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Backstepping for Nonsmooth MIMO Nonlinear Volterra Systems with Noninvertible Input-Output Maps and Controllability of Their Large Scale Interconnections ◇

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Abstract: We prove the global controllability for a class of nonlinear MIMO Volterra systems of the triangular form as well as for their bounded perturbations. In contrast to the related preceeding work [15], we replace the condition of C^1 smoothness, which was essentially used before, with that of local Lipschitzness. Furthermore, we remove the assumption of the invertibility of the input-output interconnections, which was also essential in these preceeding results. In order to solve the problem, we revise the backstepping procedure proposed in these works, and combine it with another method of constructing discontinuous feedbacks proposed for the so-called "generalized triangular form" in the case of ODE [16, 21].

Keywords: *backstepping; Volterra nonlinear control systems; controllability, large scale systems.*

Mathematics Subject Classification (2000): 93C10, 93B51, 93B05, 93A15.

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1 Introduction

During the last two decades such recursive procedures as backstepping-like designs became very popular when solving various problems of adaptive and robust nonlinear control [5, 9, 17, 18, 23]. It is worth mentioning that, despite of the fruitfulness of the backstepping-like algorithms, the most works devoted to them address the triangular or pure-feedback form systems [13]

$$\begin{cases} \dot{x}_i = f_i(x_1, ..., x_{i+1}), \quad i = 1, \dots, n-1; \\ \dot{x}_n = f_n(x_1, ..., x_n, u) \end{cases} \tag{1}$$

that are feedback linearizable, i.e., to those which satisfy the condition $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$, $i = 1, \dots, n$; or even have the strict-feedback form

$$\begin{cases} \dot{x}_i = b_i x_{i+1} + \theta_i \varphi_i(x_1, ..., x_i), \quad i = 1, \dots, n-1; \\ \dot{x}_n = b_n u + \theta_n \varphi_n(x_1, ..., x_n) \end{cases}$$

(with $b_i \neq 0$). Indeed, whatever the problem is (Lyapunov stabilization, adaptive stabilization etc.), the classical version of the backstepping requires system (1) to satisfy the following two properties:

(A) The virtual control $x_{i+1} = \alpha_i(t, x_1, ..., x_i)$ obtained at the i-th step ($i = 1, \dots, n$) should be well-defined as an implicit function obtained from some nonlinear equation of the form $f_i(x_1, ..., x_{i+1}) = F_i(t, x_1, ..., x_i)$ to be resolved w.r.t. x_{i+1}, where $F_i(t, x_1, ..., x_i)$ is some function of the previous coordinates $x_1, ..., x_i$ (and maybe of t).

(B) Each virtual control $x_{i+1} = \alpha_i(t, x_1, ..., x_i)$ obtained at the i - th step should be smooth enough because one needs to take its derivatives at the next steps $i = 1, \dots, n$.

This necessarily leads to the conditions like $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$, $i = 1, \dots, n$, (to comply with (A)) and like $f_i \in C^n$ or $f_i \in C^{n-i+1}$ (to comply with (B)).

Works [3, 4, 18, 22, 25, 26] were devoted to the issue of how to obviate the first restriction $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$, at least for some special cases: when $f_i(x_1, ..., x_{i+1})$ are polynomials w.r.t. x_{i+1} of odd degree (see work [22]); when $f_i = x_{i+1}^p + \varphi_i(x_1, ..., x_i)$ (see works [18,26] devoted to the problem of global stabilization of such systems into the origin as well as further works by some of these authors devoted to various adaptive and robust control problems for this class); partial-state stabilization under the assumption that the "controllable part" satisfies some additional "growth conditions" (see work [25] and conditions (A3),(i),(ii),(iii)); the problem of feedback triangulation under the assumption that the set of regular points is open and dense in the state space (see [3]).

A natural generalization of these cases is the so-called "generalized triangular form" (GTF), when the only assumption is that $f_i(t, x_1, ..., x_i, \cdot)$ is a surjection whereas x_i and u are vectors not necessarily of the same dimension (and the dynamics is of class C^1 or C^n depending of the problem to be explored). In works [16, 21] it was proved that, first, the systems of this class are globally robustly controllable, in particular, their bounded perturbations are globally controllable as well (see [16]) and, second, they are globally asymptotically stabilizable into every regular point (see [21]). Note that, although the methods proposed in [14–16, 21] are called "backstepping", their only common feature with the classical backstepping designs is the induction over the dimension of the system and treatment x_{i+1} as the virtual control at the i-th step. As to the construction, the approach proposed in [14–16, 21] is absolutely different. This especially applies to [16] and to the preceeding related works [14, 15] devoted to the problem of global robust controllability.

It is worth mentioning that, despite of the importance of the Volterra equations in applications, the controllability problem for the Volterra systems was investigated in few works only. Works [1, 2] are devoted to the complete controllability of perturbations of linear Volterra systems. In these papers, some natural analogs of the integral criterion of the controllability for linear ODE systems were obtained.

