le \bar{\theta} \le 0$, in (4.12), and taking supremum over $\bar{\theta}$ on $[-\tau, 0]$, we get

$$
\sup_{-\tau \le \theta \le 0} \|u(t+h+\theta) - u(t+\theta)\|_X
$$
$$
\le 2C_3 \left[h + \int_0^t \sup_{-\tau \le \theta \le 0} \|u(s+h+\theta) - u(s+\theta)\|_X ds \right]. \tag{4.13}
$$

Applying Gronwall's inequality in (4.13), we obtain

$$
\|u(t+h) - u(t)\|_X \quad \le \quad \sup_{-\tau \le \theta \le 0} \|u(t+h+\theta) - u(t+\theta)\|_X \le C_4 h.
$$

Thus, u is Lipschitz continuous on $[-\tau, T]$.

The function $\bar{F} : [0, T] \to X$ given by $\bar{F}(t) = F(t, u(t), u_t)$, is Lipschitz continuous and therefore differentiable a.e. on $[0, T]$ and $\bar{F}'$ is in $L^1((0, T); X)$. Consider the Cauchy problem

$$
\begin{cases} v'(t) + Av(t) = \bar{F}(t), \ t \in (0, T], \\ v(0) = u(0), \end{cases} \tag{4.14}
$$

By Corollary 2.10 on page 109 in Pazy [14], there exists a unique strong solution v of (4.14) on $[0, T]$. Clearly, $\bar{v}$ defined by

$$
\bar{v}(t) = \begin{cases} u(t), & t \in [-\tau, 0], \\ v(t), & t \in [0, T], \end{cases}
$$

is a strong solution of (1.4) on $[-\tau, T]$. But this strong solution is also a mild solution of (1.4) and $\bar{v} \in \mathcal{W}(\psi, T) := \{\Psi \in \mathcal{C}_T : \Psi = \psi \text{ on } [-\tau, 0]\}$. By the uniqueness of such a function in $\mathcal{W}(\psi, T)$, we get $\bar{v}(t) = u(t)$ on $[-\tau, T]$. Thus u is a strong solution of (1.4). If $S(t)$ is analytic semigroup in X then we may use Corollary 3.3 on page 113 in Pazy [14] to obtain that u is a classical solution of (1.4). Clearly, if $\psi \in \mathcal{C}_T$ satisfying $h(\psi) = \Phi$ on $[-\tau, 0]$ is unique on $[-\tau, 0]$, then u is unique. If there are two ψ and $\tilde{\psi}$ in $\mathcal{C}_T$ satisfying $h(\psi) = h(\tilde{\psi}) = \Phi$ on $[-\tau, 0]$, with $\psi \ne \tilde{\psi}$ on $[-\tau, 0]$, then $\mathcal{W}(\psi, T) \cap \mathcal{W}(\tilde{\psi}, T) = \emptyset$ and hence the corresponding solutions u and $\tilde{u}$ of (1.4) belonging to $\mathcal{W}(\psi, T)$ and $\mathcal{W}(\tilde{\psi}, T)$, respectively, are different. This completes the proof of Theorem 4.1. $\square$

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Liapunov Functionals, Convex Kernels, and Strategy

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Abstract: We study an integral equation of the form $x(t) = a(t) - \int_0^t C(t,s)g(x(s))ds$ where C is convex and g has the sign of x. In earlier work we treated the case of $\sup \int_s^t C^2(u,s)du =: \Gamma < \infty$. Here, we study the case of $\Gamma = \infty$ by looking at a new equation formed from $x' + kx$ with k a positive constant. This enables us to define a Liapunov functional which will give a bound on $\int_0^t g^2(x(s))ds$ and a parallel bound on one of the resolvents in the linear case. Equations of this type have been used since the early work of Volterra in a number of real-world problems.

1 Introduction

We are concerned here with an integral equation

$$x(t) = a(t) - \int_0^t C(t,s)g(x(s))ds, \tag{1}$$

where $a : [0,\infty) \to \Re$ is continuous, while C is continuous for $0 \le s \le t < \infty$, and $g : \Re \to \Re$ is continuous with $xg(x) > 0$ if $x \ne 0$. Continuity of a, C, g will ensure the existence of a solution. If the solution remains bounded, then it can be continued on $[0,\infty)$. See [5; pp. 178-180], for example.

It is always assumed that the kernel, $C(t,s)$, is convex in the sense that

$$C(t,s) \ge 0, \quad C_s(t,s) \ge 0, \quad C_{st}(t,s) \le 0, \quad C_t(t,s) \le 0. \tag{2}$$

Convolution problems of this type are seen in Levin [10] and Londen [11], for example.

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In the classical theory of integral equations we generally need to ask that the kernel be very small in order to obtain global stability results. In 1928, Volterra [13] noticed that a great many real world problems were being modeled by integral and integro-differential equations with convex kernels which inherently suggested a fading memory. He conjectured that there is a Liapunov functional for such kernels which would yield much qualitative information about solutions and which would allow very large kernels. Today we see problems in biology, nuclear reactors, viscoelasticity, and neural networks being modeled using convex kernels.

In 1963, Levin followed Volterra's idea and constructed such a Liapunov functional for a convolution form of the integro-differential equation

$$x' = -\int_0^t C(t,s)g(x(s))ds$$

with C convex and in 1992 [2] we constructed one for integral equations in the form of (1). For the linear integral equation there is also a Liapunov functional for the resolvent equation and we discussed this in some detail in [6] when $\sup_{0 \le s \le t < \infty} \int_s^t C^2(u,s)du =: \Gamma < \infty$. This paper seeks to extend some of that work to the case $\Gamma = \infty$. In the nonlinear integral equation there is a severe technical problem in dealing with the derivative of the Liapunov functional and the investigator must make some undesirable assumptions about the nonlinearity. This paper offers an alternative to those assumptions. Here is some detail concerning the two difficulties which we study.

Our Liapunov functional

$$V_1(t) = \int_0^t C_s(t,s)\left(\int_s^t g(x(u))du\right)^2 ds + C(t,0)\left(\int_0^t g(x(u))du\right)^2$$

for (1) has a derivative satisfying

$$V_1'(t) \le 2g(x)[a(t) - x(t)].$$

Owing to the absence of a chain rule, that differentiation is not simple so we want to give the details. It would be a distraction to give them here, so we offer them in the appendix.

In order to relate $g(x)$ to $a(t)$ we need to be able to separate that relation into

$$V_1'(t) \le |p(a(t))| - |q(x(t)|$$

for some functions p and q with q positive definite with respect to x or $g(x)$ and p positive definite with respect to $a(t)$ so that

$$0 \le V_1(t) \le V_1(0) + \int_0^t |p(a(s))|ds - \int_0^t |q(x(s))|ds.$$

That separation has proved to be very cumbersome and investigators ([5; pp. 190-191], [4], [14]) have resorted to ad hoc assumptions, as well as stringent conditions on g in order to use Young's inequality. A definite example will show the need for the theory which is to follow.

Example 1.1 Consider the scalar equation

$$x(t) = a(t) - \int_0^t [1 + t - s]^{-1/4}g(x(s))ds$$

where g is an arbitrary continuous function satisfying $xg(x) > 0$ if $x \neq 0$. For $a \in L^2[0, \infty)$ if x is a solution on $[0, \infty)$ then we know of no result or technique in the literature that will yield $g(x) \in L^p[0, \infty)$. The Liapunov functional mentioned above will yield the indicated derivative and we find no way to perform the required separation. The difficulty will vanish using Theorem 3.1, (14). We will immediately find $g(x) \in L^2[0, \infty)$ without further restriction on g.

In the linear case, $g(x) = x$, we have

$$V_1'(t) \leq a^2(t) - x^2(t)$$

so that

$$\int_0^t x^2(s)ds \leq \int_0^t a^2(s)ds,$$

a very useful relation. Moreover, it extends to the resolvent equation ([5; pp. 130-131])

$$R(t, s) = C(t, s) - \int_s^t C(t, u)R(u, s)du$$

as

$$V_2(t) = \int_s^t C_v(t, v)\left(\int_s^t R(u, s)du\right)^2 dv + C(t, s)\left(\int_s^t R(u, s)du\right)^2$$

with a derivative satisfying

$$V_2'(t) \leq -R^2(t, s) + C^2(t, s)$$

as may be seen following the details in the appendix. This yields

$$\int_s^t R^2(u, s)du \leq \int_s^t C^2(u, s)du$$

which is so useful in the variation of parameters formula

$$x(t) = a(t) - \int_0^t R(t, s)a(s)ds.$$

But we have a difficulty here also. If $\sup_{0 \leq s \leq t < \infty} \int_s^t C^2(u, s)du =: \Gamma < \infty$ then we have a very useful parallel property for R. On the other hand, if $\Gamma = \infty$ then the property is lost and we are left with the obvious fact that if $a \in L^2$ then $x \in L^2$ so by default $\int_0^t R(t, s)a(s)ds \in L^2$ and $x - a \in L^2$, but we can not extract from that any essential properties of R itself.

Example 1.2 We can continue Example 1.1 with $g(x) = x$ and study $x(t) = a(t) - \int_0^t [1 + t - s]^{-1/4}x(s)ds$. The Liapunov functional of the appendix will yield $\int_0^t x^2(s)ds \leq \int_0^t a^2(s)ds$ and $\int_0^t R(t, s)a(s)ds \in L^2[0, \infty)$ when $a \in L^2$ without any independent property of R. Our second goal is to obtain basic properties of a resolvent independent of $a(t)$. That resolvent will not be R but it will serve in a parallel way to R.

Thus, we encounter fundamental problems in both the nonlinear and linear cases. These two unsolved problems will drive this paper.

In an effort to avoid the difficulties just mentioned we consider the old technique of differentiating (1) to obtain

$$x'(t) = a'(t) - C(t,t)x(t) - \int_0^t C_t(t,s)x(s)ds$$

which seems promising since for $C(t,t) \geq \alpha > 0$ we have a perturbation of the uniformly asymptotically stable equation
$$x' + C(t,t)x = 0.$$

However, that gain pales in comparison to our great loss in that $C_t(t,s)$ is no longer convex; hence, we would require some restrictions on the magnitude of $C_t(t,s)$ in order to use standard results on qualitative properties. To avoid all of those problems we develop a strategy which yields very good results.

Moreover, there is an added benefit, uncommon in the theory of convex kernels. If we can find a function $f : [0,\infty) \to [0,\infty)$ with $\int_0^t \frac{ds}{f(s)}$ continuous for $t \geq 0$,

$$|g(x)| \leq f\left(\int_0^x g(s)ds \right), \ x \in \Re, \ f \in \nearrow, \tag{3}$$

and

$$\int_0^\infty \frac{ds}{f(s)} = \infty, \tag{4}$$

then we prove that the solution has certain integral properties.

The work is based on four Liapunov functionals, a differential inequality, and a strategy for finding a strongly stable equation which has a solution of (1) as one of its solutions.

2   The Strategy

We will now employ a very simple device which seems to have been totally overlooked in the literature until recently. It was introduced in Burton [5], further developed in Burton-Haddock [7], and has significant applications in the existence of periodic solutions, a project to be presented later.

A classic strategy is to differentiate (1), turning it into a differential equation

$$x'(t) = a'(t) - C(t,t)g(x) - \int_0^t C_t(t,s)g(x(s))ds. \tag{5}$$

Among other techniques, we can then apply Liapunov's direct method, as discussed in Miller [12; p. 337]. While it sometimes is effective, it is usually a disaster since differentiation tends to have a very non-smoothing effect. But under some general conditions, if $C(t,s)$ is convex and if k is a sufficiently large positive constant, then it is true that

$$D(t,s) := kC(t,s) + C_t(t,s) \tag{6}$$

is convex. For example, it is readily verified that if r is a positive constant, then $k = r+3$ is a suitable constant for $C(t,s) = [1 + t - s]^{-r}$ (this pertains to Example 1.1), while $k = r + 1$ is suitable for $C(t,s) = e^{-r(t-s)}$.

If we form $x' + kx$ then we have

$$x'(t) = a'(t) + ka(t) - [kx + C(t,t)g(x)] - \int_0^t D(t,s)g(x(s))ds. \tag{7}$$

This is a one-parameter family of totally different equations having exactly one property in common: a solution of (1) satisfies every one of those equations. If all solutions of (7) satisfy a certain property, so does a solution of (1). Two things have happened. Since C is convex, $C(t,t) \geq 0$ and, hence, $x' + kx + C(t,t)g(x) = 0$ is uniformly asymptotically stable. If $a'(t) + ka(t) \in L^2[0,\infty)$ and $C(t,t) \geq \alpha > 0$, then Levin's original Liapunov functional will yield $g(x(t)) \in L^2[0,\infty)$. In addition, if (3) and (4) hold and if $a' + ka \in L^1[0,\infty)$, then we will obtain an L^2 result for x.

We have used (5) to introduce a differential equation, but we have overwhelmed it with the integral equation by taking k large. In the parallel work with Haddock [7] the technique was different in that a very careful selection of an exact value for k was made. An entirely different selection is made in the aforementioned work with periodic solutions. But all three projects stem from the same idea.

What is, perhaps, more interesting is the fact that when $g(x) = x$, then Becker's [1] resolvent equation for (7) is

$$Z_t(t,s) = -[k + C(t,t)]Z(t,s) - \int_s^t D(t,u)Z(u,s)du \tag{8}$$

and a slight modification of Levin's [9] Liapunov functional will yield

$$\sup_{0 \leq s \leq t < \infty} \int_s^t Z^2(u,s)du < \infty. \tag{9}$$

This is a critical result in the variation of parameters formula

$$x(t) = Z(t,0)x(0) + \int_0^t Z(t,s)[a'(s) + ka(s)]ds \tag{10}$$

where x solves (1) if $x(0) = a(0)$. These ideas will be developed in the coming sections.

3   The Nonlinear Problem

In 1963, Levin [9] considered a convolution form of

$$x' = -\int_0^t D(t,s)g(x(s))ds \tag{11}$$

with $xg(x) > 0$ for $x \neq 0$ and D convex. He constructed the Liapunov functional

$$V_3(t) = \int_0^x g(s)ds + \frac{1}{2}\int_0^t D_s(t,s)\left(\int_s^t g(x(u))du\right)^2 ds$$

$$+ \frac{1}{2}D(t,0)\left(\int_0^t g(x(u))du\right)^2 \tag{12}$$

and found that $V_3'(t) \leq 0$ along a solution of (11). This means that

$$\int_0^{x(t)} g(s)ds \leq V_3(t) \leq V_3(0) = \int_0^{x(0)} g(s)ds$$

so that if $\int_0^{\pm\infty} g(s)ds = \infty$, then every solution of (11) is bounded.

We are going to use the same Liapunov functional on (7). In (16) below recall that $xg(x) > 0$ if $x \neq 0$ and that $C(t,t) \geq 0$.

Theorem 3.1 *Suppose that D is convex,*

$$D(t,s) \geq 0, \quad D_s(t,s) \geq 0, \quad D_{st}(t,s) \leq 0, \quad D_t(t,s) \leq 0, \tag{13}$$

that $xg(x) > 0$ if $x \neq 0$, and that V_3 is defined in (12). Then along a solution of (7) we have

$$V_3'(t) \leq g(x)[a'(t) + ka(t)] - g(x)[kx + C(t,t)g(x)]. \tag{14}$$

If, in addition, $|kx + C(t,t)g(x)| \geq \mu|g(x)|$, for some $\mu > 0$, then $a' + ka \in L^2[0,\infty)$ implies $g(x(t)) \in L^2[0,\infty)$. In particular, any solution x in Example 1.1 satisfies $g(x) \in L^2[0,\infty)$.

If (14) and (3) hold then along a solution of (7) we have

$$V_3'(t) \leq f(V_3(t))|a'(t) + ka(t)|. \tag{15}$$

If, in addition, (4) holds, $\int_0^{\pm\infty} g(s)ds = \infty$, and $a' + ka \in L^1[0,\infty)$, then every solution of (7) is bounded and

$$\int_0^\infty g(x(s))[kx(s) + C(s,s)g(x(s))]ds < \infty. \tag{16}$$

Proof Along a solution of (7) we have

$$V_3'(t) \leq g(x)[a'(t) + ka(t)] - g(x)[kx + C(t,t)g(x)] - g(x)\int_0^t D(t,s)g(x(s)ds$$

$$+ g(x)D(t,0)\int_0^t g(x(u))du + g(x)\int_0^t D_s(t,s)\int_s^t g(x(u))duds.$$

Integrating the last term by parts yields

$$g(x)[D(t,s)\int_s^t g(x(u))du\Big|_0^t + \int_0^t D(t,s)g(x(s))ds]$$

$$= g(x)[-D(t,0)\int_0^t g(x(u))du + \int_0^t D(t,s)g(x(s))ds]$$

so that

$$V_3'(t) \leq g(x)[a'(t) + ka(t)] - g(x)[kx + C(t,t)g(x)].$$

Now, if $|kx + C(t,t)g(x)| \geq \mu|g(x)|$ then

$$V_3'(t) \leq \alpha[a'(t) + ka(t)]^2 - \beta g^2(x)$$

for some positive α and β, from which $a' + ka \in L^2$ implies $g(x) \in L^2$.

Remark 3.1 The obvious and usual condition is that $C(t,t)$ be greater than a positive constant, entirely consistent with the convexity. Indeed, in the convolution case $C(t) \geq 0$ and $C'(t) \leq 0$ so if $C(0) = 0$ then $C(t) \equiv 0$. Even if this fails, in the next step

we get x bounded. None of the ad hoc assumptions on g needed in Young's inequality found in earlier work (e.g., [4]) are needed.

Next, if (14) and (3) hold, then

$$V_3'(t) \le f\left(\int_0^x g(s)ds\right)|a'(t) + ka(t)| \le f(V_3)|a'(t) + ka(t)|$$

so

$$\int_{V_3(0)}^{V_3(t)} \frac{du}{f(u)} \le \int_0^t |a'(s) + ka(s)|ds.$$

If (4) holds and $a' + ka \in L^1$, then $V_3(t)$ is bounded and, hence, $x(t)$ is bounded. This means that $g(x)[a' + ka] \in L^1$ so from (14) we see that (16) follows. The proof is complete.

These results raise questions for the linear case. For we then see that $a' + ka \in L^1$ yields $x \in L^2$, but $a' + ka \in L^2$ also yields $x \in L^2$. Linear theory shows that $x \in L^1[0, \infty)$ is intimately related to uniform asymptotic stability. The next result shows that for certain choices of g we approximate $x \in L^1[0, \infty)$.

Theorem 3.2 *Suppose that D is convex and that $g(x) = x^{1/n}$ where n is an odd positive integer. If $a' + ka \in L^1[0, \infty)$, then $\int_0^\infty |x(s)|^{\frac{1+n}{n}}ds < \infty$.*

Proof Note that there is a positive number p with

$$|g(x)| = |x^{1/n}| = \left(|x^{1/n}|^{(n+1)}\right)^{\frac{1}{n+1}} = p\left(\int_0^x s^{1/n}ds\right)^{\frac{1}{n+1}}.$$

Hence $f(r) = p(r)^{\frac{1}{n+1}}$. We then have

$$V_3'(t) \le p(V_3(t))^{\frac{1}{n+1}}|a'(t) + ka(t)|$$

and

$$\frac{1}{p}\int_{V_3(0)}^{V_3(t)} \frac{ds}{s^{\frac{1}{n+1}}} = \frac{1}{p}s^{1-\frac{1}{n+1}}\Big|_{V_3(0)}^{V_3(t)}$$

so

$$\frac{1}{p}V_3(t)^{\frac{n}{n+1}} \le \frac{1}{p}V_3(0)^{\frac{n}{n+1}} + \int_0^t |a'(s) + ka(s)|ds.$$

Hence, $V_3(t)$ is bounded so $x(t)$ is bounded and $x(t)[a'(t) + ka(t)] \in L^1[0, \infty)$. But

$$V_3'(t) \le |g(x)||a'(t) + ka(t)| - kxg(x)$$

with $xg(x) = xx^{1/n} = x^{1+\frac{1}{n}}$ so $\int_0^\infty |x(s)|^{\frac{1+n}{n}}ds < \infty$.

Notice that $V_3'(t) \le -\beta g^2(x) + \alpha(a'(t) + ka(t))^2$ with $a' + ka \in L^2$ would yield $\int_0^\infty x^{2/n}(s)ds < \infty$, an entirely different property. Suppose now that $n > 2$ and that $C(t, t) \ge \alpha > 0$ so that both of our integral relations hold. Note that if $|x(t)| \ge 1$ then $|x(t)| \le |x|^{1+\frac{1}{n}}$. If $|x(t)| < 1$ then $|x(t)| \le |x(t)|^{2/n}$. Hence, we conclude that $\int_0^\infty |x(s)|ds < \infty$.

4   The Linear Case

If $g(x) = x$ then (7) becomes

$$x'(t) = a'(t) + ka(t) - [k + C(t,t)]x(t) - \int_0^t D(t,s)x(s)ds \tag{17}$$

and Becker's [1] resolvent equation is

$$Z_t(t,s) = -[k + C(t,t)]Z(t,s) - \int_s^t D(t,u)Z(u,s)du, \ Z(s,s) = 1, \tag{18}$$

(where $Z_t = \frac{\partial Z}{\partial t}$) with variation of parameters formula

$$x(t) = Z(t,0)x(0) + \int_0^t Z(t,s)[a'(s) + ka(s)]ds. \tag{19}$$

The Grossman-Miller [8] resolvent equation is

$$H_s(t,s) = H(t,s)[k + C(s,s)] + \int_s^t H(t,u)D(u,s)du, \ H(t,t) = 1, \tag{20}$$

and it is true that

$$H(t,s) = Z(t,s). \tag{21}$$

With D convex, a Liapunov functional for (18) is

$$V_4(t) = Z^2(t,s) + \int_s^t D_u(t,u)\left(\int_u^t Z(v,s)dv\right)^2 du + D(t,s)\left(\int_s^t Z(v,s)dv\right)^2. \tag{22}$$

Theorem 4.1 *If D is convex and $k > 0$ then the derivative of V_4 along a solution of (18) satisfies*

$$V_4'(t) \leq -2[k + C(t,t)]Z^2(t,s), \text{ and } \sup_{0 \leq s \leq t < \infty} \int_s^t Z^2(u,s)du < \infty. \tag{24}$$

Proof We have

$$V_4'(t) \leq -2[k + C(t,t)]Z^2(t,s) - 2Z(t,s)\int_s^t D(t,u)Z(u,s)du$$

$$+ 2Z(t,s)D(t,s)\int_s^t Z(v,s)dv + 2Z(t,s)\int_s^t D_u(t,u)\int_u^t Z(v,s)dvdu.$$

An integration of the last term by parts yields

$$2Z(t,s)\left[D(t,u)\int_u^t Z(v,s)dv\Big|_s^t + \int_s^t D(t,u)Z(u,s)du\right]$$

$$= 2Z(t,s)\left[-D(t,s)\int_s^t Z(v,s)dv + \int_s^t D(t,u)Z(u,s)du\right].$$

Cancellation of terms yields the required conclusion.

We then see that

$$Z^2(t,s) \le V_4(t) \le V_4(s) - 2k \int_s^t Z^2(u,s)du$$

with $Z^2(s,s) = 1$ yielding

$$Z^2(t,s) + 2k \int_s^t Z^2(u,s)du \le 1. \tag{25}$$

This is a significant difference from the integral equation resolvent which requires $\int_s^t C^2(u,s)du$ bounded in order to get the parallel conclusion for the resolvent. Notice that $\int_0^t Z^2(u,0)du \le 1/(2k)$; as $k \to \infty$, the integral tends to zero.

It is most direct to obtain $x \in L^2[0,\infty)$ in the linear case from (1) with the Liapunov functional

$$V_1(t) = \int_0^t C_s(t,s)\left(\int_s^t x(u)du \right)ds + C(t,0)\left(\int_0^t x(u)du \right)^2,$$

yielding

$$V_1'(t) \le -x^2(t) + a^2(t).$$

We are coming to one of our central issues. From $a \in L^2$ we obtain $x \in L^2$ and, hence, from (29) we have by default that

$$\int_0^t R(t,s)a(s)ds \in L^2 \text{ and } x - a \in L^2.$$

However, we have no independent property of R which can be used without $a(t)$. We seek integral properties on R alone and the following is a typical way in which we would use them. Recall that in Section 1 we found that for C convex, then

$$\int_s^t R^2(u,s)ds \le \int_s^t C^2(u,s)du \le \Gamma \le +\infty.$$

We just noted that $a \in L^2$ yields $x - a \in L^2$ by default. But $\Gamma < \infty$ yields $x - a \in L^2$ by direct computation, not by default, and that is such a desirable property in other contexts.

Proposition 4.1 *If $\Gamma < \infty$ then $a \in L^1[0,\infty)$ implies $x - a \in L^2[0,\infty)$.*

We will give a proof of a parallel result below, but it is sketched as follows.

$$(x(t) - a(t))^2 = \left(-\int_0^t R(t,s)a(s)ds \right)^2$$

so integration, followed by the Schwarz inequality and interchange of the order of integration will yield the result.

Our focus here is on the case of $\Gamma = +\infty$ and we attempt to obtain an integrability property of a resolvent. The first step is to note that (25) did not require $\Gamma < \infty$.

Proposition 4.2 *If (25) holds, then $a' + ka \in L^1[0,\infty)$ implies $x \in L^2[0,\infty)$ and $x - Z(t,0)x(0) \in L^2[0,\infty)$.*

Proof Let $a'(t) + ka(t) =: p(t)$ and from (19) we have

$$(1/2)x^2(t) \leq Z^2(t,0)x^2(0) + \left(\int_0^t Z(t,s)p(s)ds \right)^2.$$

The last term is in L^1, not by default, but by the nonconvolution extension of the classical theorem that the convolution of an L^1-function with an L^2-function is an L^2-function. Here are the details. We have

$$
\begin{aligned}
(1/2) \int_0^t x^2(u)du &\leq \int_0^t Z^2(u,0)x^2(0)du + \int_0^t \left(\int_0^u Z(u,s)p(s)ds \right)^2 du \\
&\leq \int_0^t Z^2(u,0)x^2(0)du + \int_0^t \int_0^u |p(s)|ds \int_0^u Z^2(u,s)|p(s)|dsdu \\
&\leq \int_0^t Z^2(u,0)x^2(0)du + \int_0^\infty |p(s)|ds \int_0^t \int_s^t Z^2(u,s)du|p(s)|ds \\
&\leq \int_0^t Z^2(u,0)x^2(0)du + \left(\int_0^\infty |p(s)|ds \right)^2 (1/2k).
\end{aligned}
$$

We will now see how this applies to $R(t,s)$.

5   Relations Between Resolvents

If we begin with C convex and

$$x(t) = a(t) - \int_0^t C(t,s)x(s)ds, \tag{26}$$

we have the resolvent equation

$$R(t,s) = C(t,s) - \int_s^t C(t,u)R(u,s)du \tag{27}$$

and the variation of parameters formula

$$x(t) = a(t) - \int_0^t R(t,s)a(s)ds. \tag{28}$$

For (26) there is the Liapunov functional

$$V_1(t) = \int_0^t C_s(t,s) \left(\int_s^t x(u)du \right)^2 ds + C(t,0) \left(\int_0^t x(u)du \right)^2 \tag{29}$$

and a calculation given in the appendix yields

$$V_1'(t) \leq -x^2(t) + a^2(t) \tag{30}$$

and

$$\int_0^t x^2(s)ds \leq \int_0^t a^2(s)ds. \tag{31}$$

In a parallel manner we have a Liapunov functional for the resolvent equation given by

$$V_2(t) = \int_s^t C_2(t, u) \left(\int_u^t R(v, s) dv \right)^2 du + C(t, s) \left(\int_s^t R(u, s) du \right)^2 \tag{32}$$

and a calculation will yield

$$V_2'(t) \leq -R^2(t, s) + C^2(t, s) \tag{33}$$

with

$$\int_s^t R^2(u, s) du \leq \int_s^t C^2(u, s) du. \tag{34}$$

We explored consequences of these relations in [6] for the case

$$\sup_{0 \leq s \leq t < \infty} \int_s^t C^2(u, s) du < \infty. \tag{35}$$

Here, we look at the case where

$$\sup_{0 \leq s \leq t < \infty} \int_s^t C^2(u, s) du = \infty \tag{36}$$

so that V_2 yields nothing about R. We find a substitute for

$$\sup_{0 \leq s \leq t < \infty} \int_s^t R^2(u, s) du < \infty \tag{37}$$

when (36) holds.

Theorem 5.1 *If D is defined in (6), D convex, and if $\frac{d}{ds}C(s, s)$ is continuous, then*

$$R(t, s) = Z_s(t, s) - kZ(t, s). \tag{38}$$

Proof From (17), (18), and (24) we see that for (1)

$$x(t) = a(t) - \int_0^t R(t, s) a(s) ds$$

and

$$
\begin{aligned}
x(t) &= Z(t, 0) a(0) + \int_0^t Z(t, s)[a'(s) + ka(s)] ds \\
&= Z(t, 0) a(0) + Z(t, s) a(s) \Big|_0^t - \int_0^t Z_s(t, s) a(s) ds + k \int_0^t Z(t, s) a(s) ds \\
&= Z(t, 0) a(0) + a(t) - Z(t, 0) a(0) - \int_0^t [Z_s(t, s) + kZ(t, s)] a(s) ds \\
&= a(t) - \int_0^t [Z_s(t, s) - kZ(t, s)] a(s) ds.
\end{aligned}
$$

This means that for any $a(t)$ with $a'(t)$ continuous then

$$\int_0^t R(t,s)a(s)ds = \int_0^t [Z_s(t,s) - kZ(t,s)]a(s)ds. \tag{39}$$

Looking back at the Grossman-Miller resolvent (20) and noting that $H(t,s) = Z(t,s)$ we see that if $C(s,s)$ has a continuous derivative, then $H_{ss} = Z_{ss}$ is continuous. We should also note from (27) that R_s is continuous. Thus, for any fixed t we can pick $a(s) = Z_s(t,s) - kZ(t,s) - R(t,s)$ and have from (39) with t fixed that

$$\int_0^t [Z_s(t,s) - kZ(t,s) - R(t,s)]^2 ds = 0. \tag{40}$$

Thus, the integrand is identically zero and (38) holds. This completes the proof.

The variation of parameters formula for (1) now becomes

$$x(t) = a(t) - \int_0^t [Z_s(t,s) - Z(t,s)]a(s)ds. \tag{41}$$

We have independent properties of Z, as well as Z_s through (20) and through integration by parts.

