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Cone Inequalities and Stability of Dynamical Systems

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Abstract: The paper is devoted to working out new methods for stability analysis of equilibrium states of nonlinear dynamic systems in a partially ordered space. The concerned classes of differential systems are described by operator inequalities and inclusions using the notion of derivative with respect to a cone of nonlinear operator. Sufficient stability conditions of equilibrium states are formulated for sets of nonlinear and pseudolinear systems with the interval and polyhedral types operator coefficients. More general result is presented in the form of comparison principle for a finite set of differential systems.

Keywords: *dynamic system; pseudolinear system; monotone system; positive system; Lyapunov stability; cone inequality; partially ordered space.*

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1 Introduction

Stability analysis for dynamic systems with parameter or functional uncertainties is one of the fundamental issues in system and control theory. The applied researches employ continuous and discrete models of dynamic objects whose states possess certain properties with respect to a cone in the phase space (positivity, monotonicity, cooperativity, etc.). For example, these properties can be determined very often by using a cone of nonnegative vectors, a cone of symmetric nonnegatively definite matrices, an ellipsoidal cone, etc. Many important advances have been achieved on the basis of the operator theory in partially ordered spaces (see, e.g., [1–8]). In addition, classes of positive and monotone systems arise in stability theory as systems of comparison [7, 9–11].

We study generalized classes of positive and monotone dynamic systems with respect to a cone and give characterization for such systems by means of operator inequalities and

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inclusions. We formulate analogs of the Lyapunov theorem on the stability of equilibrium states of nonlinear autonomous differential systems with respect to the first approximation using the notion of derivative of nonlinear operator with respect to a cone. Finally, we propose general technique for comparison of a set of differential systems and formulate robust stability conditions for some families of nonlinear, pseudolinear and linear systems in terms of the cone and operator inequalities.

2 Definitions and Auxiliary Facts

A convex closed set $\mathcal{K}$ of a real normed space $\mathcal{E}$ is called a *wedge* if $\alpha\mathcal{K} + \beta\mathcal{K} \subseteq \mathcal{K}$ $\forall\alpha, \beta \geq 0$. A wedge $\mathcal{K}$ with *edge* $\mathcal{K} \cap -\mathcal{K} = \{0\}$ is a *cone*. A space with a wedge is partially ordered: $X \overset{\mathcal{K}}{\leq} Y \Leftrightarrow Y - X \in \mathcal{K}$. A *solid* cone contains nonempty sets of *interior* points $\operatorname{int}\mathcal{K}$ and *boundary* $\partial\mathcal{K}$. A cone $\mathcal{K}$ is *normal* if $0 \overset{\mathcal{K}}{\leq} X \overset{\mathcal{K}}{\leq} Y$ implies $\|X\| \leq \nu\|Y\|$, where ν is a universal constant. The least of these numbers ν is the *normality constant* of $\mathcal{K}$. If $\mathcal{E} = \mathcal{K} - \mathcal{K}$, then the cone $\mathcal{K}$ is *reproducing*. A reproducing cone $\mathcal{K}$ is *non-flat*, i.e. $X = X_+ - X_-$ and $X_\pm \in \mathcal{K}$ imply $\|X_\pm\| \leq \mu\|X\|$, where μ is a universal constant. The *dual* cone $\mathcal{K}^*$ consists of linear nonnegative functionals. Moreover,

$$\mathcal{K} = \{X \in \mathcal{E} : \varphi(X) \geq 0, \forall\varphi \in \mathcal{K}^*\}, \quad \mathcal{K}^* = \{\varphi \in \mathcal{E}^* : \varphi(X) \geq 0, \forall X \in \mathcal{K}\},$$

$$\operatorname{int}\mathcal{K} = \{X \in \mathcal{K}\colon \varphi(X) > 0, \forall\varphi \neq 0 \in \mathcal{K}^*\}, \quad \partial\mathcal{K} = \{X \in \mathcal{K}\colon \exists\varphi \neq 0 \in \mathcal{K}^*, \varphi(X) = 0\}.$$

A functional $\varphi \in \mathcal{E}^*$ is *uniformly positive* if $\varphi(X) \geq \gamma\|X\|$ for some $\gamma > 0$ and $\forall X \in \mathcal{K}$.

A *convex shell* of $X_1, \ldots, X_n \in \mathcal{E}$ is defined by

$$\operatorname{Co}\{X_1, \ldots, X_n\} = \Big\{X : X = \sum_{i=1}^{n} \alpha_i X_i, \ \sum_{i=1}^{n} \alpha_i = 1, \ \alpha_i \geq 0, \ i = \overline{1, n}\Big\}.$$

A set $\mathcal{D} \subset \mathcal{E}$ is *$\mathcal{K}$–convex* if $X \overset{\mathcal{K}}{\leq} Y$ implies $\operatorname{Co}\{X, Y\} \subseteq \mathcal{D}$ for $X, Y \in \mathcal{D}$.

Let $\mathcal{E}(\mathcal{E}_1)$ be a Banach space with a cone $\mathcal{K}(\mathcal{K}_1)$. An operator $M : \mathcal{E} \to \mathcal{E}_1$ is positive if $M\mathcal{K} \subseteq \mathcal{K}_1$. The operator is *monotone* if $X \overset{\mathcal{K}}{\leq} Y \Rightarrow MX \overset{\mathcal{K}_1}{\leq} MY$. The operator inequality $M_2 \trianglerighteq M_1$ means that $M_2 - M_1$ is positive. A linear invertible operator M is *positive invertible* if $\mathcal{K}_1 \subseteq M\mathcal{K}$. Since $(M^{-1})^* = (M^*)^{-1}$, positive invertibility of M leads to positive invertibility of M^*. If $\mathcal{K}_1$ is a normal reproducing cone and $M_1 \trianglelefteq M \trianglelefteq M_2$, then positive invertibility of M_1 and M_2 yields positive invertibility of M, furthermore $M_2^{-1} \trianglelefteq M^{-1} \trianglelefteq M_1^{-1}$ [1]. An operator $M : \mathcal{E} \to \mathcal{E}$ is called *positive-off-diagonal*, if $X \in \mathcal{K}$ and $\varphi \in \mathcal{K}^*$ with $\varphi(X) = 0$ imply $\varphi(MX) \geq 0$. Obviously, if $M \trianglerighteq \alpha I$ for a certain real α, where I is the identity operator, then M is positive-off-diagonal. The inverse statement holds under certain additional conditions with $\alpha \leq -\nu\mu\|M\|$, where ν and μ are normality and non-flatness constants of M, respectively [4].

A linear operator of the form $M = L - P$, $P\mathcal{K} \subseteq \mathcal{K}_1 \subseteq L\mathcal{K}$, with a normal reproducing cone $\mathcal{K}_1$ is positive invertible if and only if $\rho(T) < 1$, where $\rho(T)$ is the spectral radius of the operator pencil of $T(\lambda) = P - \lambda L$. If $\mathcal{K}_1$ is solid, then $\rho(T) < 1 \Leftrightarrow M\mathcal{K} \cap \operatorname{int}\mathcal{K}_1 \neq \emptyset$ [7].

A linear bounded operator $F'(X)$ is called the Gâteaux derivative of a nonlinear operator $F(X)$ at X, if $\lim_{\varepsilon\to 0} \varepsilon^{-1}\big[F(X + \varepsilon H) - F(X)\big] = F'(X)H$ exists in the sense of strong convergence. If this relation holds only for $H \in \mathcal{K}$, then F' is *the Gâteaux derivative of F with respect to a cone* $\mathcal{K}$ [13]. The Fréchet derivative F' with respect to $\mathcal{K}$ is defined by $F(X + H) - F(X) = F'(X)H + o(\|H\|)$, $H \in \mathcal{K}$. The Fréchet derivative is also the Gâteaux derivative. If the Gâteaux derivative is continuous in a neighborhood of X, then it is the Fréchet derivative. We denote the Gâteaux and Fréchet derivatives with respect to $\mathcal{K}$ and $-\mathcal{K}$ by $F'_+(X)$ and $F'_-(X)$, respectively. If $F'(X)$ exists, then $F'_+(X) = F'_-(X) = F'(X)$.

3 Classes of Dynamic Systems in a Partially Ordered Space

Assume that a dynamic system $\mathcal{S}$ operates in a certain domain $\mathcal{D}$ of a Banach space $\mathcal{E}$ and its states are defined by

$$X_t = E(X_\tau, \tau, t) \in \mathcal{E}, \quad \tau \in \Upsilon, \ t \in \Upsilon_\tau, \tag{1}$$

where E is an operator of the transition from initial state X_τ to state X_t and such that

$$E(X, \tau, \tau) = X, \quad E\big(E(X, \tau, t), t, s\big) = E(X, \tau, s), \quad t \in \Upsilon_\tau, \quad s \in \Upsilon_t,$$

$\Upsilon \subseteq R^1$ is an ordered set of indices, $\Upsilon_\tau = \{t \in \Upsilon : t \geq \tau\}$. The system is *continuous, discrete* or *hybrid* subject to the structure of Υ. Note that $E(\cdot, \tau, \tau) \equiv I$ is the identity operator. If $E(\Theta, \tau, t) \equiv \Theta$, then $X_t \equiv \Theta$ is the *equilibrium state* of $\mathcal{S}$. We shall consider only the isolated equilibrium states of dynamic systems.

