Nonlinear Dynamics and Systems Theory, **11** (4) (2011) 341–372

Complex Network Synchronization of Coupled Time-Delay Chua Oscillators in Different Topologies

O.R. Acosta-Del Campo [1], C. Cruz-Hernández [2*], R.M. López-Gutiérrez [1], A. Arellano-Delgado [1], L. Cardoza-Avendaño [1] and R. Chávez-Pérez [2]

[1] *Faculty of Engineering, Baja California Autonomous University (UABC), Km. 103, Carretera Tijuana-Ensenada, 22860 Ensenada, B.C., México.*
[2] *Electronics and Telecommunications Department, Scientifc Research and Advanced Studies Center of Ensenada (CICESE), Carretera Ensenada-Tijuana, No. 3918, Zona Playitas, 22860 Ensenada, B.C., México.*

Received: June 06, 2011;   Revised: September 30, 2011

Abstract: In this paper, complex network synchronization of coupled hyperchaotic nodes (described by time-delay Chua oscillators) in different topologies is reported. In particular, networks synchronization in nearest-neighbor, small-world, open ring, tree, star, and global topologies are achieved. For each topology, the number of hyperchaotic nodes is evaluated that can be connected in the dynamical networks for synchronization purpose, which is based on a particular coupling strength. In addition, complex network synchronization for the mentioned topologies with unidirectional and bidirectional coupling of hyperchaotic nodes is considerated.

Keywords: *complex networks; nearest-neighbor topology; small-world topology; open ring topology; tree topology; star topology; global topology; network synchronization; hyperchaotic time-delay Chua oscillator.*

Mathematics Subject Classification (2000): 37N35, 65P20, 68P25, 70K99, 93D20, 94A99.

* Corresponding author: `mailto:ccruz@cicese.mx`

1 Introduction

Many systems in nature, applied sciences, and technology are constituted by a large number of highly interconnected dynamical units, the so-called complex dynamical networks. Some typical examples are coupled biological and chemical systems, neural networks, social interacting species, the Internet or the World Wide Web.

Complex networks of dynamical systems have been recently proposed as models in many diverse fields of applications (see e.g. [3, 17] and references therein). Recently, particular attention has been focused on the problem of making a network of dynamical systems synchronize to a common behavior. Typically, the complex network consists of N identical nonlinear dynamical systems coupled through the edges of the network itself [3, 17].

Synchronization is an important property of dynamical systems and even more when the dynamical systems have chaotic behavior, since achieving synchronization of chaotic systems provides superior alternatives to be explored, in complex network synchronization with chaotic nodes. The most works on network synchronization is about network configurations with regular coupling, see for example [8, 34], while the study in random network synchronization has been smaller, see e.g. [7, 13].

In particular, there is an increased interest in complex network synchronization of dynamical chaotic systems, which has led many scientists to consider the phenomenon of synchronization in large-scale networks with coupled chaotic oscillators like nodes, see e.g. [1, 2, 5, 8, 9, 18–27, 30, 31, 34, 35]. This type of network synchronization has been with topologies completely regular and global networks, see for example [26]. The main benefit of these simple architectures is that we can focus on the complexity caused by the nonlinear dynamics of the nodes, without taking into account the additional complexity, characteristic of the network topology.

In this paper, we synchronize complex dynamical networks of coupled hyperchaotic nodes in different topologies. In particular, each uncoupled dynamical system is described by a nonlinear set of time-delay Chua oscillators, which generate very complex behavior including hyperchaotic motion. This study presents network synchronization in nearest-neihbor, small-world, open ring, tree, star, and global coupling topologies; which are the most widely used in network communication systems. The complex network synchronization is achieved in two different way: with unidirectional and bidirectional coupling. In addition, network synchronization is evaluated according to a particular coupling strength for each topology.

The rest of the paper is outlined as follows. Section 2 describes the mathematical preliminaries, some important definitions, description of the networks, topologies, characteristics, network synchronization conditions, probability conditions, etc. Section 3 shows network synchronization with time-delay Chua oscillator like nodes, then it's performed network synchronization with each of the topologies with unidirectional and bidirectional coupling. Section 4 gives the conclusions of the results.

2 Preliminaries

We consider a complex dynamical network of N identical nodes, linearly coupled through the first state variable of each node, each node being a n-dimensional dynamical system.

The state equations of the network are given by:

$$\begin{aligned}
\dot{x}_{i1} &= f_1(\mathbf{x}_i) + s\sum_{j=1}^{N} a_{ij}x_{j1}, \qquad i = 1, 2, ..., N. \\
\dot{x}_{i2} &= f_2(\mathbf{x}_i), \\
&\vdots \\
\dot{x}_{in} &= f_n(\mathbf{x}_i), \qquad (1)
\end{aligned}$$

where $\mathbf{x}_i = (x_{i1}, x_{i2}, ..., x_{in})^{\mathrm{T}} \in \mathbb{R}^n$ are the state variables of node i, $f_i(\mathbf{0}) = \mathbf{0}$, $s > 0$ represents the coupling strength of the network, and the coupling matrix $\mathbf{A} = (a_{ij})_{(N\times N)} \in \mathbb{R}^{N\times N}$ represents the coupling configuration of the complex dynamical network. If there is a connection between node i and node j, then $a_{ij} = 1$; otherwise, $a_{ij} = 0$ $(i \neq j)$.

In this paper, we only consider symmetric and diffusive coupling. In particular, we assume that:
(i) $\mathbf{A}$ is a symmetric and irreducible matrix.
(ii) The off-diagonal elements a_{ij} $(i \neq j)$ of coupling matrix $\mathbf{A}$, are either 1 (when a connection between node i and node j) or 0 (when a connection between node i and node j is absent).
(iii) The elements of the principal diagonal of $\mathbf{A}$ satisfy

$$a_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij} = -\sum_{\substack{j=1\\ j\neq i}}^{N} a_{ji}, \qquad i = 1, 2, ..., N. \qquad (2)$$

The above conditions imply that one eigenvalue of the coupling matrix $\mathbf{A}$ is zero, with multiplicity 1, and all the other eigenvalues of $\mathbf{A}$ are strictly negative.

Given the dynamics of an isolated node and the coupling strength, stability of the synchronization state of the complex dynamical network (1) can be characterized by those nonzero eigenvalues of the coupling matrix $\mathbf{A}$. A typical result states that the complex dynamical network (1) will synchronize if these eigenvalues are negative enough [34].

Lemma 2.1 [31] *Consider the dynamical network (1). Let λ_1 be the largest nonzero eigenvalue of the coupling matrix $\mathbf{A}$ of the network. The synchronization state of network (1) defined by $\mathbf{x}_1 = \mathbf{x}_2 = ... = \mathbf{x}_n$ is asymptotically stable, if*

$$\lambda_1 \quad \leq \quad -\frac{T}{s}, \qquad (3)$$

where $s > 0$ is the coupling strength of the network and $T > 0$ is a positive constant such that zero is an exponentially stable point of the following n-dimensional system:

$$\dot{z}_1 = f_1(z) - Tz_1, \quad \dot{z}_2 = f_2(z), \quad \dot{z}_n = f_n(z). \qquad (4)$$

System (4) corresponds an *isolated node* with self-feedback $-Tz_1$. Condition (3) means that the complex dynamical network (1) will synchronize provided that λ_1 is negative enough, e.g. it is sufficient to be less than $-T/s$, so that the self-feedback term $-Tz_1$ could stabilize the isolated node (4).

3 Complex Network Topologies

3.1 Nearest-neighbor coupled network topology

The coupling configuration in nearest-neighbor consists of N arranged nodes in ring where each node i coupled to its nearest-neighbors nodes. The corresponding coupling matrix is given by

$$\mathbf{A}_{nc} = \begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}. \qquad (5)$$

The eigenvalues of the coupling matrix $\mathbf{A}_{nc}$ are given by [31]:

$$\Big\{ -4sin^2\left(\frac{k\pi}{N}\right),\ k = 0, 1, \ldots, N-1 \Big\}. \qquad (6)$$

Therefore, according to Lemma 2.1, the nearest-neighbor coupled dynamical network will asymptotically synchronize if [31]:

$$4sin^2\left(\frac{\pi}{N}\right) \geq \frac{T}{s}. \qquad (7)$$

3.2 Small-world coupled network topology

Aiming to describe a transition from a regular network to a random network, Watts and Strogatz [32] introduced an interesting model, called the *small-world* (SW) network. The original SW model can be described as follows. Take an one-dimensional network of N arranged nodes in a ring with connections between only nearest neighbors. We "rewire" each connection with some *probability* p. Rewiring in this context means shifting one end of the connection to a new node chosen at random from the whole network, with the constraint that no two different nodes can have more than one connection among them, and no node can have a connection with itself.