6 Appendix

In Section 1, we mentioned that our Liapunov functional

$$V_1(t) = \int_0^t C_s(t,s)\left(\int_s^t g(x(u))du\right)^2 ds + C(t,0)\left(\int_0^t g(x(u))du\right)^2$$

has a derivative along a solution of (1) satisfying

$$V_1'(t) \le 2g(x)[a(t) - x(t)].$$

Owing to the absence of a chain rule, that differentiation is not simple so we want to give the details. Using Leibnitz's rule we have

$$V_1'(t) = \int_0^t C_{st}(t,s)\left(\int_s^t g(x(u))du\right)^2 ds + 2g(x)\int_0^t C_s(t,s)\int_s^t g(x(u))du\,ds$$
$$+ C_t(t,0)\left(\int_0^t g(x(s))ds\right)^2 + 2g(x)C(t,0)\int_0^t g(x(s))ds.$$

We now integrate the third-to-last term by parts to obtain

$$2g(x)\left[C(t,s)\int_s^t g(x(u))du\Big|_0^t + \int_0^t C(t,s)g(x(s))ds\right]$$
$$= 2g(x)\left[-C(t,0)\int_0^t g(x(u))du + \int_0^t C(t,s)g(x(s))ds\right].$$

Cancel terms, use the sign conditions, and use (1) in the last step of the process to unite the Liapunov functional and the equation obtaining

$$V_1'(t) = \int_0^t C_{st}(t,s)\left(\int_s^t g(x(u))du\right)^2 ds + C_t(t,0)\left(\int_0^t g(x(s))ds\right)^2$$
$$+ 2g(x)[a(t) - x(t)] \le 2g(x)[a(t) - x(t)].$$

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Homoclinic Orbits for Superquadratic Hamiltonian Systems with Small Forcing Terms

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Abstract: In this paper, we prove the existence of homoclinic orbits for the second order Hamiltonian system: $\ddot{q}(t) + \nabla V(t, q(t)) = f(t)$, where $V \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), V(t, q) = -K(t, q) + W(t, q)$ is T-periodic in t, K satisfies the "pinching" condition $b_1|q|^2 \leq K(t, q) \leq b_2|q|^2$ and W is superquadratic at the infinity and needs not satisfy the global Ambrosetti-Rabinowitz condition. A homoclinic orbit is obtained as the limit of $2kT$-periodic solutions of a certain sequence of second order differential equations.

Keywords: *homoclinic orbit; Hamiltonian system; Mountain Pass Theorem.*

Mathematics Subject Classification (2000): 34C37, 37J45, 70H05.

1 Introduction

Let us consider the second order Hamiltonian system

$$\ddot{q}(t) + \nabla V(t, q(t)) = f(t), \qquad\qquad (HS)$$

where $V(t, x) = -K(t, x) + W(t, x), \nabla V(t, x) = (\partial V / \partial x)(t, x)$, K, $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are C^1-maps, T-periodic with respect to t, $T > 0$ and $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is continuous and bounded. We will say that a solution q of (HS) is *homoclinic* (to 0) if $q(t) \longrightarrow 0$ as $t \longrightarrow \pm\infty$. In addition, if $q \not\equiv 0$ then q is called a nontrivial homoclinic solution.

The problem of finding subharmonic and homoclinic solutions for Hamiltonian systems has been the object of many works under different assumptions on the growth

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of W at infinity, see [1,3-5,8,12,13] and references therein. Most of them treat the superquadratic case. They usually suppose $K(t,x) = \dfrac{1}{2}(L(t)x,x)$ with $L(t)$ is a symmetric matrix valued function and W satisfies the global Ambrosetti-Rabinowitz condition, that is, there exists $\mu > 2$ such that

$$0 < \mu W(t,x) \le (\nabla W(t,x),x), \quad \forall\, (t,x) \in \mathbb{R} \times \mathbb{R}^n \backslash \{0\}.$$

Especially, in [13], Rabinowitz established the existence of homoclinic orbits for the Hamiltonian system (HS) under the above assumptions and $f \equiv 0$. Recently, the authors in [7] consider a more general case where K is assumed to satisfy the "pinching" condition $b_1|x|^2 \le K(t,x) \le b_2|x|^2$ and the function f may be nonzero.

In this paper, we shall study the existence of homoclinic orbits for (HS) when W satisfies the following superquadratic condition:

$$W(t,x)/|x|^2 \longrightarrow +\infty \text{ as } |x| \to \infty \quad uniformly\ in\ t \in \mathbb{R}, \tag{1}$$

and needs not satisfy the global Ambrosetti-Rabinowitz condition.

The superquadratic condition (1) was used in many recent works to study the existence of periodic and subharmonic solutions for Hamiltonian systems (see for example [6,12]). Subsequently, this condition was applied among other conditions in [9,11] to look for homoclinic orbits. Our approach is different from the last ones, in fact, similarly to [13], a homoclinic orbit will be obtained as a limit, as $k \longrightarrow \infty$, of sequence q_k of subharmonics for second order differential equations. The sequence q_k is obtained via a standard version of the Mountain Pass Theorem (Theorem 2.2 in [14]). Part of the difficulty in applying this theorem is in verifying the Palais-Smale condition. However, as it's shown in [2], a deformation lemma can be proved with the (C) condition, replacing the usual Palais-Smale condition, and it turns out that the Mountain Pass Theorem still holds true.

We make the following assumptions :
(H_1) *there exist* $a_1, a_2 > 0$ *such that*

$$a_1|x|^2 \le K(t,x) \le a_2|x|^2,\ \forall\, (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

(H_2) $\quad K(t,x) \le (x, \nabla K(t,x)) \le 2K(t,x), \forall\, (t,x) \in \mathbb{R} \times \mathbb{R}^n,$
(H_3) $\quad W(t,0) \equiv 0 \quad and \quad \nabla W(t,x) = o(|x|)\ as\ x \longrightarrow 0\ uniformly\ in\ t,$
(H_4) *there exist constants* $d_1 > 0$ *and* $r > 2$ *such that*

$$W(t,x) \le d_1|x|^r,\ \forall\, (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

(H_5) *there exist constants* $d_2 > 0, \mu > 1, \mu > r-2$ *and* $\beta \in L^1(\mathbb{R}, \mathbb{R}_+)$ *such that*

$$(\nabla W(t,x),x) - 2W(t,x) \ge d_2|x|^\mu - \beta(t),\ \forall\, (t,x) \in \mathbb{R} \times \mathbb{R}^n.$$

Here $(.,.)$ denotes the standard inner product in $\mathbb{R}^n$ and $|\,.\,|$ is the induced norm.

For each $k \in \mathbb{N}$, let $E_k = W_{2kT}^{1,2}(\mathbb{R}, \mathbb{R}^n)$, the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm

$$\|q\|_{E_k} = \left(\int_{-kT}^{kT} (|\dot{q}(t)|^2 + |q(t)|^2)dt \right)^{\frac{1}{2}}.$$

Furthermore, let $L_{2kT}^\infty(\mathbb{R}, \mathbb{R}^n)$ denote the space of $2kT$-periodic essentially bounded (measurable) functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$||q||_{L_{2kT}^\infty} = ess \ \sup\{|q(t)|; t \in [-kT, kT]\}.$$

The following result was proved by Rabinowitz in [13].

Proposition 1.1 *There is a positive constant C such that for each $k \in \mathbb{N}$, and $q \in E_k$ the following inequality holds:*

$$||q||_{L_{2kT}^\infty} \leq C||q||_{E_k}. \tag{2}$$

Set $b_1 := \min\{1, 2a_1\}$, $b_2 := \max\{1, 2a_2\}$ and suppose that

(H_6) $2d_1 < b_1$, $f \in L^2(\mathbb{R}, \mathbb{R}^n) \cap L^\gamma(\mathbb{R}, \mathbb{R}^n)$ *and* $||f||_{L^2} < \dfrac{b_1 - 2d_1}{2C}$, *where* $\dfrac{1}{\gamma} + \dfrac{1}{\mu} = 1$.

Our main result is the following :

Theorem 1.1 *Suppose $(H_1) - (H_6)$ and (1) are satisfied then the system (HS) possesses a nontrivial homoclinic solution $q \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ such that $\dot{q}(t) \longrightarrow 0$ as $t \longrightarrow \pm\infty$.*

Remark 1.1 Consider the functions

$$K(t, x) = (1 + \frac{1}{1 + x^2})x^2, \quad W(t, x) = h(t)|x|^2 \ln(1 + |x|^2),$$

where h is positive, continuous and T-periodic function. A straightforward computation shows that W satisfies the assumptions $(H_3) - (H_5)$ of Theorem 1.1 but does not satisfy the global Ambrosetti–Rabinowitz condition essentially. Moreover, $K(t, x)$ satisfies the assumptions (H_1) and (H_2) but can not be written in the form $1/2(L(t)x, x)$. Hence, Theorem 1.1 extends the results in [7,13] mainly. Furthermore, contrary to [7,13], the conditions of our result permit to W to change sign near the origin. Theorem 1.1 is also related to those in [9,11,15], where $K(t, x)$ has the form $1/2(L(t)x, x)$ without periodicity assumption on V and $f \equiv 0$.

2   Proof of Theorem 1.1

For each $k \in \mathbb{N}$, let $L_{2kT}^2(\mathbb{R}, \mathbb{R}^n)$ denote the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm $||q||_{L_{2kT}^2} = (\int_{-kT}^{kT} |q(t)|^2 dt)^{1/2}$. Let $f_k : \mathbb{R} \longrightarrow \mathbb{R}^n$ be the $2kT$-periodic extension of the restriction of f to the interval $[-kT, kT]$ and $\eta_k : E_k \longrightarrow [0, +\infty[$ given by

$$\eta_k(q) = \Big(\int_{-kT}^{kT} \Big[|\dot{q}(t)|^2 + 2K(t, q(t)) \Big] dt \Big)^{1/2}.$$

By (H_1) we get

$$b_1||q||_{E_k}^2 \leq \eta_k^2(q) \leq b_2||q||_{E_k}^2. \tag{3}$$

Let $I_k : E_k \longrightarrow \mathbb{R}$, be defined by

$$\begin{aligned}
I_k(q) &= \int_{-kT}^{kT} \Big[\frac{1}{2}|\dot{q}(t)|^2 - V(t, q(t)) \Big] dt + \int_{-kT}^{kT} (f_k(t), q(t)) dt \\
&= \frac{1}{2}\eta_k^2(q) - \int_{-kT}^{kT} W(t, q(t)) dt + \int_{-kT}^{kT} (f_k(t), q(t)) dt. \tag{4}
\end{aligned}$$

Then $I_k \in C^1(E_k, \mathbb{R})$ and it's easy to show that

$$I'_k(q)v = \int_{-kT}^{kT} \left[(\dot{q}(t), \dot{v}(t)) - (\nabla V(t, q(t)), v(t)) \right] dt + \int_{-kT}^{kT} (f_k(t), v(t)) dt.$$

By (H_2), we get

$$I'_k(q)q \leq \eta_k^2(q) - \int_{-kT}^{kT} (\nabla W(t, q(t)), q(t)) dt + \int_{-kT}^{kT} (f_k(t), q(t)) dt. \tag{5}$$

Moreover, it is well known that critical points of I_k are classical $2kT$-periodic solutions of the second order Hamiltonian system

$$\ddot{q}(t) + \nabla V(t, q(t)) = f_k(t). \tag{HS_k}$$

Lemma 2.1 *If V and f satisfy $(H_1) - (H_6)$ and (1), then for all $k \in \mathbb{N}$ the system (HS_k) possesses a $2kT$-periodic solution.*

Proof It suffices to prove that the functional I_k satisfies all the assumptions of the Mountain Pass Theorem (Theorem 2.2 in [14]) with the (C) condition replacing the usual Palais-Smale condition. This will be done by a sequence of lemmas. $\square$

Lemma 2.2 *I_k satisfies the (C) condition, i.e., for every constant c and sequence $\{u_n\} \subset E_k, \{u_n\}$ has a convergent subsequence if $I_k(u_n) \longrightarrow c$ and $(1+||u_n||)I'_k(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Proof Assume that $\{u_n\} \subset E_k$ is a (C) sequence of I_k, that is, $I_k(u_n)$ is bounded and $(1 + ||u_n||)||I'_k(u_n)|| \longrightarrow 0$ as $n \longrightarrow \infty$. Then there exists $M_k > 0$ such that

$$M_k \geq 2I_k(u_n) - I'_k(u_n)u_n$$

$$\geq \int_{-kT}^{kT} \left[(\nabla W(t, u_n(t)), u_n(t)) - 2W(t, u_n(t)) \right] dt + \int_{-kT}^{kT} (f_k(t), u_n(t)) dt.$$

So, by (H_5), we get

$$M_k \geq d_2 \int_{-kT}^{kT} |u_n(t)|^\mu dt - \int_{-kT}^{kT} \beta(t) dt + \int_{-kT}^{kT} (f_k(t), u_n(t)) dt.$$

Then, by Hölder inequality

$$d_2 ||u_n||_{L_{2kT}^\mu}^\mu \leq M_k + \int_{-kT}^{kT} \beta(t) dt + ||f_k||_{L_{2kT}^\gamma} ||u_n||_{L_{2kT}^\mu},$$

where γ is the conjugate exponent of μ. Since $\mu > 1$, there exists a constant C_k such that

$$||u_n||_{L_{2kT}^\mu} \leq C_k. \tag{6}$$

On the other hand, by (3), (4) and (H_4), one has

$$b_1 ||u_n||_{E_k}^2 \leq 2I_k(u_n) + 2d_1 \int_{-kT}^{kT} |u_n(t)|^r dt - 2 \int_{-kT}^{kT} (f_k(t), u_n(t)) dt$$

$$\leq 2I_k(u_n) + 2d_1 \int_{-kT}^{kT} |u_n(t)|^r dt + 2C_k ||f_k||_{L_{2kT}^\gamma}. \tag{7}$$

If $\mu \geq r$, by Hölder inequality

$$\int_{-kT}^{kT} |u_n(t)|^r dt \leq (2kT)^{\frac{\mu-r}{\mu}} \left(\int_{-kT}^{kT} |u_n(t)|^\mu dt \right)^{\frac{r}{\mu}}.$$

Combining the above with (6) and (7), we obtain that $||u_n||_{E_k}$ is bounded.
If $\mu < r$, by (2), we have

$$\int_{-kT}^{kT} |u_n(t)|^r dt = \int_{-kT}^{kT} |u_n(t)|^{r-\mu} |u_n(t)|^\mu dt$$

$$\leq ||u_n||_{L_{2kT}^\infty}^{r-\mu} \int_{-kT}^{kT} |u_n(t)|^\mu dt$$

$$\leq C^{r-\mu} ||u_n||_{E_k}^{r-\mu} \int_{-kT}^{kT} |u_n(t)|^\mu dt. \tag{8}$$

Hence, by (6) and (8) there exists a constant C_k' such that

$$b_1 ||u_n||_{E_k}^2 \leq 2I_k(u_n) + C_k' ||u_n||_{E_k}^{r-\mu} + 2\, C_k ||f_k||_{L_{2kT}^\gamma}.$$

Since $r - \mu < 2$ and $I_k(u_n)$ is bounded, then $||u_n||_{E_k}$ will be bounded too.
In a similar way to Proposition B.35 in [14], we can prove that $\{u_n\}$ has a convergent subsequence. Hence I_k satisfies the (C) condition. □

Lemma 2.3 *The functional I_k satisfies the condition (I_1) of the Mountain Pass Theorem.*

Proof Let $q \in E_k$, such that $0 < ||q||_{L_{2kT}^\infty} \leq 1$. By (H_4) we have

$$\int_{-kT}^{kT} W(t, q(t)) dt \leq d_1 \int_{-kT}^{kT} |q(t)|^2 dt \leq d_1 ||q||_{E_k}^2. \tag{9}$$

Then, by (3), (4), (9) and (H_6) it follows that

$$I_k(q) \geq \frac{b_1}{2} ||q||_{E_k}^2 - d_1 ||q||_{E_k}^2 - ||f_k||_{L_{2kT}^2} ||q||_{L_{2kT}^2}$$

$$\geq \frac{b_1}{2} ||q||_{E_k}^2 - d_1 ||q||_{E_k}^2 - ||f||_{L^2} ||q||_{E_k}$$

$$\geq \frac{1}{2} (b_1 - 2d_1 - 2C||f||_{L^2}) ||q||_{E_k}^2 + C||f||_{L^2} \left(||q||_{E_k}^2 - \frac{||q||_{E_k}}{C} \right).$$

Set

$$\rho = \frac{1}{C}, \quad \alpha = \frac{b_1 - 2d_1 - 2C||f||_{L^2}}{2C^2}.$$

By (2), if $||q||_{E_k} = \rho$, then $0 < ||q||_{L^\infty} \leq 1$ and $I_k(q) \geq \alpha$. □

Lemma 2.4 *Under the assumption (1), I_k satisfies the condition (I_2) of the Mountain Pass Theorem.*

Proof Let $q \in E_1, q \not\equiv 0$ such that $q(T) = q(-T) = 0$ and $A > \dfrac{b_2||q||_{E_1}^2}{2||q||_{L_{2T}^2}^2}$. By (1),
there exists $B > 0$ such that for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $W(t, x) \geq A|x|^2 - B$.
Hence, for all $\zeta \in \mathbb{R}$ the following inequality holds :

$$I_1(\zeta q) \leq \frac{b_2}{2}\zeta^2||q||_{E_1}^2 - A\zeta^2||q||_{L_{2T}^2}^2 + |\zeta|||f_1||_{L_{2T}^2}||q||_{L_{2T}^2} + 2TB. \tag{10}$$

Then by (10) and the choice of A there exists $\zeta \in \mathbb{R}$ satisfying $||\zeta q||_{E_1} > \rho$ and $I_1(\zeta q) < 0$.
For $k > 1$, set $e_1(t) = \zeta q(t)$ and

$$e_k(t) = \begin{cases} e_1(t) & \text{for} & |t| \leq T, \\ 0 & \text{for} & T < |t| \leq kT. \end{cases} \tag{11}$$

Then $e_k \in E_k, ||e_k||_{E_k} = ||e_1||_{E_1} > \rho$ and $I_k(e_k) = I_1(e_1) < 0$ for every $k \in \mathbb{N}$. $\square$

For our setting, clearly $I_k(0) = 0$, so, by applying the Mountain Pass Theorem, I_k
possesses a critical value $c_k \geq \alpha$. Hence, for every $k \in \mathbb{N}$, there is $q_k \in E_k$ such that

$$I_k(q_k) = c_k, \qquad I_k'(q_k) = 0. \tag{12}$$

This completes the proof of Lemma 2.4.

Lemma 2.5 *Let $(q_k)_{k \in \mathbb{N}}$ be the sequence given by (12). Then there exists a subse-
quence $(q_{k_j})_{j \in \mathbb{N}}$ convergent to a certain function q_0 in $C_{loc}^1(\mathbb{R}, \mathbb{R}^n)$.*

Proof First of all we show that the sequences $\{c_k\}_{k \in \mathbb{N}}$ and $\{||q_k||_{E_k}\}_{k \in \mathbb{N}}$ are bounded.
For every $k \in \mathbb{N}$, let $g_k : [0, 1] \longrightarrow E_k$ be a curve given by $g_k(s) = se_k$, where e_k is defined
by (11). Then $g_k \in \Gamma_k$ and $I_k(g_k(s)) = I_1(g_1(s))$ for all $k \in \mathbb{N}$ and $s \in [0, 1]$. Therefore,
by the Mountain Pass Theorem,

$$c_k \leq \max_{s \in [0,1]} I_1(g_1(s)) \equiv M_0 \tag{13}$$

independent of $k \in \mathbb{N}$. As $I_k'(q_k) = 0$, we receive from (4), (5) and (H_5) that

$$2c_k = 2I_k(q_k) - I_k'(q_k)q_k$$
$$\geq \int_{-kT}^{kT} \Big[(\nabla W(t, q_k(t)), q_k(t)) - 2W(t, q_k(t)) \Big] dt + \int_{-kT}^{kT} (f_k(t), q_k(t))dt$$
$$\geq d_2 \int_{-kT}^{kT} |q_k(t)|^\mu dt - \int_{-kT}^{kT} \beta(t)dt + \int_{-kT}^{kT} (f_k(t), q_k(t))dt. \tag{14}$$

By Hölder inequality, (13) and (14) we get

$$d_2||q_k||_{L_{2kT}^\mu}^\mu \leq 2M_0 + \beta_0 + \alpha_0||q_k||_{L_{2kT}^\mu},$$

where $\alpha_0 = ||f||_{L_{\mathbb{R}}^\gamma}$ and $\beta_0 = \displaystyle\int_{-\infty}^{+\infty} \beta(t)dt$. Since $\mu > 1$ and all the constants in the above
inequality are independent of k, then there exists a constant L such that

$$||q_k||_{L_{2kT}^\mu} \leq L. \tag{15}$$

On the other hand, by (3), (4) and (H_4), one has

$$b_1||q_k||_{E_k}^2 \le 2M_0 + 2d_1 \int_{-kT}^{kT} |q_k(t)|^r dt - 2 \int_{-kT}^{kT} (f_k(t), q_k(t))dt. \tag{16}$$

If $r \ge \mu$, by (1), (15) and Hölder inequality we obtain

$$b_1||q_k||_{E_k}^2 \le 2M_0 + 2d_1||q_k||_{L_{2kT}^\infty}^{r-\mu} \int_{-kT}^{kT} |q_k(t)|^\mu dt - 2 \int_{-kT}^{kT} (f_k(t), q_k(t))dt$$

$$\le 2M_0 + 2cL^\mu||q_k||_{E_k}^{r-\mu} + 2\alpha_0 L. \tag{17}$$

Since $r - \mu < 2$ and all coefficients of (17) are independent of k, we see that there is $M_1 > 0$ independent of k such that

$$||q_k||_{E_k} \le M_1. \tag{18}$$

If $r < \mu$, we have

$$\int_{-kT}^{kT} |q_k(t)|^r dt = \int_{\{t\in[-kT,kT];|q_k(t)|\le 1\}} |q_k(t)|^r dt + \int_{\{t\in[-kT,kT];|q_k(t)|>1\}} |q_k(t)|^r dt$$

$$\le \int_{\{t\in[-kT,kT];|q_k(t)|\le 1\}} |q_k(t)|^2 dt + \int_{\{t\in[-kT,kT];|q_k(t)|>1\}} |q_k(t)|^\mu dt$$

$$\le \int_{-kT}^{kT} |q_k(t)|^2 dt + \int_{-kT}^{kT} |q_k(t)|^\mu dt. \tag{19}$$

By (16) and (19) we get

$$b_1||q_k||_{E_k}^2 \le 2M_0 + 2d_1||q_k||_{E_k}^2 + 2d_1 L^\mu + 2\alpha_0 L.$$

Hence

$$(b_1 - 2d_1)||q_k||_{E_k}^2 \le 2M_0 + 2d_1 L^\mu + 2\alpha_0 L.$$

Since $b_1 > 2d_1$, (18) remains true.

Now, we observe that the sequences $\{q_k\}_{k\in\mathbb{N}}$, $\{\dot{q}_k\}_{k\in\mathbb{N}}$ and $\{\ddot{q}_k\}_{k\in\mathbb{N}}$ are uniformly bounded. By (2) and (18),

$$||q_k||_{L_{2kT}^\infty} \le CM_1 \equiv M_2 \tag{20}$$

for every $k \in \mathbb{N}$. Since q_k satisfies (HS_k), if $t \in [-kT, kT]$ we have

$$|\ddot{q}_k(t)| \le |f_k(t)| + |\nabla V(t, q_k(t))| \le \sup_{t\in\mathbb{R}} |f(t)| + |\nabla V(t, q_k(t))|,$$

so, by (20), there exists $M_3 > 0$ independent of k such that

$$||\ddot{q}_k||_{L_{2kT}^\infty} \le M_3. \tag{21}$$

From the Mean Value Theorem it follows that for every $k \in \mathbb{N}$ and $t \in \mathbb{R}$ there exists $\tau_k \in [t-1, t]$ such that

$$\dot{q}_k(\tau_k) = \int_{t-1}^t \dot{q}_k(s)ds = q_k(t) - q_k(t-1).$$

Combining the above with (20) and (21) we obtain

$$|\dot{q}_k(t)| = |\int_{\tau_k}^{t} \ddot{q}_k(s)ds + \dot{q}_k(\tau_k)|$$

$$\leq \int_{t-1}^{t} |\ddot{q}_k(s)|ds + |q_k(t) - q_k(t-1)| \leq M_3 + 2M_2 \equiv M_4,$$

and hence for every $k \in \mathbb{N}$

$$||\dot{q}_k||_{L_{2kT}^{\infty}} \leq M_4. \tag{22}$$

To finish the proof it is sufficient to note that the sequences $\{q_k\}_{k\in\mathbb{N}}$ and $\{\dot{q}_k\}_{k\in\mathbb{N}}$ are equicontinuous. Indeed, for every $k \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$, we have by (22)

$$|q_k(t_1) - q_k(t_2)| = |\int_{t_1}^{t_2} \dot{q}_k(s)ds| \leq \int_{t_1}^{t_2} |\dot{q}_k(s)|ds \leq M_4|t_1 - t_2|,$$

and similarly, by (21), we have

$$|\dot{q}_k(t_1) - \dot{q}_k(t_2)| \leq M_3|t_1 - t_2|.$$

Applying now the Arzelà-Ascoli theorem, we receive the claim. □

Lemma 2.6 *Let $q_0 : \mathbb{R} \longrightarrow \mathbb{R}^n$ be the function given by Lemma 2.5. Then q_0 is the desired homoclinic solution of (HS).*

Proof The proof of this lemma is based on the two following facts.
Fact 1 *Let $q : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a continuous map. If $\dot{q} : \mathbb{R} \longrightarrow \mathbb{R}^n$ is continuous at t_0 then*

$$\lim_{t \longrightarrow t_0} \frac{q(t) - q(t_0)}{t - t_0} = \dot{q}(t_0).$$

Fact 2 *Let $q : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a continuous map such that $\dot{q}$ is locally square integrable. Then, for all $t \in \mathbb{R}$, we have*

$$|q(t)| \leq \sqrt{2}\Big(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|q(s)|^2 + |\dot{q}(s)|^2)ds \Big)^{\frac{1}{2}}. \tag{23}$$

The proofs of these facts are elementary and can be found in [7, p 385].

First, we show that q_0 is a solution of (HS). By Lemma 2.1 and Lemma 2.5, we have $q_{k_j} \longrightarrow q_0$ in $C_{loc}^1(\mathbb{R}, \mathbb{R}^n)$, as $j \longrightarrow \infty$, and

$$\ddot{q}_{k_j}(t) + \nabla V(t, q_{k_j}(t)) = f_{k_j}(t)$$

for every $j \in \mathbb{N}$, and $t \in [-k_jT, k_jT]$. Take $a, b \in \mathbb{R}$ with $a < b$. There exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$ and $t \in [a, b]$, we have

$$\ddot{q}_{k_j}(t) = -\nabla V(t, q_{k_j}(t)) + f(t).$$

Hence, $\ddot{q}_{k_j}$ is continuous in $[a, b]$ and $\ddot{q}_{k_j}(t) \longrightarrow -\nabla V(t, q_0(t)) + f(t)$ uniformly on $[a, b]$. Fact 1 implies that $\ddot{q}_{k_j}$ is a classical derivative of $\dot{q}_{k_j}$ in (a, b) for all $j > j_0$. Moreover, since $\dot{q}_{k_j} \longrightarrow \dot{q}_0$ uniformly on $[a, b]$, we obtain

$$\ddot{q}_0(t) = -\nabla V(t, q_0(t)) + f(t)$$

for every $t \in (a, b)$. Since a and b are arbitrary, we conclude that q_0 satisfies (HS).

Now we prove that $q_0(t) \longrightarrow 0$, as $|t| \longrightarrow \infty$. First of all remark that for all $l \in \mathbb{N}$ there exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$, we have

$$\int_{-lT}^{lT} (|q_{k_j}(t)|^2 + |\dot{q}_{k_j}(t)|^2)dt \leq ||q_{k_j}||^2_{E_{k_j}} \leq M_1^2.$$

By Lemma 2.5, we get

$$\int_{-lT}^{lT} (|q_0(t)|^2 + |\dot{q}_0(t)|^2)dt \leq M_1^2.$$

Letting $l \longrightarrow \infty$, we obtain $\int\limits_{-\infty}^{\infty} (|q_0(t)|^2 + |\dot{q}_0(t)|^2)dt \leq M_1^2$, and so

$$\int_{|t| \geq r} (|q_0(t)|^2 + |\dot{q}_0(t)|^2)dt \longrightarrow 0, \tag{24}$$

as $r \longrightarrow \infty$. Combining (23) and (24), we receive our claim.

In the next step we show that $\dot{q}_0(t) \longrightarrow 0$, as $|t| \longrightarrow \infty$. To do this, applying (23), we obtain

$$|\dot{q}_0(t)| \leq \sqrt{2} \Big(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|\dot{q}_0(s)|^2 + |\ddot{q}_0(s)|^2)ds \Big)^{\frac{1}{2}}.$$

From (24), we get

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{q}_0(s)|^2 ds \longrightarrow 0,$$

as $|t| \longrightarrow \infty$. Hence, it suffices to prove that

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{q}_0(s)|^2 ds \longrightarrow 0, \tag{25}$$

as $|t| \longrightarrow \infty$. Since q_0 is a solution of (HS), we obtain

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{q}_0(s)|^2 ds = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\nabla V(s, q_0(s))|^2 ds + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |f(s)|^2 ds$$

$$-2 \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (\nabla V(s, q_0(s)), f(s)) ds,$$

and then

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{q}_0(s)|^2 ds \leq \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\nabla V(s, q_0(s))|^2 ds + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |f(s)|^2 ds$$

$$+2 \Big(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\nabla V(s, q_0(s)|^2 ds \Big)^{\frac{1}{2}} \Big(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |f(s)|^2 ds \Big)^{\frac{1}{2}}. \tag{26}$$

By (H_6), we have

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |f(s)|^2 ds \longrightarrow 0, \tag{27}$$

as $|t| \longrightarrow \infty$. On the other hand, since $\nabla V(t,0) = 0$ for all $t \in \mathbb{R}$ and $q_0(t) \longrightarrow 0$, as $|t| \longrightarrow \infty$, (25) follows from (26) and (27).

Finally, it remains to show that q_0 is nontrivial. Obviously, this will be the case when $f \not\equiv 0$, otherwise, using (H_3), the proof is the same as in [13]. $\square$

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Painlevé Test to a Reduced System of Six Coupled Nonlinear ODEs

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Abstract: In this paper we investigate the complete integrability of the system of six coupled nonlinear ODEs, which arises in the ODE reduction of rotating stratified Boussinesq equations. We use Painlevé test to investigate the complete integrability of the system. And we conclude that the system is completely integrable only if the Rayleigh number $Ra = 0$. The singular solution of the system admits the movable pole type singularity in complex domain.

Keywords: *Painlevé test; rotating stratified Boussinesq equations; integrable system.*

Mathematics Subject Classification (2000): Primary 37K10, Secondary 76B70.

1 Introduction

We undertake the Painlevé analysis of the system of six coupled nonlinear ODEs arising as a reduction of rotating stratified Boussinesq equations. The rotating stratified Boussinesq equations form a system of partial differential equations modelling the movement of planetary atmosphere. In their study of instability in stratified fluids at large *Richardson number*, Majda and Shefter [1] analyzed certain system of ODE reduction of stratified Boussinesq equations. Srinivasan et al [2] gave the complete analysis of reduced system of ODEs and discussed the stability of degenerate critical point. In their paper Desale and Srinivasan [3] examine the same system in the light of the ARS (Ablowitz, Ramani and Segur [4]) conjecture. Ablowitz, Ramani and Segur have conjectured that a system of PDEs is completely integrable if all its ODE reductions are of Painlevé type. The conjecture has been tested on large class of differential equations and has since been

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employed as a popular test of integrability. Whereas in the basin scale dynamics Maas [5], has considered the flow of fluid contained in rectangular basin of dimension $L \times L \times H$, which is temperature stratified with the fixed zeroth order moments of mass and heat. The container is assumed to be steady, uniformly rotating on an f-plane. With this assumption Maas [5] reduces the rotating stratified Boussinesq equations to an interesting six coupled system of ODEs. Further, Desale [6] has given the complete analysis of the system and also tested the system for complete integrability by determining the four first integrals and uses the Jacobi's theorem. In their recent paper Desale and Sharma [7] have reduced the rotating stratified Boussinesq equations into the system of six coupled ODEs that are also in similar nature with the system which we are looking in this paper.

