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Existence and Uniqueness for Nonlinear Multi-variables Fractional Differential Equations

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Abstract: The existence and uniqueness of solutions of nonlinear multi-variables fractional differential equations have been investigated. Using Schauder fixed points theorems and Global contraction mapping theory, we obtain two results concerning the existence and uniqueness of solutions respectively. Moreover, our results are more general than in [8].

Keywords: *existence and uniqueness; nonlinear multi-variables fractional differential equations; Schauder fixed points theorems.*

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1 Introduction

In recent years, interest has increased concerning the fractional differential equations [5, 13, 26]. Most of works are devoted to the solvability of linear fractional equations in terms of special functions [1, 7] and to problems of analyticity in the complex domain [6]. There are also some studies on the solution of nonlinear differential equations [8]–[11] and [20]. D. Delbosco argues nonlinear fractional equation [11]. E. Ahmed has investigated the fractional-order Lotka–Volterra predator-prey system [20]. Very few contributions exist, as far as we know, concerning nonlinear multi-variables fractional equations of the form

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$$\begin{cases} {}_0D_t^{s_1} u_1(t) = f_1(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_i} u_1(t) = f_i(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_n} u_1(t) = f_n(t, u_1(t), u_2(t), \cdots, u_n(t)), \end{cases}$$

where $0 < s_i < 1$, and $i = 1, \cdots, n$ and ${}_0D_t^{s_i}$ is the standard Riemann–Liouville fractional derivative, considered in $\mathbb{R}^+$ or in an interval $(0, a)$, with $a > 0$.

Fractional-order calculus will play an important role in mechatronic and biological systems. It has been found that the behavior of many physical systems can be properly described by using the fractional order system theory. For example, heat conduction, dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, visco-elastic systems, quantum evolution of complex systems, quantitative finance and diffusion wave are among the known dynamical systems that were modeled using fractional order equations. In fact, real world processes generally or most likely are nonlinear multi-variable fractional order systems. In the last 6 years, considerable attention has also been paid to obtain analytical existence conditions for nonlinear fractional order systems [27]. Our aim is to analyze uniqueness conditions further more. The paper is organized as follows. In Section 2 we recall the definitions of fractional integral and derivative and related basic properties used in the text. Section 3 contains results for solutions which are continuous at the origin. Conclusions are given in Section 4.

2 Definitions and Preliminary Results

The definitions and the results of the fractional calculus reported below are not exhaustive but rather oriented to the subject of this paper. For the proofs, which are omitted, we refer the reader to Miller and Ross [7] or other texts on basic fractional calculus.

Definition 2.1 The uniform formula of a fractional integral with $\alpha \in (0, 1)$ is defined as

$${}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau)\, d\tau,$$

where $f(t) : \mathbb{R}^+ \longrightarrow \mathbb{R}$, is an arbitrary integrable function, ${}_0D_t^{-\alpha}$ is the fractional integral of order α on $[0, t]$, $t > 0$ and $\Gamma(\cdot)$ denotes the Gamma function. For an arbitrary real number, the Riemann–Liouville fractional derivative is defined as

$${}_0D_t^p f(t) = \tfrac{d^{[p]+1}}{dt^{[p]+1}} \left[{}_0D_t^{-[p]-p+1} f(t) \right].$$

The following properties are some of the main ones of the fractional derivatives and integrals [12, 18, 20].

Property 1. ${}_0D_t^p t^{\upsilon} = \frac{\Gamma(1+\upsilon)}{\Gamma(1+\upsilon-p)} t^{\upsilon - p}$, where $p \in \mathbb{R}$, $\upsilon > -1$.

Property 2. ${}_0D_t^p \left({}_0D_t^q f(t)\right) = {}_0D_t^{p+q} f(t)$, where $p < 0, q < 0$.

Property 3. ${}_0D_t^p \left({}_0D_t^{-p} f(t)\right) = f(t)$, where $p \in \mathbb{R}^2$, $t > 0$.

Property 4. ${}_0D_t^{-\alpha} f(0) = 0$, where $f \in C[0,a]$, $\alpha \in (0,1)$.

Property 5. ${}_0D_t^{p} f(t) = 0$, where $f \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $\alpha \in (0,1)$, then $f(x) = cx^{p-1}$, $c \in \mathbb{R}$.