In works [14, 15] the problem of global robust controllability was successively solved for the nonlinear Volterra systems of the triangular form

$$\dot{x}_i = f_i(t, x_1, ..., x_{i+1}) + \int_{t_0}^{t} g_i(t, s, x_1(s), ..., x_{i+1}(s))ds, \quad i = 1, \dots, n,$$

(where $x_{n+1} = u$ is the control, and $(x_1, ..., x_n)$ is the state) including the global controllability of their bounded perturbations. Although, as we highlighted above, the inductive construction proposed in these works differs totally from the classical backstepping designs, the following two assumptions, which are similar to (A) and (B), are essential in this construction:

(A') For every $x_1(\cdot), ..., x_i(\cdot)$ of class C^1 the integral equation

$$\dot{x}_i = f_i(t, x_1(t), ..., x_{i+1}(t)) + \int_{t_0}^{t} g_i(t, s, x_1(s), ..., x_{i+1}(s))ds,$$

should be resolvable w.r.t. $x_{i+1}(\cdot)$ on the whole time interval $[t_0, T]$.

(B') The properties of the linearized control systems (and those of the Frechet derivative of the input-output map) are essential, which is why f_i and g_i should be of class C^1 at least.

The goal of the current paper is to remove these restrictions (A') and (B') and to show how a modification of the methods proposed in [16, 21] can be applied to the problem of global controllability of the Volterra systems. In many modern applications one has to deal with large scale interconnected systems - see, for instance [6, 10, 19]. Developing our technique, we solve the problem of global controllability for large scale interconnections of generalized triangular non-smooth Volterra systems.

2 Preliminaries

The first result of the current paper (Theorem 3.1 below) is concerned with the control systems of the Volterra integro-differential equations:

$$\dot{x}(t) = f(t, x(t), u(t)) + \int_{t_0}^{t} g(t, s, x(s))ds, \quad t \in I = [t_0, T], \tag{2}$$

where $u \in \mathbb{R}^m = \mathbb{R}^{m_{\nu+1}}$ is the control, $x = (x_1, ..., x_\nu)^T \in \mathbb{R}^n$ is the state with $x_i \in \mathbb{R}^{m_i}$, $m_i \le m_{i+1}$ and $n = m_1 + \dots, +m_\nu$, functions f and g have the form

$$f(t, x, u) = \begin{pmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2, x_3) \\ \dots \\ f_\nu(t, x_1, ..., x_\nu, u) \end{pmatrix} \quad \text{and} \quad g(t, s, x) = \begin{pmatrix} g_1(t, s, x_1) \\ g_2(t, s, x_1, x_2) \\ \dots \\ g_\nu(t, s, x_1, ..., x_\nu) \end{pmatrix} \tag{3}$$

with $f_i \in \mathbb{R}^{m_i}$, $g_i \in \mathbb{R}^{m_i}$ and satisfy the conditions:

(i) $f \in C(I \times \mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$, $g \in C(I^2 \times \mathbb{R}^n; \mathbb{R}^n)$,

(ii) *f and g satisfy the* local *Lipschitz condition w.r.t.* (x, u), *i.e., for every compact set $K \subset \mathbb{R}^n \times \mathbb{R}^m$ there is $l_K > 0$ such that, for every $(x^1, u^1) \in K$ and every $(x_2, u_2) \in K$ it holds*

$$|f(t, x^1, u^1) - f(t, x^2, u^2)| \leq l_K(|x^1 - x^2| + |u^1 - u^2|) \quad \text{and}$$

$$|g(t, s, x^1) - g(t, s, x^2)| \leq l_K|x^1 - x^2| \ \text{ for all } t \in I,\ s \in I$$

(iii) *For each $i = 1, \ldots, \nu$, each $t \in I$ and each $(x_1, ..., x_i)^T$ in $\mathbb{R}^{m_1+\ldots+m_i}$, we have* $f_i(t, x_1, ..., x_i, \mathbb{R}^{m_{i+1}}) = \mathbb{R}^{m_i}$.

Given $x^0 \in \mathbb{R}^n$, and $u(\cdot) \in L_\infty(I; \mathbb{R}^m)$, let $t \mapsto x(t, x^0, u(\cdot))$ denote the trajectory of (2), defined by this control $u(\cdot)$ and by the initial condition $x(t_0) = x^0$ on the maximal interval $J \subset I$ of the existence of the solution. As in [15], we say that a system of the Volterra integro-differential equations is globally controllable in time $I = [t_0, T]$ in class $C^\mu(I; \mathbb{R}^m)$ ($\mu \geq 0$), iff for each initial state $x^0 \in \mathbb{R}^n$ and each terminal state $x^T \in \mathbb{R}^n$ there is a control $u(\cdot)$ in $C^\mu(I; \mathbb{R}^m)$ which "steers x^0 into x^T w.r.t. the system", i.e., the trajectory $x(\cdot)$ of the system with this control $u(\cdot)$ such that $x(t_0) = x^0$ is well-defined on I and satisfies $x(T) = x^T$.

In our second result (Theorem 3.2 in the next Section) we consider a large scale interconnection of systems like (2) in the form

$$\dot{X}_i(t) = F_i(t, X_i(t), U_i(t)) + \int_{t_0}^{t} G_i(t, s, X_i(s))ds + H(t, X(t), U(t)) +$$

$$+ \int_{t_0}^{t} R(t, s, X(s), U(s))ds, \quad i = 1, \ldots, q, \quad t \in I = [t_0, T], \tag{4}$$

where $X = [X_1, \ldots, X_q]^T \in \mathbb{R}^N$ is the state with $X_i = [x_{i,1}, \ldots, x_{i,\nu_i}]^T \in \mathbb{R}^{n_i}$ and with $x_{i,j} \in \mathbb{R}^{m_{i,j}}$ and $U = [U_1, \ldots, U_q]^T \in \mathbb{R}^M$ is the control with $U_i \in \mathbb{R}^{m_{i,\nu_i+1}}$ (and $N = \sum\limits_{i=1}^{q} n_i = \sum\limits_{i=1,j=1}^{q,\nu_i} m_{i,j}$; $M = \sum\limits_{i=1}^{q} \nu_{i+1}$).