In this paper we have tested the system of six coupled nonlinear ODEs for its complete integrability via Painlevé analysis. Here we state that our analysis follows similar kind of techniques as used by Desale and Srinivasan in their paper [3]. But our system includes additional terms due to the effects of rotation so that in calculations we are far apart from Desale and Srinivasan [3].

This paper is organized as follows. Section 2 gives the ODE reduction of rotating stratified Boussinesq equations. We implement the Painlevé test to determine the singular solution of the system in Section 3. In Section 4, we illustrate two systems that also exhibit the similar kind of solutions. Finally, we conclude the results in Section 5.

2   Reduced System of Nonlinear ODEs

We now begin by describing the rotating stratified Boussinesq equations (see Majda [8], p. 1)

$$
\begin{aligned}
\frac{D\mathbf{v}}{Dt} + f(\hat{\mathbf{e}}_3 \times \mathbf{v}) &= -\nabla p + \nu(\Delta \mathbf{v}) - \frac{g\tilde{\rho}}{\rho_b}\hat{\mathbf{e}}_3, \\
\operatorname{div}\mathbf{v} &= 0, \\
\frac{D\tilde{\rho}}{Dt} &= \kappa \Delta \tilde{\rho},
\end{aligned}
\tag{1}
$$

where $\mathbf{v}$ denotes the velocity field, ρ is the density of fluid which is the sum of constant reference density ρ_b and perturb density $\tilde{\rho}$, p is the pressure, g is the acceleration due to gravity that points in $-\hat{\mathbf{e}}_3$ direction, f is the rotation frequency of earth, ν is the coefficient of viscosity, κ the coefficient of heat conduction and $\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$ is a convective derivative. For more about rotating stratified Boussinesq equations one may consult with Majda [8].

In the frame of reference of an uniformly stratified fluid contained in rotating rectangular box of dimension $L \times L \times H$, which is temperature stratified with fixed zeroth order moments of mass and heat (so that there is no net evaporation or precipitation, nor any net river input or output, and neither a heating nor cooling). The container is assumed to be in steady uniform rotation on an f-plane. Maas [5] reduces the system of equations (1) into the following system of six coupled ODEs:

$$
\begin{aligned}
Pr^{-1}\frac{d\mathbf{w}}{dt} + f'\hat{\mathbf{e}}_3 \times \mathbf{w} &= \hat{\mathbf{e}}_3 \times \mathbf{b} - (w_1,\, w_2,\, rw_3) + \hat{T}\mathbf{T}, \\
\frac{d\mathbf{b}}{dt} + \mathbf{b} \times \mathbf{w} &= -(b_1,\, b_2,\, \mu b_3) + Ra\mathbf{F}.
\end{aligned}
\tag{2}
$$

In these equations, $\mathbf{b} = (b_1,\, b_2,\, b_3)$ is the center of mass, $\mathbf{w} = (w_1,\, w_2,\, w_3)$ is the basin's averaged angular momentum vector, $\mathbf{T}$ is the differential momentum, $\mathbf{F}$ are buoyancy

fluxes, $f' = f/2r_h$ is the earth's rotation, $r = r_v/r_h$ is the friction ($r_{v,h}$ are the Rayleigh damping coefficients), Ra is the Rayleigh number, Pr is the Prandtl number, μ the diffusion coefficient and $\hat{T}$ is the magnitude of the wind stress torque.

Neglecting diffusive and viscous terms, Maas [5] consider the dynamics of an ideal rotating, uniformly stratified fluid in response to forcing. He assumes this to be due solely to differential heating in the meridional (y) direction $\mathbf{F} = (0, 1, 0)$; the wind effect is neglected i.e. $\mathbf{T} = 0$. For Prandtl number, Pr, equal to one the system of equations (2) reduces to the following ideal rotating, uniformly stratified system of six coupled ODEs

$$
\begin{aligned}
\frac{d\mathbf{w}}{dt} &= -f'\hat{\mathbf{e_3}} \times \mathbf{w} + \hat{\mathbf{e_3}} \times \mathbf{b}, \\
\frac{d\mathbf{b}}{dt} &= -\mathbf{b} \times \mathbf{w} + Ra\mathbf{F}.
\end{aligned}
\tag{3}
$$

In his paper Desale [6] has demonstrated the complete integrability of the system (3) for $Ra = 0$. Our approach to discuss the integrability of above system is quite different than Desale has used in his paper [6]. In the following section we deploy the Painlevé test for complete integrability of the system (3).

3 Singular Solution of the System

We can write the system of six coupled ODEs (3) component-wise as:

$$
\begin{aligned}
&\dot{w}_1 = f'w_2 - b_2, \quad \dot{w}_2 = -f'w_1 + b_1, \quad \dot{w}_3 = 0, \\
&\dot{b}_1 = w_2b_3 - w_3b_2, \quad \dot{b}_2 = w_3b_1 - w_1b_3 + Ra, \quad \dot{b}_3 = w_1b_2 - w_2b_1.
\end{aligned}
\tag{4}
$$

Since $\dot{w}_3 = 0$, hence we get $w_3 = \text{constant} = k_1$ say and consequently we have the system of five ODEs

$$
\begin{aligned}
&\dot{w}_1 = f'w_2 - b_2, \quad \dot{w}_2 = -f'w_1 + b_1, \\
&\dot{b}_1 = w_2b_3 - k_1b_2, \quad \dot{b}_2 = k_1b_1 - w_1b_3 + Ra, \quad \dot{b}_3 = w_1b_2 - w_2b_1.
\end{aligned}
\tag{5}
$$

We are looking for the solution of system (5) in the form of power series as given below

$$
\begin{aligned}
w_1(t) &= \sum_{j=0}^{\infty} w_{1j}\tau^{j+m_1}, \quad w_2(t) = \sum_{j=0}^{\infty} w_{2j}\tau^{j+m_2}, \\
b_1(t) &= \sum_{j=0}^{\infty} b_{1j}\tau^{j+n_1}, \quad b_2(t) = \sum_{j=0}^{\infty} b_{2j}\tau^{j+n_2}, \quad b_3(t) = \sum_{j=0}^{\infty} b_{3j}\tau^{j+n_3},
\end{aligned}
\tag{6}
$$

where $\tau = t - t_0$ and t_0 is the arbitrary position of singularity. As per the Painlevé algorithm there are three main steps in determination of singular solution. These steps are:

1. Determination of dominant behavior.

2. Determination of resonances.

3. Examining the compatibility conditions at the resonances.

It is natural that the algorithm may stop at the first step, second step or third step. For more details about this algorithm one may consult with Ablowitz et al [4]. The

convergence of the series solution by use of this algorithm is guaranteed by Kichenassamy and Littman [9, 10].

Now we proceed for implementation of algorithm so in the first step we determine dominant behavior of the system (5). There are the several possible cases for dominant balance but the system of ODEs (5) admits the singular solution only in the following case of principle dominant balance

$$\dot{w}_1 = -b_2, \quad \dot{w}_2 = b_1, \quad \dot{b}_1 = w_2 b_3, \quad \dot{b}_2 = -w_1 b_3, \quad \dot{b}_3 = w_1 b_2 - w_2 b_1. \qquad (7)$$

In the following subsection we determine exponents and leading order coefficients.

3.1   Determination of exponents

To determine the singular exponents m_1, m_2, n_1, n_2 & n_3, which appear in (6), it is sufficient to truncate the expansions up to the leading order and then substituting these truncated expansions into (7) we obtain the following system of equations

$$\begin{aligned}
&m_1 w_{10} \tau^{m_1-1} = -b_{20} \tau^{n_2}, \quad m_2 w_{20} \tau^{m_2-1} = b_{10} \tau^{n_1}, \\
&n_1 b_{10} \tau^{n_1-1} = w_{20} b_{30} \tau^{m_2+n_3}, \quad n_2 b_{20} \tau^{n_2-1} = -w_{10} b_{30} \tau^{m_1+n_3}, \\
&n_3 b_{30} \tau^{n_3-1} = \left(w_{10} b_{20} \tau^{m_1+n_2} - w_{20} b_{10} \tau^{m_2+n_1} \right).
\end{aligned} \qquad (8)$$

Equating the powers of τ so that equations (8) get satisfied we have the following linear equations

$$\begin{aligned}
&m_1 - 1 = n_2, \quad m_2 - 1 = n_1, \quad n_1 - 1 = m_2 + n_3, \\
&n_2 - 1 = m_1 + n_3, \quad n_3 - 1 = m_1 + n_2 = m_2 + n_1.
\end{aligned} \qquad (9)$$

From equations (9) the exponents can be uniquely determined as given below.

$$m_1 = m_2 = -1, \quad n_1 = n_2 = n_3 = -2. \qquad (10)$$

Substituting the values of m_1, m_2, n_1, n_2 & n_3 into equations (8) and then equating the coefficients of like powers of τ on both sides of each equation, we get the following system of equations to determine the leading order coefficients

$$\begin{aligned}
&w_{10} = b_{20}, \quad w_{20} = -b_{10}, \\
&b_{10} = -\tfrac{1}{2} w_{20} b_{30}, \quad b_{20} = \tfrac{1}{2} w_{10} b_{30}, \\
&b_{30} = -\tfrac{1}{2}(w_{10} b_{20} - w_{20} b_{10}).
\end{aligned} \qquad (11)$$

Solving these equations we find that there are two possible branches of leading order involving one leading order coefficient to be an arbitrary constant. Suppose that $w_{20} = k_2$ is an arbitrary constant. The possible branches of leading order are as given below

$$w_{10} = \pm\sqrt{-4 - k_2^2},\ w_{20} = k_2,\ b_{10} = -k_2,\ b_{20} = \pm\sqrt{-4 - k_2^2},\ b_{30} = 2. \qquad (12)$$

Here we notice that there are two possible branches of leading order. Hence, we will get two different singular solutions in complex domain. The next step of Painlevé algorithm is to determine the resonances. In the following section we proceed to determine the resonances.

3.2   Determination of resonances

As per the Painlevé algorithm this is the second step. Here we determine the resonances. So we rewrite the equations (6) by substituting the values of exponents

$$w_1(t) = w_{10}\tau^{-1} + \sum_{j=1}^{\infty} w_{1j}\tau^{j-1}, \quad w_2(t) = w_{20}\tau^{-1} + \sum_{j=1}^{\infty} w_{2j}\tau^{j-1},$$

$$b_1(t) = b_{10}\tau^{-2} + \sum_{j=1}^{\infty} b_{1j}\tau^{j-2}, \quad b_2(t) = b_{20}\tau^{-2} + \sum_{j=1}^{\infty} b_{2j}\tau^{j-2}, \tag{13}$$

$$b_3(t) = b_{30}\tau^{-2} + \sum_{j=1}^{\infty} b_{3j}\tau^{j-2}.$$

Substituting the above equations into the system (5) we obtained the following recursion relations for determining the coefficients of different powers of τ in the equations (13), which are valid for $j \geq 2$,

$$\begin{pmatrix} j-1 & 0 & 0 & 1 & 0 \\ 0 & j-1 & -1 & 0 & 0 \\ 0 & -b_{30} & j-2 & 0 & -w_{20} \\ b_{30} & 0 & 0 & j-2 & w_{10} \\ -b_{20} & b_{10} & w_{20} & -w_{10} & j-2 \end{pmatrix} \begin{pmatrix} w_{1j} \\ w_{2j} \\ b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix} = \begin{pmatrix} A_j \\ B_j \\ C_j \\ D_j \\ E_j \end{pmatrix}, \tag{14}$$

where

$$A_j = f'w_{2(j-1)}, \quad B_j = -f'w_{1(j-1)}, \quad C_j = -k_1 b_{2(j-1)} + \sum_{k=1}^{j-1} w_{2k}b_{3(j-k)},$$

$$D_j = k_1 b_{1(j-1)} - \sum_{k=1}^{j-1} w_{1k}b_{3(j-k)}, \quad E_j = \sum_{k=1}^{j-1} w_{1k}b_{2(j-k)} - \sum_{k=1}^{j-1} w_{2k}b_{1(j-k)}. \tag{15}$$

Now we denote by $M(j)$ the matrix

$$M(j) = \begin{pmatrix} j-1 & 0 & 0 & 1 & 0 \\ 0 & j-1 & -1 & 0 & 0 \\ 0 & -b_{30} & j-2 & 0 & -w_{20} \\ b_{30} & 0 & 0 & j-2 & w_{10} \\ -b_{20} & b_{10} & w_{20} & -w_{10} & j-2 \end{pmatrix}. \tag{16}$$

The above recursion relations (14) determine the unknown expansion coefficients uniquely unless the determinant of matrix $M(j)$ is zero. Those values of j at which the determinant $det(M(j))$ vanishes are called *resonances*. Here we see that for both possible branches of leading orders given in equations (12) the determinant of matrix $M(j)$ is

$$det(M(j)) = (j+1)j(j-2)(j-3)(j-4). \tag{17}$$

Hence, the resonances are

$$j = -1, 0, 2, 3, 4. \tag{18}$$

Here $j = -1$ is a usual resonance and $j = 0$ is corresponding to the arbitrariness of w_{20} in leading order behavior.

For the next step in the algorithm we check the compatibility conditions at non negative resonances given in equation (18).

3.3 Compatibility conditions

In this section we check whether the compatibility conditions hold at positive resonances which are determined in previous section. The recursion relations (14) will be valid if and only if the vector appearing on the right hand side of (14) must be annihilated by every left null vector of $M(j)$ (when j is a resonance) resulting in a set of compatibility conditions to be satisfied by the previously determined coefficients. When these conditions hold, the j-th coefficient vector enters as an arbitrary coefficient vector in the expansion (13). On the other hand if the compatibility condition fails at a resonant level, logarithms need to be introduced in the expansion (see [9, 10] for details). We investigate this in each case of possible branches of leading order coefficients given by (12) and we determine the expansion coefficients in each case up to the last resonant level.

• **Case 1:** Consider the leading order coefficients

$$w_{10} = \sqrt{-4 - k_1^2}, \quad w_{20} = k_2 \text{ (arbitrary constant)},$$
$$b_{10} = -k_2, \quad b_{20} = \sqrt{-4 - k_1^2}, \quad b_{30} = 2. \tag{19}$$

• **Compatibility condition at $j = 1$.** Since the recursion relations (14) come into force when $j \geq 2$, hence, we have directly substituted equations (19) into (13) and then into the equations (5). After simplifying we equate the like powers of τ on both sides of the resulting expansion thereby obtaining the following system of linear equations for w_{11}, w_{21}, b_{11}, b_{21} and b_{31}

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 & -k_2 \\ 2 & 0 & 0 & -1 & \sqrt{-4-k_2^2} \\ -\sqrt{-4-k_2^2} & -k_2 & k_2 & -\sqrt{-4-k_2^2} & -1 \end{pmatrix} \begin{pmatrix} w_{11} \\ w_{21} \\ b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} f'k_2 \\ -f'\sqrt{-4-k_2^2} \\ -k_1\sqrt{-4-k_2^2} \\ -k_1 k_2 \\ 0 \end{pmatrix}. \tag{20}$$

The system of linear equations (20) has a unique solution, hence w_{11}, w_{21}, b_{11}, b_{21} and b_{31} are uniquely determined and these are given below

$$w_{11} = \tfrac{1}{2}(f'k_2 - k_1 k_2), \quad w_{21} = \tfrac{1}{2}(-f' + k_1)\sqrt{-4 - k_2^2},$$
$$b_{11} = f'\sqrt{-4 - k_2^2}, \quad b_{21} = f'k_2, \quad b_{31} = 0. \tag{21}$$

• **Compatibility condition at the resonance $j = 2$.** Now substituting the values of w_{ij} and b_{ij} for $i = 1, 2, 3$ and $j = 0, 1$ into the recursion relations (14) for $j = 2$, we get the following set of linear equations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 & -k_2 \\ 2 & 0 & 0 & 0 & \sqrt{-4-k_2^2} \\ -\sqrt{-4-k_2^2} & -k_2 & k_2 & -\sqrt{-4-k_2^2} & 0 \end{pmatrix} \begin{pmatrix} w_{12} \\ w_{22} \\ b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \\ E_2 \end{pmatrix}, \tag{22}$$

where

$$A_2 = \tfrac{f'}{2}(k_1 - f')\sqrt{-4 - k_2^2}, \quad B_2 = -\tfrac{f'k_2}{2}(f' - k_1), \quad C_2 = -f'k_1 k_2,$$
$$D_2 = k_1 f'\sqrt{-4 - k_2^2}, \quad E_2 = \tfrac{(f'-k_1)}{2}(f'k_2^2 - k_2^2 - 4). \tag{23}$$

Since $j = 2$ is a resonance, the coefficient matrix to the left hand side of equation (22) vanishes. Hence, we have infinitely many solutions to above system of linear equations

with one arbitrary constant say $b_{32} = k_3$. Solving the system (22) with the help of (23) we get the following set of values of w_{12}, w_{22}, b_{12}, b_{22} and b_{32}.

$$w_{12} = \tfrac{1}{2}(f'k_1 - k_3)\sqrt{-4 - k_2^2}, \quad w_{22} = \tfrac{k_2}{2}(f'k_1 - k_3),$$
$$b_{12} = \tfrac{k_2}{2}\left[(f')^2 - k_3\right], \quad b_{22} = \tfrac{1}{2}\left[k_3 - (f')^2\right]\sqrt{-4 - k_2^2}, \quad b_{32} = k_3. \tag{24}$$

• **Compatibility condition at the resonance** $j = 3$. Now we check the compatibility condition at the resonant level $j = 3$. At this resonance level we observe that recurrence relations fail to collect the additional term Ra, which is one of the terms involved in the equations (3) due to the effects of rotation. So we substitute the equations (13) into the system of differential equations (5), then equating the like powers of τ with $j = 3$ we get the following system of nonhomogeneous linear equations

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 0 & -2 & 1 & 0 & -k_2 \\ 2 & 0 & 0 & 1 & \sqrt{-4 - k_2^2} \\ -\sqrt{-4 - k_2^2} & -k_2 & k_2 & -\sqrt{-4 - k_2^2} & 1 \end{pmatrix} \begin{pmatrix} w_{13} \\ w_{23} \\ b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \\ E_3 \end{pmatrix}, \tag{25}$$

where

$$A_3 = f'w_{22}, \quad B_2 = -f'w_{12}, \quad C_2 = w_{21}b_{32} - k_1b_{22},$$
$$D_2 = k_1b_{12} - w_{11}b_{32} + Ra, \quad E_2 = w_{11}b_{22} + w_{12}b_{21} - w_{21}b_{12} - w_{22}b_{11}. \tag{26}$$

After substituting the values of w_{ij} and b_{ij} for $i = 1, 2, 3$ and $j = 0, 1, 2$ in above equation and simplifying we see that the rank of coefficient matrix is 4, whereas the rank of augmented matrix is 5. This shows the inconsistency of the system (25). This is because of the term Ra, the Rayleigh number. Hence, we reduce the augmented matrix to its triangular form by use of elementary row transformation, which is given below

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 & \frac{f'k_2(-k_3 + f'k_1)}{2} \\ 0 & 1 & 0 & -\frac{1}{2k_2}\sqrt{-4 - k_2^2} & \frac{1}{k_1} & \frac{f'}{4}(k_3 - f'k_1)\sqrt{-4 - k_2^2} \\ 0 & 0 & 1 & -\frac{1}{k_2}\sqrt{-4 - k_2^2} & \frac{2}{k_2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & Ra \end{pmatrix}.$$

From the above triangular matrix we notice that the system (25) is consistent if and only if $Ra = 0$. Hence, the compatibility condition at resonance level $j = 3$ will hold only if $Ra = 0$. Now we assume that $Ra = 0$ (Note that with this assumption we have one more term in equations (3) due to the effect of rotation), so that the linear equations (25) can be solved and we see that there are infinitely many solutions with one independent variable. We found that the variable b_{23} to be independent. We assign the arbitrary value k_4 to b_{23} that is to say $b_{23} = k_4$. The solutions of system (25) are given below

$$w_{13} = -\tfrac{k_4}{2} + \tfrac{f'k_2}{4}(-k_3 + f'k_1), \quad w_{23} = \left(\tfrac{k_4}{2k_2} + \tfrac{f'k_3}{4} - (f')^2k_1\right)\sqrt{-4 - k_2^2},$$
$$b_{13} = \tfrac{k_4}{k_2}\sqrt{-4 - k_2^2}, \quad b_{23} = k_4, \quad b_{33} = 0. \tag{27}$$

• **Compatibility condition at the resonance** $j = 4$. At the resonant level $j = 3$ we notice that compatibility conditions hold only if $Ra = 0$ and there we assume that

$Ra = 0$. Now we proceed to check the compatibility conditions at the resonance $j = 4$. We substitute the equations (27), (24), (21) and (19) into the recurrence relations given by (14) for $j = 4$; and then equating the like powers of τ with $j = 3$ we get the following system of linear equations

$$
\begin{pmatrix}
3 & 0 & 0 & 1 & 0 \\
0 & 3 & -1 & 0 & 0 \\
0 & -2 & 2 & 0 & -k_2 \\
2 & 0 & 0 & 2 & \sqrt{-4 - k_2^2} \\
-\sqrt{-4 - k_2^2} & -k_2 & k_2 & -\sqrt{-4 - k_2^2} & 2
\end{pmatrix}
\begin{pmatrix}
w_{14} \\ w_{24} \\ b_{14} \\ b_{24} \\ b_{34}
\end{pmatrix}
=
\begin{pmatrix}
A_4 \\ B_4 \\ C_4 \\ D_4 \\ E_4
\end{pmatrix}, \qquad (28)
$$

where

$$
\begin{aligned}
A_4 &= \tfrac{f'}{4k_2}\Big(2k_4 + f'k_2[k_3 - f'k_1]\Big)\sqrt{-4 - k_2^2}, \\
B_4 &= -\tfrac{f'}{4}\big(-2k_4 - f'k_2k_3 + (f')^2 k_1 k_2\big), \\
C_4 &= -k_1 k_4 + \tfrac{k_2 k_3}{2}(-k_3 + f'k_1), \\
D_4 &= \tfrac{1}{2k_2}\big(2k_1 k_4 + k_2 k_3^2 - f'k_1 k_2 k_3\big)\sqrt{-4 - k_2^2}, \\
E_4 &= \tfrac{1}{2}k_4\big(f'k_2 - k_1 k_2\big) - \tfrac{1}{4}k_2^2\big((f')^2 - k_3\big)\big(-k_3 + f'k_1\big) \\
&+ \tfrac{f'k_2}{4}\big(-2k_4 - f'k_2 k_3 + (f')^2 k_1 k_2\big) \\
&+ \tfrac{1}{4}(4 + k_2^2)\Big[k_4(f' + k_1) - [(f')^2 + k_3](-k_3 + f'k_1) \\
&\quad + \tfrac{f'}{k_2}\big(2k_4 + f'k_2 k_3 - [f']^2 k_1 k_2\big)\Big].
\end{aligned} \qquad (29)
$$

We see that the linear system (28) is consistent and admits infinitely many solutions with one independent variable. Reducing the augmented matrix to its upper triangular form we found the variable b_{24} to be an independent variable. Let $b_{24} = k_5$ be an arbitrary constant. Solving the system (28) with this independent variable we get the following solutions

$$
\begin{aligned}
w_{14} &= -\tfrac{k_5}{3} - \tfrac{\sqrt{-4 - k_2^2}}{12 k_2}\Big[-(f')^2 k_2 k_3 - 2f'k_4 + (f')^3 k_1 k_2\Big], \\
w_{24} &= -\tfrac{k_2 k_5}{3\sqrt{-4 - k_2^2}} + \tfrac{(f')^2 k_2 k_3 + 2f'k_4 - (f')^3 k_1 k_2}{12}, \\
b_{14} &= \tfrac{k_2 k_5}{4 + k_2^2}, \quad b_{24} = k_5, \\
b_{34} &= -\tfrac{4k_5}{3\sqrt{-4 - k_2^2}} - \tfrac{(f')^2 k_2 k_3 - 3k_2 k_3^2 + 2f'k_4 - (f')^3 k_1 k_2 + 3f'k_1 k_2 k_3 - k_1 k_4}{6k_1}.
\end{aligned}
$$
$$(30)$$

• **Compatibility condition for** $j \geq 5$. From the equation (16) we observe that the matrix $M(j)$ for $j \geq 5$ is nonsingular matrix in this case of leading order coefficients as given by equations (19). So the system (14) with (15) in this case of leading order coefficients possesses unique solution. For the calculations of w_{ij} and b_{ij} for $i = 1, 2, 3$ and $j \geq 5$, we substitute (30), (27), (24), (21) into the recursion relations (14) and (15) for passing successively $j = 5, 6, \ldots$. In this fashion we find all the coefficients are uniquely determined for $j \geq 5$.

As we notice the compatibility conditions hold provided that $Ra = 0$. Hence, the system (4) passes the Painlevé test implying the complete integrability of the system.

So we can write the general solution of the system (4). In that respect we substitute all these coefficients into the Laurent's series expansions as given in equations (13). The general solution of system (4) in this case of leading order coefficients consists of five arbitrary constants k_1, k_2, k_3, k_4, k_5 and an arbitrary position of t_0 singularity and the required solution is as given below

$$
\begin{aligned}
w_1(t) &= \sqrt{-4-k_2^2}\,\tau^{-1} + \tfrac{1}{2}(f'k_2 - k_1 k_2) + \left[\tfrac{1}{2}(f'k_1 - k_3)\sqrt{-4-k_2^2}\right]\tau \\
&\quad + \left[-\tfrac{k_4}{2} + \tfrac{f'k_2}{4}(-k_3 + f'k_1)\right]\tau^2 \\
&\quad + \left[-\tfrac{k_5}{3} - \tfrac{\sqrt{-4-k_2^2}}{12k_2}\left(-(f')^2 k_2 k_3 - 2f'k_4 + (f')^3 k_1 k_2\right)\right]\tau^3 \\
&\quad + \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\
w_2(t) &= k_2\tau^{-1} + \left[\tfrac{1}{2}(-f' + k_1)\sqrt{-4-k_2^2}\right] + \left[\tfrac{k_2}{2}(f'k_1 - k_3)\right]\tau \\
&\quad + \left[\left(\tfrac{k_4}{2k_2} + \tfrac{f'k_3}{4} - (f')^2 k_1\right)\sqrt{-4-k_2^2}\right]\tau^2 \\
&\quad + \left[-\frac{k_2 k_5}{3\sqrt{-4-k_2^2}} + \frac{(f')^2 k_2 k_3 + 2f'k_4 - (f')^3 k_1 k_2}{12}\right]\tau^3 \\
&\quad + \sum_{j=5}^{\infty} w_{2j}\tau^{j-1}, \\
w_3(t) &= k_1 \ (\text{arbitrary constant}), \\
b_1(t) &= -k_2\tau^{-2} + \left[f'\sqrt{-4-k_2^2}\right]\tau^{-1} + \tfrac{k_2}{2}\left[(f')^2 - k_3\right] + \left[\tfrac{k_4}{k_2}\sqrt{-4-k_2^2}\right]\tau^1 \\
&\quad + \left[\frac{k_2 k_5}{4+k_2^2}\right]\tau^2 + \sum_{j=5}^{\infty} b_{1j}\tau^{j-2}, \\
b_2(t) &= \sqrt{-4-k_1^2}\,\tau^{-2} + f'k_2\tau^{-1} + \left[\tfrac{1}{2}\left((k_3 - (f')^2)\sqrt{-4-k_2^2}\right)\right] + k_4\tau \\
&\quad + k_5\tau^2 + \sum_{j=5}^{\infty} b_{2j}\tau^{j-2}, \\
b_3(t) &= 2\tau^{-2} + k_3 + \Bigg[-\frac{4k_5}{3\sqrt{-4-k_2^2}} \\
&\quad - \frac{(f')^2 k_2 k_3 - 3k_2 k_3^2 + 2f'k_4 - (f')^3 k_1 k_2 + 3f'k_1 k_2 k_3 - k_1 k_4}{6k_1}\Bigg]\tau^2 \\
&\quad + \sum_{j=1}^{\infty} b_{3j}\tau^{j-2}.
\end{aligned}
\tag{31}
$$

Equations (31) contain five arbitrary constants k_1, k_2, k_3, k_4, k_5 and arbitrary position of t_0; these equations satisfy the system of differential equations (3) for $Ra = 0$. Hence, in the present case of leading order coefficient, equations (31) represent the general solution of (3). The convergence of such series solutions is guaranteed by Kichenassamy and Littman [9, 10]. And it seems that the solution contains the movable pole type singularity. Similar kind of steps are involved for another branch of leading order coefficients. In the following subparagraphs we listed these calculations.

- **Case 2:** Consider the leading order coefficients

$$w_{10} = -\sqrt{-4 - k_2^2}, \quad w_{20} = k_2 \text{ (arbitrary constant)},$$

$$b_{10} = -k_2, \quad b_{20} = -\sqrt{-4 - k_2^2}, \quad b_{30} = 2. \tag{32}$$

Using the same approach as in the previous case we have determined the expansion coefficients of (13) for $j = 1, j = 2, j = 3$, and $j = 4$ which are listed below.