However, there is a possibility for the SW model to be broken into unconnected clusters. This problem can be circumvented by a slight modification of the SW model, suggested by Newman and Watts [15], which is called the *NW model*. In the NW model, we do not break any connection between any two nearest neighbors. We add with *probability* p a connection between each other pair of nodes. Likewise, we do not allow a node to be coupled to another node more than once, or coupling of a node with itself. For $p = 0$, it reduces to the originally nearest-neighbor coupled network; for $p = 1$, it becomes a globally coupled network. In this paper, we are interested in probabilities with $0 < p < 1$.

From a coupling matrix point of view, a complex dynamical network (1) with new connections in small-world is determined as follows: if $a_{ij} = 0$, this element can change to $a_{ij} = a_{ji} = 1$ according to the *probability* p. Then, we recompute the diagonal elements according to Eq. (2). We denote the new *small-world coupling matrix* as $\mathbf{A}_{swc}(p; N)$ and let $\lambda_{1swc}(p; N)$ be its largest nonzero eigenvalue. According to Lemma 2.1, if

$$\lambda_{1swc}(p; N) \leq -\frac{T}{s}, \qquad (8)$$

then the corresponding complex dynamical network (1) with small-world connections will synchronize [31].

3.3 Open ring coupled network topology

Open ring configuration consists of N arranged nodes in a ring, but in this case, the last node is not connected to the first node. The corresponding coupling matrix is given by

$$\mathbf{A}_{rc} = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}. \tag{9}$$

This matrix have an eigenvalue at 0 and others $N-1$ are at -1 [30].

3.4 Star coupled network topology

A complex dynamical network with star coupling consists of a single node (called the common node or central node) of the complex dynamical network connected with the remaining $N-1$ nodes. The coupling matrix is given by

$$\mathbf{A}_{sc} = \begin{bmatrix} 1-N & 1 & \dots & \dots & 1 \\ 1 & -1 & & & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 1 & & & & -1 \end{bmatrix}. \tag{10}$$

The eigenvalues of the coupling matrix $\mathbf{A}_{sc}$ are $\{0, -N, -1, \dots, -1\}$ [30].

3.5 Globally coupled network topology

In this coupling configuration, the N nodes are connected with others; that is any two nodes are connected directly. All nodes are connected to the same number $(N-1)$ of nodes. Thus, the coupling matrix is given by

$$\mathbf{A}_{gc} = \begin{bmatrix} 1-N & 1 & 1 & \dots & 1 \\ 1 & 1-N & 1 & \dots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1-N \end{bmatrix}. \tag{11}$$

This matrix has a single eigenvalue at 0 and others $N-1$ at $-N$ [30]. Lemma 2.1 implies that the global coupled network will asymptotically synchronize, if

$$N \geq -\frac{T}{s}.$$

3.6 Tree coupled network topology

The network topology in tree can be viewed as a collection of star networks arranged in a hierarchy way, the network begins with a master node and this in turn is connected to other slave nodes which are connected with the rest of the nodes. The coupling matrix for this network is given by

$$\mathbf{A}_{tc} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & -1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & 1 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & 1 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{12}$$

This matrix has an eigenvalue at 0 and the others $N-1$ are at -1 [30].

4 Network Synchronization with Time-Delay Chua Oscillator Like Nodes

4.1 Time-delay Chua oscillator

The time-delay Chua oscillator is a physical system, which presents well-defined hyperchaotic dynamics confirmed experimentally and numerically. The state equations describing the time-delay Chua oscillator in dimensionless form are given by [4, 6, 29]:

$$\begin{aligned} \dot{x}_1 &= \alpha(-x_1 + x_2 - f(x_1)), \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -\beta x_2 - \gamma x_3 - \beta\varepsilon \sin(\sigma x_1 (t-\tau)), \end{aligned} \tag{13}$$

with nonlinear function defined by

$$f(x_1) = bx_1 + \frac{1}{2}(a-b)(|x_1+1| - |x_1-1|).$$

The parameters which are obtained hyperchaotic dynamics are: $\alpha = 10$, $\beta = 19.53$, $\gamma = 0.1636$, $a = -1.4325$, $b = -0.7831$, $\sigma = 0.5$, $\varepsilon = 0.2$, and by using the time-delay $\tau = 0.001$. The initial conditions of the oscillator are $\mathbf{x}(0) = (1.1, 0.1, 0.5)$. The generated hyperchaotic attractors by the time-delay Chua oscillator (13) are shown in Figure 1.

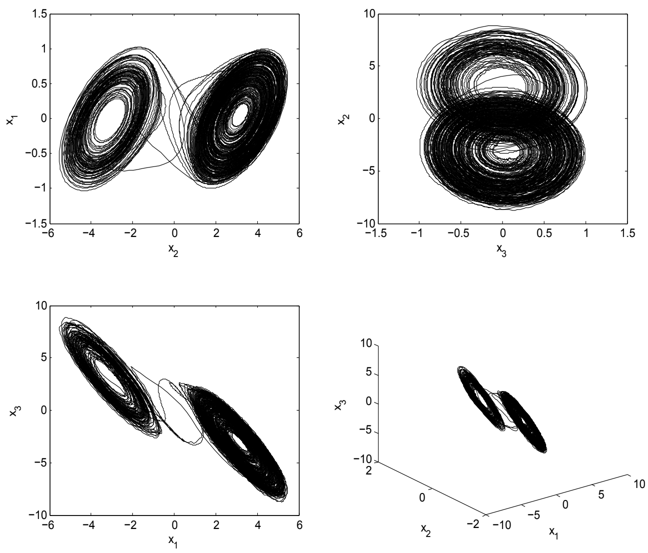

Figure 1: Hyperchaotic attractors generated by the time-delay Chua oscillator.

4.2 Synchronization in nearest-neighbor coupled networks

4.2.1 Bidirectional network synchronization

The state equations for N hyperchaotic nodes of the complex dynamical network (1) are given by

$$\begin{aligned}
\dot{x}_{i1} &= \alpha(-x_{i1} + x_{i2} - f(x_{i1})) + s\sum_{j=1}^{N}(a_{ij}x_{j1}), \qquad i = 1, 2, ..., N, \\
\dot{x}_{i2} &= x_{i1} - x_{i2} + x_{i3}, \\
\dot{x}_{i3} &= -\beta x_{i2} - \gamma x_{i3} - \beta\varepsilon \sin(\sigma x_{i1}\,(t-\tau)),
\end{aligned} \tag{14}$$

with nonlinear function defined by

$$f\,(x_1) = bx_1 + \frac{1}{2}\,(a-b)\,(|x_1+1| - |x_1-1|)\,.$$

For $T = 30$, the isolated node time-delay Chua oscillator (13) stabilizes at a point as is shown in Figure 2. The coupling strength chosen is $s = 25$ and the connection grade of the dynamical network is $K = 2$.

For example, with $N = 5$ hyperchaotic nodes (time-delay Chua oscillators), the dynamical network is shown in Figure 3(a). The bidirectional coupling matrix is given

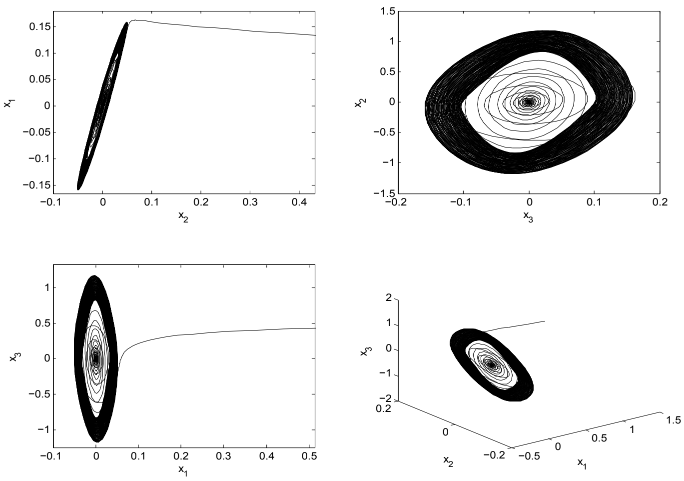

Figure 2: Attractors generated by an isolated time-delay Chua oscillator with feedback $-Tx_1$.

by

$$\mathbf{A}_{nc} = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}. \tag{15}$$

The largest nonzero eigenvalue of $\mathbf{A}_{nc}$ is defined by (6) as follows

$$-4sin^2\left(\frac{\pi}{5}\right) = -1.382. \tag{16}$$

The network synchronization condition (3) is as follows

$$-1.382 \leq -\frac{30}{25} = -1.2. \tag{17}$$

With these chosen values, the condition (3) is fulfilled and therefore the dynamical network with $N = 5$ hyperchaotic nodes in nearest-neighbor will synchronize.

Figure 4 shows the first attractor (x_{i1} vs x_{i2}) of each node. While, Figure 5 illustrates synchronization among nodes, showing the first state of each hyperchaotic node.