Let $\mathcal{K}_t$ be a constant or time-varying set in $\mathcal{E}$. If $E(\mathcal{K}_\tau, \tau, t) \subseteq \mathcal{K}_t$ for $t \in \Upsilon_\tau$, then $\mathcal{K}_t$ is an *invariant set* of system $\mathcal{S}$. The system is *positive* with respect to an invariant cone $\mathcal{K}_t$. System $\mathcal{S}$ is *monotone* with respect to a cone $\mathcal{K}_t$ if

$$X_\tau \overset{\mathcal{K}_\tau}{\leq} Y_\tau \quad \Rightarrow \quad X_t = E(X_\tau, \tau, t) \overset{\mathcal{K}_t}{\leq} Y_t = E(Y_\tau, \tau, t) \tag{2}$$

for any $\tau \in \Upsilon$ and $t \in \Upsilon_\tau$. A positive (monotone) dynamic system $\mathcal{S}$ is defined by a positive (monotone) operator E with respect to $\mathcal{K}_t$. Denote the classes of monotone and positive systems with respect to $\pm\mathcal{K}_t$ by $\mathcal{M}$ and $\mathcal{M}_0^\pm$, respectively.

Consider the sets

$$\mathcal{K}_t^+(\Theta) = \big\{X \in \mathcal{E} : X \overset{\mathcal{K}_t}{\geq} \Theta\big\}, \quad \mathcal{K}_t^-(\Theta) = \big\{X \in \mathcal{E} : X \overset{\mathcal{K}_t}{\leq} \Theta\big\},$$

where $\Theta \in \mathcal{E}$, $\mathcal{K}_t$ is a cone. For the class of systems with invariant sets $\mathcal{K}_t^\pm(\Theta)$, we use the notation $\mathcal{M}_0^\pm(\Theta)$. Denote the classes of systems which posses the property (2) with $Y_\tau \in \mathcal{K}_\tau^+(\Theta)$, $X_\tau \in \mathcal{K}_\tau^+(\Theta)$, $X_\tau \in \mathcal{K}_\tau^-(\Theta)$ and $Y_\tau \in \mathcal{K}_\tau^-(\Theta)$ by $\mathcal{M}_1^+(\Theta)$, $\mathcal{M}_2^+(\Theta)$, $\mathcal{M}_1^-(\Theta)$ and $\mathcal{M}_2^-(\Theta)$, respectively. It is obvious that

$$\mathcal{M} \subseteq \mathcal{M}_1^\pm(\Theta) \subseteq \mathcal{M}_2^\pm(\Theta), \quad \mathcal{M} \subseteq \mathcal{M}_1(\Theta) \subseteq \mathcal{M}_2(\Theta),$$

where $\mathcal{M}_1(\Theta) = \mathcal{M}_1^+(\Theta) \cap \mathcal{M}_1^-(\Theta)$, $\mathcal{M}_2(\Theta) = \mathcal{M}_2^+(\Theta) \cap \mathcal{M}_2^-(\Theta)$. A system of $\mathcal{M}_2^\pm(\Theta)$ is monotone in $\mathcal{K}_t^\pm(\Theta)$. Every system of $\mathcal{M}_2^+(\Theta)$, $\mathcal{M}_2^-(\Theta)$ or $\mathcal{M}_2(\Theta)$ with the equilibrium state $X_t \equiv \Theta$ belongs to $\mathcal{M}_0^+(\Theta)$, $\mathcal{M}_0^-(\Theta)$ or $\mathcal{M}_0(\Theta) = \mathcal{M}_0^+(\Theta) \cap \mathcal{M}_0^-(\Theta)$, respectively.

We describe the classes of systems $\mathcal{S}$ introduced above via the inclusions

$$E'_\pm(X, \tau, t)\, \mathcal{K}_\tau \subseteq \mathcal{K}_t, \quad X \in \mathcal{D}, \ \tau \in \Upsilon, \ t \in \Upsilon_\tau, \tag{3}$$

where $E'_\pm(X, \tau, t)$ are the Gâteaux derivatives of $E(X, \tau, t)$ with respect to $\pm\mathcal{K}_\tau$.

Lemma 3.1 *Suppose that $E(X, \tau, t)$ is Gâteaux differentiable with respect to $\pm\mathcal{K}_\tau$ in a $\mathcal{K}_\tau$-convex domain $\mathcal{D}$ for $\tau \in \Upsilon$, $t \in \Upsilon_\tau$. Then:* (i) *$\mathcal{S} \in \mathcal{M}$ if and only if one of the inclusions* (3) *holds;* (ii) *$\mathcal{S} \in \mathcal{M}_0^\pm(\Theta)$ if $E(\Theta, \tau, t) - \Theta \in \pm\mathcal{K}_t$ and* (3) *holds for $X \in \mathcal{K}_\tau^\pm(\Theta)$;* (iii) *$\mathcal{S} \in \mathcal{M}_2^\pm(\Theta)$ if and only if* (3) *holds for $X \in \mathcal{K}_\tau^\pm(\Theta)$.*

Proof The necessity assertions (i)–(iii) are obtained by using the definitions of the corresponding classes of systems $\mathcal{S}$ and the Gâteaux derivatives

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\big[E(X+\varepsilon H,\tau,t)-E(X,\tau,t)\big]=E'_{\pm}(X,\tau,t)H,\quad X\in\mathcal{D},\quad H\in\mathcal{K}_{\tau}^{\pm}.$$

The sufficiency assertions (i)–(iii) follow from the Lagrange type formula:

$$\varphi\big(E(X+H,\tau,t)-E(X,\tau,t)\big)=\varphi\big(E'_{\pm}(Z,\tau,t)H\big),$$

where $\varphi\in\mathcal{E}^*$, $Z=X+\mu H\in\mathrm{Co}\{X,X+H\}$, $0<\mu<1$, X and $X+H$ are arbitrary points of a certain convex set. For this purpose, we use only functionals $\varphi\in\pm\mathcal{K}_t^*$ and the property of $\mathcal{K}_\tau$–convexity of $\mathcal{D}$. Moreover, $Z=(1-\mu)X+\mu(X+H)\in\mathcal{D}$ for $X\in\mathcal{D}$ and $H\in\pm\mathcal{K}_\tau$. □

Consider the nonlinear differential system

$$\dot X=F(X,t),\quad t\geq\tau\geq 0,\tag{4}$$

where F is a continuous operator function that guarantees the existence and uniqueness of the continuously differentiable solution $X(t)=E(X_\tau,\tau,t)$ for any $\tau\geq 0$, $X_\tau\in\mathcal{D}$. Let $\mathcal{K}_t$ be a cone in the phase space $\mathcal{E}$. For example, the Lyapunov transformation $\mathcal{K}_t=L(t)\mathcal{K}$ of a given cone $\mathcal{K}$ is a cone also. In this case, we can study the solutions (4) in the form $X(t)=L(t)Z(t)$ by means of a constant cone $\mathcal{K}$ instead of $\mathcal{K}_t$ in a phase space of the transformed system

$$\dot Z=L^{-1}(t)F(L(t)Z,t)-L^{-1}(t)\dot L(t).$$

For $t\geq 0$, we introduce the following conditions:

$$X\overset{\mathcal{K}_t}{\geq}\Theta,\ \varphi\in\mathcal{K}_t^*,\ \varphi(X-\Theta)=0\ \ \Rightarrow\ \ \varphi\left(F(X,t)\right)\geq 0,\tag{5}$$

$$X\overset{\mathcal{K}_t}{\leq}Y,\ \varphi\in\mathcal{K}_t^*,\ \varphi(X-Y)=0\ \ \Rightarrow\ \ \varphi\left(F(X,t)-F(Y,t)\right)\leq 0.\tag{6}$$

Let $\mathcal{F}_0^{\pm}(\Theta)$ denote the classes of operator functions F satisfying (5) with respect to $\pm\mathcal{K}_t$. Let $\mathcal{F}$ be a class of operator functions satisfying (6). We also define the classes of operator functions $\mathcal{F}_1^{+}(\Theta)$, $\mathcal{F}_2^{+}(\Theta)$, $\mathcal{F}_1^{-}(\Theta)$ and $\mathcal{F}_2^{-}(\Theta)$, that possess property (6) with $Y\in\mathcal{K}_t^{+}(\Theta)$, $X\in\mathcal{K}_t^{+}(\Theta)$, $X\in\mathcal{K}_t^{-}(\Theta)$ and $Y\in\mathcal{K}_t^{-}(\Theta)$, respectively. Denote $\mathcal{F}_k(\Theta)=\mathcal{F}_k^{+}(\Theta)\cap\mathcal{F}_k^{-}(\Theta)$, $k=0,1,2$. It is obvious that $\mathcal{F}\subseteq\mathcal{F}_1^{\pm}(\Theta)\subseteq\mathcal{F}_2^{\pm}(\Theta)$.

