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# Existence of Almost Automorphic Solutions of Neutral Functional Differential Equation

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**Abstract:** In this work we use the theory of evolution semigroup of bounded linear operators and fixed point theorem to establish the existence and uniqueness of a mild solution of a neutral functional differential equation in a Banach space.

**Keywords:** *almost automorphic function; evolution semigroup; neutral functional differential equation; mild solution.*

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## 1 Introduction

In 1964, S. Bochner introduced almost automorphic functions in one of his landmark paper [10]. Almost automorphic functions are more general than almost periodic functions. Many authors had established the almost periodic solution of differential equations in abstract spaces ([8, 9, 13, 15], etc.). The theory has been generalized by many authors for almost automorphic solutions ([11, 12, 14], etc.). Goldstein [14] has considered the following differential equation in a Banach space $X$

$$\frac{dx(t)}{dt} = Ax(t) + f(t, x(t)), \quad t \in \mathbb{R}, \tag{1}$$

where $A$ generates an exponentially stable $C_0$- semigroup and $f$ be a jointly continuous function and shown the existence of almost automorphic solution of the problem if $f$ is almost automorphic.

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These days, functional differential equations have been of very great interest, for many mathematicians. Bahuguna [1] studied a class of partial functional differential equtations and its application to population dynamics. Analytical techniques of semigroup theory have been applied in [2], [3] and [4], which we are also going to use in this paper.

Bahuguna and Muslim [5] also considered the second order history valued delay differential equations [4] and used evolution equations and semigroup theory to find approximation of a solution. Recently, D.N. Pandey, A. Ujlayan and D. Bahuguna [6] proved existence and uniqueness of a hyperbolic integrodifferential equation with a nonlocal condition.

Abbas and Bahuguna [7] considered the following nonautonomous neutral functional differential equations

$$\frac{d}{dt}(x(t) - F_1(t, x(t-g(t)))) = A(t)x(t) + F_2(t, x(t), x(t-g(t))), \tag{2}$$

where $A(t)$ generates an exponentially stable evolution systems and $g$ is a continuous function. The authors have shown the existence of an almost periodic mild solutions using Kransnoselskii's fixed point theorem and theory of evolution operator. They also assumed the well known Acquistapace–Terreni conditions which ensure the existence of evolution family.

In the present work we study the existence of an almost automorphic solution of equation (2) using the evolution semigroup and the Banach fixed point approach.

## 2 Preliminaries

Let $X$ be a complex Banach space endowed with the norm $\|.\|_X$. $\mathbb{N}$, $\mathbb{R}$ *and* $\mathbb{C}$ stand for Natural, Real and Complex numbers respectively. Let $B(X)$ be a Banach space of all bounded linear operators from X to itself; endowed with norm $\|.\|_{B(X)}$ given by

$$\|L\|_{B(X)} = \sup\{\|Lx\|_X : x \in X \text{ and } \|x\|_X \leq 1\}.$$

Now, we will recall certain definitions to be used subsequently in this paper.

**Definition 2.1** A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence $\{s_n\}_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} f(t+\tau_n) = g(t)$ and $\lim_{n\to\infty} g(t-\tau_n) = f(t)$ for all $t \in \mathbb{R}$.

We denote by $AA(X)$ the set of all such functions.

**Definition 2.2** A continuous function $f : \mathbb{R} \times X \to X$ is said to be almost automorphic if $f(t,x)$ is almost automorphic for each $t \in \mathbb{R}$ uniformly for all $x \in Y$, where $Y$ is any bounded subset of $X$.

Equivalently, for every sequence of real numbers $\{s_n\}_{n\in\mathbb{N}}$ we can extract a subsequence $\{\tau_n\}_{n\in\mathbb{N}}$ such that $g(t,x) = \lim_{n\to\infty} f(t+\tau_n, x)$ is well defined for all $t \in \mathbb{R}$ and for all $x \in Y$ and $f(t,x) = \lim_{n\to\infty} g(t-\tau_n, x)$ is well defined for all $t \in \mathbb{R}$ and for all $x \in Y$.

**Lemma 2.1** *$(AA(X), \|.\|_{AA(X)})$ is a Banach space with supremum norm, given by $\|f\|_{AA(X)} = \sup_{t\in\mathbb{R}} \|f(t)\|$.*

**Lemma 2.2** *If $f : \mathbb{R} \to X$ is almost automorphic, then $f$ is bounded.*

For the proof of the above two lemmas, we refer to [12].