- **Leading order coefficients at $j = 1$:**

As we notice already $j = 1$ is not a resonance and hence, in this branch of leading order coefficients for $j = 1$ we can determine w_{ij} and b_{ij} uniquely for $i = 1, 2, 3 \quad j = 1$ as given below.

$$w_{11} = \frac{f'k_2 - k_1 k_2}{2}, \quad w_{21} = \frac{f' - k_1}{2}\sqrt{-4 - k_2^2},$$

$$b_{11} = -f'\sqrt{-4 - k_2^2}, \quad b_{21} = f'k_2, \quad b_{31} = 0. \tag{33}$$

- **At the resonance $j = 2$:** At this resonant level $j = 2$, we find that one of the coefficients is independent. Let b_{32} be independent. Assign the value to $b_{32} = k_3$ and consequently other expansion coefficients for $j = 2$ are given below

$$w_{12} = \frac{k_3 - f'k_1}{2}\sqrt{-4 - k_2^2}, \quad w_{22} = \frac{k_2}{2}(k_1 f' - k_3),$$

$$b_{12} = \frac{k_2}{2}[(f')^2 - k_3], \quad b_{22} = \frac{((f')^2 - k_3)}{2}\sqrt{-4 - k_2^2}, \quad b_{32} = k_3. \tag{34}$$

- **At the resonance $j = 3$:** As we noticed in previous case at this resonant level $j = 3$ is that system of linear equations (25) is inconsistent unless $Ra = 0$. Similarly in this case we also notice that a system of linear equations is inconsistent unless $Ra = 0$. Again assuming that $Ra = 0$, we determine the expansion coefficients with one independent variable. Let b_{23} be independent. Assign $b_{23} = k_4$ and other expansion coefficients for $j = 3$ are given below

$$w_{13} = \frac{1}{4}[-2k_4 + f'k_2(f'k_1 - k_3)], \quad w_{23} = \frac{\sqrt{-4 - k_2^2}}{4}\left[\frac{-2k_4}{k_2} - f'k_3 + (f')^2 k_1\right],$$

$$b_{13} = \frac{-k_4\sqrt{-4 - k_2^2}}{k_2}, \quad b_{23} = k_4, \quad b_{33} = 0. \tag{35}$$

At the resonance $j = 4$: Also, at this resonant level $j = 4$ we found that one of the expansion coefficients is independent. Let b_{24} be independent and assign the arbitrary value say $b_{24} = k_5$. Other expansion coefficients are as listed below

$$w_{14} = \frac{-k_4}{3} - \frac{\sqrt{-4 - k_2^2}}{12k_2}\left[(f')^2 k_2 k_3 + 2f'k_4 - (f')^3 k_1 k_2\right],$$

$$w_{24} = \frac{k_2 k_5}{3\sqrt{-4 - k_2^2}} + \frac{f'}{12}[2k_4 + f'k_2 k_3 - (f')^2 k_1 k_2],$$

$$b_{14} = \frac{k_2 k_5}{\sqrt{-4 - k_2^2}},$$

$$b_{24} = k_5,$$

$$b_{34} = \frac{4k_5}{3\sqrt{-4 - k_2^2}} - \frac{1}{6k_2}\left[k_2 k_3\big((f')^2 - 3k_3\big) + 2k_4(f' - 3k_1) + f'k_1 k_2(3k_3 - f'^2)\right]. \tag{36}$$

For $j \geq 5$**:** Plugging the equations (36), (35), (34), (33) and (32) into the recursion relations (14), we can uniquely determine the expansion coefficients w_{ij} and b_{ij} for $j \geq 5$. The general solution of system (4) in this case of leading order is as given below

$$
\begin{aligned}
w_1(t) &= -\sqrt{-4 - k_2^2}\,\tau^{-1} + \frac{f'k_2 - k_2k_1}{2} + \frac{\sqrt{-4 - k_2^2}}{2}(k_3 - f'k_1)\tau \\
&\quad + \left(-\frac{k_4}{2} + \frac{f'k_2}{4}\left[-k_3 + f'k_1\right]\right)\tau^2 \\
&\quad - \left(\frac{k_5}{3} + \frac{f'\sqrt{-4 - k_2^2}}{12k_1}\left[f'k_2k_3 + 2k_4 - (f')^2k_2k_1\right]\right)\tau^3 \\
&\quad + \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\
w_2(t) &= k_2\tau^{-1} + \left(\frac{\sqrt{-4 - k_2^2}}{2}\left[f' - k_1\right]\right) + \frac{-k_2k_3 + f'k_1k_2}{2}\tau \\
&\quad + \frac{\sqrt{-4 - k_2^2}}{4}\left(\frac{-2k_4}{k_2} - k_3f' + (f')^2k_1\right)\tau^2 \\
&\quad + \left(\frac{k_2k_5}{3\sqrt{-4 - k_2^2}} + \frac{f'}{12}\left[f'k_2k_3 + 2k_4 - (f')^2k_2k_1\right]\right)\tau^3 + \sum_{j=5}^{\infty} w_{1j}\tau^{j-1}, \\
w_3(t) &= k_1, \\
b_1(t) &= -k_2\tau^{-2} - \left(f'\sqrt{-4 - k_2^2}\right)\tau^{-1} - \frac{k_2k_3 - (f')^2k_2}{2} - \frac{k_4\sqrt{-4 - k_2^2}}{k_2}\tau \\
&\quad + \frac{k_2k_5}{\sqrt{-4 - k_2^2}}\tau^2 + \sum_{j=5}^{\infty} b_{1j}\tau^{j-2}, \\
b_2(t) &= -\sqrt{-4 - k_2^2}\,\tau^{-2} + f'k_2\tau^{-1} + \frac{\sqrt{-4 - k_2^2}}{2}\left(-k_3 + (f')^2\right) + k_4\tau + k_5\tau^2 \\
&\quad + \sum_{j=5}^{\infty} b_{2j}\tau^{j-2}, \\
b_3(t) &= 2\tau^{-2} + k_3 + \left[\frac{4k_5}{3\sqrt{-4 - k_2^2}}\right. \\
&\quad \left. -\frac{(f')^2k_2k_3 - 3k_2k_3^2 + 2f'k_4 - (f')^3k_1k_2 + 3f'k_1k_2k_3 - k_1k_4}{6k_1}\right]\tau^2 \\
&\quad + \sum_{j=1}^{\infty} b_{3j}\tau^{j-2}.
\end{aligned}
$$

$$\tag{37}$$

4 Examples

In this section we present two systems of ODEs that are in similar analog with our system (3) for $Ra = 0$. Hence, these systems will have the singular solutions and these solutions

will be in similar nature as we have obtained so far.

Now consider the equations for the motion under gravity of a rigid body about a fixed point

$$
\begin{aligned}
\frac{d\mathbf{l}}{dt} &= \mathbf{l} \times \omega + \mathbf{c} \times \mathbf{g}, \\
\frac{d\mathbf{g}}{dt} &= \mathbf{g} \times \omega; \quad \mathbf{l} = \mathbf{I}\omega.
\end{aligned}
\tag{38}
$$

In the above equations, $\mathbf{l}$ and ω are respectively the angular momentum and angular velocity of the body, $\mathbf{g}$ is the gravitational acceleration with respective the moving frame. The vector $\mathbf{c}$ is the center of mass and inertia tensor $\mathbf{I}$ are both constants. The explicit details about the system (38) have been discussed by Andrew Hone [11]. This system will be as similar to our system (3) for $Ra = 0$ and assigning the value $f' = 0$. So that the singular solutions of a system (38) will be obtained in similar fashion as we discussed above.

In their paper Julien et al [12] employ a multiscale expansion in both time and space. Specifically, they define the Ekman number $E \equiv \nu/2\Omega d^2$, where ν is kinematic viscosity, d is typical length scale, and $\mathbf{\Omega} \equiv \Omega\hat{\mathbf{z}}$ (which is equivalent to $f\hat{\mathbf{e_3}}$ in our equations (1)) is the rotation vector, and treat E as a small parameter. With these assumptions and in the absence of stratification the incompressible Navier-Stokes equations then become

$$
\begin{aligned}
\frac{D\mathbf{u}}{Dt} + \hat{\mathbf{\Omega}} \times \mathbf{u} &= -\nabla\pi + E\nabla^2\mathbf{u} + \mathbf{f}, \\
\nabla \cdot \mathbf{u} &= 0,
\end{aligned}
\tag{39}
$$

where $\mathbf{f}$ is an unspecified body force and π is the pressure. Further Julien et all [12] present their results for the specific case of rotating convection for which they took $\mathbf{f} = (Ra/\sigma)E^2T\hat{\mathbf{z}}$ and (39) were supplemented with the energy equation

$$
\sigma\frac{DT}{Dt} = E\nabla^2 T.
\tag{40}
$$

In equation (40), T is the temperature, Ra is the Rayleigh number, and $\sigma = \nu/\kappa$ is the Prandtl number; κ is the thermal diffusivity.

Here we observe that if we take $E \equiv 0$ and unspecified body forces to be equal to zero, and going through the local analysis as Desale and Sharma [7] deploy it to a similar equations. We can have a system of ODEs which is equivalent to system (3). Hence for $Ra = 0$ the singular solutions in this case will be in similar nature with the solutions which we have investigated in Section 3.

5   Conclusion

Now we conclude that the system of ODE reduction of rotating Stratified Boussinesq Equations (3) is completely integrable (in the light of ARS conjecture). There are several possible cases of principle dominant balance cases among these the system of ODEs (3) admits the singular solution only in the case of (7). There are two possible branches of leading orders and in both cases of leading orders system (3) passes the strong Painlevé test only if the Rayleigh number $Ra = 0$. The general solutions are given by(31) and (37). We found that these solutions are in complex domain and contain the movable pole type singularity at $t = t_0$. In Section 4 we illustrate the systems which also exhibit similar kind of solutions so far we obtained in Section 3.

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Design of Decoupling Nonlinear Controllers for Fuzzy Systems

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Abstract: The use of linear matrix inequalities and Lyapunov functions is a powerful and commonplace tool for Takagi–Sugeno fuzzy controlled system analysis and synthesis. This paper shows how to split and handle the coupling terms arising from the existence of different input matrices in the subsystems. Then, a method is proposed which allows to synthesize, for a sufficient number of subsystems, the local gains of a nonlinear parallel distributed controller. It is shown that the controller gains depend on the values of the input matrices and of the membership functions, and are thus able to relax classical stability conditions by embedding information on the fuzzy premises.

Keywords: *fuzzy control; stability; nonlinear control; linear matrix inequalities.*

Mathematics Subject Classification (2000): 93D42, 93D15, 93D21.

1 Introduction

The Takagi–Sugeno fuzzy state-space model allows to describe a nonlinear system using a set of fuzzy rules for which the consequents are a set of linear models, which are smoothly connected by fuzzy membership functions [1]. An intuitive approach to the control of T-S fuzzy systems consists of designing a fuzzy controller which shares the same fuzzy sets with the fuzzy model in the premise parts. In this parallel distributed compensation method (PDC), each control rule is distributively designed for the corresponding rule of a T-S fuzzy model [2].

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Most works considering the design of controlled Takagi–Sugeno fuzzy systems lead to express stability conditions and gain synthesis as a set of linear matrix inequalities (LMIs) which can be solved via efficient semi-definite programming optimization software [3]. These works can be extended for very complex systems such as time-delay nonlinear systems modelling and control [6]. However, very few works will consider the relevance of a nonlinear PDC controller. Original results dealing with the search for a common quadratic Lyapunov functions (CQLF) are known to be quite conservative, and, as a result, a number of methods have been proposed to relax standard stability conditions [4, 7, 8, 9], and new tools such as piecewise quadratic Lyapunov functions or fuzzy Lyapunov functions have been introduced (e.g. [10]). Extended results have allowed to consider bounds and/or shapes of the premises' membership functions considering PDC [11, 12] or non PDC [13, 14, 15] controllers (see [24] for a summary of conservativeness issues). An extension of these results to fuzzy nonlinear systems can be done using vector norm approaches [5] with the drawback of adding more conservatism.

A main difficulty to the synthesis of fuzzy controlled systems lies in the combination of closed-loop subsystems which does not result into a parallel distribution of the individual closed-loop subsystems, because of additional coupling terms. These coupling terms result from the linkage of the local subsystems to the other subsystems' local controllers, in particular when input matrices are not identical. Some works, e.g. [22, 23] allow to handle subsystems with different matrices, and a descriptor formulation along with a non quadratic Lyapunov function has been proposed [21] to decouple input and gain matrices. The whole coupling term has also been represented explicitly by a product of matrices involving a single uncertain matrix with a norm smaller than one, leading to a global Riccati equation (e.g. [2]). Finding a global bounding matrix for the coupling term is often not easy to work out, because these terms depend on the membership functions and on the control gains themselves, which prevent the use of the method for control synthesis. The exact cancelation of coupling terms has been tackled explicitly only for large-scale systems [17].

In this paper, it is shown that the closed-loop T-S fuzzy system under PDC control is the sum of distributed closed-loop fuzzy systems and of a coupling term. This coupling term is rewritten as a sum of pairwise products involving input matrices and control gains. A first method is proposed to design fuzzy control gains which attenuate the coupling effect for any of the closed-loop subsystems, considering a common CQLF. This is done by considering bounds on the coupling term, and, when a priori limitations are given for control gains, the stability conditions are resumed to a set of independent Lyapunov equations. As this method still presents high degrees of conservatism, it is shown that when the number of subsystems is large enough, the coupling terms can be canceled by proposing nonlinear control gains for the PDC control structure.

2   Analysis of Fuzzy Systems Under PDC Control

2.1   Closed-loop T-S fuzzy systems decomposition

The fuzzy model proposed by Takagi and Sugeno consists of a set of r fuzzy IF...THEN rules for which the consequents are linear state-space models:

Plant Rule R_i: IF z_1 IS M_{i1} AND $\cdots$ AND z_g IS M_{ig} THEN $\dot{x} = A_i x + B_i u$;
where $x(t)$, $u(t)$ are respectively the state and input vectors, $z_i(t)$, M_{ij} are the premise variables and the corresponding fuzzy models.

The final output of the fuzzy system is inferred as follows:

$$\dot{x} = \sum_{i=1}^{r} \mu_i \big(A_i x + B_i u\big), \tag{1}$$

where $\mu_i = \frac{\omega_i}{\sum_{i=1}^{r} \omega_i}$ and ω_i is the grade of membership function of the rule R_i.

For every subsystem S_i, a local controller can be defined as $u = K_i x$, where K_i is a control gain. The rules which describe the fuzzy controller share the same premises as the fuzzy models, hence distributing the local controllers into the global controllers according to their systems' weights. In general, the controllers are supposed to be linear, but, in this study, it will be shown that nonlinear consequents might be preferred.

Controller C_i: IF z_1 IS M_{i1} AND $\cdots$ AND z_g IS M_{ig} THEN $u = K_i x$, yielding:

$$u = \sum_{i=1}^{r} \mu_i K_i x. \tag{2}$$

Lemma 2.1 *Let the system* $\dot{x} = \sum_{i=1}^{r} \mu_i(A_i x + B_i u)$ *with PDC control* $u = \sum_{i=1}^{r} \mu_i K_i x$ *such that* $A_i + B_i K_i = G_i$ *and* $\sum_{i=1}^{r} \mu_i \leq 1$, $\mu_i \geq 0$. *The closed-loop system is:*

$$\dot{x} = \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j G_i + \sum_{i=1}^{r} \mu_i A_i \Big(1 - \sum_{j=1}^{r} \mu_j\Big) + \sum_{i,j=1, j \neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x. \tag{3}$$

Proof One has

$$\dot{x} = \sum_{i=1}^{r} \mu_i \left(A_i x + B_i \sum_{j=1}^{r} \mu_j K_j x \right) = \sum_{i=1}^{r} \mu_i \left(A_i + \mu_i B_i K_i + B_i \sum_{j=1, j\neq i}^{r} \mu_j K_j \right) x.$$

$$\dot{x} = \sum_{i=1}^{r} \left[\mu_i^2 G_i + \mu_i A_i (1 - \mu_i) + \mu_i B_i \sum_{j=1, j\neq i}^{r} \mu_j K_j \right] x.$$

Moreover,

$$\sum_{i=1}^{r} \mu_i B_i \sum_{j=1, j\neq i}^{r} \mu_j K_j = \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \mu_i \mu_j (G_i - A_i)$$

$$+ \sum_{i=1}^{r} \mu_i B_i \sum_{j=1, j\neq i}^{r} \mu_j K_j - \sum_{i=1}^{r} \sum_{j=1, j\neq i}^{r} \mu_i \mu_j B_i K_i.$$

In this equation, one can rearrange the two last sums into a sum of pairwise terms:

$$\sum_{i,j=1, j\neq i}^{r} \mu_j B_j \mu_i K_i + \mu_i B_i \mu_j K_j - \mu_i \mu_j B_i K_i - \mu_j \mu_i B_j K_j$$

$$= \sum_{i,j=1, j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i).$$

Hence,

$$\sum_{i=1}^{r}\mu_i B_i \sum_{j=1,j\neq i}^{r}\mu_j K_j = \sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\mu_i\mu_j(G_i - A_i) + \sum_{i,j=1,j\neq i}^{r}\mu_i\mu_j(B_i - B_j)(K_j - K_i).$$

One has now:

$$\dot{x} = \left(\sum_{i=1}^{r}\left(\mu_i^2 G_i + \mu_i A_i(1-\mu_i)\right) + \sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\mu_i\mu_j(G_i - A_i)\right.$$
$$\left. + \sum_{i,j=1,j\neq i}^{r}\mu_i\mu_j(B_i - B_j)(K_j - K_i)\right)x.$$

As $\sum_{i=1}^{r}\mu_i^2 G_i + \sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\mu_i\mu_j G_i = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j G_i$, and

$$\sum_{i=1}^{r}\mu_i A_i(1-\mu_i) - \sum_{i=1}^{r}\sum_{j=1,j\neq i}^{r}\mu_i\mu_j A_i = \sum_{i=1}^{r}\mu_i A_i\left(1 - \sum_{j=1}^{r}\mu_j\right),$$

we demonstrate the final result:

$$\dot{x} = \left(\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j G_i + \sum_{i=1}^{r}\mu_i A_i\left(1 - \sum_{j=1}^{r}\mu_j\right) + \sum_{i,j=1,j\neq i}^{r}\mu_i\mu_j(B_i - B_j)(K_j - K_i)\right)x.$$

2.2 Specific cases

Note that in the Lemma, the formula could also be valid for $\sum_{i=1}^{r}\mu_i \leq 1$. One can derive more specific cases.

Polytopic systems: When $\sum_{i=1}^{r}\mu_i = 1$, formula (3) is reduced to:

$$\dot{x} = \left(\sum_{i=1}^{r}\mu_i G_i + \sum_{i,j=1,j\neq i}^{r}\mu_i\mu_j(B_i - B_j)(K_j - K_i)\right)x.$$

Two-subsystems: The coupling term is now $\mu_1\mu_2(B_1 - B_2)(K_2 - K_1)$. In this case, the deviation from the polytopic closed-loop system only depends on the difference between gains K_2 and K_1, and this only degree of freedom is a limitation to the cancellation of the coupling term and of the choice of the local controllers.

Common input matrix: Suppose that $\forall i, B_i = B$, and $\sum_{i=1}^{r}\mu_i = 1$, then formula (3) is reduced to:

$$\dot{x} = \left(\sum_{i=1}^{r}\mu_i G_i\right)x.$$

As a remark, one can say that, when the system exhibits a common input matrix, the closed-loop system behavior is a polytope of closed-loop local systems, and, thus, the coupling terms vanishes. The analysis of the whole closed-loop system can be handled easily.

Proportional input matrices: Suppose that $\forall i, B_i = \alpha_i B$, where $\alpha_i \in \mathbb{R}$, then the closed-loop subsystem is

$$\dot{x} = \left(\sum_{i=1}^{r} \mu_i G_i + B \sum_{i,j=1, j \neq i}^{r} \mu_i \mu_j (\alpha_i - \alpha_j)(K_j - K_i) \right) x.$$

This case arises often in Takagi-Sugeno modelling, and it can be seen that the coupling term is strongly dependent of the membership functions and the gains amplitude.

2.3   Global stability verification

Theorem 2.1 [2] *The system $\dot{x} = \sum_{i=1}^{r} \mu_i (A_i x + B_i u)$, under PDC control $u = \sum_{i=1}^{r} \mu_i K_i x$, such that $A_i + B_i K_i = G_i$ and $A_i + B_i K_j = G_{ij}$, is stable if there exists a common positive definite matrix P such that:*

$$\begin{aligned} &\forall i = 1, \cdots, r, PG_i + G_i^T P \prec 0, \\ &\forall i < j, P\left(G_{ij} + G_{ji}\right) + \left(G_{ij} + G_{ji}\right)^T P \prec 0. \end{aligned} \tag{4}$$

Remark 2.1 Theorem (2.1) allows the determination of both the Lyapunov matrix and the controller gain, using a change of variable $N_i = K_i P^{-1}$, when being replaced in the stability conditions, leads to a set of LMIs in N_i and in P, the PDC controller being provided by $K_i = N_i P$. The existence of a common quadratic Lyapunov function is only a sufficient stability condition, and, moreover, the conditions of Theorem (2.1) are independent of the membership functions, leading to conservative results. Coupling terms are not accounted for, since any of local subsystems i under any local controller $u = K_j x$, where $j \neq i$, should be performing, whereas it cannot be expected that a system with a controller designed for another plant has necessarily a "good" behavior. Hence, the PDC controller is designed according to the "worst" case among the pairs {Plant i, Controller j}.

Corollary 2.1 *Suppose that $\forall i, j, B_i \neq B_j$ iff $\mu_i \mu_j \equiv 0$, then the closed-loop system in Theorem (2.1) is stable if there exists a common positive definite matrix P such that:*

$$\forall i = 1, \cdots, r, PG_i + G_i^T P \prec 0.$$

This corollary shows that, when there exists a common input matrix, the closed-loop systems are uncoupled. What is more interesting is that, within the coupling term, the contributions involving different input matrices can be canceled when their corresponding membership functions do not overlap, i.e. their product is identically zero.

3   Coupling Terms Attenuation

Theorem 3.1 [16] *First, we consider the linear uncertain system for which $\dot{x} = A + \sum_{i=1}^{r} D_i \delta_i E_i, \|\delta_i\| \leq 1$, and the elements of the time-varying matrices δ_i are Lebesgue measurable. Then the positive-definite matrix P is a common Lyapunov matrix for this system if there exists r positive scalars η_i such that:*

$$PA + A^T P + \sum_{i=1}^{r} \eta_i P D_i D_i^T P + \eta_i^{-1} E_i^T E_i \prec 0,$$

or, as a specific case:

$$PA + A^T P + \sum_{i=1}^{r} PD_i D_i^T P + E_i^T E_i \prec 0. \tag{5}$$

Remark 3.1 This Theorem was applied first by Tanaka et al. [2] and then by numerous authors to the whole coupling term. Note that some authors [19, 18] introduce a DδE component within the consequent part. Whereas this method provides for a rather non-conservative solution, it is clear that finding individual uncertain matrices might be a tedious task, because the rate of variation and thus the bounds of the uncertain matrix depend on the control gains themselves. It can thus be applied to analyze an existing solution (when the gains are fixed a priori) but not for gain synthesis considering models/controllers coupling. The following theorem proposes a different application of this method to every individual component of the coupling term.

Theorem 3.2 *Consider the system $\dot{x} = \sum_{i=1}^{r} \mu_i(A_i x + B_i u)$, under PDC controller:*

$$\dot{x} = \left(\sum_{i=1}^{r} \mu_i G_i + \sum_{i,j=1, j \neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x.$$

Let us suppose that: $\forall i$, there exists b_i such that:

$$\sum_{B_i \neq B_j} \mu_j (B_j - B_i) = b_i \delta_i, \text{ where } \|\delta_i\| \leq 1.$$

The matrices δ_i thus depend on membership functions μ_i and other input matrices μ_j and B_j; as μ_j may vary with time, δ_i is a matrix which may vary with time or with the state space x.
The closed-loop system is quadratically stable if:

$$\forall i = 1, \cdots, r, PG_i + G_i^T P + Pb_i b_i^T P + K_i^T K_i \prec 0. \tag{6}$$

This can be turned into:

$$\forall i = 1, \cdots, r, \begin{pmatrix} PG_i + G_i^T P & Pb_i & K_i^T \\ b_i^T P & -I & 0 \\ K_i & 0 & -I \end{pmatrix} \prec 0. \tag{7}$$

Proof

$$\dot{x} = \left(\sum_{i=1}^{r} \mu_i G_i + \sum_{i,j=1, j \neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x$$

$$= \left(\sum_{i=1}^{r} \mu_i \sum_{j=1, j \neq i}^{r} (G_i + \mu_j (B_i - B_j) K_i) \right) x.$$

One has now: $\displaystyle\sum_{j=1, j \neq i}^{r} (G_i + \mu_j (B_i - B_j) K_i) = G_i + b_i \delta_i K_i$, and one can apply the Theorem 3.1.

Remark 3.2 Results involving a CQLF are known to be conservative. However, other Lyapunov functions can be searched for the TS system represented with an explicit coupling term, e.g. piecewise or fuzzy Lyapunov functions. However, this theorem wants to show that, taking explicitly the coupling term into account, one may relax standard or existing conditions for a given method.

Uncertain matrices δ_i do not depend anymore on the control gains but only on input matrices and membership functions which are supposed to be known as a part of the fuzzy model representation. Their determination is thus quite easy and the membership functions are indeed embedded in the control synthesis. Of course, it is assumed that such matrices exist. Note also that the corresponding i Riccati equations in (6) are decoupled, i.e. the ith equation only depends on the ith control gain, the influence of the other subsystems are merged into the matrix $b_i\delta_i$. The following corollary is a simplified condition of equation (7).

Corollary 3.1 *Let us suppose that:*

$$\forall\, i = 1, \cdots, r,\ \exists\, Q_i \succ 0,\ K_i^T K_i - Q_i \prec 0.$$

Then, condition (7) can be expressed as:

$$\exists\, P \prec 0, \forall\, i = 1, \cdots, r, PG_i + G_i^T P + Q_i' \prec 0, \tag{8}$$

where $Q_i' = Pb_i b_i^T P + Q_i$, with $K_i^T K_i - Q_i \prec 0$, which can be turned into:

$$\forall\, i = 1, \cdots, r, \left\{ \begin{array}{l} \begin{pmatrix} PG_i + G_i^T P + Q_i' & Pb_i \\ b_i^T P & -I \end{pmatrix} \prec 0, \\[2mm] K_i^T K_i - Q_i \prec 0. \end{array} \right.$$

The Corollary simply reduces the search for a common Lyapunov matrix to a series of r Lyapunov equations and thus r LMIs. This is really an improvement to other methods because, now, control gains can nearly be selected independently without the need of taking care of coupling terms, at the expense of gain limitation. The synthesis gains are now completely uncoupled, the interdependence being lumped into the matrices b_i; in general, matrices b_i can be obtained from simple membership functions analysis. The following corollary focuses on the specific (and commonly encountered) case for which input matrices are proportional, and shows that the computation of matrices b_i is quite direct.

Corollary 3.2 *Suppose that the input matrices are proportional, i.e. $\forall i, B_i = \alpha_i B$, where $\alpha_i \in \mathbb{R}$, then the bounding matrices in Theorem 3.2 are given by:*

$$b_i = B \max\left(\sum_{j=1, j\neq i}^{r} \mu_i \mu_j (\alpha_j - \alpha_i) \right).$$

4 Coupling Terms Exact Compensation

In the previous section, a method has been proposed to choose control gains by balancing the effect of coupling terms resulting from other subcontrollers. The problem is that a

CQLF is still needed and that the coupling still exixsts, still yielding conservative solutions. Of course, a high number of subsystems increases the size of the set of Lyapunov equations but offers more degrees of freedom. It will be shown that, when these degrees of freedom are numerous enough, they can be used to cancel explicitly the coupling terms.

Proposition 4.1 *Let the system:*

$$\dot{x} = \left(\sum_{i=1}^{r} \mu_i G_i + \sum_{i,j=1,j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) \right) x,$$

and let $n = dim(x)$. *Let us suppose also that* $rank[B_1 \cdots B_n] = n$ *and* $\mu_i \mu_j \not\equiv 0$. *There exists a nonlinear PDC controller* $K(\mu_i)$, *such that* $\displaystyle\sum_{i,j=1,j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) = 0$ *and* $\exists i, j, K_i \neq K_j$, *only if* $r > n + 1$.

Proof There exists of course a trivial solution $K_i = K, \forall i$. The system has a solution different from this trivial solution, i.e. a true nonlinear PDC iff the system

$$\sum_{i,j=1,j\neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) = 0$$

is compatible. The weight corresponding to control gain K_i is:

$$w_i = \sum_{j=1,j\neq i}^{r} \mu_i \mu_j (B_i - B_j).$$

One can notice that $\sum_{i=1}^{r} w_i = 0$. Hence, there is a solution $K_i \not\equiv 0$ only if $r > n + 1$.

In this case, the nonlinear PDC gain is membership-function dependent and non linear; one has to check that all the subsystems share a CQLF – or some other common Lyapunov function – which can however be more complicated. The workout will be shown in the example section.

5   Examples

5.1   Example 1

Let us take the following 3 systems:

$$A_1 = \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix}, B_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$A_3 = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, B_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$

along with local gains: $K_1 = (\ 2 \ \ 1 \ ), K_2 = (\ -2 \ \ 1 \ ), K_3 = (\ 2 \ \ 0 \ ).$

In these examples, gains were fixed a priori. The grades of membership corresponding to systems 1, 2 and 3 are: $\omega_1 = z$, $\omega_2 = 1 - z$ and $\omega_3 = z$ where $z \in [-1 \cdots 1]$. For every subsystem i, it is quite easy to compute the matrices b_i such that

$$\sum_{B_i \neq B_j} \mu_i (B_j - B_i) = b_i \delta_i$$

since the upper bound depends on the fuzzy variable z. One finds: $b_1^T = (\; 1 \quad 0.25 \;)$, $b_2^T = (\; 0.75 \quad 0.25 \;)$, $b_3^T = (\; 1 \quad 1 \;)$.

The application of Theorem 3.2 allows to find a common positive definite matrix $P = \begin{pmatrix} 1.28 & -0.37 \\ -0.37 & 0.87 \end{pmatrix}$ whereas it is impossible to find one by the classical method; it is easy to check that the gain K_2 is unable to stabilize matrix A_1 and the converse for K_1 and A_2. It is quite interesting to note that the result is quite tied to the value of the matrices b_i. When all other variables keep the same values, but $b_2^T = (\; 1 \quad 1 \;)$, then Theorem 3.2 is no more applicable because a positive definite CQLF cannot be found. Thus, Theorem 3.2 is able to relax stability conditions, depending strongly on the membership functions and input matrices values. Yet, results may remain conservative with respect to other methods, but, such methods as piecewise Lyapunov or fuzzy functions can also be applied (with further insight) to the TS fuzzy system with coupling terms.

Suppose that, now, we add the following subsystem

$$A_4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, B_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

along with the grade of membership $\omega_4 = (1 - z)/2$. It is possible, in this case, to find a nonlinear PDC controller such that

$$\sum_{i,j=1, j \neq i}^{r} \mu_i \mu_j (B_i - B_j)(K_j - K_i) = 0.$$

Indeed, the solution of this system of equations is:

$$K_1 = (\; k_{31} + (1 - z)k_{41} \quad k_{32} + (1 - z)k_{42} \;)^T, K_2 = K_1,$$

$$K_3 = (\; k_{31} \quad k_{32} \;)^T, K_4 = (\; k_{41} \quad k_{42} \;)^T.$$

In this case, one only has to ensure that the local closed-loop controlled systems share a CQLF. If $A_i + B_i K_i(z) = G_i(z)$, one has to check that there exists a common positive definite matrix P such that $\forall i = 1 \cdots r, PG_i(z) + G_i(z)^T P \prec 0$, which is easy to solve since the closed-loop matrices are affine in z.

5.2 Example 2

Consider the model of a stirred tank reactor:

$$\dot{C}_A = \frac{q}{V}(C_{Af} - C_A) - k_0 C_A e^{-\frac{E}{RT}},$$

$$\dot{T} = \frac{q}{V}(T_f - T) - \frac{\Delta H k_0}{\rho C_p} C_A e^{-\frac{E}{RT}} + \frac{\rho_c C_{pc}}{\rho C_p V} q_c (1 - e^{-\frac{hA}{\rho_c C_{pc} q_c}})(T_{cf} - T), \tag{9}$$

where q, q_c are the process and coolant flowrates, C_A and C_{Af} are the ouput and feed concentrations, T, T_f, T_{cf} are the reactor, feed and coolant temperatures. V is the reactor volume, h_a a heat transfer coefficient, E/R an energy activation term, ΔH the heat of reaction, ρ_c, ρ the liquid and coolant densities, and C_{pc}, C_p their specific heats. All values can be found in [20]. The coolant flowrate q_c is the control, C_A is the measured variable, and one supposes that $C_A \in [0.06 \cdots 0.13]$, the operating points for $C_A^1 = 0.06, C_A^2 = 0.1, C_A^3 = 0.13$ have the following linear models:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -16.67 & -0.047 \\ 0 & 3133.33 & 7.42 \end{pmatrix}, B_1 = \begin{pmatrix} 0 \\ 0 \\ -0.99 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -10 & -0.047 \\ 0 & 1800 & 7.33 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ 0 \\ -0.88 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -7.69 & -0.046 \\ 0 & 1338.46 & 7.19 \end{pmatrix}, B_3 = \begin{pmatrix} 0 \\ 0 \\ -0.82 \end{pmatrix}.$$

$$T^1 = 449.47, q_c^1 = 89.03, T^2 = 438.54, q_c^2 = 103.41, T^3 = 432.92, q_c^3 = 110.03.$$

For Gaussian validity functions, the nominal T-S model is given by:

$$\begin{pmatrix} \int \dot{C}_A(t) dt \\ \dot{C}_A(t) \\ \dot{T}(t) \end{pmatrix} = \sum_{i=1}^{3} \mu_i(C_A) \left[A_i \begin{pmatrix} C_A(t) \\ C_A(t) - C_A^i \\ T(t) - T^i \end{pmatrix} + B_i(q_c(t) - q_c^i) \right],$$

where $\mu_i = \omega_i(C_A) / \sum_{j=1}^{3} (\omega_j(C_A))$, $\omega_i = exp(-\frac{1}{2}(\frac{C_A - CA^i}{\sigma_i})^2)$, and $\sigma_i = 0.01, i = 1, 2, 3$

is a reasonable choice to represent with a good accuracy the nonlinear model (see [20] for full details).