Lemma 3.2 [8] *Let $\mathcal{K}_t$ be a solid cone possessing the extension property*

$$0\leq\tau<t\ \ \Rightarrow\ \ \mathcal{K}_\tau\subseteq\mathcal{K}_t.\tag{7}$$

Then: (i) *system* (4) *is monotone with respect to $\mathcal{K}_t$ if $F\in\mathcal{F}$;* (ii) *system* (4) *belongs to $\mathcal{M}_0^{\pm}(\Theta)$ if $F\in\mathcal{F}_0^{\pm}(\Theta)$;* (iii) *system* (4) *belongs to $\mathcal{M}_0^{\pm}(\Theta)\cap\mathcal{M}_k^{\pm}(\Theta)$ if $F\in\mathcal{F}_k^{\pm}(\Theta)$, $k=1,2$;* (iv) *system* (4) *belongs to $\mathcal{M}_k^{\pm}(\Theta)$ if $F(\Theta,t)\in\pm\mathcal{K}_t$ and $F\in\mathcal{F}_k^{\pm}(\Theta)$, $k=1,2$.*

Note that the cone inequality

$$F(X,t)\overset{\mathcal{K}_t}{\geq}\alpha_{+}(X,t)\,(X-\Theta),\quad X-\Theta\in\partial\mathcal{K}_t,\quad t\geq 0,$$

where $\alpha_\pm(X,t)$ are scalar functions, yields $F \in \mathcal{F}_0^+(\Theta)$. Analogously, if

$$F(X,t) - F(Y,t) \overset{\mathcal{K}_t}{\leq} \beta(X,Y,t)\,(X-Y), \quad Y - X \in \partial\mathcal{K}_t, \quad t \geq 0,$$

where $\beta(X,Y,t)$ is a scalar function, then $F \in \mathcal{F}$. If this condition holds for $Y \in \mathcal{K}_t^+(\Theta)$, $X \in \mathcal{K}_t^+(\Theta)$, $X \in \mathcal{K}_t^-(\Theta)$ and $Y \in \mathcal{K}_t^-(\Theta)$, then $F \in \mathcal{F}_1^+(\Theta)$, $F \in \mathcal{F}_2^+(\Theta)$, $F \in \mathcal{F}_1^-(\Theta)$ and $F \in \mathcal{F}_2^-(\Theta)$, respectively.

We can describe the classes of operator function $\mathcal{F}$, $\mathcal{F}_0^\pm(\Theta)$ and $\mathcal{F}_2^\pm(\Theta)$ by means of the following operator inequalities generated by $\mathcal{K}_t$:

$$F'_\pm(X,t) \trianglerighteq \beta_\pm(X,t)I, \quad X \in \mathcal{D}, \quad t \geq 0, \tag{8}$$

where $\beta_\pm(X,t)$ are scalar function. These inequalities ensure that $F'_\pm(X,t)$ are positive-off-diagonal with respect to $\mathcal{K}_t$ for $X \in \mathcal{D}$ and $t \geq 0$. In view of Lemma 3.2, we have the following characterization of the introduced classes of differential systems (4).

Lemma 3.3 *Suppose that the operator $F(X,t)$ is Gâteaux differentiable with respect to $\pm\mathcal{K}_t$ in the $\mathcal{K}_t$-convex domain $\mathcal{D}$ for $t \geq 0$. Then:* (i) *$F \in \mathcal{F}$ if one of the operator inequalities* (8) *holds;* (ii) *$F \in \mathcal{F}_0^\pm(\Theta)$ if $F(\Theta,t) \in \pm\mathcal{K}_t$ and* (8) *holds for $X \in \mathcal{K}_t^\pm(\Theta)$;* (iii) *$F \in \mathcal{F}_2^\pm(\Theta)$ if* (8) *holds for $X \in \mathcal{K}_t^\pm(\Theta)$.*

Proof The assertions (i)–(iii) of Lemma 3.3 are obtained by using the Lagrange type formula:

$$\varphi\big(F(X+H,t) - F(X,t)\big) = \varphi\big(F'_\pm(Z,t)H\big), \quad H \in \pm\mathcal{K}_t, \ \varphi \in \pm\mathcal{K}_t^*,$$

where $Z = X + \mu H \in \mathrm{Co}\{X, X+H\}$, $0 < \mu < 1$. If $F'_\pm(Z,t)H$ is continuous, then

$$F(X+H,t) - F(X,t) = \int_0^1 F'_\pm(X+\mu H,t)H\,d\mu, \quad H \in \pm\mathcal{K}_t. \ \Box$$

Let's introduce some classes of operator functions which are used in the theory of comparison systems. We write $F \in \overline{\mathcal{F}}$, if one can establish a correspondence between solutions of (4) and solutions of the differential inequalities $\dot{Z} \overset{\mathcal{K}_t}{\leq} F(Z,t)$ such that

$$Z(\tau) \overset{\mathcal{K}_\tau}{\leq} X(\tau) \ \Rightarrow \ Z(t) \overset{\mathcal{K}_t}{\leq} X(t), \quad t > \tau \geq 0.$$

In addition, if $X(\tau) \in \mathcal{K}_\tau^+(\Theta)$ ($Z(\tau) \in \mathcal{K}_\tau^+(\Theta)$), then $F \in \overline{\mathcal{F}}_1(\Theta)$ ($F \in \overline{\mathcal{F}}_2(\Theta)$). Similarly, we introduce the classes $\underline{\mathcal{F}}$, $\underline{\mathcal{F}}_1(\Theta)$ and $\underline{\mathcal{F}}_2(\Theta)$ by using $-\mathcal{K}_t$ instead of $\mathcal{K}_t$. It is obvious that $\overline{\mathcal{F}} \subseteq \overline{\mathcal{F}}_1(\Theta) \subseteq \overline{\mathcal{F}}_2(\Theta)$ and $\underline{\mathcal{F}} \subseteq \underline{\mathcal{F}}_1(\Theta) \subseteq \underline{\mathcal{F}}_2(\Theta)$.

If $F \in \overline{\mathcal{F}} \cup \underline{\mathcal{F}}$, then system (4) is monotone with respect to $\mathcal{K}_t$. If $F \in \overline{\mathcal{F}}$ and $F(\Theta,t) \in \mathcal{K}_t$ ($F \in \underline{\mathcal{F}}$ and $F(\Theta,t) \in -\mathcal{K}_t$), then system (4) belongs to $\mathcal{M}_0^+(\Theta)$ ($\mathcal{M}_0^-(\Theta)$).

Lemma 3.4 *Under the conditions of Lemma 3.2, we have:* (i) *$\mathcal{F} \subseteq \overline{\mathcal{F}} \cap \underline{\mathcal{F}}$;* (ii) *$\mathcal{F}_k^+(\Theta) \cap \mathcal{F}_0^+(\Theta) \subseteq \overline{\mathcal{F}}_k(\Theta)$, $\mathcal{F}_k^-(\Theta) \cap \mathcal{F}_0^-(\Theta) \subseteq \underline{\mathcal{F}}_k(\Theta)$, $k = 1,2$.*

By analogy, we can introduce and study classes of difference systems in a Banach space $\mathcal{E}$ with respect to a cone $\mathcal{K}_t \subset \mathcal{E}$ (see [12]).

4 Stability of Equilibrium States of Autonomous Systems

Definition 4.1 The equilibrium state $X_t \equiv \Theta$ of system $\mathcal{S}$ is *stable in* $\mathcal{K}_t^+(\Theta)$ if, for any $\varepsilon > 0$ and $\tau \in \Upsilon$, there exists $\delta > 0$ such that $X_\tau \in \mathcal{S}_\delta(\tau) \Rightarrow X_t \in \mathcal{S}_\varepsilon(t)$ for $t \in \Upsilon_\tau$, where $\mathcal{S}_\varepsilon(t) = \{X \in \mathcal{K}_t^+(\Theta) : \|X - \Theta\| \le \varepsilon\}$. If, for a certain $\delta > 0$, $X_\tau \in \mathcal{S}_\delta(\tau) \Rightarrow \|X_t - \Theta\| \to 0$ as $t \to \infty$, then the state $X_t \equiv \Theta$ is *asymptotically stable in* $\mathcal{K}_t^+(\Theta)$.

Lemma 4.1 [8] *Let $\mathcal{K}_t$ be a normal reproducing cone. The state $X \equiv \Theta$ of system $\mathcal{S} \in \mathcal{M}_1(\Theta)$ is Lyapunov stable (asymptotically stable) if and only if it is stable (asymptotically stable) in $\mathcal{K}_t^+(\Theta)$ and $\mathcal{K}_t^-(\Theta)$.*

At first, we formulate known results for linear systems. Let $\mathcal{K} \subset \mathcal{E}$ be a normal reproducing cone. Positive system $\dot{X} = AX$ with a linear bounded operator $A : \mathcal{E} \to \mathcal{E}$ is exponentially stable if and only if $-A$ is positive invertible. If $\mathcal{K} \subseteq (\gamma I - A)\mathcal{K}$ for $\gamma \ge 0$, then the system is exponentially stable and positive with respect to $\mathcal{K}$ [14]. Moreover, the system is exponentially stable if $\mathcal{K} \subset -A\mathcal{K} \cap (\gamma_0 I - A)\mathcal{K}$ for a certain $\gamma_0 > [\rho^2(A) - r^2(A)]/[2r(A)]$, where $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, $r(A) = \min\{|\lambda| : \lambda \in \sigma(A)\}$ [15].

Now we formulate the asymptotic stability conditions for an isolated equilibrium state of nonlinear autonomous system in terms of positive invertible operators.