We assume that functions F_i and G_i have the form

$$F_i(t, X_i, U_i) = \begin{pmatrix} F_{i,1}(t, x_{i,1}, x_{i,2}) \\ F_{i,2}(t, x_{i,1}, x_{i,2}, x_{i,3}) \\ \cdots \\ F_{i,\nu_i}(t, x_{i,1}, ..., x_{i,\nu_i}, U_i) \end{pmatrix},$$

$$G_i(t, s, X_i) = \begin{pmatrix} G_{i,1}(t, s, x_{i,1}) \\ G_{i,2}(t, s, x_{i,1}, x_{i,2}) \\ \cdots \\ G_{i,\nu_i}(t, s, x_{i,1}, ..., x_{i,\nu_i}) \end{pmatrix}. \tag{5}$$

We define

$$F(t, X, U) = \begin{pmatrix} F_1(t, X_1, U_1) \\ F_2(t, X_2, U_2) \\ \cdots \\ F_q(t, X_q, U_q) \end{pmatrix}, \quad G(t, s, X) = \begin{pmatrix} G_1(t, s, X_1) \\ G_2(t, s, X_2) \\ \cdots \\ G_q(t, s, X_q) \end{pmatrix},$$

and assume that the following conditions hold:

(I) $F \in C(I \times \mathbb{R}^N \times \mathbb{R}^M; \mathbb{R}^N)$, $G \in C(I^2\mathbb{R}^N; \mathbb{R}^N)$.

(II) *There exists* $L > 0$ *such that, for every* $(X^1, U^1) \in K$ *and every* $(X_2, U_2) \in K$ *it holds*

$$|F(t, X^1, U^1) - F(t, X^2, U^2)| \le L(|X^1 - X^2| + |U^1 - U^2|),$$

$$|G(t, s, X^1) - G(t, s, X^2)| \le L|X^1 - X^2| \text{ for all } t \in I, \ s \in I$$

*(*global *Lipschitz property with respect to* (X, U)*).*

(III) *For each* $i = 1, \ldots, q$, *each* $j = 1, \ldots, \nu_i$, *each* $t \in I$ *and each* $(x_{i,1}, ..., x_{i,j})^T$ *in* $\mathbb{R}^{m_{i,1}+\ldots+m_{i,j}}$, *we have* $F_{i,j}(t, x_{i,1}, ..., x_{i,j}, \mathbb{R}^{m_{i,j+1}}) = \mathbb{R}^{m_{i,j}}$.

Also we assume that functions H and R satisfy the conditions:

(IV) $H \in C(I \times \mathbb{R}^N \times \mathbb{R}^M; \mathbb{R}^N)$, $R \in C(I^2 \times \mathbb{R}^N \times \mathbb{R}^M; \mathbb{R}^N)$, *and for each compact set* $Q \subset \mathbb{R}^N \times \mathbb{R}^M$, *there exists* $L_Q > 0$ *such that, for all* $(t, s) \in I^2$, $(X^1, U^1) \in Q$, $(X^2, U^2) \in Q$, *we have:*

$$|H(t, X^1, U^1) - H(t, X^2, U^2)| \le L_Q(|X^1 - X^2| + |U^1 - U^2|),$$

$$|R(t, s, X^1, U^1) - R(t, s, X^2, U^2)| \le L_Q(|X^1 - X^2| + |U^1 - U^2|),$$

(V) *There exists* $H_0 > 0$ *such that* H *and* R *satisfy the inequalities* $|H(t, X, U)| \le H_0$ *and* $|R(t, s, X, U)| \le H_0$ *for all* $(t, s, X, U) \in I^2 \times \mathbb{R}^N \times \mathbb{R}^M$.

Note that F_i and G_i have the "general traingular form", while H and R have an arbitrary form and are "cross terms", which characterize the interconnections of the isolated X_i-subsystems.

3 Main Results

Theorem 3.1 *Suppose that system (2) has the form (3) and satisfies conditions (i),(ii),(iii). Then system (2) is globally controllable in class* $C^\infty(I; \mathbb{R}^m)$.

Theorem 3.2 *Suppose that functions* F_i *and* G_i *have the form (5), satisfy (I),(II),(III), and suppose that* H *and* R *satisfy (IV), (V). Then system (4) is globally controllable in time* I *by means of controls of class* $C^\infty(I; \mathbb{R}^M)$.