**Lemma 2.3** *Suppose $\mathbb{Z}$ and $\mathbb{W}$ are Banach spaces. Let $F : \mathbb{R}\times\mathbb{Z} \to \mathbb{W}$ be an almost automorphic function in $t \in \mathbb{R}$, for each $z \in \mathbb{Z}$ and assume that $F$ satisfies Lipschitz condition in $z$ uniformly in $t \in \mathbb{R}$. Let $\phi : \mathbb{R} \to \mathbb{Z}$ be an almost automorphic function, then the function $\Phi : \mathbb{R} \to \mathbb{W}$, defined by $\Phi(t) = f(t, \phi(t))$ is almost automorphic.*

In [18], Acquistapace and Terreni gave conditions on $A(t),\ t \in \mathbb{R}$, which ensure the existence of unique evolution family $\{U(t,s) : t \geq s > -\infty\}$ on $X$, such that

$$u(t) = U(t,0)u(0) + \int_0^t U(t,\xi)f(\xi)d\xi,$$

where $u(t)$ satisfies

$$\frac{du(t)}{dt} = A(t)u(t) + f(t), \quad t \in \mathbb{R}.$$

**Lemma 2.4** *ATC (Acquistapace–Terreni condition). Let*

$$S_\theta = \{\lambda \in \mathbb{C} : |\arg\lambda| \leq \theta\} \cup \{0\} \subset \rho(A(t)), \quad \theta \in (\frac{\pi}{2}, \pi).$$

*If there exist a constant $K_0$ and a set of real numbers $\alpha_1, ..., \alpha_k, \beta_1, ...\beta_k$ with $0 \leq \beta_i < \alpha_i \leq 2, i = 1, 2, ...k$, such that*

$$\|A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1})\|_{B(X)} \leq K_0\ \Sigma_{i=1}^k (t-\alpha)^{\alpha_i}|\lambda_i|^{\beta_i - 1}$$

*for $t, s \in \mathbb{R}$ and $\lambda \in S_\theta\backslash\{0\}$ and there exists constant $M \geq 0$ such that*

$$\|(\lambda - A(t))^{-1}\| \leq \frac{M}{1+|\lambda|}, \quad \lambda \in S_\theta,$$

*then there exists a unique evolution family $\{U(t,s) : t \geq s > -\infty\}$ on $X$.*

These conditions resulting from Theorem 2.3 of [17] are known as "Acquistapace–Terreni conditions".

**Definition 2.3** A mild solution of (2) is a continuous function $x : \mathbb{R} \to X$, satisfying

$$\begin{aligned} x(t) - F_1(t, x(t-g(t)) &= U(t,s)(x(s) - F_1(s, x(s-g(s)))) \\ &\quad + \int_a^t U(t,\xi)F_2(\xi, x(\xi), x(\xi - g(\xi)))d\xi \end{aligned} \tag{3}$$

for $t \geq s$ all $s \in \mathbb{R}$.

**Note:** We say, an evolution family $\{U(t,s)\}_{t\geq s>-\infty}$ is exponentially stable, if $\exists\ M \geq 1$ and $\delta > 0$ such that $\|U(t,s)\| \leq Me^{-\delta(t-s)}$ for $t \geq s$. When $s \to -\infty$ the above equation takes the form

$$x(t) = F_1(t, x(t-g(t))) + \int_{-\infty}^t U(t,\xi)F_2(\xi, x(\xi), x(\xi - g(\xi)))d\xi.$$

**Assumptions:**

($\mathbf{C}_1$) : $F_1(t,x), F_2(t,x,y)$ are almost automorphic.

($\mathbf{C}_2$) : $F_1$ and $F_2$ are Lipschitz continuous that is there exist positive numbers $L_{F_1}(t)$ and $L_{F_2}(t)$ such that

$$\|F_1(t,x) - F_1(t,y)\| \leq L_{F_1}(t)\|x-y\|_{AA(X)},$$

$$\|F_2(t,x,u) - F_2(t,y,v)\| \leq L_{F_2}(t)(\|x-y\|_{AA(X)} + \|u-v\|_{AA(X)}).$$

($\mathbf{C}_3$) : $\{U(t,s) : t \geq s\}$ is an exponentially stable evolution family on $X$.