With $N = 6$ hyperchaotic nodes, the network is shown in Figure 3(b). The coupling matrix is defined by

$$\mathbf{A}_{nc} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}. \tag{18}$$

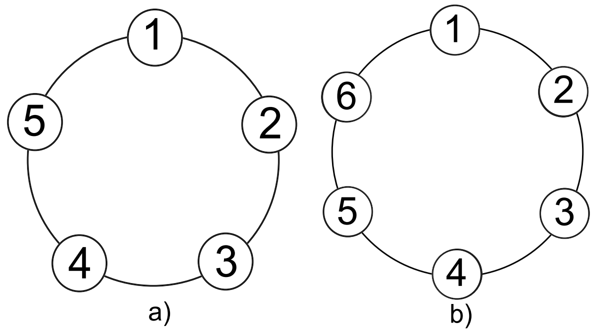

Figure 3: Network topologies in bidirectional nearest-neighbor: a) With $N = 5$ nodes. b) With $N = 6$ nodes.

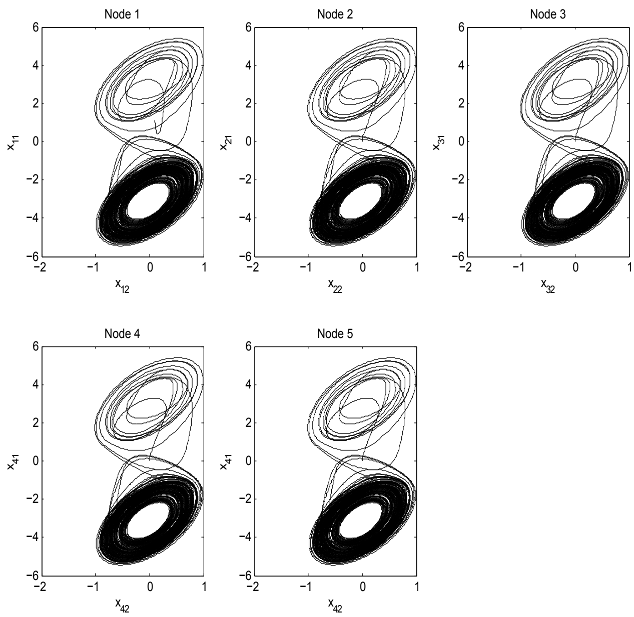

Figure 4: First attractor of each node with bidirectional synchronization for $N = 5$.

The largest nonzero eigenvalue is defined by (6),

$$-4sin^2\left(\frac{\pi}{6}\right) = -1. \tag{19}$$

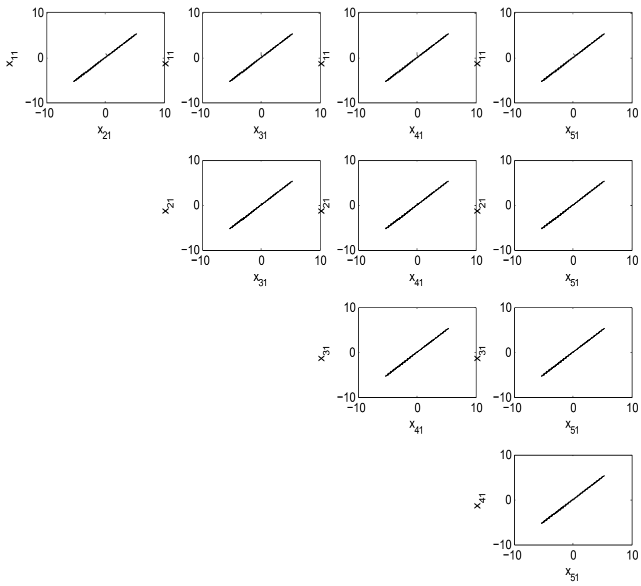

Figure 5: Synchronization among 5 hyperchaotic nodes in the bidirectional nearest-neighbor coupled network.

The network synchronization condition (3) is as follows

$$-1 \quad \leq \quad -\frac{30}{25} = -1.2.$$

Now, the network synchronization condition (3) is not fulfilled and therefore the dynamical network with 6 hyperchaotic nodes will not synchronize. Figure 6 illustrates the phase portrait among nodes, showing the first state of each node. It can be seen that there is no synchronization among nodes.

Therefore, we can say that for $s = 25$ and $N = 6$, the synchronization of the network in bidirectional nearest-neighbor configuration will synchronize up to with 5 hyperchaotic nodes.

4.2.2 Unidirectional network synchronization

With $N = 5$ hyperchaotic nodes (time-delay Chua oscillators), the dynamical network is shown in Figure 7(a). The unidirectional coupling matrix is given by

$$\mathbf{A}_{nc} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{20}$$

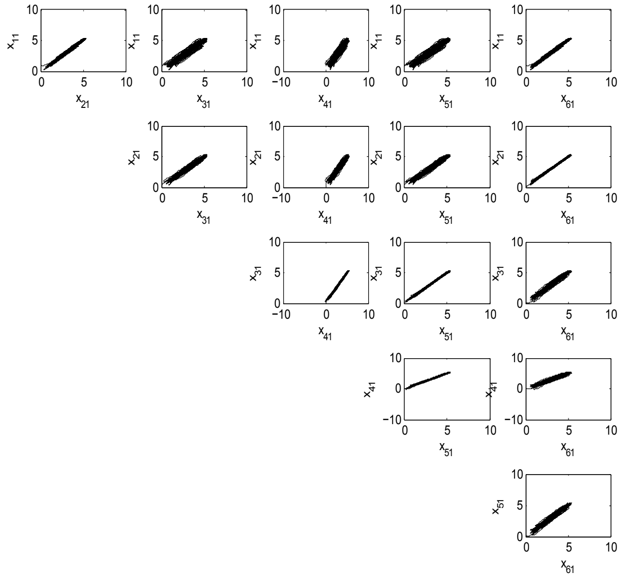

Figure 6: Phase portrait for no synchronization process among 6 hyperchaotic nodes of the dynamical network.

With a coupling strength $s = 10$ and previous values, the dynamical network with 5 nodes synchronizes. Figure 8 shows the first hyperchaotic attractors (x_{i1} vs x_{i2}) of each node. While, Figure 9 illustrates the synchronization among 5 hyperchaotic nodes, showing the first state of each node.

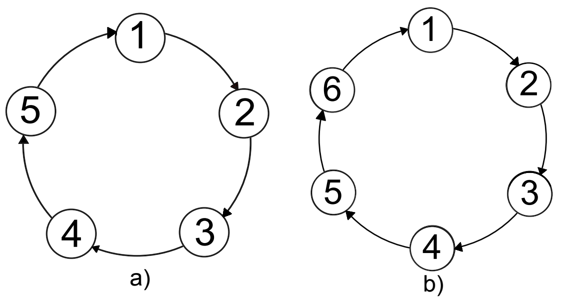

Figure 7: Network configuration in unidirectional nearest-neighbor: a) With $N = 5$ hyperchaotic nodes. b) With $N = 6$ hyperchaotic nodes.

With $N = 6$ hyperchaotic nodes, the dynamical network is shown in Figure 7(b). For

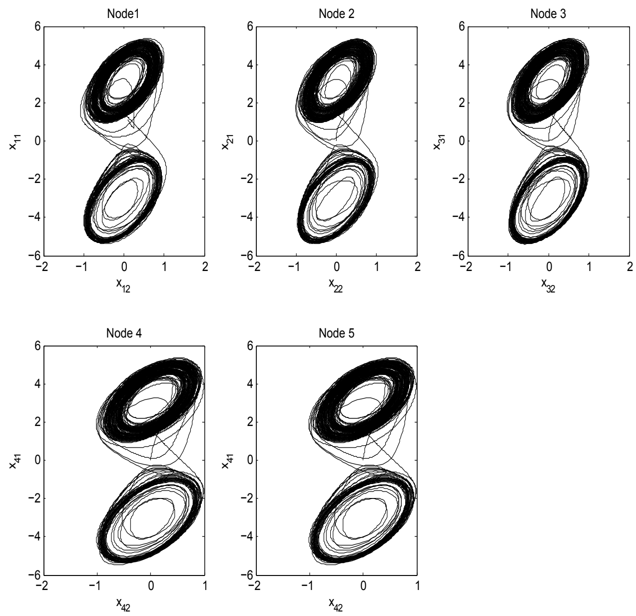

Figure 8: First attractor of each node with unidirectional synchronization for $N = 5$ hyperchaotic nodes.

this case, the coupling matrix is defined by

$$\mathbf{A}_{nc} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{21}$$

The largest nonzero eigenvalue is $-0.5 - 0.866i$. With a coupling strength $s = 10$ and previous values, the network with 6 hyperchaotic nodes will not synchronize. Figure 10 illustrates the synchronization errors among 6 hyperchaotic nodes, showing the first state of each node. It can be seen that there is no synchronization among nodes.

Therefore, we have that for coupling strength $s = 10$ and $N = 6$ hyperchaotic nodes, the dynamical network in unidirectional nearest-neighbor configuration will synchronize up to with $N = 5$ hyperchaotic nodes. In the sequel, we show how synchronize the mentioned complex dynamical network with few extra connections for $N \geq 6$ hyperchaotic nodes.