The state space is $x = (\int c_A dt, c_A, T)^T$, and the control gains have been chosen to place the poles at $\lambda = (-3.4205 + 1.8701i, -3.4205 - 1.8701i, -5)^T$.

In this case the products $\mu_1 \mu_2$ and $\mu_2 \mu_3$ are bounded by 0.25 and $\mu_1 \mu_3$ is bounded by 10^{-5}. Thus, it is easy to find bounds for b_1, b_2, b_3. It is impossible to find a common Lyapunov matrix P for the T-S system using Theorem (2.1), but it is possible to find one using Theorem (3.2) with

$$P = 10^5 . \begin{pmatrix} 10 & -0.72 & -0.23 \\ -0.72 & 2.71 & 0.014 \\ -0.23 & 0.014 & 0.0001 \end{pmatrix}.$$

The magnitude of elements of P is still important because of the small overlapping between membership functions. Of course, this result only guaranties the convergence of the Takagi–Sugeno fuzzy system and not that of the corresponding nonlinear system, for which uncertainties should be lumped into the T-S fuzzy model as for example in [19].

6   Conclusion

In this paper, the stability of a Takagi–Sugeno fuzzy system under the Parallel Distributed Compensation controller has been studied. This control strategy allocates the same weight to a local controller as the one in the fuzzy combination of local submodels. The influence of the coupling between any local subsystem and any local controller (different from the corresponding local controller designed from the local subsystem considered) in the closed-loop response has been highlighted, and it has been shown to be effective when the input matrices of the subsystems are different. It has been subsequently shown that a controller synthesis based on an analysis of each local subsystem controlled by any local compensator, would lead to conservative results. A new approach has been proposed which, for every local subsystem, takes the coupling term coming from other subsystems into account, and proposes to choose the gain in order to cope with the effect of the coupling terms. This strategy allows to minimize the number of linear matrix inequalities to be solved for controller synthesis and to take into account the shape of the membership functions. Moreover, an exact compensation using a nonlinear PDC controller has been proposed, which is tractable only if the number of subsystems is greater than the model order plus one. Further investigation will be undertaken to generalize the results for Lyapunov functions leading to less conservative results, i.e. piecewise and fuzzy Lyapunov functions.

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Existence of the Unique Solution to Abstract Second Order Semilinear Integrodifferential Equations

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Abstract: In this paper, a strongly damped semilinear integrodifferential equation has been considered and reformulated as an abstract second order integrodifferential equation in a Banach space. The local existence and uniqueness of a classical solution is estabilied. The continuation of classical solution, the maximal interval of the existence and the global existence of the classical solution have been also studied. Finally an application of the established results is demonstrated.

Keywords: *analytic semigroup; second order integrodifferential equation; mild solution; classical solution; contraction mapping theorem.*

Mathematics Subject Classification (2000): 34G20, 35A07, 35A35.

1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^N$ with sufficiently smooth boundary $\partial\Omega$ and $Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$ be a symmetric second order strongly elliptic differential operator in Ω. Consider the following initial boundary value problem for the strongly damped partial integrodifferential equation,

$$
\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} \;&+\; (aL+bI)\left(\frac{\partial u(x,t)}{\partial t}\right) \;+\; (cL+dI)u(x,t) \;=\; h\left(x,t,u(x,t),\frac{\partial u(x,t)}{\partial t}\right) \\
&+\; \int_{t_0}^{t} k(t-s)g\left(x,s,u(x,s),\frac{\partial u(x,s)}{\partial s}\right)ds, \\
&\qquad (x,t) \in \Omega \times (t_0,T), \quad 0 < T < \infty,
\end{aligned}
\tag{1}
$$

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with initial conditions

$$u(x, t_0) = x_0(x), \quad \frac{\partial u(x, t_0)}{\partial t} = x_1(x), \quad x \in \Omega,$$

and the homogeneous Dirichlet boundary conditions, where $a > 0$, b, c, d are constants and h and g are smooth nonlinear functions and k is a locally p-integrable function for $1 < p < \infty$.

Duvaut and Lions [5], Glowinski, Lions and Tremolieres [7] have studied particular case of (1) in which $L = -\triangle$ and $k \equiv 0$, in the context of the theory of viscoelastic materials.

We may rewrite (1) with initial and homogeneous Dirichlet boundary conditions in the abstract form as the following initial value problem in the Banach space $H = L^2(\Omega)$,

$$\frac{d^2 u(t)}{dt^2} + A\left(\frac{du(t)}{dt}\right) + Bu(t)$$
$$= f\left(t, u(t), \frac{du(t)}{dt}\right) + \int_{t_0}^t k(t-s)g\left(s, u(s), \frac{du(s)}{ds}\right) ds, \quad t > t_0,$$
$$u(t_0) = x_0, \quad u'(t_0) = x_1. \tag{2}$$

where operator A with domain $D(A) = H^2(\Omega) \bigcap H_0^1(\Omega)$ is given by

$$Au = aLu, \quad u \in D(A),$$

and the operator B is such that $D(A) = D(B)$ with $B = (cL + dI)$ for some constants c and d. The function f is defined from $R_+ \times H \times H$ into H given by $f(t, u, v) = h(t, u, v) - bv$. We assume that $-A$ generates an analytic semigroup $T(t)$ in X. The nonlinear maps f and g satisfy the assumptions (F) and (G), respectively, and the kernel k satisfies (K) stated in the next section.

In this paper, we concentrate on the study of the abstract second order semilinear integrodifferential equation

$$u''(t) + Au'(t) = f(t, u(t), u'(t)) + \int_{t_0}^t k(t-s)g(s, u(s), u'(s)) \, ds,$$
$$u(t_0) = x_0, \quad u'(t_0) = x_1, \tag{3}$$

as we can merge the term Bu in the function f so that the modified function f still satisfies the assumption (F).

Sandefur [10] has studied the second order semilinear differential equation

$$u''(t) + Au'(t) + Bu(t) = f(t, u(t)),$$
$$u(0) = \phi, \quad u'(0) = \psi, \tag{4}$$

in a Banach space X under the assumptions that the linear operators A and B can be decomposed as $-A = A_1 + A_2$ and $B = A_2 A_1$, where each A_k generates a C_0-semigroup $T_k(t)$, $k = 1, 2$; and the function f satisfies a locally Lipschitz condition. He has established the local existence and uniqueness of a mild solution to (4), i.e., there exists a continuous function u on $[0, c]$ for some $c > 0$ such that u satisfies the integral equation,

$$u(t) = T_1(t)\phi + \int_0^t T_1(t-\tau)T_2(\tau)(\psi - A_1\phi)d\tau$$
$$+ \int_0^t \int_0^\tau T_1(t-\tau)T_2(t-s)f(s, u(s))dsd\tau,$$

where $\phi \in D(A_1)$. Aviles and Sandefur [1] have studied the well-posedness of (4) under the similar conditions.

In [3] Bahuguna, Shukla and Singh have considered initial value problem (2) with the kernel $k \equiv 0$ and $t_0 = 0$ i.e.

$$\frac{d^2u(t)}{dt^2} + A\left(\frac{du(t)}{dt}\right) + Bu(t) = f\left(t, u(t), \frac{du(t)}{dt}\right), \quad t > 0,$$
$$u(0) = x_0, \quad u'(0) = x_1.$$

in real Banach space and used the method of semidiscretization in time to prove the existence, uniqueness and continuous dependence on initial data of a solution to this initial value problem and discussed their application to the viscoelastic models involving short and long memory effects.

Bahuguna [2] has considered the following special case of (3) with the kernel $k \equiv 0$,

$$\begin{aligned} u''(t) + Au'(t) &= f(t, u(t), u'(t)), \\ u(t_0) = x_0, \qquad u'(t) &= x_1, \end{aligned} \qquad (5)$$

and established the existence, uniqueness, continuation of a solution to the maximal interval of existence, and the global existence of a strong solution and a classical solution for this special case. He has assumed that $-A$ generates an analytic semigroup $T(t)$ in X and the nonlinear map f satisfies an assumption similar to the assumption (F).

Engler, Neubrander and Sandefur [6] have proved the local existence and uniqueness of a mild solution to (5) under the assumptions that $-A$ generates an analytic semigroup $T(t)$ in X and f satisfies a condition similar to the assumption (F), where a mild solution on $[t_0, t_1)$, for some $t_1 > t_0$, to (5) is the first component of a solution $(u(t), v(t))$ of the integral equations

$$\begin{aligned} u(t) &= x_0 + (T(t - t_0) - I)(-A)^{-1}x_1 \\ &\quad + \int_{t_0}^{t} (T(t - s) - I)(-A)^{-1}f(s, u(s), v(s))ds, \quad t_0 \le t \le t_1, \\ v(t) &= T(t - t_0)x_1 + \int_{t_0}^{t} T(t - s)f(s, u(s), v(s))ds, \quad t_0 \le t \le t_1. \end{aligned}$$

Bahuguna [2] has improved the results of [6] by showing that (5) has a unique local classical solution, i.e., there exists a unique $u \in C^1([t_0, t_1); X) \cap C^2((t_0, t_1); X)$ and satisfies (5) on $[t_0, t_1)$ for some $t_1 > t_0$. Further, he has established the continuation of this solution, the maximal interval of existence and the global existence.

In [4] Bahuguna and Shukla studied the Faedo-Galerkin approximation of solutions to the initial value problem (3) in a Hilbert space. Pandey, Ujlayan and Bahuguna considerd an abstract semilinear hyperbolic integrodifferential equation in [9] and used the theory of resolvent operators to establish the existence and uniqueness of a mild solution under local Lipschitz conditions on the nonlinear maps and an integrability condition on the kernel. Under some additional conditions on the nonlinear maps they also proved the existence of a classical solution.

In this paper we show that (3) has a unique local classical solution, i.e., there exists a unique $u \in C^1([t_0, t_1); X) \cap C^2((t_0, t_1); X)$ satisfying (3) on $[t_0, t_1)$ for some $t_1 > t_0$. Further, we discuss the continuation of this solution, the maximal interval of existence and the global existence. We achieve these objectives by extending the ideas and techniques

used in the proofs of Theorems 6.3.1 and 6.3.3 in Pazy [8], concerning a semilinear equation of the first order, to (3). For the global existence, we require a modified version of Lemma 4.1, stated and proved at the end of the fourth section in [2]. Finally in the last section we demonstrate an application of the results established in earlier sections.

2   Preliminaries and Assumptions

Let X be a Banach space and let $-A$ generate the analytic semigroup $T(t)$ in X. we note that if $-A$ is the infinitesimal generator of an analytic semigroup then $-(A + \alpha I)$ is invertible and generates a bounded analytic semigroup for $\alpha > 0$ large enough. This allows us to reduce the general case, in which $-A$ is the infinitesimal generator of an analytic semigroup, to the case where the semigroup is bounded and the generator is invertible. Hence, for convenience, without loss of generality, we assume that $T(t)$ is bounded, that is $\|T(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$, i.e., $-A$ is invertible. Here $\rho(-A)$ is the resolvent set of $-A$. It follows that, for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with its domain $D(A^\alpha)$ being dense in X. We denote by X_α the Banach space $D(A^\alpha)$ equipped with the norm

$$\|x\|_\alpha = \|A^\alpha x\|,$$

which is equivalent to the graph norm of A^α. For $0 < \alpha < \beta$, we have $X_\beta \hookrightarrow X_\alpha$ and the embedding is continuous.

We consider the problem

$$u''(t) + Au'(t) = f(t, u(t), u'(t)) + \int_{t_0}^t k(t - s)g(t, u(t), u'(t))ds, \quad t > t_0,$$

$$u(t_0) = x_0, \quad u'(t_0) = x_1. \tag{6}$$

On the kernel k we assume the following condition.

(K) The kernel $k \in L_{loc}^p(0, \infty)$ for some $1 < p < \infty$ is locally Hölder continuous on $(0, \infty)$ i.e.,

$$|k(t) - k(s)| \leq L_k|t - s|^\mu \quad \text{for} \quad s, t \in (0, \infty) \quad \text{and} \quad 0 < \mu < 1.$$

The nonlinear functions f and g satisfy the following assumptions on an open subset U of $R_+ \times X_1 \times X_\alpha$.

Assumption (F): A function f is said to satisfy the assumption (F) if for every $(t, x, \tilde{x}) \in U$ there exists a neighborhood $V \subset U$ and constant $L_f \geq 0$, $0 < \vartheta \leq 1$, such that

$$\|f(t, x_1, \tilde{x}_1) - f(t, x_2, \tilde{x}_2)\| \leq L_f[|t_1 - t_2|^\vartheta + \|x_1 - x_2\|_1 + \|\tilde{x}_1 - \tilde{x}_2\|_\alpha], \tag{7}$$

for all $(t_i, x_i, \tilde{x}_i) \in V$.

Assumption (G): A function g is said to satisfy the assumption (G) if for every $(t, x, \tilde{x}) \in U$ there exists a neighborhood $V \subset U$ and a nonnegative function $L_g \in L_{loc}^q(0, \infty)$ where $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$\|g(t, x_1, \tilde{x}_1) - g(t, x_2, \tilde{x}_2)\| \leq L_g(t)[\|x_1 - x_2\|_1 + \|\tilde{x}_1 - \tilde{x}_2\|_\alpha], \tag{8}$$

for all $(t, x_i, \tilde{x}_i) \in V$.

Definition 2.1 *By a* local classical solution *to (6) we mean a function* $u \in C^1([t_0, t_1); X) \cap C^2((t_0, t_1); X)$ *satisfying (6) on* $[t_0, t_1)$ *for some* $t_1 > t_0$.

Definition 2.2 *By a* local mild solution *to (6) we mean the first component of a solution* (u, v) *to the pair of integral equations*

$$
\begin{aligned}
u(t) &= x_0 + (T(t - t_0) - I)(-A)^{-1}x_1 + \int_{t_0}^t (T(t - s) - I)(-A)^{-1}[f(s, u(s), v(s)) \\
&\quad + \int_{t_0}^s k(s - \tau)g(\tau, u(\tau), v(\tau))d\tau]ds, \quad t_0 \le t \le t_1, \\
v(t) &= T(t - t_0)x_1 + \int_{t_0}^t T(t - s)[f(s, u(s), v(s)) \\
&\quad + \int_{t_0}^s k(s - \tau)g(\tau, u(\tau), v(\tau))d\tau]ds, \quad t_0 \le t \le t_1,
\end{aligned}
$$

$$(9)$$

on $[t_0, t_1)$ *for some* $t_1 > t_0$.

3 Local Existence of Solution

As we have already pointed out, without loss of generality, the semigroup generated by $-A$, can be assumed to be bounded and A is invertible. Under these conditions imposed on A we prove the following local existence and uniqueness theorem.

Theorem 3.1 *Suppose that* $-A$ *generates the analytic semigroup* $T(t)$ *such that* $\|T(t)\| \le M$ *and* $0 \in \rho(-A)$. *If the maps* f *and* g *satisfy assumptions (F) and (G), respectively, and the kernel* k *satisfies (K) then (6) has a unique local classical solution.*

Proof Fix (t_0, x_0, x_1) in U and choose $t_1' > t_0$ and $\delta > 0$ such that (7), with some fixed constant $L_f > 0$, $0 < \vartheta \le 1$ and (8) with the nonnegative function $L_g(t)$ hold on the set
$$
V = \{(t, x, \tilde{x}) \in U \mid t_0 \le t \le t_1', \|x - x_0\|_1 + \|\tilde{x} - x_1\|_\alpha \le \delta\}.
$$
Let
$$
B_f = \max_{t_0 \le t \le t_1'} \|f(t, x_0, x_1)\|,
$$
$$
B_g = \max_{t_0 \le t \le t_1'} \|g(t, x_0, x_1)\|
$$
and
$$
C(\delta) = [L_f + \|k\|_{L^p(t_0, t_1')}\|L_g\|_{L^q(t_0, t_1')}]\delta + B_f + B_g\|k\|_{L^p(t_0, t_1')}(t_1' - t_0)^{\frac{1}{q}}.
$$
Choose $t_1 > t_0$ such that
$$
\|T(t - t_0)x_1 - x_1\| + \|T(t - t_0)A^\alpha x_1 - A^\alpha x_1\| \le \frac{\delta}{3}
$$
and
$$
t_1 - t_0 < \min\left\{t_1' - t_0, \frac{\delta}{3}(M + 1)^{-1}C(\delta)^{-1}, [\frac{\delta}{2}C_\alpha^{-1}(1 - \alpha)C(\delta)^{-1}]^{\frac{1}{1-\alpha}}\right\},
$$

where C_α is a positive constant depending on α and satisfying

$$\|A^\alpha T(t)\| \le C_\alpha t^{-\alpha} \quad \text{for} \quad t > 0. \tag{10}$$

Let $Y = C([t_0, t_1]; X \times X)$. Then $y \in Y$ is of the form $y = (y_1, y_2)$, $y_i \in C([t_0, t_1]; X)$, $i = 1, 2$. Y, endowed with the supremum norm,

$$\|(y_1, y_2)\|_Y = \sup_{t_0 \le t \le t_1} [\|y_1(t)\| + \|y_2(t)\|]$$

is a Banach space. We define a map F on Y by $Fy = F(y_1, y_2) := (\hat{y}_1, \hat{y}_2)$ with

$$\hat{y}_1(t) = Ax_0 - (T(t - t_0) - I)x_1 - \int_{t_0}^t (T(t - s) - I)F_y(s)ds,$$

$$\hat{y}_2(t) = T(t - t_0)A^\alpha x_1 + \int_{t_0}^t T(t - s)A^\alpha F_y(s)ds, \tag{11}$$

where

$$F_y(t) = f(t, A^{-1}y_1(t), A^{-\alpha}y_2(t)) + \int_{t_0}^t k(t - \tau)g(\tau, A^{-1}y_1(\tau), A^{-\alpha}y_2(\tau))d\tau,$$

for $t \in [t_0, t]$.

For every $y \in Y$, $Fy(t_0) = (Ax_0, A^\alpha x_1)$, and the assumptions (F) and (G) on f and g, respectively, and (K) on the kernel k imply that $F : Y \to Y$. Let S be a nonempty closed and bounded set given by

$$S = \{y \in Y \mid y = (y_1, y_2), y_1(t_0) = Ax_0, y_2(t_0) = A^\alpha x_1, \|y_1(t) - Ax_0\| + \|y_2(t) - A^\alpha x_1\| \le \delta\}.$$

Let $y = (y_1, y_2)$ be any element of S. We have from (11)

$$\begin{aligned}
\|\hat{y}_1(t) - Ax_0\| &+ \|\hat{y}_2(t) - A^\alpha x_1\| \\
&\le \|(T(t - t_0) - I)x_1\| + \int_{t_0}^t \|T(t - s) - I\|\|F_y(s)\|ds \\
&\quad + \|(T(t - t_0) - I)A^\alpha x_1\| + \int_{t_0}^t \|A^\alpha T(t - s)\|\|F_y(s)\|ds. \quad (12)
\end{aligned}$$

To find the estimate for $F_y(s)$, we add and subtract $f(s, x_0, x_1)$ and $g(s, x_0, x_1)$ and using (F), (G) and (K), we get

$$\begin{aligned}
\|F_y(s)\| &\le \|f(s, A^{-1}y_1(s), A^{-\alpha}y_2(s)) - f(s, x_0, x_1)\| + B_f \\
&\quad + \int_{t_0}^s |k(s - \tau)|[\|g(\tau, A^{-1}y_1(\tau), A^{-\alpha}y_2(\tau)) - g(\tau, x_0, x_1)\| + B_g]d\tau \\
&\le [L_f + \|k\|_{L^p(t_0, t_1')}\|L_g\|_{L^q(t_0, t_1')}]\delta + B_f + B_g\|k\|_{L^p(t_0, t_1')}(t_1' - t_0)^{\frac{1}{q}} \\
&\le C(\delta). \tag{13}
\end{aligned}$$

Using the estimate (13) and the fact that $\|T(t)\| \le M$ together with (10) and (12), we get

$$\begin{aligned}
\|\hat{y}_1(t) - Ax_0\| + \|\hat{y}_2(t) - A^\alpha x_1\| &\le \frac{\delta}{3} + (M + 1)C(\delta)(t - t_0) + \frac{C_\alpha C(\delta)(t - t_0)^{1-\alpha}}{1 - \alpha} \\
&\le \delta.
\end{aligned}$$

Hence, $F : S \to S$. Now, we show that F is a contraction on S. Let (y_1, y_2) and (z_1, z_2) be any two points of S. From (11) we have

$$
\|\hat{y_1}(t) - \hat{z_1}(t)\| + \|\hat{y_2}(t) - \hat{z_2}(t)\| \leq \int_{t_0}^{t} \|T(t-s) - I\| \|F_y(s) - F_z(s)\| ds
$$
$$
+ \int_{t_0}^{t} \|T(t-s)A^\alpha\| \|F_y(s) - F_z(s)\| ds. \quad (14)
$$

Using (F), (G) and (K), we get

$$
\begin{aligned}
&\|F_y(s) - F_z(s)\| \\
&\leq \|f(s, A^{-1}y_1(s), A^{-\alpha}y_2(s)) - f(s, A^{-1}z_1(s), A^{-\alpha}z_2(s))\| \\
&\quad + \int_{t_0}^{s} |a(s-\tau)| \|g(\tau, A^{-1}y_1(\tau), A^{-\alpha}y_2(\tau)) - g(\tau, A^{-1}z_1(\tau), A^{-\alpha}z_2(\tau))\| d\tau \\
&\leq [L_f + \|k\|_{L^p(t_0, t_1')} \|L_g\|_{L^q(t_0, t_1')}] \|(y_1, y_2) - (z_1, z_2)\|_Y \\
&\leq \frac{C(\delta)}{\delta} \|(y_1, y_2) - (z_1, z_2)\|_Y. \quad (15)
\end{aligned}
$$

Using (15) in (14), we get

$$
\begin{aligned}
\|\hat{y_1}(t) - \hat{z_1}(t)\| + \|\hat{y_2}(t) - \hat{z_2}(t)\| \\
\leq \left[\frac{(M+1)C(\delta)(t-t_0)}{\delta} + \frac{C_\alpha C(\delta)(t-t_0)^{1-\alpha}}{1-\alpha} \right] \|(y_1, y_2) - (z_1, z_2)\|_Y \\
\leq \frac{2}{3} \|(y_1, y_2) - (z_1, z_2)\|_Y.
\end{aligned}
$$

Taking supremum over $[t_0, t_1]$, we have

$$
\|(\hat{y_1}, \hat{y_2}) - (\hat{z_1}, \hat{z_2})\|_Y \leq \frac{2}{3} \|(y_1, y_2) - (z_1, z_2)\|_Y.
$$

Thus, F is a contraction on S. Therefore, it has a unique fixed point in S. Let $\bar{y} = (\bar{y_1}, \bar{y_2}) \in S$ be that fixed point of F. Then

$$
\begin{aligned}
\bar{y_1}(t) &= Ax_0 - (T(t-t_0) - I)x_1 - \int_{t_0}^{t} (T(t-s) - I)F_{\bar{y}}(s) ds, \\
\bar{y_2}(t) &= T(t-t_0)A^\alpha x_1 + \int_{t_0}^{t} T(t-s)A^\alpha F_{\bar{y}}(s) ds, \quad (16)
\end{aligned}
$$

where

$$
F_{\bar{y}}(t) = f(t, A^{-1}\bar{y_1}(t), A^{-\alpha}\bar{y_2}(t)) + \int_{t_0}^{t} k(t-\tau) g(\tau, A^{-1}\bar{y_1}(\tau), A^{-\alpha}\bar{y_2}(\tau)) d\tau.
$$

We note that $(u, v) = (A^{-1}\bar{y_1}, A^{-\alpha}\bar{y_2})$ is the unique solution of the integral equations (9) on $[t_0, t_1]$. We can easily check that the assumption (F) and the continuity of $\bar{y_1}$ and $\bar{y_2}$ on $[t_0, t_1]$ imply that the map $t \mapsto F_{\bar{y}}(t)$ is continuous and hence bounded on $[t_0, t_1]$. Let $\|F_{\bar{y}}(t)\| \leq N$ for $t_0 \leq t \leq t_1$. We will now show that $t \mapsto F_{\bar{y}}(t)$ is locally Hölder continuous on $(t_0, t_1]$. For this we first show that $\bar{y_1}$ and $\bar{y_2}$ are locally Hölder

continuous on $(t_0, t_1]$. From Theorem 2.6.13 in Pazy [8], for every $0 < \beta < 1 - \alpha$ and every $0 < h < 1$, we have

$$\|(T(h) - I)A^\alpha T(t - s)\| \leq C_\beta h^\beta \|A^{\alpha + \beta} T(t - s)\| \leq Ch^\beta (t - s)^{-(\alpha + \beta)}. \qquad (17)$$

Now

$$\begin{aligned}
\|\bar{y}_2(t + h) - \bar{y}_2(t)\| &\leq \|(T(h) - I)A^\alpha T(t - t_0)x_1\| + \int_{t_0}^t \|(T(h) - I)A^\alpha T(t-s)F_{\bar{y}}(s)\| ds \\
&\quad + \int_t^{t+h} \|A^\alpha T(t + h - s)F_{\bar{y}}(s)\| ds := I_1 + I_2 + I_3 \text{ (respectively)}.
\end{aligned}$$

We use (17) to get

$$\begin{aligned}
I_1 &\leq C(t - t_0)^{-(\alpha + \beta)} h^\beta \leq M_1 h^\beta, \\
I_2 &\leq NCh^\beta \int_{t_0}^t (t - s)^{-(\alpha + \beta)} ds = \frac{NCh^\beta (t - t_0)^{1 - (\alpha + \beta)}}{1 - (\alpha + \beta)} \leq M_2 h^\beta, \\
I_3 &\leq NC_\alpha \int_t^{t+h} (t + h - s)^{-\alpha} = \frac{NC_\alpha h^{1 - \alpha}}{1 - \alpha} \leq M_3 h^\beta.
\end{aligned}$$

Here M_1 depends on t and increases to infinity as $t \downarrow t_0$, while M_2 and M_3 can be chosen independent of t. From the above estimates, it follows that there exists a positive constant C such that for every $t_0' > t_0$,

$$\|\bar{y}_2(t) - \bar{y}_2(s)\| \leq C|t - s|^\beta, \quad \text{for} \quad t_0 < t_0' \leq t, s \leq t_1.$$

Similar result holds for $\bar{y}_1$ (if we take $\alpha = 0$ in the above consideration). For $s, t \in (t_0, t_1]$ with $t > s$ we have

$$\begin{aligned}
\|F_{\bar{y}}(t) - F_{\bar{y}}(s)\| &\leq \|f(t, A^{-1}\bar{y}_1(t), A^{-\alpha}\bar{y}_2(t)) - f(s, A^{-1}\bar{y}_1(s), A^{-\alpha}\bar{y}_2(s))\| \\
&\quad + \int_{t_0}^s |k(t - \tau) - a(s - \tau)| \|g(\tau, A^{-1}\bar{y}_1(\tau), A^{-\alpha}\bar{y}_2(\tau))\| d\tau \\
&\quad + \int_s^t |k(t - \tau)| \|g(\tau, A^{-1}\bar{y}_1(\tau), A^{-\alpha}\bar{y}_2(\tau))\| d\tau.
\end{aligned}$$

Since k is Hölder continuous with the exponent μ, we have

$$\int_{t_0}^s |k(t - \tau) - k(s - \tau)| \|g(\tau, A^{-1}\bar{y}_1(\tau), A^{-\alpha}\bar{y}_2(\tau))\| d\tau \leq N(t_1 - t_0)|t - s|^\mu, \qquad (18)$$

and

$$\int_s^t |k(t - \tau)| \|g(\tau, A^{-1}\bar{y}_1(\tau), A^{-\alpha}\bar{y}_2(\tau))\| d\tau \leq Nk_0(t_1 - t_0)^\alpha |t - s|^{1 - \alpha}, \qquad (19)$$

where $k_0 = \max_{t_0 \leq t \leq t_1} |k(t)|$. The local Hölder continuity of $F_{\bar{y}}(t)$ on $(t_0, t_1]$ follows from the assumption (F), and the local Hölder continuity of $\bar{y}_1$ and $\bar{y}_2$ on $(t_0, t_1]$ and from estimates (18) and (19).

Consider the inhomogeneous initial value problem

$$\frac{dv(t)}{dt} + Av(t) = F_{\bar{y}}(t), \quad v(t_0) = x_1. \qquad (20)$$

By the corollary 4.3.3 in [8], (20) has a unique solution $v \in C^1((t_0, t_1]; X)$ given by

$$v(t) = T(t - t_0)x_1 + \int_{t_0}^{t} T(t - s)F_{\bar{y}}(s)ds, \tag{21}$$

for $t > t_0$. Each term on the right hand side belongs to $D(A)$ and hence belongs to $D(A^\alpha)$ since $D(A) \subset D(A^\alpha)$, $0 \le \alpha \le 1$. Operating on both sides of (21) with A^α, we find that

$$A^\alpha v(t) = T(t - t_0)A^\alpha x_1 + \int_{t_0}^{t} T(t - s)A^\alpha F_{\bar{y}}(s)ds. \tag{22}$$

By (16), the right hand side of (22) equals to $\bar{y}_2(t)$ and therefore $A^\alpha v(t) = \bar{y}_2(t)$, i.e., $v(t) = A^{-\alpha}\bar{y}_2(t)$. Let $u(t) = A^{-1}\bar{y}_1(t)$, then we have $u(t) = x_0 + \int_{t_0}^{t} v(s)ds$ which yields $u(t) \in C^1([t_0, t_1); X) \cap C^2((t_0, t_1); X)$. Thus, u satisfies (6) on $[t_0, t_1)$. $\square$

4 Global Existence of Solutions

In this section we will prove, under additional growth conditions on the nonlinear map f and g, the following global existence result.

Theorem 4.1 *Let $0 \in D(-A)$ and $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$ such that $\|T(t)\| \le M$ for $t \ge 0$. Let $f, g : [0, \infty) \times X_1 \times X_\alpha \mapsto X$ satisfy the assumptions (F) and (G) respectively and let k satisfy (K). If there exist a nondecreasing function $a_f : [t_0, \infty) \mapsto R_+$ and a nonnegative function $a_g \in L^q_{loc}(0, \infty)$, where q is the same as before, such that*

$$
\begin{aligned}
\|f(t, x, \tilde{x})\| &\le a_f(t)[1 + \|x\|_1 + \|\tilde{x}\|_\alpha], &\quad for \quad t \ge t_0, (x, \tilde{x}) \in X_1 \times X_\alpha, \\
\|g(t, x, \tilde{x})\| &\le a_g(t)[1 + \|x\|_1 + \|\tilde{x}\|_\alpha], &\quad for \quad t \ge t_0, (x, \tilde{x}) \in X_1 \times X_\alpha,
\end{aligned}
$$

then for each $(x_0, x_1) \in X_1 \times X_\alpha$, (6) has a unique classical solution u which exists for all $t \ge t_0$.