Remark 3.1 Let us compare the results of [15] with our Theorems 3.1 and 3.2. First, in [15], functions f and g are required not only to be continuous but also to have all their partial derivatives, w.r.t. x and u, which are required to be continuous whereas we require (i) and (ii) only ((I) and (II) respectively for Theorem 3.2); (ii) or (II) being the standard condition needed to guarantee the existence and the uniqueness of the solution of the "Cauchy problem" for the Volterra systems. Second, our system (2) is MIMO and furthermore x_i and u are vectors of different dimensions whereas, in [15], the system is SISO (i.e., x_i and u are scalar) or at least x_i and u should be of the same dimension (see Remark 3.1 from [15]). Third (and this is essential), our current Assumption (iii) is much more general than the corresponding Assumption (ii) (or (II), p. 747) from [15]. In this sense, our current Theorem 3.1 and Theorem 3.2 generalize Theorem 3.3 and Theorem 3.2 from [15] respectively. However: firstly, in our case, function g has a bit more specific form than function g from [15] (g_i does not depend on x_{i+1} in the current paper); secondly, since we replace the assumption of C^1 smoothness with that of local Lipschitzness, we do not obtain stronger results on robustness (Theorem 3.1 from [15]).

Example 3.1 Consider the system given by

$$\begin{cases} \dot{x}_1(t) = (x_2(t) + x_1(t))|\sin x_2(t)| + \int\limits_0^t \sqrt{s^2 x_1^2(s) + 1}ds, \\ \dot{x}_2(t) = u(t)|\cos u(t)| + \int\limits_0^t \sqrt{e^{ts}(x_1^2(s) + x_2^2(s)) + 1}ds, \end{cases} \tag{6}$$

$t \in [0, T]$. It is clear that system (6) satisfies our Assumptions (i)-(iii) and therefore is globally controllable by Theorem 3.1. On the other hand, system (6) does not satisfy the Assumptions from [15] and the results of [15] are not applicable to system (6).

Remark 3.2 Note that, if $g = 0$ in (2), then (2) is reduced to the class of the so-called "generalized triangular form" of ODE control systems considered in [16, 20, 21]. However, in the case of ODE, stronger results were obtained in these works: global robust controllability (Theorem 3.1 from [16]), global asymptotic stabilization by means of smooth controls (Theorem 2.1 from [21]), and global discontinuous stabilization in the sense of Clarke-Ledyaev-Sontag-Subbotin (Theorem 3.4 from [16]).

4 Backstepping in the Non-smooth Case

Let us first reduce Theorem 3.1 to a backstepping process which can be compared with that from [16].

Let p be in $\{1, \ldots, \nu\}$. Define $k := m_1 + \ldots + m_p$ and consider the following k - dimensional control system

$$\dot{y}(t) = \varphi(t, y(t), v(t)) + \int\limits_{t_0}^{t} \psi(t, s, y(s))ds, \quad t \in I = [t_0, T], \tag{7}$$

where $y := (x_1, \ldots, x_p)^T \in \mathbb{R}^k = \mathbb{R}^{m_1 + \ldots + m_p}$ is the state, $v \in \mathbb{R}^{m_{p+1}}$ is the control and

$$\varphi(t, y, v) = \begin{pmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2, x_3) \\ \cdots \\ f_p(t, x_1, ..., x_p, v) \end{pmatrix}, \quad \psi(t, s, y) = \begin{pmatrix} g_1(t, s, x_1) \\ g_2(t, s, x_1, x_2) \\ \cdots \\ g_p(t, s, x_1, ..., x_p) \end{pmatrix}, \tag{8}$$

for all (t, y, v) in $I \times \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$. Given $y^0 \in \mathbb{R}^k$, and $v(\cdot) \in L_\infty(I; \mathbb{R}^{m_{p+1}})$, let $t \mapsto y(t, y^0, v(\cdot))$ denote the trajectory, of (7), defined by the control $v(\cdot)$ and by the initial condition $y(t_0, y^0, v(\cdot)) = y^0$ on the maximal interval $J \subset I$ of the existence of the solution. We reduce the proof of Theorems 3.1 to the following theorem.

Theorem 4.1 *Let p be in $\{1, ..., \nu\}$. Suppose for each $y^0 \in \mathbb{R}^k$ and each $\delta > 0$, there is a family of functions $\{y(\xi, \cdot) = (x_1(\xi, \cdot), ..., x_p(\xi, \cdot))\}_{\xi \in \mathbb{R}^k}$ such that:*

1) The map $\xi \mapsto y(\xi, \cdot)$ is of class $C(\mathbb{R}^k; C^1(I; \mathbb{R}^k))$

2) For each $\xi \in \mathbb{R}^k$, each $t \in I$ and each $1 \leq i \leq p - 1$ we have:

$$\dot{x}_i(\xi, t) = f_i(t, x_1(\xi, t), \ldots, x_{i+1}(\xi, t)) + \int\limits_{t_0}^{t} g_i(t, s, x_1(\xi, s), \ldots, x_i(\xi, s))ds$$

(if $p = 1$, then, the set of equalities is empty and, by definition, Condition 2) holds true)

3) $y(\xi, t_0) = y^0$ and $|y(\xi, T) - \xi| < \delta$ for all $\xi \in \mathbb{R}^k$

Then, for each $(y^0, y^0_{p+1}) \in \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$, and each $\varepsilon > 0$, there exists a family of controls $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbb{R}^k\times\mathbb{R}^{m_{p+1}}}$ such that

4) The map $(\xi, \beta) \mapsto \hat{v}_{(\xi,\beta)}(\cdot)$ is of class $C(\mathbb{R}^k \times \mathbb{R}^{m_{p+1}}; C^\infty(I; \mathbb{R}^{m_{p+1}}))$

5) For each $(\xi, \beta) \in \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$, we have $\hat{v}_{(\xi,\beta)}(T) = \beta$ and $\hat{v}_{(\xi,\beta)}(t_0) = y^0_{p+1}$.