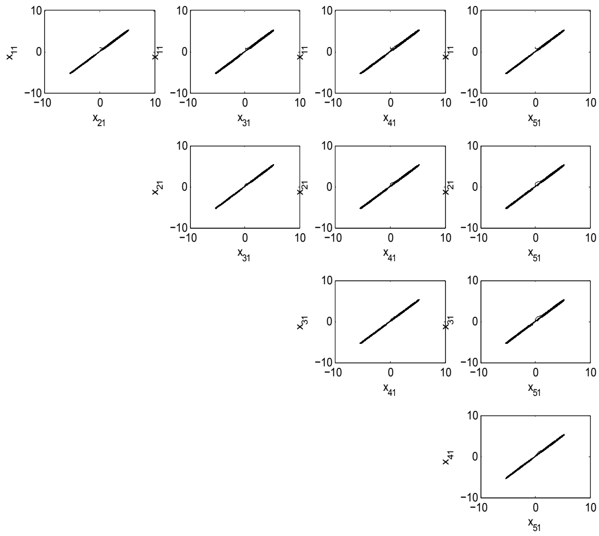

Figure 9: Synchronization among 5 hyperchaotic nodes in unidirectional nearest-neighbor coupled network.

4.3 Synchronization in small-world coupled networks

4.3.1 Bidirectional network synchronization

The nearest-neighbor coupled network with the values of previous parameters will not synchronize for $N \geq 6$, therefore, in this subsection we use the small-world configuration for synchronization of a number of hyperchaotic time-delay Chua oscillator nodes $N \geq 6$, of course without reaching the global coupling configuration, where network synchronization can be achieved without "any problem", besides having unnecessary connections that would increase construction costs and higher energy consumption, while with some new connections we can achieve complete synchronization of the complex dynamical network by using the small-world configuration.

Based on the coupling matrix (12) with $N = 6$ hyperchaotic nodes, the bidirectional nearest-neighbor configuration can build the new coupling matrix in small-world as follows [31]. In the nearest-neighbor coupling matrix $\mathbf{A}_{nc}$, the elements $a_{ij} = a_{ji} = 0$ can change to $a_{ij} = a_{ji} = 1$ according to a chosen probability p. In this case, we used a probability of connection $p = 0.15$, we choose a dynamical network with two new connections. With these new connections the dynamical network is shown in Figure 11(a). Then, we recompute the diagonal elements according to Eq. (2), the coupling matrix

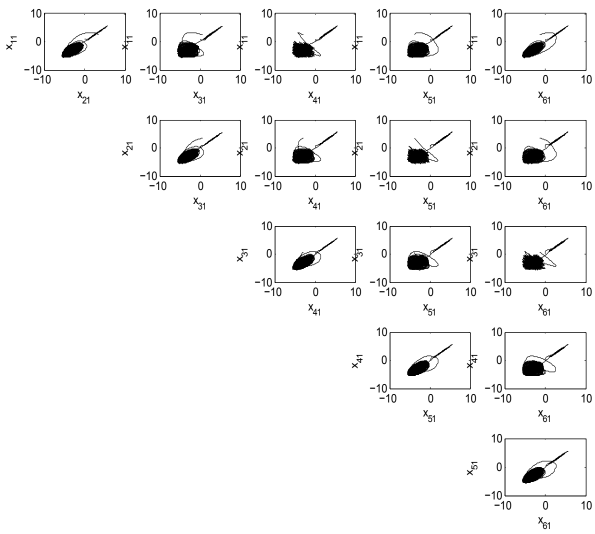

Figure 10: Phase portrait for no synchronization process among 6 hyperchaotic nodes of the unidirectional network.

with two new connections is defined by

$$\mathbf{A}_{swc} = \begin{bmatrix} -3 & 1 & 0 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & -4 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & 1 & -3 \end{bmatrix}.$$

The largest nonzero eigenvalue is -1.3088. The network synchronization condition (3) is given by

$$-1.3088 \leq -\frac{30}{25} = -1.2. \tag{22}$$

With these chosen values, the condition (3) is fulfilled and therefore the complex dynamical network with 6 hyperchaotic nodes will synchronize. Figure 12 shows the first attractor of each node.

Figure 13 illustrates the network synchronization among 6 hyperchaotic nodes, showing the first state of each node.

Figure 14 shows the numerical values of λ_{1swc} as a function of the number of N hyperchaotic nodes. In this figure each pair of values N and λ_{1swc} is obtained by averaging the results of 20 runs, implemented in the Matlab programming language. The above results imply that, for any given coupling strength s, we have: For any given

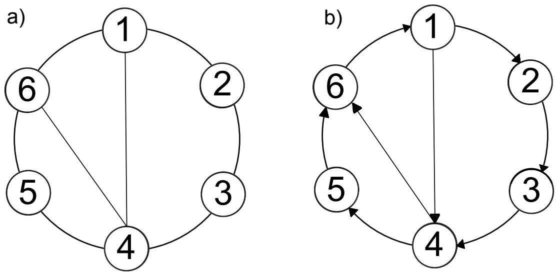

Figure 11: Small-world network configuration with $N = 6$ hyperchaotic nodes. a) Bidirectional coupling. b) Unidirectional coupling.

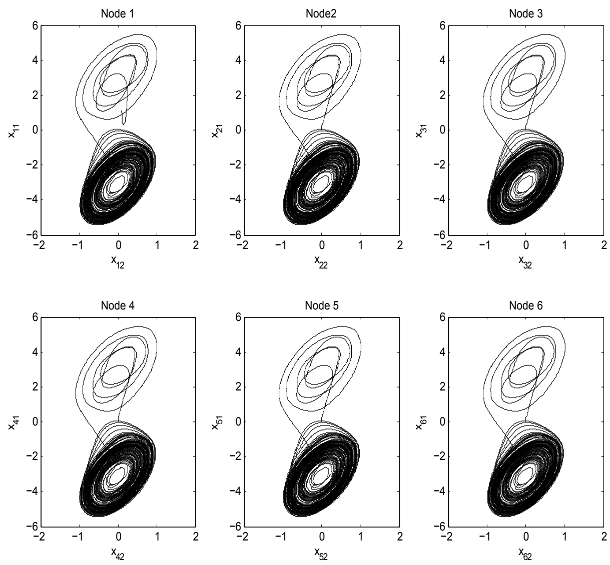

Figure 12: First hyperchaotic attractor of each node in bidirectional small-world configuration with $N = 6$ nodes.

N hyperchaotic nodes, there exists a critical value $\overline{\lambda}_{1swc}$, such that if $\overline{\lambda}_{1swc} \geq \lambda_{1swc}$, then the small-world connected network will synchronize.

4.3.2 Unidirectional network synchronization

Based on the coupling matrix (12) with $N = 6$ hyperchaotic nodes, the unidirectional nearest-neighbor configuration can build the new coupling matrix in small-world as follows [31]. In the nearest-neighbor coupling matrix $\mathbf{A}_{nc}$, the elements $a_{ij} = a_{ji} = 0$ can change to $a_{ij} = a_{ji} = 1$ according to a chosen probability p. In this case, we used a

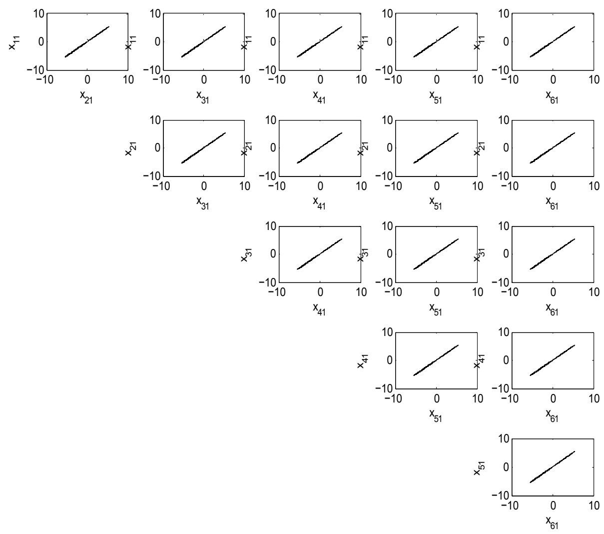

Figure 13: Synchronization in the first states among 6 hyperchaotic nodes of the network in bidirectional small-world configuration.

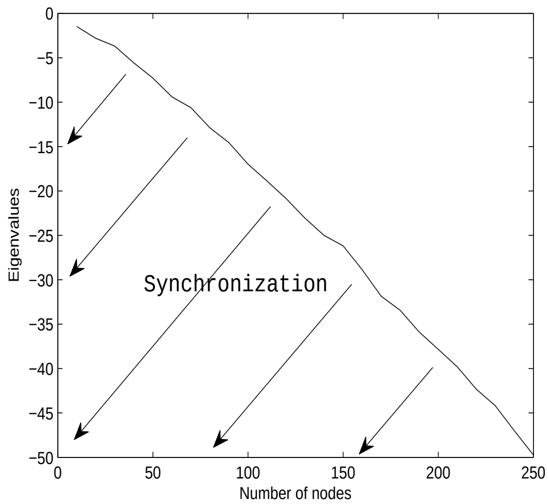

Figure 14: Number of nodes in function of the eigenvalues to achieve complex network synchronization.

probability of connection $p = 0.15$, we choose a dynamical network with two new connections. With these new connections the dynamical network is shown in Figure 11(b). Then, we recompute the diagonal elements according to Eq. (2), the coupling matrix with two new connections is defined by

$$\mathbf{A}_{swc} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{bmatrix}.$$

The largest nonzero eigenvalue is $-0.9293 + 0.7587i$.