Theorem 4.1 *Let $\mathcal{K}$ be a normal reproducing cone. The state $X \equiv \Theta$ of system*

$$\dot{X} = F(X), \quad F(\Theta) = 0, \quad t \ge 0, \tag{9}$$

is Lyapunov asymptotically stable if one of the following conditions holds:
(a) *$F \in \mathcal{F}_0^+(\Theta) \cup \mathcal{F}_0^-(\Theta)$, there exists the Fréchet derivative $F'(\Theta)$, and $-F'(\Theta)$ is positive invertible:*

$$\mathcal{K} \subseteq -F'(\Theta)\mathcal{K}. \tag{10}$$

(b) *$F \in \mathcal{F}_1(\Theta)$, there exist the Fréchet derivatives $F'_\pm(\Theta)$ with respect to $\pm\mathcal{K}$, and $-F'_\pm(\Theta)$ are positive invertible:*

$$\mathcal{K} \subseteq -F'_+(\Theta)\mathcal{K} \cap F'_-(\Theta)\mathcal{K}. \tag{11}$$

Proof (a) For $X = \Theta + H$, system (9) is represented as follows:

$$\dot{H} = F'(\Theta)H + R(\Theta, H), \quad R(\Theta, H) = o(\|H\|), \quad H \in \mathcal{E}.$$

In order to use the Lyapunov theorem on stability with respect to the first approximation, we establish the asymptotic stability of the linear system

$$\dot{H} = F'(\Theta)H. \tag{12}$$

System (12) is positive with respect to $\mathcal{K}$ and $-\mathcal{K}$. Indeed, using the relations

$$F(\Theta + \varepsilon H) = \varepsilon F'(\Theta)H + R(\Theta, \varepsilon H), \quad \frac{R(\Theta, \varepsilon H)}{\varepsilon\|H\|} \underset{\varepsilon \to 0}{\to} 0,$$

and the fact that $F \in \mathcal{F}_0^+(\Theta) \cup \mathcal{F}_0^-(\Theta)$, we have

$$H \in \pm\mathcal{K}, \;\; \varphi \in \pm\mathcal{K}^*, \;\; \varphi(H) = 0 \;\;\Rightarrow\;\; \frac{\varphi(F'(\Theta)H)}{\|H\|} + \frac{\varphi(R(\Theta, \varepsilon H))}{\varepsilon\|H\|} \ge 0.$$

This implies that $\varphi(F'(\Theta)H) \geq 0$, i.e. the positivity conditions of system (12) are satisfied (see Lemma 3.2). In view of (10), system (12) is exponentially stable. Moreover, the state $X \equiv \Theta$ of original system (9) is Lyapunov asymptotically stable.

(b) If $F \in \mathcal{F}_1(\Theta)$, then system (9) belongs to $\mathcal{M}_1(\Theta)$ and has the invariant sets $\mathcal{K}^{\pm}(\Theta)$. For $X = \Theta + H \in \mathcal{K}^{\pm}(\Theta)$, we have the systems

$$\dot{H} = F'_{\pm}(\Theta)H + R_{\pm}(\Theta, H), \quad R_{\pm}(\Theta, H) = o(\|H\|), \quad H \in \pm\mathcal{K}.$$

According to Lemma 4.1, the asymptotic stability in $\mathcal{K}$ and $-\mathcal{K}$ of the zero state $H \equiv 0$ of the systems yields the Lyapunov asymptotic stability of the state $X \equiv \Theta$ of original system (9). The linear systems $\dot{H} = F'_{\pm}(\Theta)H$ are positive with respect to $\mathcal{K}$ and $-\mathcal{K}$ and exponentially stable (see above). Therefore, the state $X \equiv \Theta$ of system (9) is Lyapunov asymptotically stable. □

Note that, in the case of a solid cone $\mathcal{K}$, conditions (10) and (11) are equivalent to consistency of the corresponding systems of cone inequalities:

$$H \overset{\mathcal{K}}{\geq} 0, \quad F'(\Theta)H \overset{\mathcal{K}}{<} 0, \tag{13}$$

$$H_- \overset{\mathcal{K}}{\leq} 0 \overset{\mathcal{K}}{\leq} H_+, \quad F'_+(\Theta)H_+ \overset{\mathcal{K}}{<} 0 \overset{\mathcal{K}}{<} F'_-(\Theta)H_-. \tag{14}$$

Conjecture 4.1 *Let system* (9) *belong to* $\mathcal{M}_1(\Theta)$ *with respect to a normal solid cone* $\mathcal{K}$ *and let the following cone inequalities be feasible:*

$$X_- \overset{\mathcal{K}}{\leq} \Theta \overset{\mathcal{K}}{\leq} X_+, \quad F(X_+) \overset{\mathcal{K}}{<} 0 \overset{\mathcal{K}}{<} F(X_-). \tag{15}$$

Then the state $X \equiv \Theta$ *of system* (9) *is Lyapunov asymptotically stable.*

Consider the pseudolinear differential system

$$\dot{X} = A(X)X, \quad t \geq 0, \tag{16}$$

where A is a continuous operator function with the values $A(X)$ that are assumed to be linear bounded operators in $\mathcal{E}$. The Gâteaux (Fréchet) derivatives and Gâteaux (Fréchet) derivatives with respect to $\pm\mathcal{K}$ of $F(X) = A(X)X$ have the form

$$F'(X) = A(X) + B(X), \quad B(X)H = [A'(X)H]X,$$

$$F'_{\pm}(X) = A(X) + B_{\pm}(X), \quad B_{\pm}(X)H = [A'_{\pm}(X)H]X,$$

where $A'(X)$ and $A'_{\pm}(X)$ are the Gâteaux (Fréchet) derivatives of $A(X)$, the values $B(X)$ and $B_{\pm}(X)$ are linear operators in $\mathcal{E}$. Since $F'(0) = F'_{\pm}(0) = A(0)$, we have the following corollary of Theorem 4.1.

Corollary 4.1 *Let one of the following off-diagonal positivity type constraints hold:*

$$A(X) \trianglerighteq \alpha_{\pm}(X)I, \quad X \in \pm\partial\mathcal{K},$$

$$A(X) + B(X) \trianglerighteq \beta(X)I, \quad X \in \pm\mathcal{K},$$

$$A(X) + B_{\pm}(X) \trianglerighteq \beta_{\pm}(X)I, \quad X \in \mathcal{D},$$

where $\mathcal{K}$ *is a solid cone,* $\alpha_{\pm}(X)$, $\beta(X)$ *and* $\beta_{\pm}(X)$ *are scalar functions. Then the zero state* $X \equiv 0$ *of system* (9) *is Lyapunov asymptotically stable if the following system of cone inequalities is feasible:*

$$H \overset{\mathcal{K}}{\geq} 0, \quad A(0)H \overset{\mathcal{K}}{<} 0. \tag{17}$$

Similarly, we can formulate the asymptotic stability conditions of the isolated equilibrium states for some classes of autonomous nonlinear and pseudolinear difference system (see [8]).

Example 4.1 Consider the pseudolinear system

$$\dot{x} = A(x)x, \quad A(x) = \mathrm{diag}\{d - Cx\}, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{18}$$

where $d \in \mathbb{R}^n$ is a vector, C is an invertible $n \times n$ matrix, $\mathrm{diag}\{\cdot\}$ denotes the diagonal $n \times n$ matrix generated by n vector components. This system is the Kolmogorov type model describing the dynamics of growth and interaction of n populations. There are two equilibrium states $\theta_0 = 0$ and $\theta_1 = C^{-1}d$.

The diagonal matrix $A(x)$ for any $x \in \mathbb{R}^n$ is positive-off-diagonal with respect to the cones $\pm\mathcal{K}$, where $\mathcal{K} = \mathbb{R}^n_+$. Therefore, (18) is positive with respect to $\pm\mathcal{K}$ and the asymptotic stability condition (17) of the state $x \equiv \theta_0$ is reduced to the inequality $d \overset{\mathcal{K}}{<} 0$.

Fréchet derivative of the vector function $F(x) = A(x)x$ has the form $F'(x) = A(x) + B(x)$, where $B(x) = -\mathrm{diag}\{x\}\, C$. The matrix $F'(x)$ is positive-off-diagonal for $x - \theta_1 \in \pm\partial\mathcal{K}$ if $B_1 = \mathrm{diag}\{\theta_1\}\, C$ is negative-off-diagonal. By virtue of Lemmas 3.2 and 3.3, system (18) belongs to $\mathcal{M}_0^{\pm}(\theta_1)$. Moreover, according to Theorem 4.1, the state $x \equiv \theta_1$ of the system is asymptotically stable if B_1 is a M-matrix, i.e. $B_1^{-1} \trianglerighteq 0$ and B_1 is negative-off-diagonal.

5 Comparison Principle for a Set of Differential Systems

Consider a set of independent systems of the type (4):

$$\mathcal{S}_i: \quad \dot{X}_i = F_i(X_i, t), \quad X_i \in \mathcal{E}_i, \quad t \geq 0, \ i = \overline{1, s}. \tag{19}$$

For simplicity, we denote $X = (X_1, \dots, X_s)$, $F(X,t) = (F_1(X_1,t), \dots, F_s(X_s,t))$, $\mathcal{E} = \mathcal{E}_1 \times \cdots \times \mathcal{E}_s$ and rewrite (19) as

$$\dot{X} = F(X,t), \quad X \in \mathcal{E}, \quad t \geq 0. \tag{20}$$

Let $\mathcal{X}$ be a space with a wedge $\mathcal{W}_t$, and let $W : \mathcal{E} \times [0, \infty) \to \mathcal{X}$ be a continuous operator function together with its partial derivatives and not everywhere positive with respect to $\mathcal{W}_t$.