Proof Let $[t_0, T)$ be the maximal interval of existence for the solution u to (6) guaranteed by Theorem (3.1). It suffices to prove that $[\|u(t)\|_1 + \|v(t)\|_\alpha] \le C$ on $[t_0, T)$ for some fixed constant $C \ge 0$ independent of t.

Now, since $u(t)$ is a solution of (6) on $[t_0, T)$, it is also a mild solution to (6) therefore from (16), we have

$$Au(t) = Ax_0 - (T(t - t_0) - I)x_1 - \int_{t_0}^{t} (T(t - s) - I)\bar{F}(s)ds,$$

$$A^\alpha u'(t) = T(t - t_0)A^\alpha x_1 + \int_{t_0}^{t} T(t - s)A^\alpha \bar{F}(s)ds, \tag{23}$$

where

$$\bar{F}(t) = f(t, u(t), u'(t)) + \int_{t_0}^{t} k(t - \tau)g(\tau, u(\tau), u'(\tau))d\tau.$$

From (23), we have

$$
\begin{aligned}
[1 + \|u(\eta)\|_1 + \|u'(\eta)\|_\alpha] &= [1 + \|Au(\eta)\| + \|A^\alpha u'(\eta)\|] \\
&\leq 1 + \|Ax_0\| + (M+1)\|x_1\| + (M+1)\int_{t_0}^{\eta} \|\bar{F}(s)\| ds \\
&\quad + M\|x_1\|_\alpha + \int_{t_0}^{\eta} C_\alpha(\eta - s)^{-\alpha} \|\bar{F}(s)\| ds. \qquad (24)
\end{aligned}
$$

The assumptions on f, g and k imply that

$$
\begin{aligned}
\|\bar{F}(s)\| &\leq \|f(t, u(t), u'(t))\| + \int_{t_0}^{s} |k(s-\tau)| \|g(\tau, u(\tau), u'(\tau))\| d\tau \\
&\leq (a_f(T) + \|k\|_{L^p(t_0,T)} \|a_g\|_{L^q(t_0,T)}) \sup_{t_0 \leq \tau \leq s} [1 + \|u(\tau)\|_1 + \|u'(\tau)\|_\alpha]. \quad (25)
\end{aligned}
$$

Using (25) in (24), we get

$$
\begin{aligned}
[1 + \|u(\eta)\|_1 + \|u'(\eta)\|_\alpha] &\leq C_1 + C_2 \int_{t_0}^{\eta} \sup_{t_0 \leq \tau \leq s} [1 + \|u(\tau)\|_1 + \|u'(\tau)\|_\alpha] ds \\
&\quad C_3 \int_{t_0}^{\eta} (\eta - s)^{-\alpha} \sup_{t_0 \leq \tau \leq s} [1 + \|u(\tau)\|_1 + \|u'(\tau)\|_\alpha] ds.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\sup_{t_0 \leq \eta \leq t} [1 + \|u(\eta)\|_1 + \|u'(\eta)\|_\alpha] &\leq C_1 + C_2 \int_{t_0}^{t} \sup_{t_0 \leq \tau \leq s} [1 + \|u(\tau)\|_1 + \|u'(\tau)\|_\alpha] ds \\
&\quad C_3 \int_{t_0}^{t} (t - s)^{-\alpha} \sup_{t_0 \leq \tau \leq s} [1 + \|u(\tau)\|_1 + \|u'(\tau)\|_\alpha] ds.
\end{aligned}
$$

Using Lemma 4.1 in [2], we obtain $\sup_{t_0 \leq \eta \leq t}[1 + \|u(\eta)\|_1 + \|u'(\eta)\|_\alpha] \leq C$. □

5   Example

Let $\Omega = (0,1)$ and $H = L^2(\Omega)$. Consider the following initial boundary value problem

$$
\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} &= F\left(x, t, u(x,t), \frac{\partial^2 u(x,t)}{\partial x^2}, \frac{\partial u(x,t)}{\partial t}\right) \\
&\quad + \int_{t_0}^{t} k(t-s) G\left(x, s, u(x,s), \frac{\partial u(x,s)}{\partial s}\right) ds, \\
&\quad (x,t) \in \Omega \times (t_0, T), \quad 0 < T < \infty \qquad (26)
\end{aligned}
$$

with the initial conditions

$$
u(x, t_0) = x_0(x), \quad \frac{\partial u(x, t_0)}{\partial t} = x_1(x), \quad x \in \Omega,
$$

and the boundary conditions

$$
u(0,t) = u(1,t) = 0, \quad t \in (t_0, T), \quad 0 < T < \infty
$$

F and G are sufficiently smooth nonlinear functions and k is a locally p-integrable function for $1 < p < \infty$.

We define the operator A with domain $D(A) = H^2(\Omega) \bigcap H_0^1(\Omega)$ as follows

$$Au = -\frac{\partial^2 u}{\partial x^2}, \quad u \in D(A).$$

Here clearly the operator A is self-adjoint with the compact resolvent and is the infinitesimal generator of an analytic semigroup $T(t)$. Now we take $\alpha = 1/2$, $D(A^{1/2})$ is the Banach space endowed with the norm

$$\|x\|_{1/2} = \|A^{1/2}x\|, \quad x \in D(A^{1/2}).$$

Using the above definition of the operator A the equation (26) can be reformulated as the following abstract equation in H

$$u''(t) + Au'(t) = f(t, u(t), u'(t)) + \int_{t_0}^t k(t-s)g(s, u(s), u'(s))\, ds,$$
$$u(t_0) = x_0, \quad u'(t_0) = x_1, \tag{27}$$

where $u(t)(x) = u(x,t)$, the function f is defined from $[t_0, T] \times D(A) \times D(A^{1/2})$ into H such that

$$f(t, u(t), u'(t))(x) = F\left(x, t, u(x,t), \frac{\partial^2 u(x,t)}{\partial x^2}, \frac{\partial u(x,t)}{\partial t}\right)$$

and g is defined from $[t_0, T] \times D(A) \times D(A^{1/2})$ into H such that

$$g(t, u(t), u'(t))(x) = G\left(x, t, u(x,t), \frac{\partial u(x,t)}{\partial t}\right).$$

It can be varified that the assumptions in earlier sections for (27) are satisfied and hence the existence of a unique classical solution is guarenteed.

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Synchronization of Chaotic Systems by the Generalized Hamiltonian Systems Approach$^\diamond$

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Abstract: In this paper, the generalized Hamiltonian system approach was applied to the synchronization of chaotic systems. The synchronization is between the transmitter and the receiver dynamics. The synchronization of several chaotic systems is studied by the method, respectively. The numerical results are in very good agreement with the theoretical analysis.

Keywords: *chaotic system; chaotic synchronization; generalized Hamiltonian system.*

Mathematics Subject Classification (2000): 37N35, 65P20, 68P25, 70K99, 93D20, 94A99.

1 Introduction

In the 17th century, the analysis of synchronization phenomena in the evolution of dynamical systems was a subject of active investigation [1]. Recently, the search for synchronization has moved to chaotic systems. Synchronization of chaos refers to a process wherein two (or many) chaotic systems adjust a given property of their motion to a common behavior due to a coupling or to a forcing.

The first thing to be highlighted is that there is a great difference in the process leading to synchronized states, depending upon the particular coupling configuration [1]. Namely, one should distinguish two main cases: unidirectional coupling and bidirectional coupling. In the former case, one subsystem evolves freely and drives the evolution of the other; in the latter case, both subsystems are coupled with each other.

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In the context of coupled chaotic elements, many different synchronization states have been studied in the past 10 years: namely complete or identical synchronization [2]-[4], phase [5, 6] and lag synchronization [7], generalized synchronization [8, 9], intermittent lag synchronization [7, 10], imperfect phase synchronization [11], and almost synchronization [12]. Complete synchronization was the first discovered and is the simplest form of synchronization in chaotic systems. It consists in a perfect hooking of the chaotic trajectories of two systems which is achieved by means of a coupling signal, in such a way that they remain in step with each other in the course of time.

The phenomena of chaotic synchronization exists widely in laboratory experiments and natural systems [13]-[22]. The natural continuation of the pioneering works was to investigate synchronization phenomena in spatially extended or infinite dimensional systems [13]-[16], to test synchronization in experiments or natural systems [17]-[22]. The synchronization has also been applied to encoding or masking where the chaotic system is called the "transmitter". Correspondingly for the decoding or unmasking, the second chaotic system is called the "receiver". The synchronization between the "transmitter" and the "receiver" means that, under the assumption of no masked signal transmission, the receiver state trajectory asymptotically tracks that of the transmitter. In [23], the authors have studied the synchronization of two chaotic systems by the generalized Hamiltonian system and observer approach. Furthermore, the method is extended to the time-delay Chua's oscillator [24].

The objective of this paper is to apply the generalized Hamiltonian system and observer approach developed in [23] to the complete synchronization of two identical chaotic systems coupled unidirectionally. The organization of the paper is as follows: In Section 2, we obtain the synchronization of chaotic systems by the generalized Hamiltonian system and observer approach. In Section 3, we present several chaotic systems and study their synchronization by this method, respectively. In Section 4, the conclusion is given.

2   The Synchronization of Chaotic Systems

A smooth system is given as follows:

$$\dot{x} = f(x,t), \quad x = (x_1, x_2 \ldots x_n)^T \in R^n, \tag{1}$$

where $f \in R^n$ is smooth.

Equation (1) may be written in the generalized Hamiltonian system:

$$\dot{x} = J_1(x)\frac{\partial H}{\partial x} + S(x)\frac{\partial H}{\partial x} + F_1(x,t), \tag{2}$$

where $H(x)$ denotes a smooth energy function and is globally positive definite in R^n, and the column gradient vector $\frac{\partial H}{\partial x}$ of $H(x)$ is assumed to exist everywhere; if the form of quadratic energy function is $H = \frac{1}{2}x^T M x$ (M is a constant symmetric positive definite matrix), $\frac{\partial H}{\partial x} = Mx$. $J_1(x) + J_1^T(x) = \theta$, $S(x) = S^T(x)$. The vector field $J(x)\frac{\partial H}{\partial x}$ exhibits the conservative part of the system and it is also referred to as the workless part; and $S(x)$ depicts the working part of the system. $F_1(x,t)$ is a locally destabilizing vector field. According to the form of $H(x)$ and the different expression of $J_1(x)$, $S(x)$, $F_1(x,t)$, the form of the Generalized Hamiltonian system (2) is not unique.

In the context of observer design, we consider a special class of Generalized Hamiltonian system with liner output map y:

$$\begin{cases} \dot{x} = J(y)\frac{\partial H}{\partial x} + (I + S)\frac{\partial H}{\partial x} + F(y,t), \\ y = C\frac{\partial H}{\partial x}, \end{cases} \qquad (3)$$

where $J(x) + J^T(x) = \theta$, I is a constant skew symmetric matrix, S is a constant symmetric matrix, and $F(x,t)$ is a locally destabilizing vector field. The vector variable y is referred to as the system output, and the matrix C is a constant matrix. Equation (3) is called the transmitter.

Let ξ and μ be the estimates of the state vector x and output y, respectively; and $\frac{\partial H}{\partial \xi} = M\xi$ is naturally the gradient of the Hamiltonian energy function $H(\xi)$). A dynamic nonlinear state observer for (3) is obtained as:

$$\begin{cases} \dot{\xi} = J(y)\frac{\partial H}{\partial \xi} + (I + S)\frac{\partial H}{\partial \xi} + F(y,t) + K(y - \eta), \\ \eta = C\frac{\partial H}{\partial \xi}, \end{cases} \qquad (4)$$

where K is a constant matrix, known as the observer gain. Equation(4) is called the receiver.

In this paper, we study mainly the synchronization of the transmitter (3) and the receiver (4). Practically, it is the complete synchronization of two identical chaotic systems coupled unidirectionally.

Let $e(t) = x(t) - \xi(t)$, $e_y = y - \eta$, then the state estimation error [23] are governed by

$$\begin{cases} \dot{e} = \left(J(y) + I - \frac{1}{2}(KC - C^T K^T)\right)\frac{\partial H}{\partial e} + \left(S - \frac{1}{2}(KC + C^T K^T)\right)\frac{\partial H}{\partial e}, \\ e_y = C\frac{\partial H}{\partial e} \qquad e_y \in R^m \quad , \end{cases} \qquad (5)$$

where $\frac{\partial H}{\partial e} = \frac{\partial H}{\partial x} - \frac{\partial H}{\partial \xi} = M(x - \xi) = Me$.

In [1], the authors point out that the transmitter (3) synchronizes with the receiver (4), if $\lim_{t \to \infty} \|x(t) - \xi(t)\| = 0$ no matter which initial conditions $x(0)$ and $\xi(0)$ have. The state estimation error $e(t) = x(t) - \xi(t)$ represents the synchronization error. So we will study the system (5) for the synchronization. In the following, two theorems about (5) give the condition under which their synchronization happens. Let $W = I + S$.

Theorem 2.1 [23] *The state $x(t)$ of the nonlinear system (3) can be globally exponentially asymptotically estimated by the state ξ of the nonlinear observer (4), if the pair of matrices (C, W) or the pair (C, S), is either observable or, at least, detectable.*

An observability condition on either of the pairs (C, W) or (C, S), is clearly a sufficient but not necessary condition for asymptotic state reconstruction. A necessary and sufficient condition for global asymptotic stability to zero of the estimation error is given by the following theorem.

Theorem 2.2 [23] *The state $x(t)$ of the nonlinear system (3) can be globally exponentially asymptotically estimated by the state ξ of the nonlinear observer (4), if and only if there exists a constant matrix K such that the symmetric matrix*

$$[W - KC] + [W - KC]^T = [S - KC] + [S - KC]^T = 2\left[S - \frac{1}{2}(KC + C^T K^T)\right] \qquad (6)$$

is negative definite.

3   Numerical Application

3.1   The forced Brusselator

The equation of this system is given as follows [25]:

$$\begin{cases} \dot{x}_1 = A - (B+1)x_1 + x_1^2 x_2 + a\cos(\omega t), \\ \dot{x}_2 = Bx_1 - x_1^2 x_2. \end{cases} \tag{7}$$

After taking as a Hamiltonian energy function the scalar function $H(x) = \frac{1}{2}(x_1^2 + x_2^2)$, we obtain:

$$J(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, I = \begin{bmatrix} 0 & \frac{-B}{2} \\ \frac{B}{2} & 0 \end{bmatrix}, S = \begin{bmatrix} -(B+1) & \frac{B}{2} \\ \frac{B}{2} & 0 \end{bmatrix}, F(x) = \begin{bmatrix} A + x_1^2 x_2 + a\cos(\omega t) \\ -x_1^2 x_2 \end{bmatrix}.$$

We choose $y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, thus $K = \begin{bmatrix} K_1 & K_3 \\ K_2 & K_4 \end{bmatrix}$. The system is in generalized Hamiltonian canonical form:

$$\dot{x} = J(x)\frac{\partial H}{\partial x} + (I+S)\frac{\partial H}{\partial x} + F(x,t), \tag{8}$$

and the receiver is

$$\dot{\xi} = J(x)\frac{\partial H}{\partial \xi} + (I+S)\frac{\partial H}{\partial \xi} + F(x,t) + K(x-\xi), \tag{9}$$

The synchronization error, corresponding to this receiver, is

$$\dot{e} = \left(J(x) + I - \frac{1}{2}(KC - C^T K^T)\right)\frac{\partial H}{\partial e} + \left(S - \frac{1}{2}(KC + C^T K^T)\right)\frac{\partial H}{\partial e}$$

$$= \begin{bmatrix} 0 & \frac{-B-K_3+K_2}{2} \\ -\frac{-B-K_3+K_2}{2} & 0 \end{bmatrix}\frac{\partial H}{\partial e} + \begin{bmatrix} -(B+1)-K_1 & \frac{B-(K_2+K_3)}{2} \\ \frac{B-(K_2+K_3)}{2} & 0 \end{bmatrix}\frac{\partial H}{\partial e}. \tag{10}$$

The pair (C, S) is observable, and hence detectable. We could prescribe K_1, K_2, K_3 and K_4, in order to ensure asymptotic stability of equation(8) and equation(9) to zero of the synchronization error. By applying Theorem 2.2, we obtain

$$2\begin{bmatrix} -(B+1)-K_1 & \frac{B-(K_2+K_3)}{2} \\ \frac{B-(K_2+K_3)}{2} & 0 \end{bmatrix}$$

is negative definite, i.e. $K_1 > -(B+1); 4K_4[(B+1)+K_1] > (B-K_2-K_3)^2$.

In Figure 1, the synchronization of two chaotic systems (8) and (9) is presented. The parameters were taken as: $A = 0.4, B = 1.2, \omega = 0.8, a = 0.05, K_1 = 0.8, K_2 = 0.2, K_3 = 1, K_4 = 1$.

3.2   The forced pendulum

The equation of this system is given as follows [26]:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -ax_2 - b\sin x_1 + \rho\cos(\omega t). \end{cases} \tag{11}$$
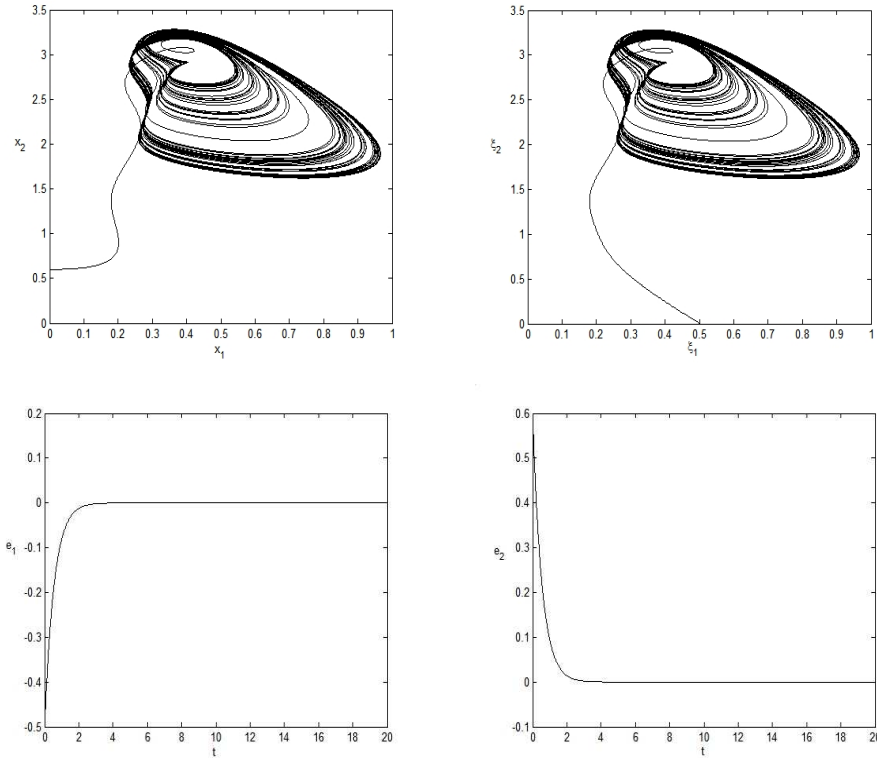
Figure 1: The synchronization of the forced Brusselator systems (8) and (9).

After taking as a Hamiltonian energy function the scalar function $H(x) = \frac{1}{2}(x_1^2 + x_2^2)$, we obtain:

$$J(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ I = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{-1}{2} & 0 \end{bmatrix}, \ S = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -a \end{bmatrix}, \ F(x) = \begin{bmatrix} 0 \\ -b\sin x_1 + \rho\cos(\omega t) \end{bmatrix}.$$

We choose $y = [x_1]$, then $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, thus $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$. The system is in generalized Hamiltonian canonical form:

$$\dot{x} = J(x)\frac{\partial H}{\partial x} + (I + S)\frac{\partial H}{\partial x} + F(x, t), \tag{12}$$

and the receiver is

$$\dot{\xi} = J(x)\frac{\partial H}{\partial \xi} + (I + S)\frac{\partial H}{\partial \xi} + F(x, t) + K(x_1 - \xi_1). \tag{13}$$

The synchronization error, corresponding to this receiver, is

$$\dot{e} = \left(J(x) + I - \frac{1}{2}(KC - C^T K^T)\right)\frac{\partial H}{\partial e} + \left(S - \frac{1}{2}(KC + C^T K^T)\right)\frac{\partial H}{\partial e}$$

$$= \begin{bmatrix} 0 & \frac{1+K_2}{2} \\ -\frac{1+K_2}{2} & 0 \end{bmatrix}\frac{\partial H}{\partial e} + \begin{bmatrix} -K_1 & \frac{1-K_2}{2} \\ \frac{1-K_2}{2} & -a \end{bmatrix}\frac{\partial H}{\partial e}. \tag{14}$$

The pair (C, S) is observable, and hence detectable. We could prescribe K_1 and K_2, in order to ensure asymptotic stability of equation (12) and equation (13) to zero of the synchronization error. By applying Theorem 2.2, we obtain

$$2 \begin{bmatrix} -K_1 & \frac{1-K_2}{2} \\ \frac{1-K_2}{2} & -a \end{bmatrix}$$

is negative definite, i.e. $K_1 > 0; 4aK_1 - (1 - K_2)^2 > 0$.

In Figure 2, the synchronization of two chaotic systems (12) and (13) is presented. The parameters were taken as: $a = 0.2, b = 1, \rho = 1.5, \omega = 0.4, K_1 = 2, K_2 = 2$.

3.3   The 3D model

The model is described by the equation as follows [27]:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -x_2 - 1.2x_3 - \mu x_1 + x_1^2 - 1.425x_2^2 + 0.2x_1x_3 - 0.01x_1^2x_3. \end{cases} \tag{15}$$

After taking as a Hamiltonian energy function the scalar function $H(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$, we obtain:

$$J(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, I = \begin{bmatrix} 0 & \frac{1}{2} & \frac{\mu}{2} \\ -\frac{1}{2} & 0 & 1 \\ \frac{-\mu}{2} & -1 & 0 \end{bmatrix}, S = \begin{bmatrix} 0 & \frac{1}{2} & -\frac{\mu}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{-\mu}{2} & 0 & -1.2 \end{bmatrix},$$

$$F(x) = \begin{bmatrix} 0 \\ 0 \\ x_1^2 - 1.425x_2^2 + 0.2x_1x_3 - 0.01x_1^2x_3 \end{bmatrix}.$$

We choose $y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, thus $K = \begin{bmatrix} K_1 & K_4 & K_7 \\ K_2 & K_5 & K_8 \\ K_3 & K_6 & K_9 \end{bmatrix}$.

The system is in generalized Hamiltonian canonical form:

$$\dot{x} = J(x)\frac{\partial H}{\partial x} + (I + S)\frac{\partial H}{\partial x} + F(x, t), \tag{16}$$

and the receiver is

$$\dot{\xi} = J(x)\frac{\partial H}{\partial \xi} + (I + S)\frac{\partial H}{\partial \xi} + F(x, t) + K(x - \xi). \tag{17}$$

The synchronization error, corresponding to this receiver, is

$$\begin{aligned} \dot{e} &= \left(J(x) + I - \tfrac{1}{2}(KC - C^T K^T) \right) \frac{\partial H}{\partial e} + \left(S - \tfrac{1}{2}(KC + C^T K^T) \right) \frac{\partial H}{\partial e} \\ &= \begin{bmatrix} 0 & \frac{1+K_2-K_4}{2} & \frac{\mu+K_3-K_7}{2} \\ -\frac{1+K_2-K_4}{2} & 0 & 1 - \frac{K_8-K_6}{2} \\ -\frac{\mu+K_3-K_7}{2} & -1 + \frac{K_8-K_6}{2} & 0 \end{bmatrix} \frac{\partial H}{\partial e} \\ &+ \begin{bmatrix} -K_1 & \frac{1-K_2-K_4}{2} & -\frac{\mu+K_3+K_7}{2} \\ \frac{1-K_2-K_4}{2} & -K_5 & \frac{-K_8-K_6}{2} \\ -\frac{\mu+K_3+K_7}{2} & \frac{-K_8-K_6}{2} & -1.2 - K_9 \end{bmatrix} \frac{\partial H}{\partial e}. \end{aligned} \tag{18}$$
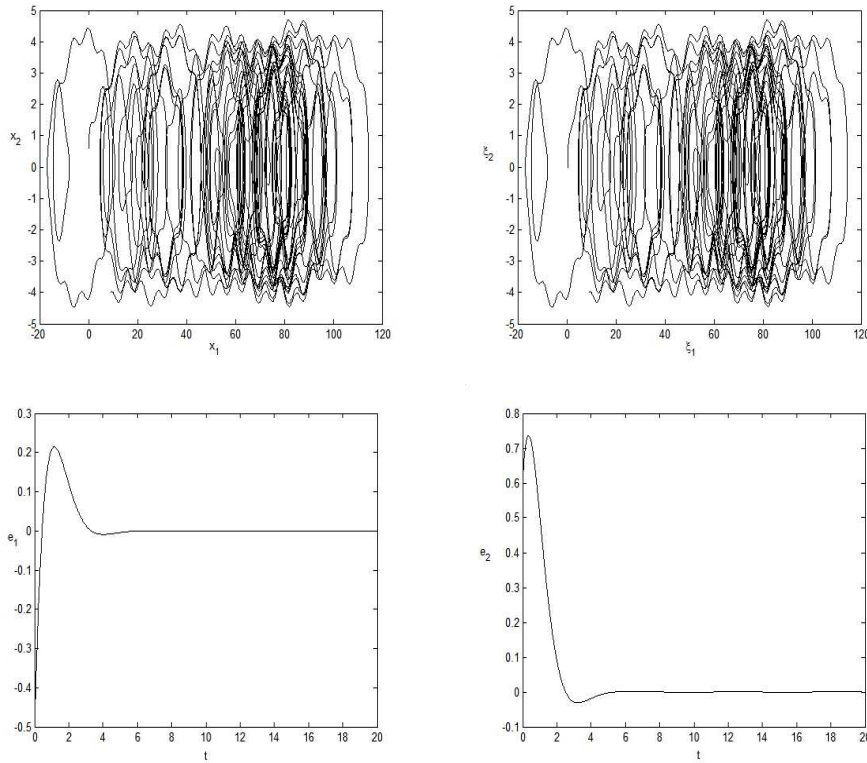
Figure 2: The synchronization of the forced pendulum systems (12) and (13).

The pair (C, S) is observable, and hence detectable. We could prescribe K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9 in order to ensure asymptotic stability of equation (16) and equation (17) to zero of the synchronization error. By applying Theorem 2.2, we obtain

$$2 \begin{bmatrix} -K_1 & \frac{1-K_2-K_4}{2} & -\frac{\mu+K_3+K_7}{2} \\ \frac{1-K_2-K_4}{2} & -K_5 & \frac{-K_8-K_6}{2} \\ -\frac{\mu+K_3+K_7}{2} & \frac{-K_8-K_6}{2} & -1.2-K_9 \end{bmatrix}$$

is negative definite, i.e.

$$K_1 > 0,$$
$$4K_1K_5 > (1-K_2-K_4)^2,$$
$$(1.2+K_9)\left[4K_1K_5 - (1-K_2-K_4)^2\right] - K_1(K_6+K_8)^2$$
$$-(1-K_2-K_4)(\mu+K_3+K_7)(K_6+K_8) - K_5(\mu+K_3+K_7)^2 > 0.$$

In Figure 3, the synchronization of two chaotic systems (16) and (17) is presented. The parameters were taken as: $\mu = 1.6, K_1 = K_4 = K_5 = 1, K_2 = 0, K_3 = K_6 = K_8 = \frac{1}{2}, K_9 = 3, K_7 = 0$.

4   Conclusion

In this paper, we apply the generalized Hamiltonian system and observer approach and obtain two chaotic systems: the "transmitter" and the "receiver" dynamics. Practically,
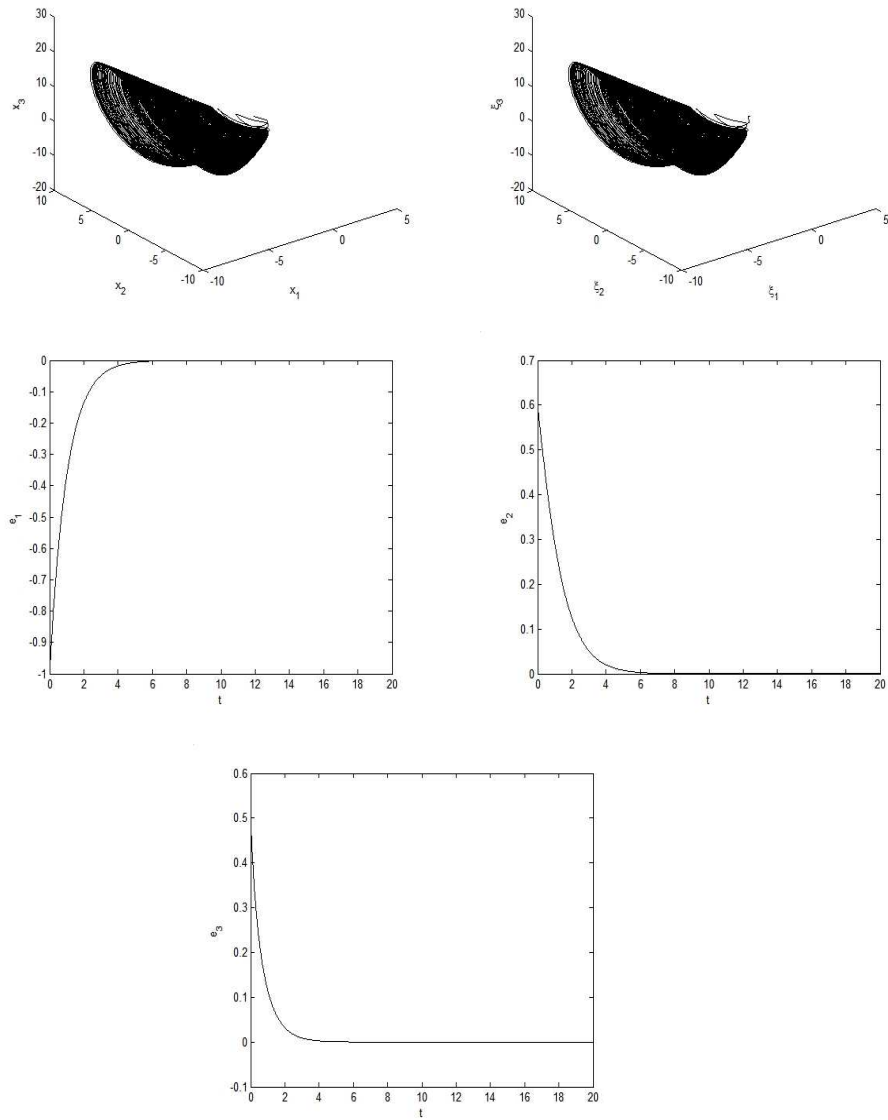
Figure 3: The synchronization of the 3D model (16) and (17).

two chaotic systems are the systems coupled unidirectionally. We study mainly the condition with which the coupling coefficient matrix K is satisfied when the complete synchronization of two coupled chaotic systems happens.

Acknowledgments

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On Stability of Hopfield Neural Network on Time Scales

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Abstract: In the paper uniform asymptotic, exponential and uniform exponential sufficient stability conditions for the neural systems on time scales are obtained. The sufficient conditions of regressivity of system's function are given.