($\mathbf{C}_4$) : For every sequence $\{s_n\}$ of real numbers there exists a subsequence $\{\tau_n\}$ and for any fixed $s \in \mathbb{R}$ , $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, it follows that

$$\|U(t+\tau_n, s+\tau_n) - U(t,s)\| \leq \epsilon e^{-\frac{\delta}{2}(t-s)}$$

and

$$\|U(t-\tau_n, s-\tau_n) - U(t,s)\| \leq \epsilon e^{-\frac{\delta}{2}(t-s)} \quad \text{for all } t \geq s \in \mathbb{R}.$$

## 3 Almost Automorphic Solution

We define the mapping $F$ by

$$(Fx)(t) = F_1(t, x(t-g(t))) + \int_{-\infty}^{t} U(t,s) F_2(s, x(s), x(s) - g(s)))ds.$$

**Lemma 3.1** *For $x(.) \in AA(X)$, we have $Fx$ is also almost automorphic.*

***Proof*** Since $F_1$ is Lipschitz and $F_1 \in AA(\mathbb{R}, X)$; by Lemma 2.3, we have

$$F_1(t, x(t-g(t))) = K(t) \in AA(X).$$

By $(C_2)$, we have $F_2(., x(.), y(.)) \in AA(\mathbb{R} \times X \times X, X)$, also we have assumed that $F_2$ is Lipschitz with respect to both variables $x$ and $y$, further using the fact that $X \times X$ is Banach space; hence from Lemma 2.3, one can easily see that $F_2(., x(.), y(.)) \in AA(X)$.

Next, we define $F_2(t, x(t), y(t)) = H(t)$, where $H(.) \in AA(X)$. Now we show that

$$\begin{aligned}
\|Fx\|_{AA(X)} &< \infty, \\
\|Fx(t)\|_X &\leq \|K(t)\|_X + \int_{-\infty}^{t} \|U(t,s)\| \|F_2(s, x(s), x(s-g(s)))\|_X ds \\
&\leq M_1 + \int_{-\infty}^{t} M e^{-\delta(t-s)} \|H(s)\|_X ds \\
&\leq M_1 + M_2 \frac{M}{\delta} < \infty. \quad \text{where } \sup_{t \in \mathbb{R}} \|H(t)\| = M_2.
\end{aligned}$$

Thus, we have shown that $Fx$ is bounded.

Now, we show that $(Fx)(t)$ is almost automorphic with respect to $t \in \mathbb{R}$. Since $H(.) \in AA(X)$ for all sequence $\{s_n\}$ of real numbers, there exists a subsequence $\{\tau_n\}$ such that

$(H_1)$ : $h(t) = \lim_{n\to\infty} H(t+\tau_n)$ is well defined for all $t \in \mathbb{R}$.

$(H_2)$ : $H(t) = \lim_{n\to\infty} h(t-\tau_n)$ is well defined for all $t \in \mathbb{R}$.

As we are going to use Lebesgue dominated convergence theorem to show that $(Fx)(t+\tau_n) \to (Gx)(t)$ as $n \to \infty$; we need to show $|Fx(t+\tau_n)| < l(t)$ for all $n \in \mathbb{N}$; where $l$ is some integrable function. Consider

$$\begin{aligned}(Fx)(t+\tau_n) &= F_1(t+\tau_n, x(t+\tau_n - g(t+\tau_n)) \\ &\quad + \int_{-\infty}^{t+\tau_n} U(t+\tau_n, s)F_2(s, x(s), x(s-g(s)))ds. \\ &= F_1(t+\tau_n, x(t+\tau_n - g(t+\tau_n)) \\ &\quad + \int_{-\infty}^{t} U(t+\tau_n, s+\tau_n)F_2(s+\tau_n, x(s+\tau_n), x(s+\tau_n - g(s+\tau_n)))ds.\end{aligned}$$

Taking the norm on both sides, we have

$$\begin{aligned}\|(Fx)(t+\tau_n)\| &\leq \|K\|_{AA(X)} + \int_{-\infty}^{t} \|U(t+\tau_n, s+\tau_n)\|\|H(s+\tau_n)\|ds \\ &\leq M_1 + \frac{M_2 M}{\delta} \quad (\|H\| \leq M_2).\end{aligned}$$