Definition 5.1 Systems (19) are called *comparable* if $W(X(t),t) \in \mathcal{W}_t$ whenever $W(X(\tau),\tau) \in \mathcal{W}_\tau$ for $t > \tau \geq 0$. Simultaneously, W is the *operator of comparison* of systems (19).

Theorem 5.1 *Let $\mathcal{W}_t$ be a solid cone satisfying* (7). *Then systems* (19) *are comparable if and only if*

$$W(X,t) \in \mathcal{W}_t, \ \ \varphi \in \mathcal{W}_t^*, \ \ \varphi\left(W(X,t)\right) = 0 \ \ \Rightarrow \ \ \varphi\left(D_t W(X,t)\right) \geq 0, \quad t \geq 0, \tag{21}$$

where D_t is the operator of differentiation along solutions of (20).

Proof We construct an invariant set of (20) in the form $\mathcal{I}_t = \{X \in \mathcal{E} : W(X,t) \in \mathcal{W}_t\}$. The operator of differentiation along solutions of the system is defined as

$$D_t W(X,t) = W'_X(X,t)\, F(X,t) + W'_t(X,t),$$

where $W'_t(X,t)$ is the strong time derivative and $W'_X(X,t)$ is the Gâteaux derivative.

Let $X(t)$ satisfy (20), $X(\tau) \in \mathcal{I}_\tau$ and $X(\xi) \in \partial\mathcal{I}_\xi$ for some $\xi \geq \tau$. Then

$$\int_\xi^t D_s W(X(s),s)ds = W(X(t),t) - W(X(\xi),\xi)$$

and $\varphi(W(X(\xi),\xi)) = 0$ for some $\varphi \neq 0 \in \mathcal{W}^*_\xi$. For $\varepsilon > 0$ and $Y \in \operatorname{int}\mathcal{W}_\xi$, we define a neighbourhood of $\mathcal{I}_t$ in the form $\mathcal{I}^\varepsilon_t = \{X \in \mathcal{E} : W_\varepsilon(X,t) \in \mathcal{W}_t\}$, where

$$W_\varepsilon(X,t) = W(X,t) + \varepsilon \arctan(t-\xi)\, Y.$$

It is obvious that $\mathcal{I}_t \subset \mathcal{I}^\varepsilon_t$, and $\mathcal{I}^\varepsilon_t \to \mathcal{I}_t$ as $\varepsilon \to 0$, $t \geq \xi$. Since $\varphi(Y) > 0$, according to (21), for some $\delta > 0$, we have

$$\varphi\left(D_t W_\varepsilon(X(t),t)\right) = \varphi\left(D_t W(X(t),t)\right) + \frac{\varepsilon}{1+(t-\xi)^2}\,\varphi(Y) > 0, \quad \xi \leq t \leq \xi + \delta,$$

$$\int_\xi^{\xi+\delta} \varphi\left(D_t W_\varepsilon(X(t),t)\right) dt = \varphi(W_\varepsilon(X(\xi+\delta),\xi+\delta)) > 0.$$

It means that the trajectory $X(t)$ at $t = \xi$ cannot leave $\mathcal{I}^\varepsilon_\xi$, i.e. $W_\varepsilon(X(t),t) \in \mathcal{W}_\xi$ for $\xi \leq t \leq \xi + \delta$. Otherwise $\varphi(W_\varepsilon(X(\xi),\xi)) = 0$ and $\varphi(W_\varepsilon(X(\xi+\delta),\xi+\delta)) < 0$ for some $\varphi \in \mathcal{W}^*_\xi$ and $\delta > 0$. According to (7), we have $X(t) \in \mathcal{I}^\varepsilon_t$ for $\xi \leq t \leq \xi+\delta$. By virtue of the closedness of $\mathcal{W}_t$, we get $W_\varepsilon(X(t),t) \to W(X(t),t) \in \mathcal{W}_t$ as $\varepsilon \to 0$, $\xi \leq t \leq \xi+\delta$. Thus, $\mathcal{I}_t$ is an invariant set of system (20).

The converse statement follows from the Lagrange type relation:

$$\varphi(W(X(\xi+\delta),\xi+\delta)) - \varphi(W(X(\xi),\xi)) = \delta\,\varphi(D_\zeta W(X(\zeta),\zeta)),$$

where $\zeta \in (\xi, \xi+\delta)$. If $\varphi(W(X(\xi),\xi)) = 0$ and $X(\xi+\delta) \in \mathcal{I}_{\xi+\delta}$, then it is necessary that the inequality $\varphi(D_\xi W(X(\xi),\xi)) \geq 0$ holds for sufficiently small $\delta > 0$. □

Note that (21) holds if

$$D_t W(X,t) \overset{\mathcal{W}_t}{\geq} \alpha(X,t) W(X,t), \quad X \in \partial\mathcal{I}_t, \quad t \geq 0,$$

where $\alpha(X,t)$ is a certain scalar function.

Now, we formulate known results of comparison for two and three systems with the zero equilibrium states. In some cases, these results can be established as corollaries of Theorem 5.1. In phase spaces of the comparison systems, we shall use normal reproducing cones with bounded normality constants. Consider the following cases.

Case 1. Let $s = 2$, $F_1(\Theta,t) \equiv 0$, $F_2(\Omega,t) \equiv 0$ and $W(X,t) = X_2 - V(X_1,t)$, where $V : \mathcal{E}_1 \times [0,\infty) \to \mathcal{E}_2$ is a continuous and everywhere positive operator function with respect to a normal reproducing cone $\mathcal{K}_t \subset \mathcal{E}_2$. If

$$D_t V(X_1,t) \overset{\mathcal{K}_t}{\leq} F_2(V(X_1,t),t), \quad F_2 \in \overline{\mathcal{F}}_2(\Omega), \quad t \geq 0, \tag{22}$$

then $\mathcal{S}_2$ is an *upper comparison system* for system $\mathcal{S}_1$ in the sense that:

$$\Omega \overset{\mathcal{K}_\tau}{\leq} V(X_1(\tau),\tau) \overset{\mathcal{K}_\tau}{\leq} X_2(\tau) \;\Rightarrow\; \Omega \overset{\mathcal{K}_t}{\leq} V(X_1(t),t) \overset{\mathcal{K}_t}{\leq} X_2(t), \;\; t > \tau \geq 0.$$

Thus, systems (19) are comparable with the operator of comparison W.

We assume that an operator V has the following additional properties:

$$V(\Theta,t) \equiv \Omega, \quad \|V(X,t)-\Omega\| \geq v(X) > 0, \; X \neq \Theta, \quad v(\Theta)=0, \quad t \geq 0, \tag{23}$$

where v is a continuous function such that $\|X-\Theta\| \leq \|Y-\Theta\|$ whenever $v(X) \leq v(Y)$.

Theorem 5.2 *Let an everywhere positive operator V satisfy* (22) *and* (23)*. Then the solution $X_1 \equiv \Theta$ of $\mathcal{S}_1$ is Lyapunov stable (asymptotically stable) if the solution $X_2 \equiv \Omega$ of $\mathcal{S}_2$ is stable (asymptotically stable) in $\mathcal{K}_t^+(\Omega)$.*

Case 2. Let $s=3$, $F_1(\Omega,t) \equiv F_3(\Omega,t) \equiv 0$, $F_2(\Theta,t) \equiv 0$, $\mathcal{E}_1 = \mathcal{E}_3$ and $W(X,t) = [V(X_2,t)-X_1, X_3 - V(X_2,t)]$, where $V : \mathcal{E}_2 \times [0,\infty) \to \mathcal{E}_1$ is a continuous operator function and $\mathcal{K}_t \subset \mathcal{E}_1$ is a normal reproducing cone. If

$$F_1(V(X_2,t),t) \overset{\mathcal{K}_t}{\leq} D_t V(X_2,t) \overset{\mathcal{K}_t}{\leq} F_3(V(X_2,t),t), \quad F_1 \in \underline{\mathcal{F}}_1(\Omega), \; F_3 \in \overline{\mathcal{F}}_1(\Omega), \tag{24}$$

then, for $X_1(\tau) \in \mathcal{K}_\tau^-(\Omega)$ and $X_3(\tau) \in \mathcal{K}_\tau^+(\Omega)$, we have

$$X_1(\tau) \overset{\mathcal{K}_\tau}{\leq} V(X_2(\tau),\tau) \overset{\mathcal{K}_\tau}{\leq} X_3(\tau) \;\Rightarrow\; X_1(t) \overset{\mathcal{K}_t}{\leq} V(X_2(t),t) \overset{\mathcal{K}_t}{\leq} X_3(t), \quad t > \tau \geq 0. \tag{25}$$

It means that three systems (19) are comparable with the operator of comparison W and cone $\mathcal{W}_t = \mathcal{K}_t \times \mathcal{K}_t$. Then $\mathcal{S}_1$ ($\mathcal{S}_3$) is a *lower (upper) comparison system* for $\mathcal{S}_2$.