6) $|y(T, y^0, \hat{v}_{(\xi,\beta)}(\cdot)) - \xi| < \varepsilon$ for all $(\xi, \beta) \in \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$.

Let us prove that Theorem 3.1 follows from Theorem 4.1. Indeed, suppose Theorem 4.1 holds true.

Suppose $p = 1$ and $k = m_1$, and take an arbitrary $y^0_1 \in \mathbb{R}^{m_1}$. Given an arbitrary $\delta > 0$, find any family $\{y(\eta, \cdot)\}_{\eta\in\mathbb{R}^{m_1}} = \{x_1(\eta, \cdot)\}_{\xi\in\mathbb{R}^{m_1}}$ such that Conditions 1)-3) of Theorem 4.1 hold. Then, for $p = 1$, we have: for every $\varepsilon > 0$ and every $(y^0_1, y^0_2) \in \mathbb{R}^{m_1+m_2}$, there exists a family of controls $\{\hat{v}_{(\eta,\beta)}(\cdot)\}_{(\eta,\beta)\in\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}}$ such that Conditions 4), 5), 6) of Theorem 4.1 hold with $p = 1$.

Suppose $p = 2$. Given any $y^0 = (y^0_1, y^0_2) \in \mathbb{R}^{m_1+m_2}$, and any $\delta > 0$, define $\varepsilon := \delta$, and for this $\varepsilon > 0$ find the family $\{\hat{v}_{(\eta,\beta)}(\cdot)\}_{(\eta,\beta)\in\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}}$ obtained at the previous step (with $p = 1$). From Conditions 4)-6) applied to $p = 1$ it follows that the family $\{y(\xi, \cdot)\}_{\xi=(\eta,\beta)\in\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}}$ defined by

$$y(\eta, \beta, t) := (y(t, y^0_1, \hat{v}_{(\eta,\beta)}(\cdot)), \hat{v}_{(\eta,\beta)}(t)) \text{ for all } \quad t \in I, \quad \xi = (\eta, \beta) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$$

satisfies the Conditions 1), 2), 3) of Theorem 4.1 with $p = 2$. Then we can apply Theorem 4.1 to $p = 2$, etc. Arguing by induction over $p = 1, \ldots, \nu$, we obtain for $p = \nu$ that for each $\varepsilon > 0$, each $x^0 \in \mathbb{R}^n$, and each $\alpha = y^0_{\nu+1} \in \mathbb{R}^{m_{\nu+1}}$ there exists a family of controls $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbb{R}^n\times\mathbb{R}^{m_{\nu+1}}}$ such that Conditions 4), 5), 6) of Theorem 4.1 hold for $p = \nu$. Fix an arbitrary $\beta \in \mathbb{R}^{m_{\nu+1}}$ and define the family of controls $\{u_\xi(\cdot)\}_{\xi\in\mathbb{R}^n}$ as follows: $u_\xi(t) := \hat{v}_{(\xi,\beta)}(t)$ for all $t \in I$, $\xi \in \mathbb{R}^n$. Then $\{u_\eta(\cdot)\}_{\eta\in\mathbb{R}^n}$ satisfies the conditions:

(a) $\xi \mapsto u_\xi(\cdot)$ is of class $C(\mathbb{R}^n; C^\infty(I; \mathbb{R}^{m_{\nu+1}}))$

(b) For each $\xi \in \mathbb{R}^n$, the trajectory $t \mapsto x(t, x^0, u_\xi(\cdot))$ is well-defined and $|x(T, x^0, u_\xi(\cdot)) - \xi| < \varepsilon$.

Given any $\varepsilon > 0$, an arbitrary $x^0 \in \mathbb{R}^n$, and an arbitrary $x^T \in \mathbb{R}^n$, let $\{u_\xi(\cdot)\}_{\xi\in\mathbb{R}^n}$ be a family of controls such that (a), (b) hold. By conditions (a),(b) the map $\xi \mapsto \xi - x(T, x^0, u_\xi(\cdot)) + x^T$ is well-defined and of class $C(\mathbb{R}^n; \mathbb{R}^n)$. From condition (b), it follows that this continuous function maps the compact convex set $\overline{B_\varepsilon(x^T)}$ into $\overline{B_\varepsilon(x^T)}$. Then, by the Brouwer fixed-point theorem, there exists $\xi^* \in \overline{B_\varepsilon(x^T)} \subset \mathbb{R}^n$ such that $\xi^* = \xi^* - x(T, x^0, u_{\xi^*}(\cdot)) + x^T$, i.e., $x(T, x^0, u_{\xi^*}(\cdot)) = x^T$. Thus, for every $x^0 \in \mathbb{R}^n$, and every $x^T \in \mathbb{R}^n$, there is a control $u_{\xi^*}(\cdot) \in C^\infty(I; \mathbb{R}^{m_{\nu+1}})$ such that $x^T = x(T, x^0, u_{\xi^*}(\cdot))$, i.e., Theorem 3.1 follows from Theorem 4.1.