By $(H_1)$, for any fixed $s \in \mathbb{R}, \epsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$ we have

$$\|H(s+\tau_n) - h(s)\| \leq \epsilon.$$

In addition by $(C_4)$ for $s$ and $\epsilon$ as above there exists $N_2 \in \mathbb{N}$ such that for all $n > N_2$

$$\|U(t+\tau_n, s+\tau_n) - U(t,s)\| < \epsilon e^{\frac{-\delta}{2}(t-s)}.$$

Let $N = \max\{N_1, N_2\}$, then

$$\begin{aligned}&\|U(t+\tau_n, s+\tau_n)H(s+\tau_n) - U(t,s)h(s)\| \\ &\leq \|U(t+\tau_n, s+\tau_n) - U(t,s)\|\|H(s+\tau_n)\| + \|U(t,s)\|\|H(s+\tau_n) - h(s)\| \\ &\leq M_2 \epsilon e^{\frac{-\delta}{2}(t-s)} + M\epsilon e^{\frac{-\delta}{2}(t-s)}\end{aligned}$$

$$\Rightarrow U(t+\tau_n, s+\tau_n)H(s+\tau_n) \to U(t,s)h(s)$$

as $n \to \infty$ for all fixed $s \in \mathbb{R}$ and $t \geq s$. Since $K(.) \in AA(X)$, for any sequence $\{s_n\}$ of real numbers there exists a subsequence $\{\tau_n\}$ such that

$$\lim_{n\to\infty} K(t+\tau_n) = k(t), \quad \lim_{n\to\infty} k(t-\tau_n) = K(t).$$

Thus, we have $K(t+\tau_n) \to k(t)$ as $n \to \infty$. By Lebesgue dominated convergence theorem we get $(Fx)(t+\tau_n) \to Gx(t)$ as $n \to \infty$. In a similar way we can show that $(Gx)(t-\tau_n) \to (Fx)(t)$ as $n \to \infty$ for all $t \in \mathbb{R}$ $\quad \Rightarrow Fx \in AA(X)$.

**Theorem 3.1** *Let $x(.)$ be an almost automorphic function and $F_1, F_2$ and $U(t,s)$ satisfy all conditions from $(C_1)$ to $(C_4)$. Then equation (2) has unique almost automorphic mild solution, whenever $(L_{F_1} + 2L_{F_2}\frac{M}{\delta}) < 1$.*

***Proof*** It follows by Lemma 3.1, that $Fx \in AA(X)$, whenever $x$ does. Let us assume that

$$L_{F_1} = \sup_{t\in\mathbb{R}} L_{F_1}(t), \quad L_{F_2} = \sup_{t\in\mathbb{R}} L_{F_2}(t).$$

For $x, y \in AA(X)$, we have:

$$\begin{aligned}
&\|(Fx)(t) - (Fy)(t)\| \\
&\le \|F_1(t, x(t-g(t))) - F_1(t, y(t-g(t)))\| \\
&+ \int_{-\infty}^{t} \|U(t,s)F_2(s, x(s), x(s-g(s))) - U(t,s)F_2(s, y(s), y(s-g(s)))\| ds \\
&\le L_{F_1}(s)\|x-y\| \\
&+ L_{F_2}(s)\{\|x(s)-y(s)\| + \|x(s-g(s)) - y(s-g(s))\|\} \int_{-\infty}^{t} Me^{-\delta(t-s)} ds \\
&\le L_{F_1}\|x-y\|_{AA(X)} + 2L_{F_2}\|x-y\|_{AA(X)} \int_{-\infty}^{t} Me^{-\delta(t-s)} ds \\
&\le L_{F_1}\|x-y\|_{AA(X)} + 2L_{F_2}\frac{M}{\delta}.
\end{aligned}$$

By Banach contraction principle, F has a unique fixed point $x \in AA(X)$ such that $Fx = x$.