With these two new connections and previous values, the unidirectional small-world complex dynamical network with 6 nodes will synchronize. Figure 15 shows the first hyperchaotic attractor of each node. Figure 16 illustrates the complex network synchronization among 6 hyperchaotic nodes, showing the first state of each node.

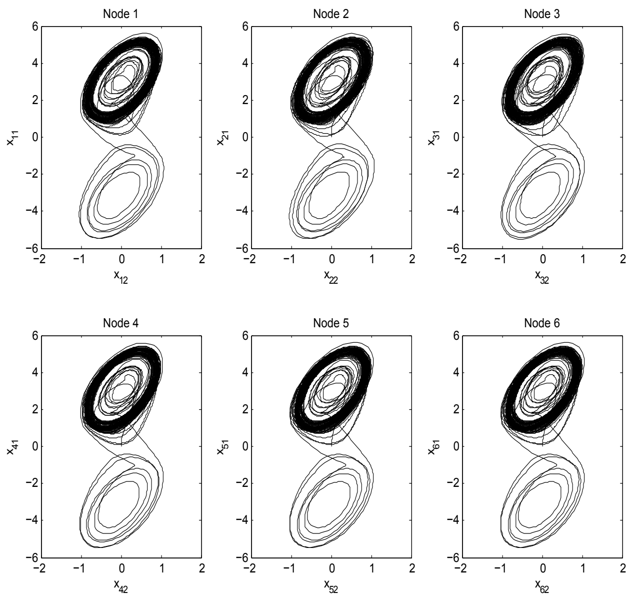

Figure 15: First hyperchaotic attractor of each node in unidirectional small-world configuration with $N = 6$.

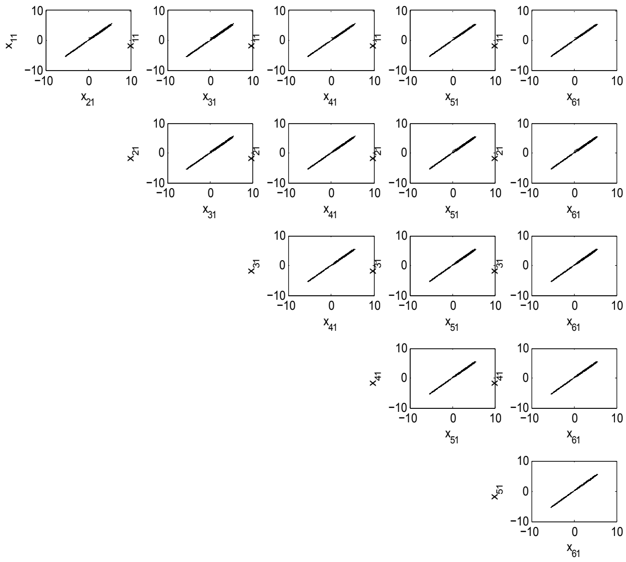

Figure 16: Synchronization in the first states among 6 hyperchaotic nodes of the network in unidirectional small-world configuration.

4.4 Synchronization in open ring coupled networks

4.4.1 Bidirectional network synchronization

With $N = 6$ hyperchaotic nodes, the dynamical network is shown in Figure 17(a). The coupling matrix is given by

$$\mathbf{A}_{rc} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{23}$$

The largest nonzero eigenvalue is -0.2679. In this case, Lemma 2.1 is not valid, however this dynamical network synchronizes with a coupling strength $s = 50$.

Figure 18 shows the first attractor (x_{i1} vs x_{i2}) of each node. Figure 19 illustrates the network synchronization among 6 hyperchaotic nodes, showing the first state of each node.

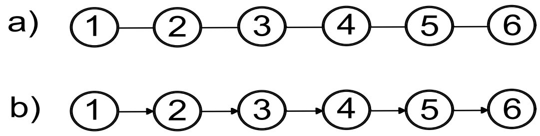

Figure 17: Network configuration in open ring with $N = 6$ hyperchaotic nodes: a) Bidirectional coupling. b) Unidirectional coupling.

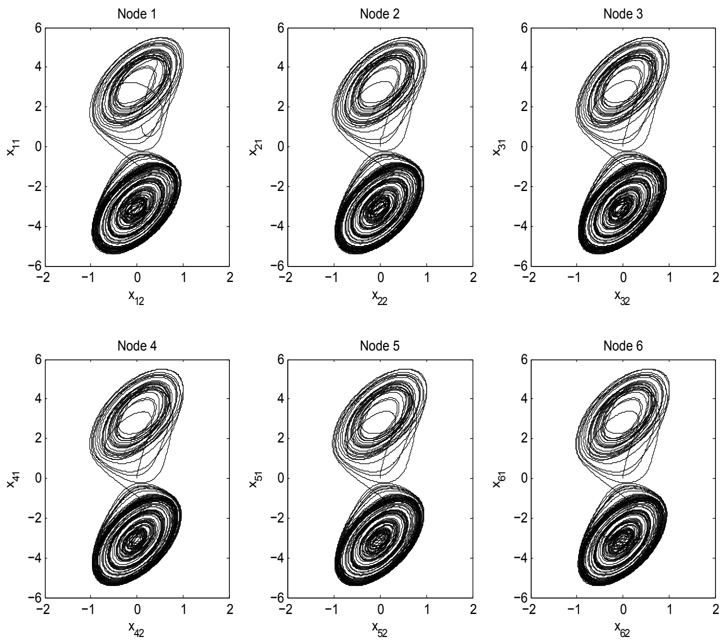

Figure 18: First attractor of each node in bidirectional open ring configuration with $N = 6$ hyperchaotic nodes.

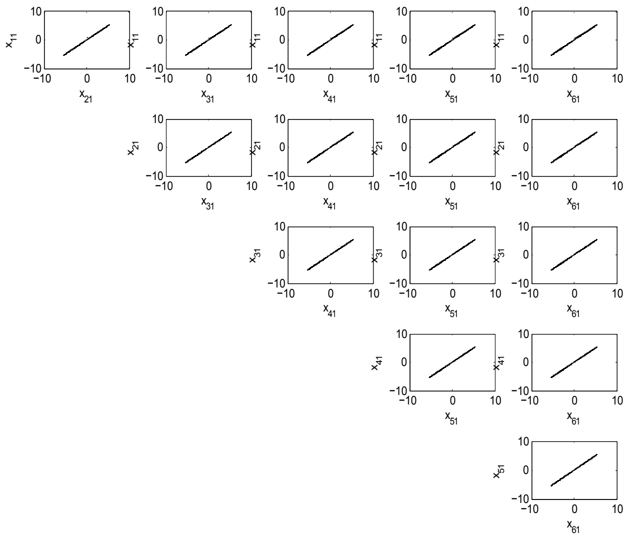

Figure 19: Synchronization among 6 nodes of the network in bidirectional open ring configuration.

4.4.2 Unidirectional synchronization

With a number $N = 6$ hyperchaotic nodes, the dynamical network is shown in Figure 17(b). The corresponding coupling matrix is defined by

$$\mathbf{A}_{rc} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{24}$$

The largest nonzero eigenvalue is -1. In this case, Lemma 2.1 is not valid, however this dynamical network synchronizes with a coupling strength $s = 50$.

Figure 20 illustrates the first attractor (x_{i1} vs x_{i2}) of each node. Figure 21 shows the network synchronization among 6 hyperchaotic nodes, showing the first state of each node.

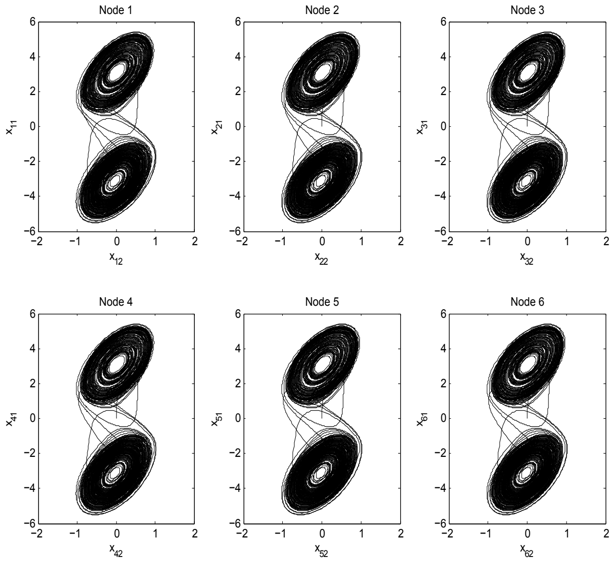

Figure 20: First attractor of each hyperchaotic node in unidirectional open ring configuration with $N = 6$ hyperchaotic nodes.