Proposition 2.1 *Assume that* $f \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ *with a fractional derivative order* $0 < \alpha < 1$ *that belongs* $C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$*. Then*

$$ {}_aD_t^{-\alpha}\left({}_aD_t^{\alpha} f(t)\right) = f(t) + cx^{\alpha-1} $$

for some $\mathrm{c} \in \mathbb{R}$*.*

When the function $f(t)$ is in $C(\mathbb{R}^+)$, then $c = 0$.

In all the definitions and results of this section the set $\mathbb{R}^+$ can be substituted by the intervals $(0,a)$ or $(0,a]$, $a > 0$. For simplicity, in the next sections we shall often limit arguments to the choice $a = 1$. A more precise analysis of the operators ${}_0D_t^{-\alpha}, {}_0D_t^{\alpha}$ can be given in the frame of the spaces $C_r(\mathbb{R}^+)$, $r > 0$, of all functions $f \in C\left(\mathbb{R}_0^+\right)$ such that $x^r\ f \in C\left(\mathbb{R}_0^+\right)$.

Let $0 < \alpha < 1$; if $f \in C\left(\mathbb{R}_0^+\right)$ with $r < \alpha$, then ${}_0D_t^{-\alpha} f \in C\left(\mathbb{R}_0^+\right)$ with ${}_0D_t^{-\alpha} f(0) = 0$. If $f \in C_\alpha(\mathbb{R}^+)$, then ${}_0D_t^{-\alpha} f$ is bounded at the origin if $f \in C\left(\mathbb{R}_0^+\right)$. With $\alpha < r < 1$, then we may expect ${}_0D_t^{-\alpha} f$ to be unbounded at the origin. Concerning Proposition 2.1, the last part can now be stated more precisely. If $f \in C_r\left(\mathbb{R}_0^+\right)$ with $r < 1 - \alpha$ and ${}_0D_t^{\alpha} f \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, then ${}_0D_t^{-\alpha}\left({}_0D_t^{\alpha} f(t)\right) = f(t)$.

3 Existence and Uniqueness

Consider the fractional differential equations

$$ \begin{cases} {}_0D_t^{s_1} u_1(t) = f_1(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_i} u_1(t) = f_i(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_n} u_1(t) = f_n(t, u_1(t), u_2(t), \cdots, u_n(t)), \end{cases} \tag{1} $$

where $0 < s_i < 1$, and $i = 1, \cdots, n$ and $f_i : [0,a] \times \mathbb{R} \to \mathbb{R}, 0 < a < +\infty$ are given functions, continuous in $(0,a) \times \mathbb{R}$.

We introduce the following definition of a solution for (1).

Definition 3.1 Let $C^*[0,a]$ be the class of continuous column vector $U(t) = (u_1(t), u_2(t), \cdots u_n(t))$ whose components $u_1(t), u_2(t), \cdots u_n(t) \in C[0,a]$ the class of continuous functions on the interval $[0,a]$. The norm of $U \in C^*[0,a]$ is given by

$$ \|U\| = \max_{1 \le i \le n} \left\{ \sup_{0 \le x \le a} u(x) \right\}. $$

Definition 3.2 By a solution of the fractional differential equations (1) we mean a column vector $U \in C^*[0,a]$. This vector satisfies (1).

Remark 3.1 We may apply the results of Section 2, in particular Proposition 2.1 and the subsequent remarks, to reduce(1) to integral equations. In fact, if $U(t) = (u_1(t), u_2(t), \cdots u_n(t)) \in C^*[0,a]$ or more generally $U \in C_r^*[0,a]$, with $r < 1 - s$,where $s = \min\limits_{1\le i\le n} \{s_i\}$, and further assumptions guarantee $f_i(t, u_1(t), u_2(t), \cdots u_n(t)) \in C[0,a] \cap L^1[0,a]$, then equations (1) are equivalent to the integral equations

$$\begin{cases} u_1(t) = {}_0D_t^{-s_1} f_1(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ u_i(t) = {}_0D_t^{-s_i} f_i(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ u_n(t) = {}_0D_t^{-s_n} f_n(t, u_1(t), u_2(t), \cdots, u_n(t)). \end{cases} \tag{2}$$

Such a reduction will be systematically used in this section. We first present Schauder fixed point theorem. It can be easily proved [25].

Theorem 3.1 *Let* E *be a closed bounded convex subset of a normed space* X*. If $f : E \to E$ is a compact map such that $f(E)$ is contained in* E*, then there is an x in E such that $f(x) = x$.*

Then, we give a local existence theorem.