Let us prove Theorem 3.2. Given any $U(\cdot) = [U_1(\cdot), \ldots, U_q(\cdot)]^T$ in $L_\infty(I; \mathbb{R}^N)$ and $X^0 \in \mathbb{R}^N$ let $t \mapsto X(t, X^0, U(\cdot))$ denote the trajectory of system

$$\dot{X}_i(t) = F_i(t, X_i(t), U_i(t)) + \int\limits_{t_0}^{t} G_i(t, s, X_i(s))ds \;\; i = 1, \ldots, q, \;\; t \in I = [t_0, T],$$

defined by the initial condition $X(t_0) = X^0$ and by the control $U = U(\cdot)$. Then arguing as above (for each X_i-subsystem separately), we construct a family $\{U_\xi(\cdot)\}_{\xi\in\mathbb{R}^N}$ such that the following conditions hold:

(c) $\xi \mapsto U_\xi(\cdot)$ is of class $C(\mathbb{R}^N; C^\infty(I; \mathbb{R}^M))$

(d) For each $\xi \in \mathbb{R}^N$, the trajectory $t \mapsto X(t, x^0, U_\xi(\cdot))$ is well-defined and $|X(T, X^0, U_\xi(\cdot)) - \xi| < \varepsilon$.

For each $\xi \in \mathbb{R}^N$, by $X(\xi, \cdot)$ denote the trajectory, of (4), defined by the control $U_\xi(\cdot)$ and by the initial condition $X(\xi, t_0) = X^0$. Using the Gronwall-Bellmann lemma, we easily obtain that $t \mapsto X(\xi, t)$ is well-defined for all $t \in I$, $\xi \in \mathbb{R}^N$ and there exists $D > 0$ such that $|X(\xi, t) - X(t, X^0, u_\xi(\cdot))| \le D$ for all $t \in I$ and $\xi \in \mathbb{R}^N$, and therefore, by condition (d), we obtain: $|X(\xi, T) - \xi| \le D + \varepsilon$ for all $\xi \in \mathbb{R}^N$. Taking an arbitrary $X^T \in \mathbb{R}^N$ and applying the Brouwer fixed-point theorem to the map $\xi \mapsto \xi - X(\xi, T) + X^T$, which maps the closed ball $\overline{B_{D+\varepsilon}(X^T)}$ into $\overline{B_{D+\varepsilon}(X^T)}$, we obtain the existence of $\xi^* \in \overline{B_{D+\varepsilon}(X^T)} \subset \mathbb{R}^N$ such that $X^T = X(\xi^*, T)$, which means that the control $U_{\xi^*}(\cdot) \in C^\infty(I; \mathbb{R}^M)$ steers X^0 into X^T in time I w.r.t. system (4). Since X^0 and X^T are chosen arbitrarely, the proof of Theorem 3.2 is complete.

5 Proof of Theorem 4.1

Fix an arbitrary p in $\{1, \dots, \nu\}$, an arbitrary $(y^0, y^0_{p+1}) \in \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$, and an arbitrary $\varepsilon > 0$. Define $\delta := \frac{\varepsilon}{4}$ and assume that $\{y(\xi, \cdot)\}_{\xi\in\mathbb{R}^k}$ satisfies Assumptions 1)-3) of Theorem 4.1.

To prove Theorem 4.1, we change the approach from [15] and [16] as follows. Along with system (7), we consider the following k-dimensional control system of the Volterra equations

$$\begin{cases} \dot{x}_i(t) = f_i(t, x_1(t), ..., x_{i+1}(t)) + \int\limits_{t_0}^{t} g_i(t, s, x_1(s), ..., x_i(s))ds, \quad i = 1, \dots, p-1, \\ \dot{x}_p(t) = w(t) + \int\limits_{t_0}^{t} g_p(t, s, x_1(s), ..., x_p(s))ds, \end{cases} \quad t \in I \tag{9}$$

with states $y = (x_1, ..., x_p)^T \in \mathbb{R}^k$ and controls $w \in \mathbb{R}^{m_p}$. Given $y \in \mathbb{R}^k$, and $w(\cdot) \in L_\infty(I; \mathbb{R}^{m_p})$, let $t \mapsto z(t, y, w(\cdot))$ denote the trajectory, of (9), defined by the control $w(\cdot)$ and by the initial condition $z(t_0, y, w(\cdot)) = y$ on some maximal interval $J \subset I$ of the existence of the solution.

For all $\xi \in \mathbb{R}^k$, define

$$\omega(\xi, t) = \dot{x}_p(\xi, t) - \int\limits_{t_0}^{t} g_p(t, s, x_1(\xi, s), ..., x_p(\xi, s))ds,\ t \in I. \tag{10}$$

Then

$$y(\xi, t) = z(t, y^0, \omega(\xi, \cdot)) \text{ for all } \quad t \in I, \ \ \xi \in \mathbb{R}^k. \tag{11}$$

Then, using the Gronwall-Bellmann lemma, we get the existence of $\delta(\cdot)$ in $C(\mathbb{R}^k;]0, +\infty[)$ such that, for each $\xi \in \mathbb{R}^k$ and each $w(\cdot) \in L_\infty(I; \mathbb{R}^{m_p})$, we have:

$$\forall\, t \in I \quad |z(t, y^0, w(\cdot)) - y(\xi, t)| < \delta,$$

$$\text{whenever} \quad \| w(\cdot) - \omega(\xi, \cdot)\|_{L_\infty(I;\mathbb{R}^{m_p})} < \delta(\xi). \tag{12}$$

In order to complete the proof of Theorem 4.1, it suffices to prove the following Proposition, which is similar to Lemma 5.1 from [16].