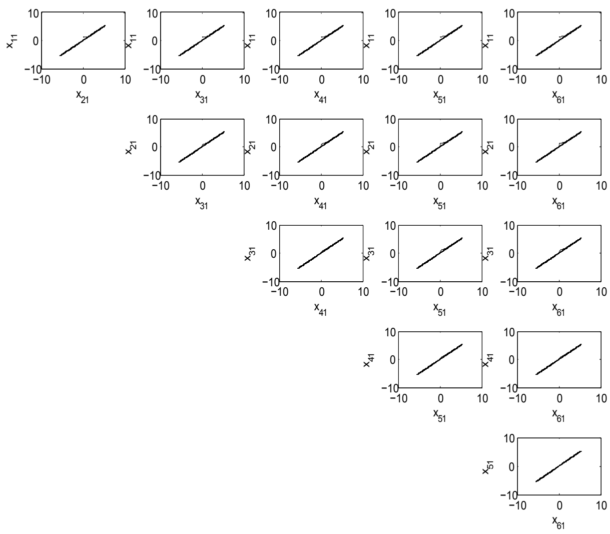

Figure 21: Synchronization among 6 hyperchaotic nodes of the network in unidirectional open ring configuration.

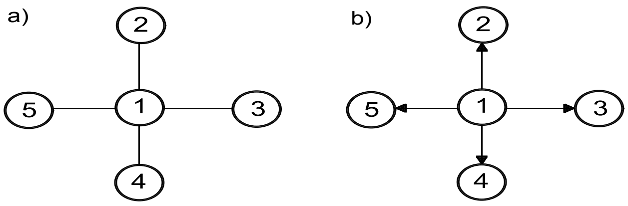

Figure 22: Network configuration in star with $N = 5$ hyperchaotic nodes: a) Bidirectional coupling. b) Unidirectional coupling.

4.5 Synchronization in star coupled networks

4.5.1 Bidirectional network synchronization

With $N = 5$ hyperchaotic nodes, the dynamical network is shown in Figure 22(a). The coupling matrix is given by

$$\mathbf{A}_{sc} = \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{25}$$

The largest nonzero eigenvalue is -1. In this case, Lemma 2.1 is not valid, however this network synchronizes with a coupling strength $s = 20$.

Figure 23 shows the first attractor (x_{i1} vs x_{i2}) of each node. Figure 24 illustrates the synchronization among nodes, showing the first state of each node.

4.5.2 Unidirectional network synchronization

With $N = 5$ hyperchaotic nodes, the dynamical network is shown in Figure 22(b). The coupling matrix is given by

$$\mathbf{A}_{sc} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{26}$$

The largest nonzero eigenvalue is -1. In this case, Lemma 2.1 is not valid, however this dynamical network synchronizes with a coupling strength $s = 20$.

Figure 25 shows the first attractor (x_{i1} vs x_{i2}) of each hyperchaotic node. Figure 26 illustrates the synchronization among nodes, showing the first state of each node.

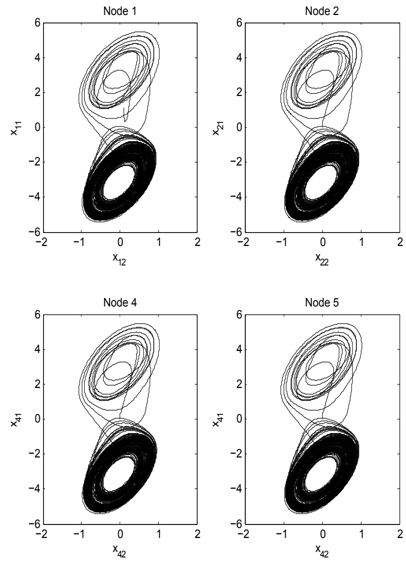

Figure 23: First hyperchaotic attractor of each node in bidirectional star configuration with $N = 5$.

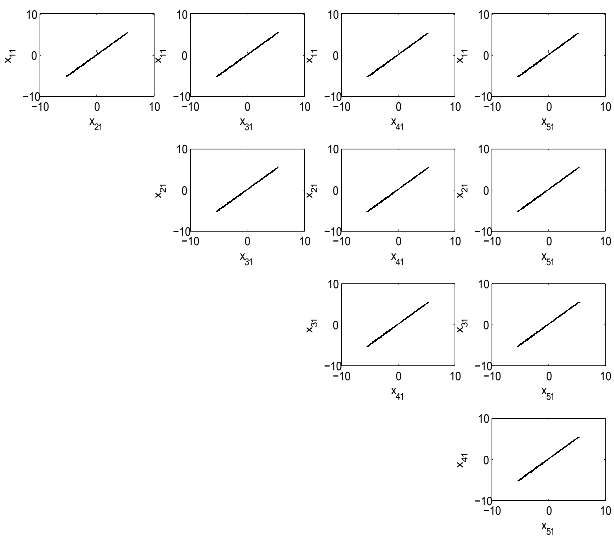

Figure 24: Synchronization among 5 hyperchaotic nodes of the network in bidirectional star configuration.

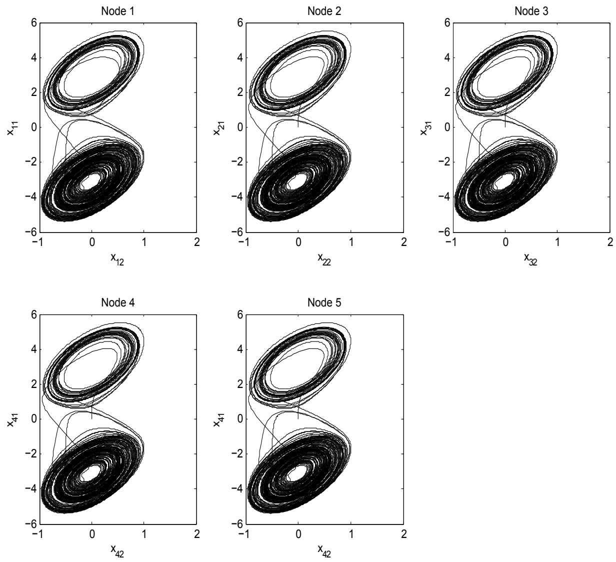

Figure 25: First hyperchaotic attractor of each node in unidirectional star configuration with $N = 5$ nodes.

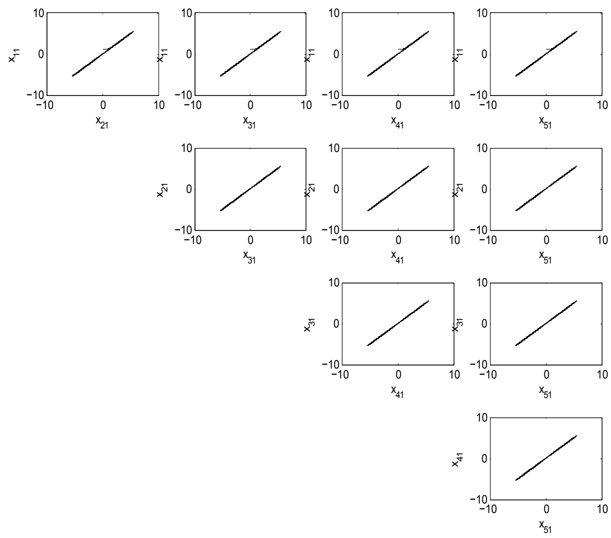

Figure 26: Synchronization among 5 hyperchaotic nodes of the network in unidirectional star configuration.

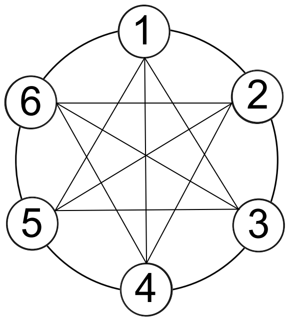

Figure 27: Network in global coupling with $N = 6$ hyperchaotic nodes.

4.6 Synchronization in global coupled networks

With $N = 6$ hyperchaotic nodes, the network is shown in Figure 27. The coupling matrix is defined by

$$\mathbf{A}_{gc} = \begin{bmatrix} -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & -5 & 1 & 1 & 1 & 1 \\ 1 & 1 & -5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -5 & 1 & 1 \\ 1 & 1 & 1 & 1 & -5 & 1 \\ 1 & 1 & 1 & 1 & 1 & -5 \end{bmatrix}. \tag{27}$$

The largest nonzero eigenvalue is -6. In this case, Lemma 2.1 is not valid, however this dynamical network synchronizes with a coupling strength $s = 10$. Figure 28 shows the first attractor (x_{i1} vs x_{i2}) of each 6 hyperchaotic nodes. Figure 29 illustrates the synchronization among 6 hyperchaotic nodes, showing the first state of each hyperchaotic node.