1 Introduction

The theory of time scale was introduced by Stefan Hilger [10] in order to unify continuous and discrete cases and was intensively developed in many papers (see [1, 4] and references therein). Recently the theory of dynamic systems on time scale have received special attention from many authors, some of them focused their interest on the stability theory for such systems [2, 3, 13].

Proposed in [11] the Hopfield-type neural networks and their generalizations [7, 8] is a special but important case of general differential systems. It derives from biological models in practical investigations and has extensive applications in many different fields such as parallel computation, signal processing, pattern recognition, optimization and associative memories (see [5, 8, 14]).

However, as the theory of dynamic systems on time scale is widely studied the corresponding theory of neural systems is still at an initial stage of its development. In [6], the authors got some stability results for delayed bidirectional associative memory neural networks on time scales. Also in [12], some criteria of stability and existence

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of periodic solutions for delayed bidirectional associative memory neural networks with impulses on time scales were obtained.

Motivated by the above we consider a neural network on time scale the dynamics of which is described by the equation of the type

$$x^{\Delta}(t) = -Bx(t) + Ts(x(t)) + u, \quad t \in \mathbb{T}_{\tau}, \tag{1}$$

whose solution $x(t; t_0, x_0)$ for $t = t_0$ takes the value x_0, i.e.

$$x(t_0; t_0, x_0) = x_0, \quad t_0 \in \mathbb{T}_{\tau}, \quad x_0 \in \mathbb{R}^n, \tag{2}$$

where $\mathbb{T}$ is an arbitrary time scale, $\sup \mathbb{T} = +\infty$, $\mathbb{T}_{\tau} = \{t \in \mathbb{T}: t \geq \tau\}$, $\tau \in \mathbb{T}$. In system (1) $x^{\Delta}(t)$ is a Δ-derivative on time scale $\mathbb{T}$, $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, x_i is the activation of the i-th neuron, $T = \{t_{ij}\} \in \mathbb{R}^{n \times n}$, the components t_{ij} describe the interaction between the i-th and j-th neurons, $s \colon \mathbb{R}^n \to \mathbb{R}^n$, $s(x) = (s_1(x_1), s_2(x_2), \ldots, s_n(x_n))^{\mathrm{T}}$, the activation function s_i describes response of the i-th neuron, $B \in \mathbb{R}^{n \times n}$, $B = \mathrm{diag}\{b_i\}$, $b_i > 0$ represents the rate with which the i-th neuron shell resets its potential to the resting state in isolation when it is disconnected from the network and the external inputs, $i = 1, 2, \ldots, n$, n corresponds to the number of neurons in layers, $u \in \mathbb{R}^n$ is a constant external input vector. All needed notations on time scales according to [4] will be given in Section 2.

System (1) is general and unifies two well known neural models. If $\mathbb{T} = \mathbb{R}$ then $x^{\Delta} = d/dt$ and the initial problem (1), (2) is equivalent to the initial problem for a continuous Hopfield type neural network [11]

$$\frac{dx(t)}{dt} = -Bx(t) + Ts(x(t)) + u, \quad t \geq \tau, \tag{3}$$
$$x(t_0; t_0, x_0) = x_0, \quad t_0 \geq \tau, \quad x_0 \in \mathbb{R}^n.$$

If $\mathbb{T} = \mathbb{N}$ then $x^{\Delta}(k) = x(k+1) - x(k) = \Delta x(k)$, $\mathbb{T}_{\tau} = \{\tau, \tau+1, \tau+2, \ldots\}$ and the initial problem (1)–(2) is equivalent to the initial problem for a discrete Hopfield type neural network [9]

$$\Delta x(k) = -Bx(k) + Ts(x(k)) + u, \quad k \in \mathbb{T}_{\tau}, \tag{4}$$
$$x(k_0; k_0, x_0) = x_0, \quad k_0 \in \mathbb{T}_{\tau}, \quad x_0 \in \mathbb{R}^n.$$

Dynamics of continuous system (3) and discrete systems (4) and their generalizations are widely studied by many authors [7, 8, 9, 11, 15, 17], but there are no stability results for system (1) on time scales. Our purpose in the paper is by using the direct Lyapunov method to study the stability of equilibrium of (1).

The outline of the paper is as follows. In Section 2 we shall give some notations and basic definitions concerning the calculus on time scale and some required assertions. In Section 3 we shall present some new sufficient conditions ensuring the asymptotic and exponential stability of the equilibrium of system (1). Also we shall offer the criteria of regressivity of function $f(x) = -Bx + Ts(x) + u$. In Section 4 we shall give one example to illustrate our results obtained in the previous sections.

2 Notations and Preliminaries

In this section all facts concerning time scale calculus are given according to book [4].

Definition 2.1 An arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$ is referred to as a *time scale* and denoted by $\mathbb{T}$.

Definition 2.2

- The *forward* and *backward jump operators* $\sigma : \mathbb{T} \to \mathbb{T}$ and $\rho : \mathbb{T} \to \mathbb{T}$ are respectively defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.

- If $\sigma(t) = t$, $\rho(t) = t$, $\sigma(t) > t$, and $\rho(t) < t$, then the element $t \in \mathbb{T}$ is called *right-dense, left-dense, right-scattered,* and *left-scattered,* respectively. Here it is assumed that $\inf \varnothing = \sup \mathbb{T}$ (i.e. $\sigma(t) = t$, if $\mathbb{T}$ contains the maximal elements t) and $\sup \varnothing = \inf \mathbb{T}$ (i.e. $\rho(t) = t$, if $\mathbb{T}$ contains the minimal elements t).

- In addition to the set $\mathbb{T}$, the set $\mathbb{T}^k$ is defined as follows

$$
\mathbb{T}^k = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}
$$

- The distance from an arbitrary element $t \in \mathbb{T}$ to its follower is called the *graininess* of the time scale $\mathbb{T}$ and is given by the formula

$$
\mu(t) = \sigma(t) - t.
$$

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and $\mu(t) = 0$, if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1$, $\rho(t) = t-1$ and $\mu(t) = 1$.

Definition 2.3

- The function $f : \mathbb{T} \to \mathbb{R}$ is called Δ-differentiable at a point $t \in \mathbb{T}^k$ if there exists $\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a W-neighborhood of t satisfying

$$
| \, [f(\sigma(t)) - f(s)] - \gamma[\sigma(t) - s] | < \varepsilon |\sigma(t) - s|
$$

for all $s \in W$. In this case we shall write $f^\Delta(t) = \gamma$.

- if the function $f : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable for any $t \in \mathbb{T}^k$, then f is called Δ-differentiable on $\mathbb{T}^k$.

Theorem 2.1 *Assume that the functions $f, g : \mathbb{T} \to \mathbb{R}$ are Δ-differentiable at $t \in \mathbb{T}^k$. Then the following assertions are valid:*

(1) *the sum $f + g$ is Δ-differentiable at t and $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$;*

(2) *for any $\alpha \in \mathbb{R}$, the function $\alpha f(t)$ is Δ-differentiable at t and $\alpha f^\Delta(t) = \alpha f^\Delta(t)$;*

(3) *the product fg is Δ-differentiable at t and*

$$
(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t));
$$

(4) $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$.

Note that, if $\mathbb{T} = \mathbb{R}$, then $f^\Delta = f'$, which is the Euler derivative of f, and if $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$, which is the forward difference of $f(t)$.

Definition 2.4

- A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dence points in $\mathbb{T}$ and its left-sided limit exists (finite) at left-dence points in $\mathbb{T}$. The set of all *rd*-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

- A function $f : \mathbb{T} \to \mathbb{R}$ is called *regressive,* if $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}^k$ and *positive regressive,* if $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}^k$.

- A function $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is called *regressive,* if the mapping $I + \mu(t)f(t, \cdot)$ is invertible at each $t \in \mathbb{T}^k$. Here $I : \mathbb{R}^n \to \mathbb{R}^n$ is identity mapping.

- The set of all regressive and *rd*-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$.

We define the function

$$\beta_k(t) = \begin{cases} \mu^{-1}(t) \log |1 + \mu(t)k(t)|, & \text{if} \quad \mu(t) > 0, \\ k(t), & \text{if} \quad \mu(t) = 0, \end{cases}$$

where $k \in \mathcal{R}$, $t \in [t_0, +\infty)_{\mathbb{T}}$. Here and bellow $[a, +\infty)_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t < +\infty\}$, $a \in \mathbb{T}$.

Definition 2.5 We recall that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class K, if it is continuous, strictly increasing on $\mathbb{R}_+$ and $\psi(0) = 0$.

Definition 2.6 We recall that the matrix $A \in \mathbb{R}^{n \times n}$ is called *M-matrix* if its all non-diagonal elements are non-positive and all principle minors are positive.

Definition 2.7 We recall that the mapping $H : \mathbb{R}^n \to \mathbb{R}^n$ is called a *homeomorphism* of $\mathbb{R}^n$ onto itself, if H is continuous, bijective, H is onto itself and the inverse mapping H^{-1} is also continuous.

For convenience, we introduce some notations. We denote by $\|x\|$ a vector norm of vector $x \in \mathbb{R}^n$ defined by $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$, $\|A\|$ denotes a matrix norm of matrix $A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$ defined by $\|A\| = (\lambda_M(A^{\mathrm{T}}A))^{1/2}$, $\lambda_m(A)$, $\lambda_M(A)$ are minimal and maximal eigenvalues of matrix A respectively. In addition A^{-1} denotes the inverse of A, $|A|$ denotes absolute-value matrix given by $|A| = \{|a_{ij}|\}$.

We assume on system (1) as follows.

S$_1$. The vector-function $f(x) = -Bx + Ts(x) + u$ is regressive.

S$_2$. There exist positive constants $M_i > 0$, $i = 1, 2, \ldots, n$, such that $|s_i(r)| \leq M_i$ for all $r \in \mathbb{R}$.

S$_3$. There exist positive constants $l_i > 0$, $i = 1, 2, \ldots, n$, such that $|s_i(r) - s_i(v)| \leq l_i|r - v|$ for all $r, v \in \mathbb{R}$.

S$_4$. $0 < \mu(t) \in \mathcal{M}$ for all $t \in \mathbb{T}_\tau$, where $\mathcal{M} \subset \mathbb{R}$ is a compact set.

Note that under conditions S_1–S_3 there exists a unique solution of problem (1), (2) on $[t_0, +\infty)_{\mathbb{T}}$ for all initial data $(t_0, x_0) \in \mathbb{T}_\tau \times \mathbb{R}^n$ [4].

We denote by $r_0 = \left(\sum_{i=1}^{n}(\sum_{j=1}^{n} M_j|T_{ij}| + |u_i|)^2/b_i^2\right)^{1/2}$ and $\Lambda = \text{diag}\{l_i\} \in \mathbb{R}^{n \times n}$. Similar to Theorem 3.1 from [16] and Theorem 1 from [17] we can easily obtain the following assertion.

Theorem 2.2 *If for system* (1) *conditions* S_1–S_3 *are satisfied then there exists an equilibrium state* $x = x^*$ *of system* (1) *and moreover,* $\|x^*\| \leq r_0$. *Besides, if the matrix* $B\Lambda^{-1} - |T|$ *is an* M-*matrix, this equilibrium state is unique.*

Definition 2.8 The equilibrium state $x = x^*$ of the system (1) is:

(1) *uniformly stable* if for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that $\|x_0 - x^*\| < \delta$ implies $\|x(t; t_0, x_0) - x^*\| < \varepsilon$ for all $t \in [t_0, +\infty)_{\mathbb{T}}$, $t_0 \in \mathbb{T}_\tau$;

(2) *uniformly asymptotically stable* if it is uniformly stable and there exists $\Delta > 0$ such that $\|x_0 - x^*\| < \Delta$ implies $\lim_{t \to +\infty} \|x(t; t_0, x_0) - x^*\| = 0$ for all $t_0 \in \mathbb{T}_\tau$;

(3) *exponentially stable* if there exist $\beta > 0$ and $\lambda > 0$ such that for all $t_0 \in \mathbb{T}_\tau$ there exists $N = N(t_0) > 0$ such that $\|x_0 - x^*\| < \beta$ implies $\|x(t; t_0, x_0) - x^*\| \leq N e^{-\lambda(t-t_0)}\|x_0 - x^*\|$ for all $t \in \mathbb{T}_\tau$;

(4) *uniformly exponentially stable* if it is exponentially stable and N does not depend on t_0.

Let x^* be the equilibrium state of system (1). We perform the change of variables $y(t) = x(t) - x^*$ and rewrite the initial problem (1), (2) as

$$y^\triangle(t) = -By(t) + Tg(y(t)), \quad t \in \mathbb{T}_\tau, \tag{5}$$

$$y(t_0; t_0, y_0) = y_0, \quad t_0 \in \mathbb{T}_\tau, \quad y_0 \in \mathbb{R}^n, \tag{6}$$

where $y \in \mathbb{R}^n$, $g \colon \mathbb{R}^n \to \mathbb{R}^n$, $g(y) = (g_1(y_1), g_2(y_2), \ldots, g_n(y_n))^{\mathrm{T}}$, $g(y) = s(y + x^*) - s(x^*)$.

If for system (1) assumptions S_1–S_3 are valid, then for system (5) the following assertions hold true.

G_1. The vector-function $\tilde{g}_1(y) = -By + Tg(y)$ is regressive.

G_2. For all $r \in \mathbb{R}$ $|g_i(r)| \leq 2M_i$, $i = 1, 2, \ldots, n$.

G_3. For all $r, v \in \mathbb{R}$ $|g_i(r) - g_i(v)| \leq l_i|r - v|$, $i = 1, 2, \ldots, n$.

Note that under conditions G_1–G_3 there exists a unique solution of problem (5), (6) on $[t_0, +\infty)_{\mathbb{T}}$ for all initial data $(t_0, x_0) \in \mathbb{T}_\tau \times \mathbb{R}^n$ [4].

Futher we shall need the following result.

Lemma 2.1 *Assume that* $g_i \in C^2(\mathbb{R})$, $g_i(0) = 0$, $i = 1, 2, \ldots, n$, *and constants* $K_i > 0$, $i = 1, 2, \ldots, n$, *exist so that* $|g_i''(u)| \leq K_i$ *for all* $u \in \mathbb{R}$. *Then the vector-function* $g(y)$ *can be represented as* $g(y) = Hy + \tilde{g}_2(y)$, *where* $H = \text{diag}\{g_i'(0)\} \in \mathbb{R}^{n \times n}$, $\tilde{g}_2 \colon \mathbb{R}^n \to \mathbb{R}^n$ *and the estimate*

$$\|\tilde{g}_2(y)\| \leq K\|y\|^2, \tag{7}$$

holds true, where $K = \max_i\{K_i\}/2$.

Proof Decomposing the functions $g_i(y_i)$ by the Maclaurin formula we easily prove the Lemma.

3 Main Results

In this section we consider stability of a *neural network on time scale*. Let x^* be the equilibrium state of system (1). Designate by $\underline{b} = \min\{b_i\}$, $\overline{b} = \max\{b_i\}$, $L = \max\{l_i\}$.

Theorem 3.1 *For system* (1) *assume that assumptions* S_1–S_4 *are valid and there exists a constant* $\mu^* \in \mathcal{M}$ *such that* $\mu(t) \le \mu^*$ *for all* $t \in \mathbb{T}_\tau$. *If the inequality*

$$2\underline{b} - 2L\|T\| - \mu^*(\overline{b} + L\|T\|)^2 > 0,$$

is satisfied, the equilibrium state $x = x^*$ *of system* (1) *is uniformly asymptotically stable.*

Proof It is clear that the behavior of solution $x(t)$ of system (1) in the neighborhood of the equilibrium state x^* is equivalent to the behavior of solution $y(t)$ of system (5) in the neighborhood of zero. For the proof we shall apply the Lyapunov function $V(y) = y^\mathrm{T} y$. If $y(t)$ is Δ-differentiable in the point $t \in \mathbb{T}^k$, for the derivative of function $V(y(t))$ we have the expression

$$V^\Delta(y(t)) = (y^\mathrm{T}(t)\,y(t))^\Delta = y^\mathrm{T}(t)\,y^\Delta(t) + [y^\mathrm{T}(t)]^\Delta y(\sigma(t))$$
$$= y^\mathrm{T}(t)\,y^\Delta(t) + [y^\mathrm{T}(t)]^\Delta[y(t) + \mu(t)y^\Delta(t)].$$

For the derivative of function V along solutions of system (5) we get

$$V^\Delta(y(t))|_{(5)} = 2y^\mathrm{T}(t)\,y^\Delta(t) + \mu(t)[y^\Delta(t)]^\mathrm{T} y^\Delta(t)$$
$$= 2y^\mathrm{T}(t)[-By(t) + Tg(y(t))] + \mu(t)\| - By(t) + Tg(y(t))\|^2$$
$$\le -2\lambda_m(B)\|y(t)\|^2 + 2\|y(t)\|\,\|T\|\,\|g(y(t))\| + \mu^*(\|B\|\,\|y(t)\| + \|T\|\,\|g(y(t))\|)^2$$
$$= -2\underline{b}\,\|y(t)\|^2 + 2\|T\|\,\|y(t)\|\,\|y(t)\| + \mu^*(\overline{b}\,\|y(t)\| + \|T\|\,\|g(y(t))\|)^2.$$

Using obvious estimation $\|g(y(t))\| \le L\,\|y(t)\|$ as a result we have

$$V^\Delta(y(t))|_{(f-11)} \le -2\,\underline{b}\,\|y(t)\|^2 + 2L\|T\|\,\|y(t)\|^2 + \mu^*\left(\overline{b}\,\|y(t)\| + L\|T\|\,\|y(t)\|\right)^2$$
$$= -\left(2\underline{b} - 2L\|T\| - \mu^*(\overline{b} + L\|T\|)^2\right)\|y(t)\|^2.$$

Hence it follows that all conditions of Corollary 4.2 from the paper [3] are satisfied. Therefore, the equilibrium state $y = 0$ of system (5) is uniformly asymptotically stable. This is equivalent to the uniform asymptotic stability of the equilibrium state $x = x^*$ of system (1).

Theorem 3.2 *Let the following conditions be satisfied:*

(1) *for system* (1) *on time scale* $\mathbb{T}$ *assumptions* S_1–S_4 *are valid;*

(2) *functions* $s_i \in C^2(\mathbb{R})$ *and there exist constants* $K_i > 0$ *such that* $|s_i''(r)| \le K_i$ *for all* $r \in \mathbb{R}$, $i = 1, 2, \ldots, n$;

(3) *there exists a constant* $\mu^* \in \mathcal{M}$ *such that* $\mu(t) \le \mu^*$ *for all* $t \in \mathbb{T}_\tau$;

(4) *there exists a positive definite symmetric matrix* $P \in \mathbb{R}^{n \times n}$ *such that the inequality* $\lambda_M(PB_1 + B_1^\mathrm{T}P) + \mu^*\|P\|\|B_1\|^2 < 0$ *holds true, where* $B_1 = -B + TH$, $H = \mathrm{diag}\{s_i'(0)\} \in \mathbb{R}^{n \times n}$.

Then the equilibrium state $x = x^*$ of system (1) is uniformly asymptotically stable.

Proof We apply the function $V(y) = y^{\mathrm{T}} P y$. For the derivative of function V along solutions of system (5) we have

$$V^{\Delta}(y(t))|_{(5)} = y^{\mathrm{T}}(t) P y^{\Delta}(t) + [y^{\mathrm{T}}(t)]^{\Delta} P y(\sigma(t)) = y^{\mathrm{T}}(t) P y^{\Delta}(t) + [y^{\mathrm{T}}(t)]^{\Delta} P y(t)$$

$$+ \mu(t)[y^{\Delta}(t)]^{\mathrm{T}} P y^{\Delta}(t) = y^{\mathrm{T}}(t) P [B_1 y(t) + T \tilde{g}_2(y(t))] + [B_1 y(t) + T \tilde{g}_2(y(t))]^{\mathrm{T}} P y(t)$$

$$+ \mu(t)[B_1 y(t) + T \tilde{g}_2(y(t))]^{\mathrm{T}} P [B_1 y(t) + T \tilde{g}_2(y(t))] \leq y^{\mathrm{T}}(t) [PB_1 + B_1^{\mathrm{T}} P] y(t)$$

$$+ 2y^{\mathrm{T}}(t) P T \tilde{g}_2(y(t)) + \mu(t)\|P\| \|B_1 y(t) + T \tilde{g}_2(y(t))\|^2 \leq (\lambda_M(PB_1 + B_1^{\mathrm{T}} P)$$

$$+ \mu(t)\|P\| \|B_1\|^2)\|y(t)\|^2 + 2\|P\| \|T\| \|\tilde{g}_2(y(t))\| \|y(t)\| + \mu(t)\|P\| \|\tilde{g}_2(y(t))\|^2 \|T\|^2$$

$$+ 2\mu(t)\|P\| \|B_1\| \|T\| \|\tilde{g}_2(y(t))\| \|y(t)\|.$$

Using inequality (7) and condition (3) of Theorem 3.2 we get

$$V^{\Delta}(y(t))|_{(5)} \leq (\lambda_M(PB_1 + B_1^{\mathrm{T}} P) + \mu^* \|P\| \|B_1\|^2)\|y(t)\|^2 + 2K\|P\| \|T\| \|y(t)\|^3$$

$$+ 2\mu^* K\|P\| \|B_1\| \|T\| \|y(t)\|^3 + \mu^* K^2 \|P\| \|T\|^2 \|y(t)\|^4.$$

Designate

$$\psi(\|y\|) = a\|y\|^2,$$

$$a = -(\lambda_M(PB_1 + B_1^{\mathrm{T}} P) + \mu^* \|B_1\| \|P\|^2) > 0,$$

$$m(\psi) = 2a^{-\frac{1}{3}} K\|P\| \|T\| (1 + \mu^* \|B_1\|) \psi^{\frac{1}{3}} + \mu^* a^{-2} K^2 \|P\| \|T\|^2 \psi.$$

For the derivative of function V along solutions of system (5) we obtain the inequality

$$V^{\Delta}(y(t))|_{(5)} \leq -\psi(\|y\|) + m(\psi(\|y\|)).$$

Since the function $\psi \in K$-class, $\lim\limits_{\psi \to 0} m(\psi) = 0$, all conditions of Corollary 4.2 from [3] are satisfied and therefore, the equilibrium state $y = 0$ of system (5) is uniformly asymptotically stable. This is equivalent to the uniform asymptotic stability of the equilibrium state $x = x^*$ of system (1).

Theorem 3.3 *Let the following conditions be satisfied*

(1) *for system (1) assumptions S_1-S_3 hold true.*

(2) *functions $s_i \in C^2(\mathbb{R})$ and there exist constants $K_i > 0$ such that $|s_i''(r)| \leq K_i$ for all $r \in \mathbb{R}$, $i = 1, 2, \ldots, n$.*

(3) *there exist a positive definite symmetric matrix $P \in \mathbb{R}^{n \times n}$ and a constant $M > 0$ such that $|1 + \mu(t)A(t)| \geq M$ for all $t \in \mathbb{T}_\tau$, where $B_1 = -B + TH$, $H = \mathrm{diag}\{s_i'(0)\} \in \mathbb{R}^{n \times n}$, $A(t) = \lambda_M(PB_1 + B_1^{\mathrm{T}} P) + \mu(t)\|P\| \|B_1\|^2$.*

Then, if

(a) *$\limsup\limits_{t \to \infty} \beta_A(t) = q < 0$, the equilibrium state $x = x^*$ of system (1) is exponentially stable;*

(b) $\sup\{\beta_A(t):\ t\in\mathbb{T}_\tau\}=\overline{q}<0$, *the equilibrium state* $x=x^*$ *of system* (1) *is uniformly exponentially stable.*

Proof We shall apply function $V(y)=y^{\mathrm{T}}Py$ and for the derivative of function V along solutions of system (5) we shall use the expression obtained in the previous theorem

$$
\begin{aligned}
V^\Delta(y(t))|_{(5)} &\le \big(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\,\|B_1\|^2\big)\|y(t)\|^2 \\
&\quad + 2\|P\|\,\|T\|\,\|\tilde{g}_2(y(t))\|\,\|y(t)\| + 2\mu(t)\|P\|\,\|B_1\|\,\|T\|\,\|\tilde{g}_2(y(t))\|\,\|y(t)\| \\
&\quad + \mu(t)\|P\|\,\|\tilde{g}_2(y(t))\|^2\|T\|^2 \le \big(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\,\|B_1\|^2\big)\|y(t)\|^2 \\
&\quad + \big(2K\|P\|\,\|T\|\,\|y(t)\| + 2\mu(t)K\|P\|\,\|B_1\|\,\|T\|\,\|y(t)\| \\
&\quad + \mu(t)K^2\|P\|\,\|T\|^2|y(t)\|^2\big)\|y(t)\|^2 = A(t)\|y(t)\|^2 + \Phi(t,V(y)),
\end{aligned}
$$

where $\Phi(t,V)=\big[2K\|P\|\,\|T\|(1+\mu(t)\|B_1\|)\sqrt{V}+\mu(t)K^2\|P\|\,\|T\|^2V\big]V.$

Consider the set $\mathcal{T}=\{t\in\mathbb{T}_\tau:\ \mu(t)\neq 0\}$. If there exists $\sup\mathcal{T}<+\infty$ then there exists $t_1\in\mathbb{T}_\tau$ such that $\mu(t)=0$ for all $t\in[t_1,+\infty)_\mathbb{T}$. If the set $\mathcal{T}$ is not bounded, the condition $\limsup_{t\to\infty}\beta_A(t)=q<0$ implies that there exists a sufficiently large $t_2\in\mathbb{T}_\tau\cap\mathcal{T}$ such that for all $t\in[t_2,+\infty)_\mathbb{T}\cap\mathcal{T}$ inequality $\beta_A(t)<0$ holds true. This yields that for all $t\in[t_2,+\infty)_\mathbb{T}\cap\mathcal{T}$ the inequality

$$
\log\big|1+\mu(t)(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\|B_1\|^2)\big|<0
$$

is true. Then

$$
\mu(t)(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\,\|B_1\|^2)-1<1,
$$
$$
\|P\|\,\|B_1\|^2\mu^2(t)+\lambda_M(PB_1+B_1^{\mathrm{T}}P)\mu(t)-2\le 0.
$$

Since $D=\lambda_M(PB_1+B_1^{\mathrm{T}}P)^2+8\|P\|\|B_1\|^2\ge 0$, we obtain the estimate $\mu(t)\le\mu_1$ for all $t\in[t_2,+\infty)\cap\mathcal{T}$, where $\mu_1=(-\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\sqrt{D})/2\|P\|\|B_1\|^2\ge 0$. Hence, one can conclude that $\mu(t)\le\mu_1$ for all $t\in[t_3,+\infty)_\mathbb{T}$, $t_3=\max\{t_1,t_2\}$. If $t\in[\tau,\rho(t_3)]\cap\mathbb{T}$ then $\mu(t)\le t_3$. This implies the estimate $\mu(t)\le\mu^*=\max\{\mu_1,t_3\}$ for all $t\in\mathbb{T}_\tau$. Since

$$
\begin{aligned}
\frac{\Phi(t,V)}{V} &= 2K\|P\|\,\|T\|(1+\mu(t)\|B_1\|)\sqrt{V}+\mu(t)K^2\|P\|\,\|T\|^2V \\
&\le 2K\|P\|\,\|T\|(1+\mu^*\|B_1\|)\sqrt{V}+\mu^*K^2\|P\|\,\|T\|^2V,
\end{aligned}
$$

we get $\Phi(t,V)/V\to 0$ for $V\to 0$ uniformly in t. According to Theorem 2 from the paper [13] we conclude that the equilibrium state $y=0$ of system (5) is exponentially stable. This is equivalent to the exponential stability of the equilibrium state $x=x^*$ of system (1).

Now we shall prove the second part of the theorem. Condition $\sup\{\beta_A:\ t\in\mathbb{T}_\tau\}=\overline{q}<0$ for $t\in\mathcal{T}$ implies

$$
\log|1+\mu(t)(\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\mu(t)\|P\|\|B_1\|^2)|\le\mu(t)\overline{q}<0
$$

for all $t\in\mathcal{T}$. Hence, we get

$$
\mu(t)\le\frac{-\lambda_M(PB_1+B_1^{\mathrm{T}}P)+\sqrt{D}}{2\|P\|\|B_1\|^2}=\mu^*,\quad \mu^*\ge 0,\ \ t\in\mathcal{T}.
$$

That is that $\mu(t) \leq \mu^*$ for all $t \in \mathbb{T}_\tau$. Then, similar to the above, we have $\Phi(t,V)/V \to 0$ for $V \to 0$ uniformly in t.

Therefore, all conditions of Theorem 2 from the paper [13] are satisfied and the equilibrium state $y = 0$ of system (5) is uniformly exponentially stable. This is equivalent to the uniform exponential stability of the equilibrium state $x = x^*$ of system (1).

Remark 3.1 Consider the scale $\mathbb{T} = \mathbb{N}$ ($\mu(t) \equiv 1$). In this case system of equations (1) is equivalent to system (4) and the condition of uniform asymptotic stability of the equilibrium state of system (1) established in Theorem 3.1 for $\mu^* = 1$ becomes

$$2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2 > 0.$$

This result coincides completely with the below result for discrete system (4).

Theorem 3.4 *For neural discrete system* (4) *let assumptions* S_2, S_3 *be satisfied. Then the equilibrium state* $x = x^*$ *of system* (4) *is uniformly asymptotically stable, provided that*

$$2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2 > 0.$$

Proof Consider function $y(k) = x(k) - x^*$ and rewrite equations (4) as

$$y(k+1) = (-B + I)y(k) + Tg(x(k)), \quad k \in \mathbb{T}_\tau, \tag{8}$$

where I is an identity $n \times n$-matrix and for the first difference of function $V(y) = y^{\mathrm{T}}y$ we get the estimate

$$
\begin{aligned}
\Delta V(y(k))|_{(8)} &= y^{\mathrm{T}}(k+1)y(k+1) - y^{\mathrm{T}}(k)y(k) \\
&= [(-B+I)y(k) + Tg(y(k))]^{\mathrm{T}}[(-B+I)y(k) + Tg(y(k))] - y^{\mathrm{T}}(k)y(k) \\
&= y^{\mathrm{T}}(k)B^{\mathrm{T}}By(k) - 2y^{\mathrm{T}}(k)B^{\mathrm{T}}y(k) - 2y(k)^{\mathrm{T}}BTg(y(k)) \\
&\quad + 2y^{\mathrm{T}}(k)Tg(y(k)) + G^{\mathrm{T}}(y(k))T^{\mathrm{T}}Tg(y(k)) \\
&\leq \|B\|^2\|y(k)\|^2 - 2\lambda_m(B)\|y(k)\|^2 + 2L\|B\|\,\|T\|\,\|y(k)\|^2 \\
&\quad + 2L\|T\|\,\|y(k)\|^2 + \|T\|^2\|g(y(k))\|^2 \\
&\leq \left[\overline{b}^2 - 2\underline{b} + 2L\overline{b}\|T\| + 2L\|T\| + \|T\|^2 L^2\right]\|(y(k))\|^2 \\
&= -\left[2\underline{b} - 2L\|T\| - (\overline{b} + L\|T\|)^2\right]\|(y(k))\|^2.
\end{aligned}
$$

This yields the assertion of the theorem.

The regressivity of function $f(x) = -Bx + Ts(x) + u$ is one of conditions for existence of solution of problem (1), (2). Here we give some sufficient regressivity conditions for the function $f(x)$.

Theorem 3.5 *Let assumption* S_3 *be fulfilled. If for every fixed* $t \in \mathbb{T}$ *the matrix* $(I - \mu(t)B)\Lambda^{-1} - \mu(t)|T|$ *is an* M-matrix, the function $f(x) = -Bx + Ts(x) + u$ is regressive.