Proposition 5.1 *Assume that* $\{y(\xi, \cdot)\}_{\xi \in \mathbb{R}^k}$ *is a family such that Conditions 1)-3) of Theorem 4.1 hold. Then, for system (7), there exist functions* $M(\cdot) \in C(\mathbb{R}^k;]0, +\infty[)$ *and a family* $\{u(\xi, \cdot)\}_{\xi \in \mathbb{R}^k}$ *of controls defined on* I *such that:*

1) For each $\xi \in \mathbb{R}^k$, *the control* $u(\xi, \cdot)$ *is a piecewise constant function on* I *and the map* $\xi \mapsto u(\xi, \cdot)$ *is of class* $C(\mathbb{R}^k; L_1(I; \mathbb{R}^{m_{p+1}}))$.

2) For each $\xi \in \mathbb{R}^k$, *the trajectory* $t \mapsto y(t, y^0, u(\xi, \cdot))$ *is defined for all* $t \in I$, *and for each* $\xi \in \mathbb{R}^k$ *we have*

$$|\omega(\xi, t) - f_p(t, y(t, y^0, u(\xi, \cdot)), u(\xi, t))| < \delta(\xi), \ \ t \in I$$

3) For each $\xi \in \mathbb{R}^k$, *we have:* $\| u(\xi, \cdot)\|_{L_\infty(I;\mathbb{R}^{m_{p+1}})} \le M(\xi)$.

Indeed, if Proposition 5.1 is proved, then, combining (10), (11), (12) with the form of the dynamics of (7),(9), we get

$$|y(t, y^0, u(\xi, \cdot)) - y(\xi, t)| < \delta \quad \text{for all } \ t \in I, \ \xi \in \mathbb{R}^k. \tag{13}$$

Using partitions of unity and arguing as in [15], [16], we get the existence of a family $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbb{R}^k\times\mathbb{R}^{m_{p+1}}}$ of controls such that Conditions 4) and 5) of Theorem 4.1 hold and such that for each $(\xi, \beta) \in \mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$ we have

$$|y(t, y^0, \hat{v}_{(\xi,\beta)}(\cdot)) - y(t, y^0, u(\xi, \cdot))| < \delta \ \text{ for all } t \in I, \tag{14}$$

($t \mapsto y(t, y^0, \hat{v}_{(\xi,\beta)}(\cdot))$ being defined on I for all (ξ, β) in $\mathbb{R}^k \times \mathbb{R}^{m_{p+1}}$). Since $\delta = \frac{\varepsilon}{4}$, from (13), (14) and from Assumption 3) of Theorem 4.1 it follows that the family $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbb{R}^k\times\mathbb{R}^{m_{p+1}}}$ also satisfies Condition 6) of Theorem 4.1. This completes the proof of Theorem 4.1.

Remark 5.1 The main distinctions of the proof of Proposition 5.1 in comparison with that of Lemma 5.1 from [16] are as follows:

($\star$) In the current paper, we deal with the Volterra systems whereas [16] is devoted to the case of ODE.

($\star\star$) In the current work, the parameter ξ characterizes the terminal state and the system should be steered to starting from the initial point $y^0 \in \mathbb{R}^k$. In [16], the construction starts with the initial condition $z(\xi, T) = \xi$ given at the *terminal* instant T, and then the control strategy is adjusted inductively ([16], Lemma 6.1) while time is decreasing (from $t = T$ until the initial instant $t = t_1$) in order to reach a certain small neighborhood of the initial state. However, for the Volterra systems, such an invertion of time is not possible in general (and one cannot consider the Cauchy initial condition at terminal instant T). Therefore the direct repetition of the argument from [16], Section 6 would not suit.

($\star\star\star$) In the current work, we consider the non-smooth case (the right-hand side of (2) satisfies the local Lipschitz condition only).

5.1 Proof of Proposition 5.1

Following [16], choose any sequence $\{R_q\}_{q=1}^{\infty}\subset\mathbb{N}$ such that $R_1=1$, $R_{q+1}>R_q+1$, $q\in\mathbb{N}$. Define

$$\delta_q := \frac{1}{2}\min_{\xi\in\overline{B_{R_{q+1}}(0)}}\delta(\xi), \qquad M_q := \max_{\xi\in\overline{B_{R_q}(0)}}\| y(\xi,\cdot)\|_{C(I;\mathbb{R}^k)}+4\delta+1, \quad q\in\mathbb{N}; \tag{15}$$

$$K_q := \{y\in\mathbb{R}^k |\ |y|\le M_q\};\ \ d_q := M_{q+2}+1,\ \ q\in\mathbb{N}; \tag{16}$$

$$W_q := \{\omega\in\mathbb{R}^{m_p}\ |\ |\omega|\le \max_{\xi\in\overline{B_{R_q}(0)}}\|\omega(\xi,\cdot)\|_{C(I;\mathbb{R}^{m_p})}+1\}, \quad q\in\mathbb{N}; \tag{17}$$

$$\Xi_1 := \overline{B_{R_1}(0)};\ \ \Xi_{q+1}=\overline{B_{R_{q+1}}(0)}\backslash\overline{B_{R_q}(0)},\ \ q\in\mathbb{N}; \tag{18}$$

$$E_1 := \overline{B_{R_1}(0)}\times I\times K_1;$$

$$E_{q+1} := E_q\bigcup\left(\left(\overline{B_{R_{q+1}}(0)}\setminus B_{R_q}(0)\right)\times I\times K_{q+1}\right),\ \ q\in\mathbb{N}; \tag{19}$$

$$E := \bigcup_{q=1}^{\infty}E_q. \tag{20}$$

Given an arbitrary $q\in\mathbb{N}$, and arbitrary $N\in\mathbb{N}$, define

$$\Lambda_N^q := \{(t,y,v)\in I\times K_{q+1}\times\mathbb{R}^{m_p}\ |\ \exists\overline{v}\in\mathbb{R}^{m_{p+1}}(|\overline{v}|\le N)\wedge(|\omega-f_p(t,y,\overline{v})|<\frac{\delta_q}{3}\}.$$