4.7 Synchronization in tree coupled networks

4.7.1 Bidirectional network synchronization

With $N = 7$ hyperchaotic nodes, the dynamical network is shown in Figure 30(a). The coupling matrix is given by

$$\mathbf{A}_{tc} = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{28}$$

The largest nonzero eigenvalue is -0.2679. In this case, Lemma 2.1 is not valid, however this dynamical network synchronizes with a coupling strength $s = 10$.

Figure 31 shows the first attractor (x_{i1} vs x_{i2}) of each node. Figure 32 illustrates the network synchronization among 7 hyperchaotic nodes, showing the first state of each node.

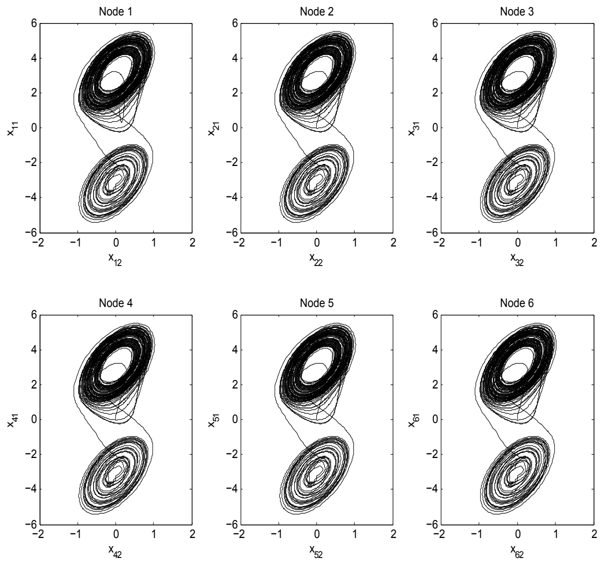

Figure 28: First hyperchaotic attractor of each node in global coupling with $N = 6$ nodes.

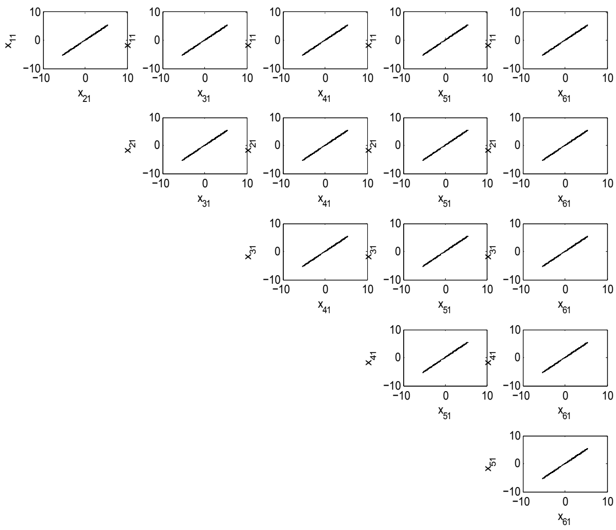

Figure 29: Synchronization among 6 hyperchaotic nodes of the dynamical network in global coupling.

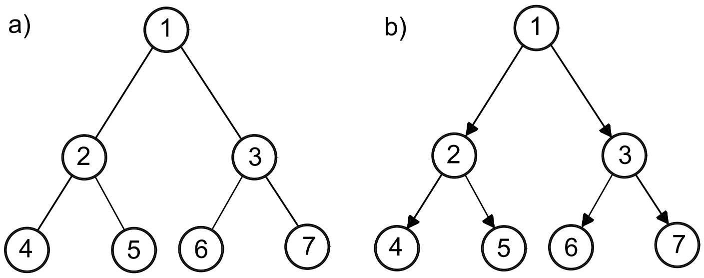

Figure 30: Network configuration in tree with $N = 7$ hyperchaotic nodes: a) Bidirectional coupling. b) Unidirectional coupling.

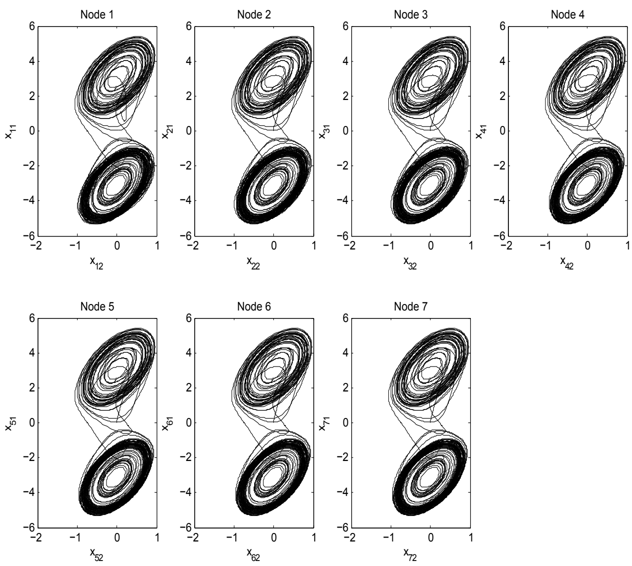

Figure 31: First hyperchaotic attractor of each node in bidirectional tree configuration with $N = 7$ nodes.

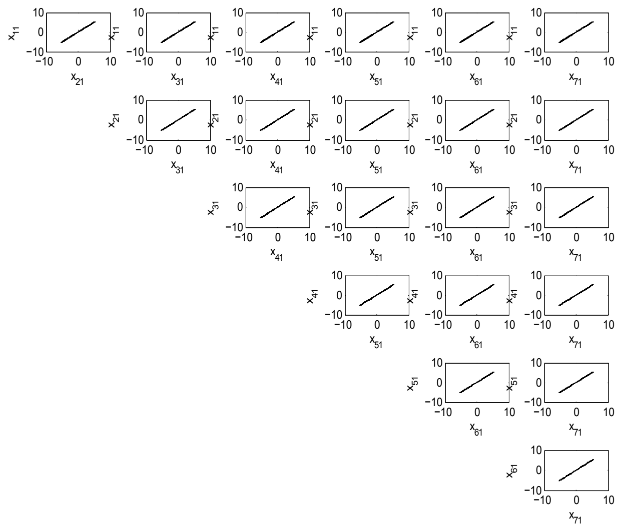

Figure 32: Synchronization among 7 hyperchaotic nodes of the network in bidirectional tree configuration.

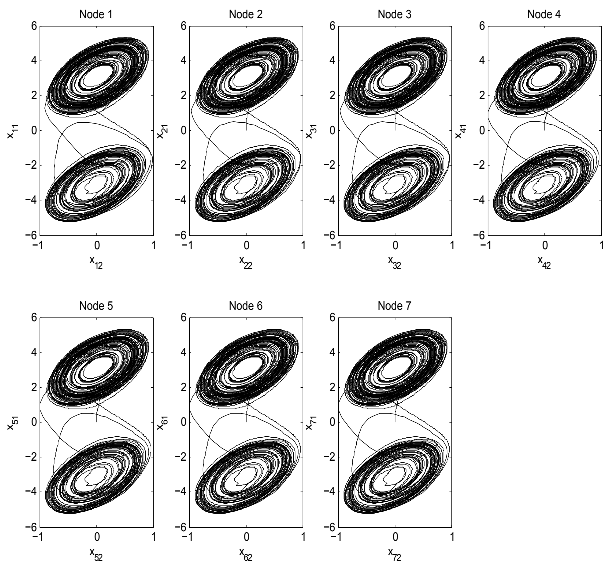

Figure 33: First hyperchaotic attractor of each node in unidirectional tree configuration with $N = 7$ nodes.

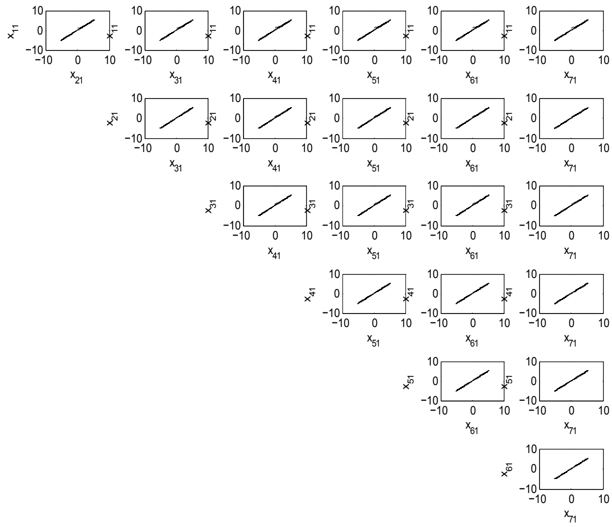

Figure 34: Synchronization among 7 hyperchaotic nodes of the network in unidirectional tree configuration.