Proof We fix $t \in \mathbb{T}$ and consider the mapping $R: \mathbb{R}^n \to \mathbb{R}^n$ given by the formula

$$R(x) = x + \mu(t)f(t,x) = (I - \mu(t)B)x + \mu(t)Ts(x) + \mu(t)u.$$

Designate by $\widetilde{B} = (I - \mu(t)B)$, $\widetilde{T} = \mu(t)T$ and $\widetilde{u} = \mu(t)u$. Then we get

$$R(x) = \widetilde{B}x + \widetilde{T}s(x) + \widetilde{u}.$$

Since the matrix $\widetilde{B}\Lambda^{-1} - |\widetilde{T}|$ is an M-matrix, the mapping $R: \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism [17]. Hence follows the reversibility of the mapping $R(x)$ which is equivalent to the reversibility of the operator $I + \mu(t)f(t, \cdot): \mathbb{R}^n \to \mathbb{R}^n$.

4   Example

On the time scale $\mathbb{P}_{1,\gamma} = \bigcup_{j=0}^{\infty}[j(1+\gamma), j(1+\gamma) + 1]$, $\gamma > 0$, we consider a neural network

$$\begin{aligned}
x_1^{\Delta}(t) &= -b_1 x_1(t) + t_{11}s(x_2(t)) + t_{12}s(x_2(t)) + u_1, \\
x_2^{\Delta}(t) &= -b_2 x_1(t) + t_{21}s(x_1(t)) + t_{22}s(x_2(t)) + u_2,
\end{aligned} \tag{9}$$

where $x_1, x_2 \in \mathbb{R}$, $u_1, u_2 \in \mathbb{R}$, $b_1 = b_2 = 1$, $T = \left(\begin{smallmatrix} 0.1 & -0.5 \\ 0.5 & 0.1 \end{smallmatrix}\right)$, $s(u) = \tanh u$.

For the time scale $\mathbb{P}_{1,\gamma}$ the granularity function

$$\mu(t) = \begin{cases} 0, & t \in \bigcup_{j=0}^{\infty} [j(1+\gamma), j(1+\gamma)+1), \\ \gamma, & t \in \bigcup_{j=0}^{\infty} \{j(1+\gamma)+1\}. \end{cases}$$

We take matrix $P = \operatorname{diag}\{0.5, 0.5\}$ and write out all the functions and constants mentioned in the conditions of Theorem 3.3

$$M_1 = M_2 = L_1 = L_2 = 1, \quad A(t) = -0.9 + 0.53\gamma,$$

$$K_1 = K_2 = 8\left|e^{\frac{2+\sqrt{3}}{2}} - e^{-\frac{2+\sqrt{3}}{2}}\right|\bigg/\left(e^{\frac{2+\sqrt{3}}{2}} + e^{-\frac{2+\sqrt{3}}{2}}\right)^3,$$

$$\beta_A(t) = \begin{cases} \gamma^{-1}\log|1 + \gamma(-0.9 + 0.53\gamma)|, & t \in \bigcup_{j=0}^{\infty} \{j(1+\gamma)+1\}, \\ -0.9 + 0.53\gamma, & t \in \bigcup_{j=0}^{\infty} [j(1+\gamma), j(1+\gamma)+1). \end{cases}$$

The regressivity condition has the form of the inequalities

$$\begin{cases} 1 - 1.1\gamma > 0, \\ (1 - 1.1\gamma)^2 - 0.25\gamma^2 > 0, \end{cases}$$

which yields $\gamma < 0.625$. Since $1 + \gamma(-0.9 + 0.53\gamma) \geq 1 + \gamma_0(-0.9 + 0.53\gamma_0)$, $\gamma_0 = 0.9/(2 \cdot 0.53)$ for any γ, we can take for the constant M the following value: $M = 1 + \gamma_0(-0.9 + 0.53\gamma_0) = 0.61$.

For $\gamma < 1.69$ the system of inequalities

$$\begin{cases} M \leq |1 + \gamma(-0.9 + 0.53\gamma)| < 1, \\ -0.9 + 0.53\gamma < 0 \end{cases}$$

is satisfied. This implies that $\sup_t \beta_A(t) = \max\{\gamma^{-1}\log|1 + \gamma(-0.9 + 0.53\gamma)|, -0.9 + 0.53\} < 0$. Since the matrix $B\Lambda^{-1} - |T| = \left(\begin{smallmatrix} 0.9 & -0.5 \\ -0.5 & 0.9 \end{smallmatrix}\right)$ is an M-matrix, for $0 < \gamma < 0.625$ system (9) possesses a unique equilibrium state for any $u_1, u_2 \in \mathbb{R}$ and this equilibrium state is uniformly exponentially stable.
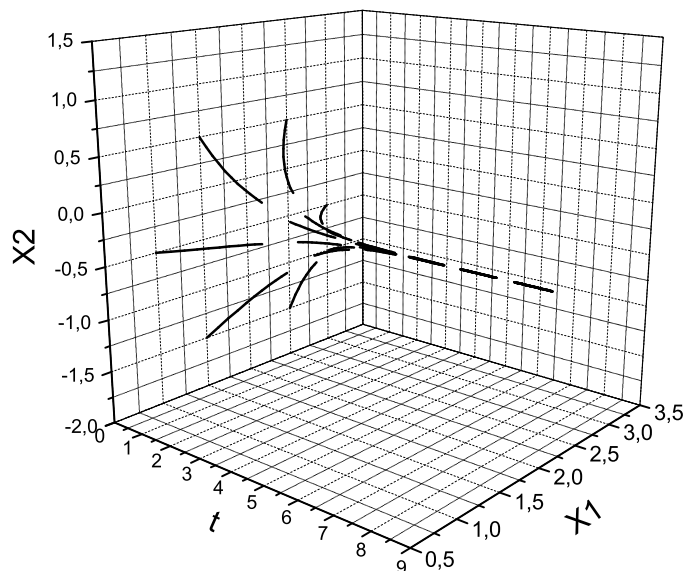
Figure 1: Dependence of the function $x(t)$ on time t obtained by numerical solution of system of equations (9).

We shall consider a model example for this problem. We take the following values of the constants: $u_1 = 2$, $u_2 = -1$, $\gamma = 0.5$. The result of numerical solution of system (9) is shown in Figure 1. It is seen from the figure, for arbitrary chosen initial conditions $(1, -0.5)$, $(1.5, -1.5)$, $(2.5, -1.5)$, $(3, -0.5)$, $(2.5, 0.5)$, $(1.5, 0.5)$ the function $x(t)$ approaches asymptotically with time t to the equilibrium state $(x_1^*, x_2^*)^{\mathrm{T}} = (2.35, -0.56)^{\mathrm{T}}$.

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Equilibrium States for Pre-image Pressure

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Abstract: In this paper equilibrium states for pre-image pressure are considered. We study the ergodic decomposition of Cheng–Newhouse metric pre-image entropy. Moreover, for a topological dynamical system (X, T) with finite topological pre-image entropy and upper semi-continuous metric pre-image entropy function $h_{\{pre,\bullet\}}(T)$, we obtain a way to describe a kind of continuous dependence of equilibrium states, and show that all functions with unique equilibrium state is dense in $C(X)$. Last, we also discuss the uniformity of equilibrium states for pre-image pressure.

Keywords: *pre-image pressure, equilibrium states, metric pre-image entropy.*

Mathematics Subject Classification (2000): Primary: 37A35, 37B40; Secondary: 47A35, 54H20.

1 Introduction

Entropies are fundamental to our current understanding of dynamical systems, and topological pressure is a generalization to topological entropy for a dynamical system (see [1] and [2]). Recently, the pre-image structure of maps has become deeply characterized via entropies and pressures, and several important pre-image entropy and pressure invariants have been introduced (see [3, 4, 5, 6, 7]).

In [3], F. Zeng, K. Yan and G. Zhang studied the topological pre-image pressure of topological dynamical systems, and proved a variational principle for it. They considered a compact metric space X and a continuous map $T : X \to X$. The pre-image pressure is defined as a real-valued continuous convex function $P_{pre}(T, \bullet)$ on $C(X)$, where $C(X)$

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denotes the Banach space of all real-valued continuous functions on X with the supremum norm. They showed that $P_{pre}(T, f) = \sup_{\mu \in \mathcal{M}(X,T)}(h_{pre,\mu}(T) + \mu(f))$, where $\mathcal{M}(X,T)$ denotes the collection of all T-invariant probability measures on X, $\mu(f) = \int_X f d\mu$ and $h_{pre,\mu}(T)$ the pre-image entropy of μ with respect to T (see [3, 4] for definition). An $\mu \in \mathcal{M}(X,T)$ such that $h_{pre,\mu}(T) + \mu(f)$ attains its supremum is called equilibrium state. For each $f \in C(X)$, there exist tangent functionals to $P_{pre}(T, \bullet)$ at f, whereas there may be no equilibrium states for f. If $\mathcal{T}_f(X,T)$ denotes the set of tangent functionals to $P_{pre}(T, \bullet)$ at f and $\mathcal{M}_f(X,T)$ the set of equilibrium states for f then one has $\mathcal{M}_f(X,T) \subset \mathcal{T}_f(X,T) \subset \mathcal{M}(X,T)$ and $\mathcal{T}_f(X,T) = \mathcal{M}_f(X,T)$ if and only if the pre-image entropy function $h_{\{pre,\cdot\}}(T)$ is upper semi-continuous at the members of $\mathcal{T}_f(X,T)$(see § 2 for definitions and [3] for some results).

The purpose of this note is to consider equilibrium states for pre-image pressure of the topological dynamical system (X,T) with finite pre-image entropy. In Section 2, we concentrate on the ergodic decomposition of measure pre-image entropy, and review some definitions and some basic properties.

In Section 3, we consider a kind of continuous dependence of the equilibrium states $\mathcal{M}_f(X,T)$ on the function f.

In Section 4, we discuss uniqueness and uniformity of equilibrium states for pre-image pressure. We obtained the collection of continuous functions which has unique equilibrium state relative to pre-image pressure and is a dense G_δ-set of $C(X)$. We also show that for any finite collection of ergodic measures, we can find some continuous function such that they contain its equilibrium states set.

2   Preliminaries

In this section, we will recall some definitions and give some useful lemmas.

For a given topological dynamical system (X,T) (where X is a compact metric space and T is a continuous map from X to itself), denote by $\mathcal{B}(X)$ the collection of all Borel subsets. A *partition* of X is a finite disjoint collection of Borel subsets of X whose union is X. For finite partitions α, β, we set $\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}$ and $T^{-1}\alpha = \{T^{-1}(A) : A \in \alpha\}$. If $0 \le j \le n$ are positive integers, we let $\alpha_j^n = \bigvee_{i=j}^n T^{-i}\alpha$ and $\alpha^n = \alpha_0^{n-1}$. Set $\mathcal{B}^- = \bigcap_{n=0}^\infty T^{-n}\mathcal{B}(X)$, then $\mathcal{B}^-$ is a T-invariant sub-σ algebra. We call $\mathcal{B}^-$ the *infinite past σ-algebra related to $\mathcal{B}(X)$*.

Denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X, $\mathcal{M}(X,T) \subset \mathcal{M}(X)$ is the set of T-invariant measures, and $\mathcal{M}^e(X,T) \subset \mathcal{M}(X,T)$ is the set of ergodic measures. Then both $\mathcal{M}(X)$ and $\mathcal{M}(X,T)$ are convex, compact metric spaces endowed with the weak*-topology (see Chapter 6 in [1]).

Given partitions α, β of X, $\mu \in \mathcal{M}(X)$ and a σ-algebra $\mathcal{A} \subset \mathcal{B}(X)$, define

$$H_\mu(\alpha|\mathcal{A}) := \sum_{A \in \alpha} \int_X -\mathbb{E}(1_A|\mathcal{A}) \log \mathbb{E}(1_A|\mathcal{A}) d\mu,$$

$$H_\mu(\alpha|\beta \vee \mathcal{A}) := H_\mu(\alpha \vee \beta|\mathcal{A}) - H_\mu(\beta|\mathcal{A}),$$

where $\mathbb{E}(1_A|\mathcal{A})$ is the expectation of 1_A with respect to $\mathcal{A}$. It is well-known that $H_\mu(\alpha|\mathcal{A})$ increases with respect to α and decreases with respect to $\mathcal{A}$.

When $\mu \in \mathcal{M}(X,T)$ and $\mathcal{A}$ is a T-invariant measurable sub-σ-algebra of X, it is not hard to see that $a_n = H_\mu(\alpha^n|\mathcal{A})$ is a non-negative sub-additive sequence for a given

partition α, i.e. $a_{n+m} \leq a_n + a_m$ for all positive integers n and m. It is well known that

$$\lim_{n \to \infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n}.$$

The *conditional entropy of α with respect to $\mathcal{A}$* is then defined by

$$h_\mu(T, \alpha|\mathcal{A}) := \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha^n|\mathcal{A}) = \inf_{n \geq 1} \frac{1}{n} H_\mu(\alpha^n|\mathcal{A}).$$

Moreover, the *metric conditional entropy of (X, T) with respect to $\mathcal{A}$* is defined by

$$h_\mu(T, X|\mathcal{A}) = \sup_\alpha h_\mu(T, \alpha|\mathcal{A}).$$

Note that if $\mathcal{N}$ is a trivial σ-algebra, we recover the metric entropy, and we write $h_\mu(T, \alpha|\mathcal{N})$ and $h_\mu(T, X|\mathcal{N})$ simple as $h_\mu(T, \alpha)$ and $h_\mu(T)$.

Particularly, if $\mathcal{A}$ is the infinite past σ-algebra $\mathcal{B}^-$, we define the *measure-theoretic (or metric) pre-image entropy of α with respect to (X, T)* by

$$h_{pre,\mu}(T, \alpha) := h_\mu(T, \alpha|\mathcal{B}^-) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha^n|\mathcal{B}^-).$$

Moreover, we define the *metric pre-image entropy of (X, T)* by

$$h_{pre,\mu}(T) := \sup_\alpha h_{pre,\mu}(T, \alpha).$$

In [4], Cheng-Newhouse have shown that the quantity $h_{pre,\mu}(T)$ satisfied power and product rules analogous to the standard metric entropy, that the map $\mu \to h_{pre,\mu}(T)$ was affine, and that there was an analog of the Shannon-Breiman-McMillan theorem for the metric pre-image entropy. In [5], Wen-Chiao Cheng obtained a method for calculating the metric pre-image entropy, which is similar to the Kolmogorov-Sinai theorem for the metric entropy.

Now we discuss the ergodic decomposition of metric pre-image entropy. Given a partition α of X, put $\alpha^- = \bigvee_{n=1}^\infty T^{-n}\alpha$ and $\alpha^T = \bigvee_{n=-\infty}^{+\infty} T^{-n}\alpha$. The following lemma is a classical result in ergodic theory (see for example [8]).

Lemma 2.1 *(Pinsker formula) Let α, β be two partitions of X. Then*

$$h_\mu(T, \alpha \vee \beta) = h_\mu(T, \beta) + H_\mu(\alpha|\beta^T \vee \alpha^-).$$

Lemma 2.2 *(Ergodic decomposition of metric entropy, [1, Theorem 8.4]) Let (X, T) be a topological dynamical system and α be a partition of X. If $\mu \in \mathcal{M}(X, T)$ and $\mu = \int_{\mathcal{M}^e(X,T)} m d\tau(m)$ is the ergodic decomposition of μ, then we have:*

$$h_\mu(T, \alpha) = \int_{\mathcal{M}^e(X,T)} h_m(T, \alpha) d\tau(m).$$

Lemma 2.3 *([5, Lemma 4.13]) Let (X, T) be a topological dynamical system, $\mu \in \mathcal{M}(X, T)$ and α be a partition of X. Then*

$$h_{pre,\mu}(T, \alpha) = H_\mu(\alpha|\alpha^- \vee \mathcal{B}^-).$$

Theorem 2.1 *(Ergodic decomposition of metric pre-image entropy). Let (X,T) be a topological dynamical system, $\mu \in \mathcal{M}(X,T)$ and α be a partition of X. If $\mu = \int_{\mathcal{M}^e(X,T)} m d\tau(m)$ is the ergodic decomposition of μ, then*

$$h_{pre,\mu}(T,\alpha) = \int_{\mathcal{M}^e(X,T)} h_{pre,m}(T,\alpha)d\tau(m),$$

and

$$h_{pre,\mu}(T) = \int_{\mathcal{M}^e(X,T)} h_{pre,m}(T)d\tau(m).$$

Proof Take an increasing sequence of finite Borel partitions β_j of X with $diam(\beta_j) \to 0$. Then using the Pinsker formula, the ergodic decomposition of metric entropy, Lemma 2.3 and dominated convergence theorem, we have

$$h_{pre,\mu}(T,\alpha) = H_\mu(\alpha|\alpha^- \vee \mathcal{B}^-) = \lim_{k\to\infty} H_\mu(\alpha|\alpha^- \vee T^{-k}\mathcal{B}(X))$$

$$= \lim_{k\to\infty} \lim_{j\to\infty} H_\mu(\alpha|\alpha^- \vee (T^{-k}\beta_j)^T)$$

$$= \lim_{k\to\infty} \lim_{j\to\infty} [h_\mu(T, \alpha \vee T^{-k}\beta_j) - h_\mu(T, T^{-k}\beta_j)]$$

$$= \lim_{k\to\infty} \lim_{j\to\infty} \int_{\mathcal{M}^e(X,T)} [h_m(T, \alpha \vee T^{-k}\beta_j) - h_m(T, T^{-k}\beta_j)]d\tau(m)$$

$$= \lim_{k\to\infty} \lim_{j\to\infty} \int_{\mathcal{M}^e(X,T)} H_m(\alpha|\alpha^- \vee (T^{-k}\beta_j)^T)d\tau(m)$$

$$= \int_{\mathcal{M}^e(X,T)} \lim_{k\to\infty} \lim_{j\to\infty} H_m(\alpha|\alpha^- \vee (T^{-k}\beta_j)^T)d\tau(m)$$

$$= \int_{\mathcal{M}^e(X,T)} h_{pre,m}(T,\alpha)d\tau(m).$$

Moreover, we can get

$$h_{pre,\mu}(T) = \lim_{j\to\infty} h_{pre,\mu}(T,\beta_j) = \lim_{j\to\infty} \int_{\mathcal{M}^e(X,T)} h_{pre,m}(T,\beta_j)d\tau(m)$$

$$= \int_{\mathcal{M}^e(X,T)} \lim_{j\to\infty} h_{pre,m}(T,\beta_j)d\tau(m)$$

$$= \int_{\mathcal{M}^e(X,T)} h_{pre,m}(T)d\tau(m).$$

Theorem 2.1 is proved.   □

Following the idea of topological pressure (see [1]), F.Zeng etc. defined a new notion of pre-image pressure, which extends Cheng-Newhouse pre-image entropy [4]. For a given topological dynamical system (X,T), the pre-image pressure of T is a map $P_{pre}(T, \bullet) : C(X) \to \mathbb{R}$ which is convex, Lipschitz continuous, increasing, with $P_{pre}(T,0) = h_{pre}(T)$ (see [3] for definition).

Given $f \in C(X)$. A member $\mu \in \mathcal{M}(X, T)$ is called an *equilibrium state* for f if $P_{pre}(T, f) = h_{pre,\mu}(T) + \mu(f)$. By the variational principle (Theorem 3.1 in [3])this is equivalent to requiring

$$h_{pre,\mu}(T) + \mu(f) = \sup\{h_{pre,m}(T) + m(f) : m \in \mathcal{M}(X, T)\}.$$

Let $\mathcal{M}_f(X, T)$ denote the collection of all equilibrium states for f. Note that this set could be empty (see Example 5.1 in [3]).

A *tangent functional* to $P_{pre}(T, \bullet)$ at f is a finite signed Borel measure μ on X such that

$$P_{pre}(T, f + g) - P_{pre}(T, f) \geq \mu(g), \ \forall \ g \in C(X).$$

Let $\mathcal{T}_f(X, T)$ denote the collection of all tangent functionals to $P_{pre}(T, \bullet)$ at f. An application of the Hahn-Banach theorem gives $\mathcal{T}_f(X, T) \neq \emptyset$. It is easy to see that $\mu \in \mathcal{T}_f(X, T)$ if and only if

$$P_{pre}(T, f) - \mu(f) = \inf\{P_{pre}(T, h) - \mu(f) : h \in C(X)\}.$$

Also we have $\mathcal{T}_f(X, T) \subset \mathcal{M}(X, T)$ (see [3] for details).

Proposition 2.1 *The following holds.*
(1) $\mathcal{M}_f(X, T)$ is convex;
(2) if the pre-image entropy map $h_{pre,\bullet}(T)$ is upper semi-continuous then $\mathcal{M}_f(X, T)$ is compact and non-empty;
(3) the extreme points of $\mathcal{M}_f(X, T)$ are precisely the ergodic members of $\mathcal{M}_f(X, T)$;
(4) If $\mu \in \mathcal{M}_f(X, T)$ and $\mu = \int_{\mathcal{M}^e(X,T)} m d\tau(m)$ is the ergodic decomposition of μ, then for τ-a.e. $m \in \mathcal{M}^e(X, T)$, $m \in \mathcal{M}_f(X, T)$.

Proof (1)-(3) can see Theorem 5.1 in [3].
(4) This follows from the following two facts: (i) $h_{pre,m}(T) + m(f) \leq P_{pre}(T, f)$ for each $m \in \mathcal{M}^e(X, T)$; (ii) $\int_{\mathcal{M}^e(X,T)}[h_{pre,m}(T) + m(f)]d\tau(m) = h_{pre,\mu}(T) + \mu(f) = P_{pre}(T, f)$ by Theorem 2.1. □

Proposition 2.2 *Let (X, T) be a topological dynamical system with $h_{pre}(T) < \infty$ and $f \in C(X)$. Then the following holds.*
(1) $\mathcal{M}_f(X, T) \subset \mathcal{T}_f(X, T) \subset \mathcal{M}(X, T)$;
(2) $\mathcal{T}_f(X, T) = \bigcap_{n=1}^{\infty} \overline{\{\mu \in \mathcal{M}(X, T) : h_{pre,\mu}(T) + \mu(f) > P_{pre}(T, f) - 1/n\}}$;
(3) $\mathcal{M}_f(X, T) = \mathcal{T}_f(X, T)$ if and only if $h_{pre,\bullet}(T)$ is upper semi-continuous at the members of $\mathcal{T}_f(X, T)$.

Proof Theorem 5.2 in [3]. □

3　Continuous Dependence of Equilibrium State

Let (X, T) be a topological dynamical system. Throughout the following sections, we assume the topological pre-image entropy $h_{pre}(T) < \infty$, and the metric pre-image entropy function $h_{\{pre,\bullet\}}(T) : \mathcal{M}(X, T) \to \mathbb{R}$ is upper semi-continuous.

In this section, we prove a theorem to describe a kind of continuous dependence of the set $\mathcal{M}_f(X, T)$ on the function $f \in C(X)$.

Theorem 3.1 *Consider $f, g_n \in C(X)$ and $t_n \in (-1, 1)$ such that $t_n \to 0$ and $||g_n||_\infty \to 0$. Let $\mu_n \in \mathcal{M}_{(1+t_n)f+g_n}(X, T)), n > 0$. Then the following holds.*

(1) If $\{\mu_n\}_{n \geq 1}$ converges weakly to some $\mu \in \mathcal{M}(X, T)$ (i.e. $\mu_n(h) \to \mu(h)$ for all $h \in C(X)$), then $\mu \in \mathcal{M}_f(X, T)$;

(2) If $\mathcal{M}_f(X, T) = \{\mu\}$, then $\lim_{n \to \infty} \mu_n = \mu$.

Proof (1) Observe that

$$P_{pre}(T, (1+t_n)f + g_n)$$

$$= \sup_{\mu \in \mathcal{M}(X,T)} (h_{pre,\mu}(T) + \mu((1+t_n)f + g_n))$$

$$= \sup_{\mu \in \mathcal{M}(X,T)} ((1+t_n)(h_{pre,\mu}(T) + \mu(f)) - t_n h_{pre,\mu}(T) + \mu(g_n)) \qquad (1)$$

$$\geq (1+t_n)P_{pre}(T, f) - |t_n|h_{pre}(T) - ||g_n||_\infty$$

Since the metric pre-image entropy function $h_{pre,\bullet}(T)$ is upper semi-continuous,

$$h_{pre,\mu}(T) + \mu(f)$$

$$\geq \limsup_{n \to \infty} h_{pre,\mu_n}(T) + \limsup_{n \to \infty} \mu_n(f)$$

$$\geq \limsup_{n \to \infty} (h_{pre,\mu_n}(T) + \mu_n((1+t_n)f + g_n) - \mu_n(t_n f + g_n))$$

$$\geq \limsup_{n \to \infty} (P_{pre}(T, (1+t_n)f + g_n) - |t_n \mu_n(f)| - ||g_n||_\infty)$$

$$\geq \limsup_{n \to \infty} ((1+t_n)P_{pre}(T, f) - |t_n|h_{pre}(T) - |t_n \mu_n(f)| - 2||g_n||_\infty) \quad (by (1))$$

$$\geq P_{pre}(T, f) - \limsup_{n \to \infty} |t_n \mu_n(f)|$$

$$\geq P_{pre}(T, f) - \limsup_{n \to \infty} |t_n| \mu_n(|f|)$$

$$= P_{pre}(T, f) \quad (Since \ \limsup_{n \to \infty} \mu_n(|f|) = \mu(|f|) < \infty).$$

Therefore, $\mu \in \mathcal{M}_f(X, T)$.

(2) If ω is a limit point of $\{\mu_n\}_{n \geq 1}$, then $\omega = \mu$ by (1). It follows that $\mu_n \to \mu$ as $n \to \infty$. $\qquad \square$

4   Uniqueness and Uniformity of Equilibrium State

In this section, we study uniqueness and uniformity of equilibrium state for pre-image pressure. First, we have the following lemma.

Lemma 4.1 *For a given topological dynamical system (X, T), there is a dense subset $C(X)$ such that each function in this set has a unique equilibrium state for pre-image pressure.*

Proof It follows directly from (3) in Proposition 2.2 and the fact that a convex continuous function on a separable Banach space has a unique tangent functional at a dense set of points (can see [9, page 450] or [10, Appendix A.3.6]). $\qquad \square$

Denote by $2^{\mathcal{M}(X,T)}$ the hyperspace of compact metric space $\mathcal{M}(X, T)$. Define $\Phi : C(X) \to 2^{\mathcal{M}(X,T)}$ by

$$\Phi(f) = \mathcal{M}_f(X, T), \quad \forall f \in C(X).$$

Lemma 4.2 Φ *is upper semi-continuous.*

Proof If $f_n \in C(X)$ with $f_n \to f \in C(X)$ and $\mu_n \in \mathcal{M}_{f_n}(X,T)$ with $\mu_n \to \mu$ for some $\mu \in \mathcal{M}(X,T)$, then for each n we have

$$h_{pre,\mu_n}(T) + \mu_n(f_n) = P_{pre}(T, f_n).$$

Letting $n \to \infty$, then by the continuity of pre-image pressure function $P_{pre}(T, \bullet)$ (see [3, Lemma 4.1 (3)]) and the upper semi-continuity of $h_{pre,\bullet}(T)$, we have

$$h_{pre,\mu}(T) + \mu(f) \geq P_{pre}(T, f).$$

Using the variational principle of pre-image pressure, $\mu \in \mathcal{M}_f(X,T)$.   □

Theorem 4.1 *Let (X,T) be a topological dynamical system. Then the following holds.*
(1) $f \in C(X)$ has a unique equilibrium state relative to pre-image pressure if and only if Φ is continuous at f;
(2) $\mathcal{C} \subset C(X)$ is a dense G_δ set, where each $f \in \mathcal{C}$ has unique equilibrium state for pre-image pressure.

Proof (1) It follows directly from Lemma 4.2 that Φ is continuous at f whenever $\mathcal{M}_f(X,T)$ has only one element.

Now we let Φ be continuous at f. By Lemma 4.1, there is a sequence $f_n \in C(X)$ such that $f_n \to f$ and each $\mathcal{M}_{f_n}(X,T)$ is a single point set. Since Φ is continuous at f, $\mathcal{M}_f(X,T)$ also has only one element.

(2) It follows directly from Lemma 4.1, Lemma 4.2 and (1) above.   □

Now we discuss uniformity of equilibrium states for pre-image pressure. Set

$$\mathcal{M}_{pre}(X,T) = \bigcup_{f \in C(X)} \mathcal{M}_f(X,T),$$

which denote the set of all equilibrium states for pre-image pressure.

Lemma 4.3 *Given $f \in C(X)$. Then for any $\mu \in \mathcal{M}(X,T)$ and $\epsilon > 0$, there is $f' \in C(X)$ and $\mu' \in \mathcal{M}_{f'}(X,T)$ such that*

$$||\mu - \mu'|| = \sup_{g \in C(X), ||g||=1} |\mu(g) - \mu'(g)| \leq \epsilon,$$

and

$$||f - f'|| \leq \frac{1}{\epsilon}[P_{pre}(T, f) - h_{pre,\mu}(T) - \mu(f)].$$

Proof The proof follows the arguments of the proof of [10, Theorem 3.16]. First we have $P_{pre}(T, \bullet) : C(X) \to \mathbb{R}$ is convex and continuous (see [3, Lemma 4.1 (3) and (4)]). Since $\mu(g) \leq P_{pre}(T, g)$ for all $g \in C(X)$, it follows from [10, Appendix A.3.6] that there is $f' \in C(X)$ and $\mu' \in \mathcal{T}_{f'}(X,T) = \mathcal{M}_{f'}(X,T)$ such that $||\mu - \mu'|| \leq \epsilon$, and

$$||f - f'|| \leq \frac{1}{\epsilon}[P_{pre}(T, f) - \mu(f) - \inf\{P_{pre}(T, g) - \mu(g) : g \in C(X)\}]$$

$$= \frac{1}{\epsilon}[P_{pre}(T, f) - \mu(f) - h_{pre,\mu}(T)] \quad (By \ [3, \ Theorem \ 4.2]).$$

The lemma is proved.   □

Theorem 4.2 *The following holds.*
(1) The set $\mathcal{M}_{pre}(X,T)$ is dense in $\mathcal{M}(X,T)$;
(2) For any finite collection of ergodic measures $\{\mu_1, \mu_2, \cdots, \mu_n\}$, there is a $f \in C(X)$ such that $\{\mu_1, \mu_2, \cdots, \mu_n\} \subset \mathcal{M}_f(X,T)$.

Proof (1) It follows directly from Lemma 4.3.
(2) Use (1), we know that there is $f \in C(X)$ and $\mu \in \mathcal{M}_f(X,T)$ such that

$$||\mu - \frac{1}{n}(\mu_1 + \mu_2 + \cdots + \mu_n)|| < \frac{1}{n}.$$

Let $\mu = \int_{\mathcal{M}^e(X,T)} m d\tau(m)$ be the ergodic decomposition of μ. Then we have

$$||\tau - \frac{1}{n}(\delta_{\mu_1} + \delta_{\mu_2} + \cdots + \delta_{\mu_n})|| < \frac{1}{n},$$

(see [10, Appendix A.5.5]), and therefore $\tau(\{\mu_1\}) > 0, \cdots, \tau(\{\mu_n\}) > 0$. Thus $\{\mu_1, \mu_2, \cdots, \mu_n\} \subset \mathcal{M}_f(X,T)$ by (4) in Proposition 2.1.   □

Acknowledgment

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