Theorem 3.2 *Let* $0 < s_i < 1$, $i = 1, \cdots, n$,$s = \min\limits_{1\le i\le n} \{s_i\}$, $0 \le \sigma < s < 1$ *and* $f_i(t, u_1(t), u_2(t), \cdots, u_n(t)) \in C(0,1]$*. Assume that* $t^\sigma f_i(t, u_1(t), u_2(t), \cdots, u_n(t)) \in C(0,1]$ $t^\sigma f_i(t, u_1(t), u_2(t), \cdots u_n(t)) \in C(0,1]$*. Then the fractional differential equations*

$$\begin{cases} {}_0D_t^{s_1} u_1(t) = f_1(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_i} u_1(t) = f_i(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ {}_0D_t^{s_n} u_1(t) = f_n(t, u_1(t), u_2(t), \cdots, u_n(t)), \end{cases} \tag{3}$$

have a least continuous solution $U \in C^*[0,1]$, for a suitable $\delta \le 1$.

Proof According to Remark 3.1, we are reduced to consider the following nonlinear integral equations

$$\begin{cases} u_1(t) = {}_0D_t^{-s_1} f_1(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ u_i(t) = {}_0D_t^{-s_i} f_i(t, u_1(t), u_2(t), \cdots, u_n(t)), \\ \vdots \\ u_n(t) = {}_0D_t^{-s_n} f_n(t, u_1(t), u_2(t), \cdots, u_n(t)). \end{cases}$$

Let $T : C^*[0,1] \to C^*[0,1]$ be the operator defined as

$$(TU)(t) = \begin{pmatrix} {}_0D_t^{-s_1} f_1(t, u_1(t), u_2(t), \cdots, u_n(t)) \\ \vdots \\ {}_0D_t^{-s_i} f_i(t, u_1(t), u_2(t), \cdots, u_n(t)) \\ \vdots \\ {}_0D_t^{-s_n} f_n(t, u_1(t), u_2(t), \cdots, u_n(t)) \end{pmatrix}^T .$$

We claim that the operator T is compact. Indeed, the operator is the composition of two simple operators in this way

$$T = A \circ N,$$

where

$$(NU)(t) = \begin{pmatrix} t^{\sigma} f_1(t, U(t)) \\ \vdots \\ t^{\sigma} f_i(t, U(t)) \\ \vdots \\ t^{\sigma} f_n(t, U(t)) \end{pmatrix}^T$$

is a continuous and bounded operator (Nemytskii operator) and

$$(AV)(t) = \begin{pmatrix} \frac{1}{\Gamma(s_1)} \int_0^t (t-\tau)^{s_1-1} \tau^{-\sigma} v_1(\tau, U(\tau))\, d\tau \\ \vdots \\ \frac{1}{\Gamma(s_i)} \int_0^t (t-\tau)^{s_i-1} \tau^{-\sigma} v_i(\tau, U(\tau))\, d \\ \vdots \\ \frac{1}{\Gamma(s_n)} \int_0^t (t-\tau)^{s_n-1} \tau^{-\sigma} v_n(\tau, U(\tau))\, d \end{pmatrix}^T$$

is a compact operator, since $s - \sigma > 0$ as for example in [5].

Moreover, from Section 2, we have for $0 < t \le \delta \le 1$.

$$(AV)(t) \prec \begin{pmatrix} |Av_1(t, U(t))| \\ \vdots \\ |Av_i(t, U(t))| \\ \vdots \\ |Av_n(t, U(t))| \end{pmatrix}^T$$

$$\prec \begin{pmatrix} \sup\limits_{0\le t\le\delta} |v_1(t, U(t))| \frac{1}{\Gamma(s_1)} \int_0^t (t-\tau)^{s_1-1} \tau^{-\sigma} v_1(\tau, U(\tau))\, d\tau \\ \vdots \\ \sup\limits_{0\le t\le\delta} |v_i(t, U(t))| \frac{1}{\Gamma(s_i)} \int_0^t (t-\tau)^{s_i-1} \tau^{-\sigma} v_i(\tau, U(\tau))\, d\tau \\ \vdots \\ \sup\limits_{0\le t\le\delta} |v_n(t, U(t))| \frac{1}{\Gamma(s_n)} \int_0^t (t-\tau)^{s_n-1} \tau^{-\sigma} v_n(\tau, U(\tau))\, d\tau \end{pmatrix}^T$$

$$\prec \begin{pmatrix} \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_1)} \delta^{s_1-\sigma} \sup\limits_{0\le t\le\delta} |v_1(t, U(t))| \\ \vdots \\ \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_i)} \delta^{s_i-\sigma} \sup\limits_{0\le t\le\delta} |v_i(t, U(t))| \\ \vdots \\ \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_n)} \delta^{s_n-\sigma} \sup\limits_{0\le t\le\delta} |v_n(t, U(t))|\, \tau \end{pmatrix}^T .$$

Let

$$\varepsilon = \max_{1\le i\le n}\left\{\tfrac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_i)}\delta^{s_i-\sigma}\right\},$$

therefore, taking the norms in $C^*[0,\delta]$,

$$\|AV\| \le \varepsilon\,\|V\|,$$

where we may assume $\varepsilon > 0$ as small as we want by shrinking $\delta > 0$.