Theorem 5.3 *Let V satisfy* (23) *and* (24)*. Then the solution $X_2 \equiv \Theta$ of system $\mathcal{S}_2$ is Lyapunov stable (asymptotically stable) if the solution $X_1 \equiv \Omega$ of $\mathcal{S}_1$ and the solution $X_3 \equiv \Omega$ of $\mathcal{S}_3$ are stable (asymptotically stable) in $\mathcal{K}_t^-(\Omega)$ and $\mathcal{K}_t^+(\Omega)$, respectively.*

Proof Since $\mathcal{K}_t$ is reproducing and non-flat, we have

$$V(X_2(\tau),\tau) - \Omega = U_+ - U_-, \quad \|U_\pm\| \leq \gamma \|V(X_2(\tau),\tau)-\Omega\|, \quad U_\pm \in \mathcal{K}_\tau,$$

where $\gamma > 0$ is a universal constant. Let $X_1(t)$ and $X_3(t)$ be the solutions of systems $\mathcal{S}_1$ and $\mathcal{S}_3$ with the initial conditions $X_1(\tau) = \Omega - U_- \in \mathcal{K}_\tau^-(\Omega)$ and $X_3(\tau) = \Omega + U_+ \in \mathcal{K}_\tau^+(\Omega)$, respectively. Then $X_1(t) \in \mathcal{K}_t^-(\Omega)$, $X_3(t) \in \mathcal{K}_t^+(\Omega)$ and

$$\|X_1(\tau)-\Omega\| \leq \gamma \|V(X_2(\tau),\tau)-\Omega\|, \quad \|X_3(\tau)-\Omega\| \leq \gamma \|V(X_2(\tau),\tau)-\Omega\|.$$

By virtue of (25) and the normality of $\mathcal{K}_t$, we get

$$\|V(X_2(t),t)-\Omega\| \leq \alpha \|X_1(t)-\Omega\| + \beta \|X_3(t)-\Omega\|, \quad t \geq \tau.$$

where $\alpha > 0$ and $\beta > 0$ depend on the normality constant of $\mathcal{K}_t$.

It follows from (23) and the continuity of $V(X,t)$ that, for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $\|X_2(t)-\Theta\| \leq \varepsilon$ whenever $\|V(X_2(t),t)-\Omega\| \leq \delta_0$ for $t \geq \tau$.

Now we use the stability properties of the solution $X_1 \equiv \Omega$ of $\mathcal{S}_1$ and the solution $X_3 \equiv \Omega$ of $\mathcal{S}_3$ in $\mathcal{K}_t^-(\Omega)$ and $\mathcal{K}_t^+(\Omega)$, respectively. We choose $\delta_\pm > 0$ so that the inequalities $\|X_1(\tau)-\Omega\| \leq \delta_-$ and $\|X_3(\tau)-\Omega\| \leq \delta_+$ yield the corresponding inequalities $\|X_1(t)-\Omega\| \leq \delta_0/(2\alpha)$ and $\|X_3(t)-\Omega\| \leq \delta_0/(2\beta)$ for $t \geq \tau$.

Finally, we choose $\delta > 0$ so that

$$\|X_2(\tau)-\Theta\| \leq \delta \;\Rightarrow\; \|V(X_2(\tau),\tau)-\Omega\| \leq \min\{\delta_-,\delta_+\}/\gamma.$$

Then, according to the arguments presented above, we get $\|X_2(t) - \Theta\| \le \varepsilon$ for $t > \tau$, i.e., the solution $X_2 \equiv \Theta$ of system $\mathcal{S}_2$ is Lyapunov stable. In this case, $X_2(t) \to \Theta$ if $X_1(t) \to \Omega$ and $X_3(t) \to \Omega$ as $t \to \infty$. □

The proofs of Theorems 5.2 and 5.3 are analogous.

Case 3. Let $s \ge 2$. The *arrangement* problems for systems (19) can be formulated in the form of a general comparison problem using the block operator

$$W(X,t) = \left[\; V_2(X_2,t) - V_1(X_1,t), \quad \dots \quad , V_s(X_s,t) - V_{s-1}(X_{s-1},t) \;\right].$$

If $\mathcal{S}_i$ are comparable with $\mathcal{W}_t = \mathcal{K}_t \times \cdots \times \mathcal{K}_t$, where $\mathcal{K}_t$ is a wedge in $\mathcal{X}_1$, then

$$V_1(X_1(t),t) \overset{\mathcal{K}_t}{\le} \cdots \overset{\mathcal{K}_t}{\le} V_s(X_s(t),t), \quad t > \tau \ge 0,$$

provided that this ordering takes place at an arbitrary initial time $t = \tau$. In particular, if $V_i(X_i,t) = \|X_i\|_{\mathcal{E}_i}$, then the solutions of comparable systems (19) are ordered by norms:

$$\|X_1(\tau)\|_{\mathcal{E}_1} \le \cdots \le \|X_s(\tau)\|_{\mathcal{E}_s} \;\Rightarrow\; \|X_1(t)\|_{\mathcal{E}_1} \le \cdots \le \|X_s(t)\|_{\mathcal{E}_s}, \quad t > \tau \ge 0.$$

Example 5.1 Consider a set of pseudolinear systems

$$\dot{x}_i = A_i(x_i,t)x_i, \quad x_i \in \mathbb{C}^{n_i}, \quad t \ge 0, \quad i = \overline{1,s}, \tag{26}$$

where $A_i(x_i,t)$ are continuous $n_i \times n_i$ matrices. We specify an operator of comparison of the systems with respect to the cone $\mathcal{W} = \mathbb{R}_+^{s-1}$:

$$W(X,t) = \left[\; x_2^* Q_2 x_2 - x_1^* Q_1 x_1, \quad \dots \quad , x_s^* Q_s x_s - x_{s-1}^* Q_{s-1} x_{s-1} \;\right],$$

where $Q_i(t) = Q_i^*(t) > 0$ are Hermitian positive definite matrices. Then

$$D_t W(X,t) = \left[\; x_2^* H_2 x_2 - x_1^* H_1 x_1, \quad \dots \quad , x_s^* H_s x_s - x_{s-1}^* H_{s-1} x_{s-1} \;\right],$$

where $H_i(x_i,t) = A_i^*(x_i,t)Q_i(t) + Q_i(t)A_i(x_i,t) + \dot{Q}_i(t)$, $i = \overline{1,s}$. Using Theorem 5.1 and the two-sided estimations

$$[\lambda_{\min}(H_i - \lambda Q_i) - \alpha]\, x_i^* Q_i x_i \le x_i^*(H_i - \alpha Q_i)x_i \le [\lambda_{\max}(H_i - \lambda Q_i) - \alpha]\, x_i^* Q_i x_i,$$

one can establish that the solutions of (26) are ordered in the form $x_1(t)^* Q_1(t) x_1(t) \le \cdots \le x_s(t)^* Q_s(t) x_s(t)$, $t \ge \tau \ge 0$, if the following relations hold:

$$\lambda_{\max}(H_i - \lambda Q_i) \le \lambda_{\min}(H_{i+1} - \lambda Q_{i+1}), \quad i = \overline{1, s-1}.$$

In particular, in the case $Q_i \equiv I$, the inequalities $\lambda_{\max}(A_i^* + A_i) \le \lambda_{\min}(A_{i+1}^* + A_{i+1})$, $i = \overline{1, s-1}$, ensure the ordering of systems (26) with respect to the Hermitian norm. Here, for matrix pencils and Hermitian matrices, $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denote the maximum and minimum eigenvalues, respectively.

6 Robust Stability Analysis of Differential Systems

Consider the family of differential systems

$$\dot{X} = F(X,t), \quad F(\Theta,t) \equiv 0, \quad t \ge 0, \tag{27}$$

$$\underline{F}(X,t) \overset{\mathcal{K}_t}{\leq} F(X,t) \overset{\mathcal{K}_t}{\leq} \overline{F}(X,t), \quad X \in \mathcal{E}, \quad t \geq 0, \tag{28}$$

under the conditions of existence and uniqueness of solutions $X(t)$, $t > \tau \geq 0$, in a Banach space $\mathcal{E}$ contained a normal reproducing cone $\mathcal{K}_t$ with a bounded normality constant. We isolate the extreme systems

$$\dot{\underline{X}} = \underline{F}(\underline{X},t), \quad \underline{F}(\Theta,t) \equiv 0, \quad t \geq 0, \tag{29}$$

$$\dot{\overline{X}} = \overline{F}(\overline{X},t), \quad \overline{F}(\Theta,t) \equiv 0, \quad t \geq 0. \tag{30}$$

If $\underline{F} \in \underline{\mathcal{F}}_1(\Theta)$ and $\overline{F} \in \overline{\mathcal{F}}_1(\Theta)$, then, for $\underline{X}(\tau) \in \mathcal{K}_\tau^-(\Theta)$, $\overline{X}(\tau) \in \mathcal{K}_\tau^+(\Theta)$, we have

$$\underline{X}(\tau) \overset{\mathcal{K}_\tau}{\leq} X(\tau) \overset{\mathcal{K}_\tau}{\leq} \overline{X}(\tau) \ \Rightarrow\ \underline{X}(t) \overset{\mathcal{K}_t}{\leq} X(t) \overset{\mathcal{K}_t}{\leq} \overline{X}(t), \quad t > \tau \geq 0.$$

In this case, (29) ((30)) is a lower (upper) comparison system for any system (27), (28).
Assumed that $V(X,t) \equiv X$ and $\Theta = \Omega$ in Theorem 5.3, we have the following result.