4.7.2 Unidirectional network synchronization

With $N = 7$ hyperchaotic nodes, the dynamical network is shown in Figure 30(b). The coupling matrix is given by

$$A_{tc} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{29}$$

The largest nonzero eigenvalue is -1. In this case, Lemma 2.1 is not valid, however this network synchronizes with a coupling strength $s = 20$. Figure 33 shows the first attractor (x_{i1} vs x_{i2}) of each hyperchaotic node. Figure 34 illustrates the synchronization among 7 hyperchaotic nodes, showing the first state of each node.

5 Conclusions

In this paper, synchronization of complex dynamical networks in various topologies was performed. Numerical results were obtained for network synchronization in nearest-neighbor configuration, small-world configuration, open ring configuration, tree configuration, star configuration, and global configuration topologies, by using the hyperchaotic time-delay Chua oscillator like nodes. One can see that each coupling configuration requires a different coupling strength, each topology also has its own characteristics with implication that this coupling strength is different for each case. As an example, we mention the first topology, the nearest-neighbor, where with a number of $N = 5$ hyperchaotic nodes, it was required a coupling strength $s = 25$ for bidirectional synchronization; instead, for unidirectional network synchronization and the same number of nodes a coupling strength $s = 10$ is enough to achieve network synchronization. For tree topology, the bidirectional network synchronization requires a coupling strength $s = 10$ to synchronize, however, the unidirectional synchronization requires a coupling strength $s = 20$ for synchronization, which is twice as large as that of bidirectional. However, for all topologies the synchronization of the network was achieved unidirectionally or bidirectionally.

Acknowledgment

This work was supported by the CONACYT, México under Research Grants Nos. J49593, P50051-Y, and UABC 402/6/C/14/15.

References

[1] Acosta-del Campo, O.R., Cruz-Hernández, C., López-Gutiérrez, R.M. and García-Guerrero, E.E. Synchronization of modified Chua's circuits in star coupled networks. In: *Procs. of 6th International Conference on Informatics in Control, Automation and Robotics.* Milan, Italy, 2009, P. 162–167.

[2] Blasius, B., Huppert, A. and Stone, L. Complex dynamics and phase synchronization in spatially extended ecologicall systems. *Nature* **399** (1999) 354–359.

[3] Boccaletti, S., Latora, V., Moreno, Y., Chavez, M. and Hwang, D.U. Complex networks: Structure and dynamics. *Physics Reports* **424** (2006) 175-308.

[4] Cruz-Hernández, C. Synchronization of Time-Delay Chua's Oscillator with Application to Secure Communication. *Nonlinear Dynamics and Systems Theory* **4** (1) (2004) 1–13.

[5] Cruz-Hernández, C., López-Gutiérrez, R.M., Inzunza-González, E. and Cardoza-Avendaño, L. Network synchronization of unified chaotic systems in master-slave coupling. In: *Procs. of the 3th International Conference on Complex Systems and Applications.* Le Havre, Normandy, France, 2009, P. 56–60.

[6] Cruz-Hernández, C. and Martynyuk, A. A. *Advances in Chaotic Dynamics and Applications, Stability, Oscillations and Optimization of Systems.* Cambridge Scientific Publishers, Vol. 4, 2010.

[7] Gade, P. M. Synchronization of oscillators with random nonlocal connectivity. *Phys. Rev.* **E54** (1996) 64–70.

[8] Heagy, J. F., Carroll, T. L. and Pecora, L. M. Synchronous chaos in coupled oscillator systems. *Phys. Rev.* **E50** (3) (1994) 1874–1885.

[9] Hu, G., Yang, J. and Liu, W. Instability and controllability of linearly coupled oscillators: Eigenvalue analysis. *Phys. Rev. Lett.* **74** (21) (1998) 4185–4188.

[10] Lago-Fernández, L. F., Huerta, R., Corbacho, F. and Siguenza, J. A. Fast response and temporal coherent oscillations in small-world networks. *Phys. Rev. Lett.* **84** (12) (2000) 2758–2761.

[11] López-Gutiérrez, R.M., Posadas-Castillo, C., López-Mancilla, D. and Cruz-Hernández, C. Communicating via robust synchronization of chaotic lasers. *Solitons and Fractals* **41** (2009) 277–285.

[12] Lu, J. and Cao, J. Adaptive synchronization of uncertain dynamical networks with delayed coupling. *Nonlinaer Dynamics* **53** (2008) 107–115.

[13] Manrubia, S. C. and Mikhailov, S. M. Mutual synchronization and clustering in randomly coupled chaotic dynamical networks. *Phys. Rev.* **E60** (1999) 1579–1589.

[14] Milgram, S. The small-world problem. *Psychol. Today* **2** (1967) 60–67.

[15] Newman, M. E. J. and Watts, D. J. Renormalization group analysis of the small-world network model. *Phys. Lett.* **A263** (1999) 341–346.

[16] Newman, M. E. J. and Watts, D. J. Scaling and percolation in the small-world network model. *Phys. Rev.* **E60** (1999) 7332–7342.

[17] Newman, M.E.J., Barabsi, A.L. and Watts, D.J. *The structure and dynamics of complex networks*. Princeton University Press, 2006.

[18] Pecora, L. M., Carroll, T. J., Johnson, G., Mar, D. and Fink, K. S. Synchronization stability in coupled oscillator arrays: Solution for arbitrary configurations. *Int. J. Bifurcation and Chaos* **10** (2) (2000) 273–290.

[19] Posadas-Castillo, C., Cruz-Hernández, C. and López-Gutiérrez, R.M. Synchronization of chaotic neural networks with delay in irregular networks. *Applied Mathematics and Computation.* **205** (2008) 487–496.

[20] Posadas-Castillo, C., López-Gutiérrez, R.M. and Cruz-Hernández, C. Synchronization in a network of chaotic solid-state Nd:YAG lasers. In: *Procs of the 17th IFAC World Congress.* Seoul, Korea, 2008, P. 1565–1570.

[21] Posadas-Castillo, C., López-Gutiérrez, R.M. and Cruz-Hernández, C. Synchronization of chaotic solid-state Nd:YAG lasers: Application to secure communications. *Communications in Nonlinear Science and Numerical Simulation* **13** (2008) 1655–1667.

[22] Posadas-Castillo, C., Cruz-Hernández, C. and López-Gutiérrez, R.M. Synchronization of 3D CNNs in Irregular Array. In: *Procs. of the 16th Mediterranean Conference on Control and Automation Congress Center.* Ajaccio, France, 2008, P. 321–325.

[23] Posadas-Castillo, C., Cruz-Hernández, C. and López-Gutiérrez, R.M. Synchronization in a network of Chua's circuits. In: *Procs. of the 4th IASTED, International Conference on Circuits, Signals and Systems*, 2006, P. 236–241.

[24] Posadas-Castillo, C., Cruz-Hernández, C. and López-Mancilla, D. Synchronization of chaotic neural networks: A generalized Hamiltonian systems approach. In: *Hybrid Intelligent Systems: Analysis and Design.* (Eds.: O. Castillo, P. Melin, J. Kacprzyk and W. Pedrycz). Springer-Verlag, 2007, Vol. 208.

[25] Posadas-Castillo, C., Cruz-Hernández, C. and López-Mancilla, D. Sincronización de múltiples osciladores de Róssler, Congreso Anual de la Asociación de Control Automático, October 19–21, 2005. (In Spanish)

[26] Posadas-Castillo, C., Cruz-Hernández, C. and López-Gutiérrez, R.M. Experimental realization of synchronization in complex networks with Chua's circuits like nodes. *Chaos, Solitons and Fractals* **40** (2009) 1963–1975.

[27] Serrano-Guerrero, H., Cruz-Hernández, C., López-Gutiérrez, R.M., Posadas-Castillo, C. and Inzunza-González, E. Chaotic synchronization in Star Coupled Networks of 3D CNNs and its Application in Communications. *International Journal of Nonlinear Sciences and Numerical Simulation* **11** (8) (2010) 571–580.

[28] Strogatz, S.H. Exploring complex netwoks. Nature **410** (2001) 268–276.

[29] Wang, X.F., Zhong, G.Q., Tang, K.F. and Liu, Z.F. Generating chaos in Chua's circuit via time-delay feedback. *IEEE Trans. Circuits Syst.* **I 48** (9) (2001) 1151–1156.

[30] Wang, X. F. Complex networks: Topology, dynamics and synchronization. *International Journal of Bifurcation and Chaos* **12** (5) (2002) 885–916.

[31] Wang, X. F. and Chen, G. Synchronization in small-world dynamical networks. *Int. J. Bifurcation and Chaos* **12** (2002) 187–192.

[32] Watts, D. J. and Strogatz, S. H. Collective dynamics of small world networks. *Nature* **393** (1998) 440–442.

[33] Watts, D. J. *Small Worlds: The Dynamics of Networks Between Order and Randomness.* Princeton University Press, RI, 1999.

[34] Wu, C. W. and Chua, L. O. Synchronization in an array of linearly coupled dynamical systems. *IEEE Trans. Circuits Syst.* **I 42** (8) (1995) 430–447.

[35] Wu, C. W. *Synchronization in complex networks of nonlinear dynamical systems.* World scientific, Singapur, 2007.