Fixing $s \in \mathbb{R}$, we have

$$x(t) = F_1(t, x(t-g(t))) + \int_{-\infty}^{t} U(t,s)F_2(s, x(s), x(s-g(s))) ds.$$

Since $U(t,s) = U(t,r)U(r,s)$ for $t \ge r \ge s$, let

$$x(\xi) = F_1(\xi, x(\xi - g(\xi))) + \int_{-\infty}^{\xi} U(\xi,s)F_2(s, x(s), x(s-g(s))) ds$$

so

$$U(t,\xi)x(\xi) = U(t,\xi)F_1(\xi, x(\xi-g(\xi))) + \int_{-\infty}^{\xi} U(t,s)F_2(s, x(s), x(s-g(s))) ds.$$

For $t \ge \xi$,

$$\begin{aligned}
\int_{\xi}^{t} U(t,s)F_2(s, x(s), x(s-g(s))) ds &= \int_{-\infty}^{t} U(t,s)F_2(s, x(s), x(s-g(s))) ds \\
&\quad - \int_{-\infty}^{\xi} U(t,s)F_2(s, x(s), x(s-g(s))) ds \\
&= x(t) - U(t,\xi)x(\xi) - F_1(t, x(t-g(t))) \\
&\quad + U(t,\xi)F_1(\xi, x(\xi - g(\xi))).
\end{aligned}$$

Hence we get

$$\begin{aligned}
x(t) &= F_1(t, x(t-g(t))) - U(t,\xi)F_1(\xi, x(\xi-g(\xi))) \\
&\quad + U(t,\xi)x(\xi) + \int_{\xi}^{t} U(t,s)F_2(s, x(s), x(s-g(s))) ds. \qquad (4)
\end{aligned}$$

**Remark 3.1** Consider the following differential equation

$$\frac{d}{dt}(x(t) - F_1(t, x(t - g(t)))) = A(t)x(t) + F_2(t, x(t), \int_{-\infty}^{t} G(t-s)f(s, x(s))ds), \tag{5}$$

where $G \in L^1(\mathbb{R})$ and $f$ is almost automophic, Lipschitz with respect to second variable. Now $f \in AA(\mathbb{R} \times X, X)$ and $f$ is Lipshitz by Lemma 2.3, we have $f \in AA(X)$.
Let $f(t, x(t)) = \psi(t)$.

If we can show $\int_{-\infty}^{t} G(t-s)f(s, x(s))$ is almost automorphic, then as a consequence of the above theorem, equation (5) has a unique almost automorphic solution.
As $\psi$ is almost automorphic for every sequence of real numbers $\{t_n\}$ there exists a subsequence $\{\tau_n\}$ such that $\lim_{n\to\infty} \psi(t + \tau_n) = \psi_1(t)$ is well defined for all $t \in \mathbb{R}$ and $\psi(t) = \lim_{n\to\infty} \psi_1(t - \tau_n)$ is well defined for all $t \in \mathbb{R}$.

Consider

$$\begin{aligned}
&\| \int_{-\infty}^{t+\tau_n} G(t + \tau_n - s)\psi(s)ds - \int_{-\infty}^{t} G(t-s)\psi_1(s)ds\| \\
&= \| \int_{-\infty}^{t} G(t-s)\psi(s + \tau_n)ds - \int_{-\infty}^{t} G(t-s)\psi_1(s)ds\| \\
&\le (\|\psi(s + \tau_n) - \psi_1(s)\|) \int_{-\infty}^{t} |G(t-s)|ds \\
&\le M'(\|\psi(s + \tau_n) - \psi_1(s)\|)
\end{aligned}$$

for some $M' < \infty \to 0$ as $n \to \infty$. Thus, $\int_{-\infty}^{t} G(t-s)\psi(s)ds$ is almost automorphic and we have the result.

## 4 Example

Consider the following equation

$$u'' + (\varepsilon_2 u^2 + 1)u' + u = \varepsilon_1 \frac{d}{dt}\big(\sin\big(\frac{1}{\sin t + \sin\sqrt{2}t}\big)u^2(t - g(t))\big) - \varepsilon_2(\cos t + \cos\sqrt{2}t).$$

Let $u = u_1$ and $u_1' = u_2$, then we can write the above equation in matrix form as follows

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \times \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{d}{dt}F_1(t, U(t - g(t))) + F_2(t, U(t), U(t - g(t))),$$

where

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$F_1(t, U(t - g(t))) = \begin{pmatrix} 0 \\ \sin(\frac{1}{\sin t + \sin\sqrt{2}t})u_1^2 \end{pmatrix},$$

$$F_2(t, U(t), U(t - g(t))) = \begin{pmatrix} 0 \\ \varepsilon_2(\cos t + \cos\sqrt{2}t) - \varepsilon_2 u_1^2 u_2 \end{pmatrix}.$$

This is of the form (2). Thus we can apply our results to ensure the existence and uniqueness of almost automorphic solutions.

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