Theorem 6.1 *Let $\underline{F} \in \underline{\mathcal{F}}_1(\Theta)$ and $\overline{F} \in \overline{\mathcal{F}}_1(\Theta)$. Then the solution $X \equiv \Theta$ of any system* (27), (28) *is Lyapunov stable (asymptotically stable), if the solution $\underline{X} \equiv \Theta$ of* (29) *and the solution $\overline{X} \equiv \Theta$ of* (30) *are stable (asymptotically stable) in $\mathcal{K}_t^-(\Theta)$ and $\mathcal{K}_t^+(\Theta)$, respectively.*

Now, we consider instead of (28) the conditions

$$\underline{F}(X,t) \overset{\mathcal{K}_t}{\leq} F(X,t) \overset{\mathcal{K}_t}{\leq} \overline{F}(X,t), \quad X \in \mathcal{K}_t^+(\Theta), \quad t \geq 0, \tag{31}$$

$$\underline{F}(X,t) \overset{\mathcal{K}_t}{\leq} F(X,t) \overset{\mathcal{K}_t}{\leq} \overline{F}(X,t), \quad X \in \mathcal{K}_t^-(\Theta), \quad t \geq 0. \tag{32}$$

Theorem 6.2 *Let $\mathcal{K}_t$ be a normal solid cone satisfying* (7). *If* (31) *holds with $\underline{F} \in \mathcal{F}_0^+(\Theta)$ and $\overline{F} \in \mathcal{F}_2^+(\Theta)$, then stability (asymptotic stability) in $\mathcal{K}_t^+(\Theta)$ of the solution $\overline{X} \equiv \Theta$ of* (30) *involves stability (asymptotic stability) in $\mathcal{K}_t^+(\Theta)$ of the solution $X \equiv \Theta$ of any system* (27), (31). *By analogy, if* (32) *holds with $\underline{F} \in \mathcal{F}_2^-(\Theta)$ and $\overline{F} \in \mathcal{F}_0^-(\Theta)$, then stability (asymptotic stability) in $\mathcal{K}_t^-(\Theta)$ of the solution $\underline{X} \equiv \Theta$ of* (29) *involves stability (asymptotic stability) in $\mathcal{K}_t^-(\Theta)$ of the solution $X \equiv \Theta$ of any system* (27), (32).

Note that under the conditions of Theorem 6.2, we have $\Theta \overset{\mathcal{K}_t}{\leq} X(t) \overset{\mathcal{K}_t}{\leq} \overline{X}(t)$ and $\underline{X}(t) \overset{\mathcal{K}_t}{\leq} X(t) \overset{\mathcal{K}_t}{\leq} \Theta$ for $t > \tau \geq 0$ as soon as these inequalities hold at $t = \tau$ (see Section 3). If (31) holds, then $\underline{F} \in \mathcal{F}_0^+(\Theta)$ implies $F \in \mathcal{F}_0^+(\Theta)$. Similarly, if (32) holds, then $\overline{F} \in \mathcal{F}_0^-(\Theta)$ implies $F \in \mathcal{F}_0^-(\Theta)$.

Consider the pseudolinear system

$$\dot{X} = A(X,t)X, \quad t \geq 0, \tag{33}$$

with the isolated equilibrium state $X \equiv \Theta$ under one of the following conditions:

$$\underline{A}(X,t) \trianglelefteq A(X,t) \trianglelefteq \overline{A}(X,t), \quad X \overset{\mathcal{K}_t}{\geq} \Theta \overset{\mathcal{K}_t}{\geq} 0, \quad t \geq 0, \tag{34}$$

$$\underline{A}(X,t) \trianglelefteq A(X,t) \trianglelefteq \overline{A}(X,t), \quad X \overset{\mathcal{K}_t}{\leq} \Theta \overset{\mathcal{K}_t}{\leq} 0, \quad t \geq 0. \tag{35}$$

The values of continuous operator functions $A(X,t)$, $\underline{A}(X,t)$ and $\overline{A}(X,t)$ are linear bounded operators in $\mathcal{E}$. If $X \equiv \Theta$ is an equilibrium state of the system, then either $\Theta = 0$ or $\Theta \neq 0$ and $\Theta \in \ker A(\Theta,t)$ at $t \geq 0$. The extreme systems

$$\dot{\underline{X}} = \underline{A}(\underline{X},t)\underline{X}, \quad t \geq 0, \tag{36}$$

$$\dot{\overline{X}} = \overline{A}(\overline{X},t)\overline{X}, \quad t \geq 0, \tag{37}$$

have the equilibrium states $\underline{X} \equiv \Theta$ and $\overline{X} \equiv \Theta$, respectively. So, we formulate the corollaries of Theorem 6.2 and Lemma 3.3 using the following constraints:

$$\underline{A}(X,t) + \underline{B}_+(X,t) \trianglerighteq \underline{\beta}_+(X,t)I, \quad X \in \mathcal{K}_t^+(\Theta), \quad t \geq 0, \tag{38}$$

$$\overline{A}(X,t) + \overline{B}_+(X,t) \trianglerighteq \overline{\beta}_+(X,t)I, \quad X \in \mathcal{K}_t^+(\Theta), \quad t \geq 0, \tag{39}$$

$$\underline{A}(X,t) + \underline{B}_-(X,t) \trianglerighteq \underline{\beta}_-(X,t)I, \quad X \in \mathcal{K}_t^-(\Theta), \quad t \geq 0, \tag{40}$$

$$\overline{A}(X,t) + \overline{B}_-(X,t) \trianglerighteq \overline{\beta}_-(X,t)I, \quad X \in \mathcal{K}_t^-(\Theta), \quad t \geq 0, \tag{41}$$

where $\underline{B}_\pm(X,t)H = [\underline{A}'_\pm(X,t)H]X$, $\overline{B}_\pm(X,t)H = [\overline{A}'_\pm(X,t)H]X$, $\underline{A}'_\pm(X,t)$ and $\overline{A}'_\pm(X,t)$ are the Gâteaux (Fréchet) derivatives with respect to $\pm\mathcal{K}_t$, $\underline{\beta}_\pm(X,t)$ and $\overline{\beta}_\pm(X,t)$ are scalar functions.

Corollary 6.1 *Let $\mathcal{K}_t$ be a normal solid cone satisfying* (7). *If* (38) *and* (39) *hold, then stability (asymptotic stability) in $\mathcal{K}_t^+(\Theta)$ of the solution $\overline{X} \equiv \Theta$ of* (37) *involves stability (asymptotic stability) in $\mathcal{K}_t^+(\Theta)$ of the solution $X \equiv \Theta$ of any system* (33), (34). *By analogy, if* (40) *and* (41) *hold, then stability (asymptotic stability) in $\mathcal{K}_t^-(\Theta)$ of the solution $\underline{X} \equiv \Theta$ of* (36) *involves stability (asymptotic stability) in $\mathcal{K}_t^-(\Theta)$ of the solution $X \equiv \Theta$ of any system* (33), (35).

Note that in Corollary 6.1, we can use the constraints

$$\underline{A}(X,t) \trianglerighteq \underline{\alpha}_+(X,t)I, \quad \underline{A}(X,t)\Theta \overset{\mathcal{K}_t}{\geq} 0, \quad X - \Theta \in \partial\mathcal{K}_t, \quad t \geq 0, \tag{42}$$

$$\overline{A}(X,t) \trianglerighteq \overline{\alpha}_-(X,t)I, \quad \overline{A}(X,t)\Theta \overset{\mathcal{K}_t}{\leq} 0, \quad \Theta - X \in \partial\mathcal{K}_t, \quad t \geq 0, \tag{43}$$

instead of (38) and (41), respectively.

Example 6.1 Consider the family of pseudolinear systems

$$\dot{x} = A(x,t)x, \quad \underline{A}(x,t) \trianglelefteq A(x,t) \trianglelefteq \overline{A}(x), \quad x \in \mathbb{R}^n_+, \quad t \geq 0, \tag{44}$$

where $\underline{A}(x,t) = \underline{A}_0(t) + \sum_{j=1}^n x_j \underline{A}_j(t)$, $\overline{A}(x) = \overline{A}_0 + \sum_{j=1}^n x_j \overline{A}_j$, $\underline{A}_i(t) \trianglelefteq \overline{A}_i$, $\underline{A}_i(t) = \|\underline{a}_{ks}^{(i)}(t)\|_{k,s=1}^n$ and $\overline{A}_i = \|\overline{a}_{ks}^{(i)}\|_{k,s=1}^n$ are $n \times n$ matrices, $i = \overline{0,n}$. Here $\mathcal{K} = \mathbb{R}^n_+$ is a cone of nonnegative vectors and $\trianglelefteq$ denotes the elementwise matrix inequality.