Now fix B_r as a domain of the operator T, where $B_r = \{V \in C^*[0,\delta] : \|V\| < r\}$, which is a convex, bounded, and closed subset of the Banach space $C^*[0,\delta]$.

For δ sufficiently small, we have $T(B_r) \subseteq B_r$ The Schauder fixed point theorem assures that operator T has at least one fixed point and then (3) has at least one continuous solution. U defined on $C^*[0,\delta]$, where $\delta \le 1$.

Example 3.1 Observe that we cannot expect uniqueness for such solutions, in general. Consider for example the equations

$$\begin{cases} {}_0D_t^{1/2}u_1 = \frac{3\mathrm{T}(3/4)}{\Gamma(1/4)}u_2^{1/2}, \\ {}_0D_t^{1/4}u_2 = \frac{2\mathrm{T}(1/2)}{\Gamma(1/4)}u_1^{1/3}, \end{cases}$$

which admit the two solutions $(0,0)$ and $\left(x^{3/4}, x^{1/2}\right)$.

The following theorem shows that uniqueness and global existence can be obtained under an uniform Lipschitz-type assumption.

Theorem 3.3 *Let* $0 < s_i < 1$, $i = 1,\cdots,n$, $s = \min\limits_{1\le i\le n}\{s_i\}$, $0 \le \sigma < s < 1$ *and* $F(t,U) = (f_1(t,U), f_2(t), \cdots f_n(t)) \in C_\sigma^*[0,1]$. *Assume further*

$$\|F(t,U) - F(t,V)\| \le \frac{L}{t^\sigma}\|U - V\| \tag{4}$$

for some positive constant L *independent of* $U, V \in \mathbb{R}^n$, $t \in (0,1]$. *Then the fractional differential equations (3) have a unique solution* $U \in C^*[0,1]$.

Proof As in the proof of Theorem 3.3, we are reduced to studying the operator

$$(TU)(t) = \begin{pmatrix} {}_0D_t^{-s_1} f_1(t, u_1(t), u_2(t), \cdots, u_n(t)) \\ \vdots \\ {}_0D_t^{-s_i} f_i(t, u_1(t), u_2(t), \cdots, u_n(t)) \\ \vdots \\ {}_0D_t^{-s_n} f_n(t, u_1(t), u_2(t), \cdots, u_n(t)) \end{pmatrix}^T$$

which is well defined and continuous as a map $T : C^*[0,1] \to C^*[0,1]$, in the view of the assumption of continuity on $t^\sigma f_i(t)$. Let us define the k iterates of the operator T as is standard

$$T^1 = T, T^K = T \circ T^{K-1}.$$

It will be sufficient to prove that T^K is a contraction operator for K being sufficiently larger. Actually, we have for $U, V \in C^*[0,1]$.

$$\left\|T^k U(t) - T^k V(t)\right\| \le \frac{(HL)^K}{\Gamma(k(s^* - \sigma) + 1)} x^{k(s^* - \sigma)} \|U - V\|, \tag{5}$$

where the constant H depends only on $s^* \in \{s_i\}$ and σ, in fact

$$TU(t) - TV(t) = \begin{pmatrix} {}_0D_t^{-s_1}(f_1(t,U) - v_1(t,U)) \\ \vdots \\ {}_0D_t^{-s_i}(f_i(t,U) - v_i(t,U)) \\ \vdots \\ {}_0D_t^{-s_n}(f_n(t,U) - v_n(t,U)) \end{pmatrix}^T \prec \begin{pmatrix} \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_1)} x^{s_1-\sigma}\|U-V\| \\ \vdots \\ \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_i)} x^{s_i-\sigma}\|U-V\| \\ \vdots \\ \frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_n)} x^{s_n-\sigma}\|U-V\| \end{pmatrix}^T.$$

Let

$$\frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s^*)} x^{s_*-\sigma} = \max_{1\le i\le n}\left\{\sup_{0\le x\le 1}\left|\frac{\Gamma(1-\sigma)}{\Gamma(1-\sigma+s_i)} x^{s_i-\sigma}\right|\right\},$$

therefore (5) is proved for $k = 1$, if $H \ge \Gamma(1-\sigma)$. Assuming by induction that(5) is valid for k, we obtain similarly

$$\begin{aligned} & T^{k+1}U(t) - T^{k+1}V(t) \\ \prec & \begin{pmatrix} \frac{(HL)^K}{\Gamma(k(s^*-\sigma)+1)\mathrm{T}(s_1)}\|U-V\|\int_a^t (t-\tau)^{s_1}\tau^{k(s_1-\sigma)-\sigma}d\tau \\ \vdots \\ \frac{(HL)^K}{\Gamma(k(s^*-\sigma)+1)\Gamma(s_i)}\|U-V\|\int_a^t (t-\tau)^{s_i}\tau^{k(s_i-\sigma)-\sigma}d\tau \\ \vdots \\ \frac{(HL)^K}{\Gamma(k(s^*-\sigma)+1)\mathrm{T}(s_n)}\|U-V\|\int_a^t (t-\tau)^{s_n}\tau^{k(s_n-\sigma)-\sigma}d\tau \end{pmatrix}^T \\ \prec & \begin{pmatrix} \frac{\Gamma(k(s_1-\sigma)-\sigma)H^kL^{k+1}}{\Gamma(k(s^*-\sigma)+1)\mathrm{T}((k+1)(s_1-\sigma)+1)}\|U-V\|t^{(k+1)(s_1-\sigma)} \\ \vdots \\ \frac{\Gamma(k(s_i-\sigma)-\sigma)H^kL^{k+1}}{\Gamma(k(s^*-\sigma)+1)\mathrm{T}((k+1)(s_i-\sigma)+1)}\|U-V\|t^{(k+1)(s_i-\sigma)} \\ \vdots \\ \frac{\Gamma(k(s_n-\sigma)-\sigma)H^kL^{k+1}}{\Gamma(k(s^*-\sigma)+1)\mathrm{T}((k+1)(s_n-\sigma)+1)}\|U-V\|t^{(k+1)(s_n-\sigma)} \end{pmatrix}^T \end{aligned}$$

and then (5) is proved for $k+1$, if H is given by

$$H = \max_k H_k,\ H_k = \max_{1\le i\le n}\left\{\frac{\Gamma(k(s_i-\sigma)-\sigma)}{\mathrm{T}(k(s_i-\sigma)+1)}\right\}. \tag{6}$$

Note that (6) defines actually a finite H, since $H_k \le 1$, for $k \ge (1+\sigma)/(s-\sigma)$, Taking k sufficiently large in (6), we have, say, $(HL)^k \big/ \Gamma(k(s^*-\sigma)+1) \le 1/2$. and therefore $\left\|T^kU(t) - T^kV(t)\right\| \le \frac{1}{2}\|U-V\|$ which gives the proof.

Similarly, the existence and uniqueness for initial value problem of nonlinear multi-variables fractional differential equations also can be proved. In particular, for one-dimensional case ${}_0D_t^s u(t) = f(t,u)$, we obtain identical results in [8].

Example 3.2 Consider for example the equations

$$\begin{cases} {}_0D_t^{1/2}u_1 = u_2, \\ {}_0D_t^{1/4}u_2 = u_1, \end{cases}$$

which admit a unique solution $(0,0)$, defined on $[0,1]$. Since it suffices (4) for $L = 2, \sigma = 1/5$.

Example 3.3 Consider he following general nonlinear system

$${}_aD_t^{\alpha}y(t) + N(y(t)) = g(t), \quad t \in [0,1],$$

where N represents a nonlinear operator with $N(0) = 0$, g(t) is a function with respect to t.

For $L = \max\limits_{y}(N'(y)), \sigma = 0$, where $F(t,U) = g(t) - N(U)$, we have

$$\|F(t,U) - F(t,V)\| = |N(U) - N(V)| \leq \frac{L}{t^{\sigma}} \|U - V\|,$$

i.e. the system admits a unique solution $(0,0)$ defined on $[0,1]$.

4 Conclusion

In this paper, we prove existence and uniqueness theorems for some classes of nonlinear multi-variables fractional differential equations. It extends the original results for fractional differential equations and provides convenience for our further work on nonlinear multi-variable fractional equations.

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