The Gâteaux (Fréchet) derivative of $F(x,t) = A(x,t)x$ has the form

$$F'(x,t) = A(x,t) + B(x,t), \quad B(x,t) = [(\partial A/\partial x_1)\, x, \ldots, (\partial A/\partial x_n)\, x],$$

So, for $\underline{F}(x,t) = \underline{A}(x,t)x$ and $\overline{F}(x) = \overline{A}(x)x$, we have

$$\underline{F}'(x,t) = \underline{F}'_\pm(x,t) = \underline{A}_0(t) + \textstyle\sum_{j=1}^n x_j \underline{B}_j(t), \quad \underline{B}_j(t) = \|\underline{a}_{ks}^{(j)}(t) + \underline{a}_{kj}^{(s)}(t)\|_{k,s=1}^n,$$
$$\overline{F}'(x) = \overline{F}'_\pm(x) = \overline{A}_0 + \textstyle\sum_{j=1}^n x_j \overline{B}_j, \quad \overline{B}_j = \|\overline{a}_{ks}^{(j)} + \overline{a}_{kj}^{(s)}\|_{k,s=1}^n.$$

Then conditions (38) and (42) with $\Theta = 0$ are reduced to the form

$$\underline{a}_{ks}^{(0)}(t) \geq 0, \quad \underline{a}_{ks}^{(j)}(t) + \underline{a}_{kj}^{(s)}(t) \geq 0, \quad k \neq s, \quad t \geq 0, \quad j = \overline{1, n};$$
$$\underline{a}_{ks}^{(i)}(t) \geq 0, \quad k \neq s, \quad t \geq 0, \quad i = \overline{0, n},$$

respectively. If one of these conditions holds and in addition

$$\overline{A}_0^{-1} \trianglelefteq 0, \quad \overline{a}_{ks}^{(0)} \geq 0, \quad \overline{a}_{ks}^{(j)} + \overline{a}_{kj}^{(s)} \geq 0, \quad k \neq s, \quad j = \overline{1, n},$$

then according to Corollary 6.1 the zero equilibrium state of any system (44) is asymptotically stable in $\mathcal{K}$ (see also assertion (a) of Theorem 4.1 and Corollary 4.1).

Consider the parameter family of autonomous pseudolinear systems

$$\dot{X} = A(X, p)X, \quad A(X, p) = \sum_{i=1}^{s} p_i A_i(X), \ X \in \mathcal{E}, \ t \geq 0, \tag{45}$$

where $p = [p_1, \ldots, p_s]^\top \in \mathbb{R}_+^s$ is a vector of nonnegative scalar parameters. The values of operator functions $A_i(X)$ and $A(X, p)$ are linear bounded operators in $\mathcal{E}$.

Corollary 6.2 *Let all the operators $A_i(X)$ satisfy one of the off-diagonal positivity type constraints of Corollary 4.1 with a normal solid cone $\mathcal{K}$, and the system of cone inequalities $H \overset{\mathcal{K}}{\geq} 0$ and $A_i(0)H \overset{\mathcal{K}}{<} 0$ for $i = \overline{1, s}$ is feasible. Then the zero solution $X \equiv 0$ of any system* (45) *for $p \in \mathbb{R}_+^s$ is Lyapunov asymptotically stable.*

Consider the family of linear differential systems

$$\dot{X} = A(t)X, \quad \underline{A}(t) \trianglelefteq A(t) \trianglelefteq \overline{A}(t), \quad t \geq 0, \tag{46}$$

where the inequality $\trianglelefteq$ between linear operators is generated by a normal reproducing cone $\mathcal{K}_t$. In (46), we isolate the extreme systems:

$$\dot{X} = \underline{A}(t)X, \tag{47}$$

$$\dot{X} = \overline{A}(t)X. \tag{48}$$

Theorem 6.3 *Any system* (46) *is positive with respect to $\mathcal{K}_t$ if*

$$e^{\underline{A}(\theta)\delta}\mathcal{K}_\tau \subseteq \mathcal{K}_t, \quad t \geq \theta \geq \tau \geq 0, \quad t - \tau \geq \delta \geq 0. \tag{49}$$

Moreover, if system (48) *is asymptotically stable, then any positive system* (46) *is asymptotically stable.*

Proof Note that $\mathcal{K}_t$ in (49) with $\delta = 0$ satisfies (7). The evolutional and exponential operators of system (46) are connected by [16]

$$E(t, \tau) = \lim_{n \to \infty} \left[e^{A(\vartheta_n)h_n} \ldots e^{A(\vartheta_1)h_n} \right], \quad e^{A(\vartheta)h} = \lim_{n \to \infty} \left[E(\vartheta, \vartheta - h/n) \right]^n,$$

where $\vartheta_k \in [t_k, t_{k+1}]$, $t_k = \tau + kh_n$, $h_n = (t - \tau)/n$, $k = \overline{0, n}$, $t \geq \tau$, $\vartheta \geq 0$, $h \geq 0$. Therefore, (49) ensures positivity of (46). In the case of a constant cone, the inverse statement holds also. If $A(t)\mathcal{K}_t \subseteq \mathcal{K}_t$ and (7) hold, then

$$e^{A(\vartheta)h}\mathcal{K}_\tau = \sum_{k=0}^{\infty} (h^k/k!) A^k(\vartheta)\, \mathcal{K}_\tau \subseteq \mathcal{K}_t, \quad \tau \leq \vartheta \leq t, \quad 0 \leq h \leq t - \tau.$$

Let $A(t) = A_1(t) + A_2(t)$ and $e^{A_s(\vartheta)h}\mathcal{K}_\tau \subseteq \mathcal{K}_t$, $s = 1, 2$. Then

$$e^{A(\vartheta)h} = \lim_{n\to\infty} 2^{-n}\left[e^{A_1(\vartheta)\frac{h}{n}}e^{A_2(\vartheta)\frac{h}{n}} + e^{A_2(\vartheta)\frac{h}{n}}e^{A_1(\vartheta)\frac{h}{n}}\right]^n, \quad e^{A(\vartheta)h}\mathcal{K}_\tau \subseteq \mathcal{K}_t,$$

and consequently $E(t,\tau)\mathcal{K}_\tau \subseteq \mathcal{K}_t$. Assuming that $A_1(t) = \underline{A}(t)$ and $A_2(t) = A(t) - \underline{A}(t)$, we have the positivity of any system (46) with respect to $\mathcal{K}_t$.

Let $X(t)$ and $\overline{X}(t)$ be the solutions of (46) and (48) with initial conditions $X(\tau) = X_\tau$ and $\overline{X}(\tau) = \overline{X}_\tau$, respectively. Since $0 \overset{\mathcal{K}_\tau}{\leq} X_\tau \overset{\mathcal{K}_\tau}{\leq} \overline{X}_\tau$ implies $0 \overset{\mathcal{K}_t}{\leq} X(t) \overset{\mathcal{K}_t}{\leq} \overline{X}(t)$, $t \geq \tau \geq 0$, and $\mathcal{K}_t$ is normal, the asymptotic stability of system (48) ensures the asymptotic stability in $\pm\mathcal{K}_t$ of any positive system (46). Moreover, if $\mathcal{K}_t$ is reproducing, then any system (46) is Lyapunov asymptotically stable. □

Remark 6.1 Note that any system (46) is positive with respect to $\mathcal{K}_t$ if the operator inequality $\underline{A}(t) \trianglerighteq \alpha(t)I$ holds for some scalar function $\alpha(t)$ (see Corollary 6.1 and the notation below). This inequality ensures (49) subject to (7). Indeed,

$$e^{\underline{A}(\vartheta)\delta} = e^{\alpha(\vartheta)\delta}\, e^{[\underline{A}(\vartheta)-\alpha(\vartheta)I]\delta}, \quad e^{\underline{A}(\vartheta)\delta}\mathcal{K}_\tau = \sum_{k=0}^{\infty}(\delta^k/k!)[\underline{A}(\vartheta) - \alpha(\vartheta)I]^k\,\mathcal{K}_\tau \subseteq \mathcal{K}_\vartheta \subseteq \mathcal{K}_t.$$

Example 6.2 Consider the family of linear systems

$$\dot{x} = A(t)x, \quad \underline{A}(t) \trianglelefteq A(t) \trianglelefteq \overline{A}, \quad \overline{A}^{-1} \trianglelefteq 0, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{50}$$

where $\underline{A}(t)$ is a matrix function with nonnegative off-diagonal entries, $-\overline{A}$ is an M-matrix and $\trianglelefteq$ denotes the elementwise matrix inequality. The system $\dot{x} = \underline{A}(t)x$ is positive with respect to the cone $\mathbb{R}^n_+$, and the system $\dot{x} = \overline{A}x$ is asymptotically stable. Thus, any system (50) is asymptotically stable and positive with respect to $\mathbb{R}^n_+$.

Example 6.3 Consider the family of linear systems in a matrix space $\mathbb{C}^{n\times n}$

$$\dot{X} = M(t)X, \quad \underline{M}(t) \trianglelefteq M(t) \trianglelefteq \overline{M}(t), \quad X \in \mathbb{C}^{n\times n}, \quad t \geq 0, \tag{51}$$

where $\underline{M}(t)X = A^*(t)X + XA(t)$, $\overline{M}(t)X = A^*(t)X + XA(t) + \sum_{i=1}^{s} B^*(t)XB(t)$, $\trianglelefteq$ is an operator inequality generated by the cone of Hermitian positive semidefinite matrices $\mathbb{K}_n$. Since $e^{\underline{M}(\vartheta)\delta}X = e^{A^*(\vartheta)\delta}Xe^{A(\vartheta)\delta}$, the Lyapunov equation

$$\dot{X} = A^*(t)X + XA(t)$$

and any system (51) are positive with respect to $\mathbb{K}_n$. If the system

$$\dot{X} = A^*(t)X + XA(t) + \sum_{i=1}^{s} B^*(t)XB(t) \tag{52}$$

is asymptotically stable, then any system (51) is positive and asymptotically stable. Autonomous system of the type (52) is asymptotically stable, if the linear matrix inequality

$$A^*X + XA + \sum_{i=1}^{s} B^*XB < 0$$

has a solution $X = X^* > 0$.

Note that the matrix differential equation (52) is known as the second-moment equation for the Itô stochastic system. This equation is positive and monotone with respect to $\mathbb{K}_n$.

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