Then every Λ_N^q is open as a subset of the metric space $I\times K_{q+1}\times\mathbb{R}^{m_p}$ whose metric is generated by the norm of $\mathbb{R}\times\mathbb{R}^k\times\mathbb{R}^{m_p}$. Since $I\times K_{q+1}\times W_q$ is compact w.r.t. this metric space, using condition (iii) and the inclusions $\Lambda_N^q\subset\Lambda_{N+1}^q$ and $I\times K_{q+1}\times W_q\subset\bigcup\limits_{N=1}^{\infty}\Lambda_N^q$, we obtain the existence of $N_0(q)\in\mathbb{N}$ such that $I\times K_{q+1}\times W_q\subset\Lambda_{N_0(q)}^q$. Without loss of generality, we assume that $N_0(q)\le N_0(q+1)$.

Define

$$U_q := \{v\in\mathbb{R}^{m_{p+1}}\ |\ \ |v|\le N_0(q)\}. \tag{21}$$

Then $U_q\subset U_{q+1}$, $q\in\mathbb{N}$ and, by the construction, for each $(t,y,\omega)\in I\times K_{q+1}\times W_q$ there exists $v\in U_q$ such that $|\omega-f_p(t,y,v)|<\frac{\delta_q}{3}$. Let $\{L_q\}_{q=1}^{\infty}\subset\mathbb{R}$ and $L(\cdot)\in C(\mathbb{R}^k;]0,+\infty[)$ be such that $0<L_{q+1}\le L_q$, $q\in\mathbb{N}$ and

$$2L_q(|\varphi(t,y,v)|+(T-t_0)|\psi(t,s,y,v)|+1)\le 1\ \ \forall(t,s,y,v)\in I^2\times\overline{B_{d_q}(0)}\times U_{q+2},\ q\in\mathbb{N}, \tag{22}$$

$$L_{q+1}\le L(\xi)\le L_q, \qquad \text{whenever } \xi\in\Xi_q,\ \ q\in\mathbb{N}. \tag{23}$$

Then we denote by F the following semi-ring of sets ([12, vol. 2, p. 17])

$$\Sigma_{\Theta(\cdot),\vartheta(\cdot),A_\Theta,A_\vartheta} := \{(\eta,s,z)\in\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^k\ |\ \vartheta(\eta,z)\le s\le\Theta(\eta,z)\}\backslash\{(\eta,s,z)\in\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^k$$

$$|(s=\Theta(\eta,z))\wedge((\eta,z)\in A_\Theta)\text{ or }(s=\vartheta(\eta,z))\wedge((\eta,z)\in A_\vartheta)\},$$

where $\Theta(\cdot)$, and $\vartheta(\cdot)$ range over the set of all the functions from class $C(\mathbb{R}^k\times\mathbb{R}^k;I)$ such that for all $(\xi,y,z)\in\mathbb{R}^k\times\mathbb{R}^k\times\mathbb{R}^k$

$$|\Theta(\xi,y)-\Theta(\xi,z)|\le L(\xi)|y-z|\ \text{ and } |\vartheta(\xi,y)-\vartheta(\xi,z)|\le L(\xi)|y-z|,$$

and $A_\Theta \subset \mathbb{R}^k \times \mathbb{R}^k$, $A_\vartheta \subset \mathbb{R}^k \times \mathbb{R}^k$ range over the set of all subsets of $\mathbb{R}^k \times \mathbb{R}^k$.

For each $(\xi, t, y) \in E$, let $q \in \mathbb{N}$ be such that $\xi \in \Xi_q$. From (18)-(20) it follows that $y \in K_{q+1}$. By (17), and by the definition of U_q, there exists $v_{\xi,t,y} \in U_q$ such that $|\omega(\xi, t) - f_p(t, y, v_{\xi,t,y})| < \frac{\delta_q}{3}$.

Using the compactness of each E_q in $\mathbb{R}^k \times I \times \mathbb{R}^k$ and the properties of semirings of sets (see Lemma 2 in [12, vol.2, p. 18]), we repeat the construction from [16, p.1435-1436] and obtain the existence of a sequence $\{(\xi_r, t_r, y_r)\}_{r=1}^\infty$, sequences $\{S_r\}_{r=1}^\infty \subset \mathcal{F}$ and $\{\Sigma_l\}_{l=1}^\infty \subset \mathcal{F}$ of sets from $\mathcal{F}$ and sequences of natural indices $1 \le r_1 < r_2 < \ldots < r_q < \ldots$ and $1 \le l_1 < l_2 < \ldots < l_q < \ldots$ such that first

$$(\xi_r, t_r, y_r) \in S_r \quad \text{and} \quad \forall (\eta, s, z) \in S_r \ \ (|\eta - \xi| < \frac{1}{4}) \wedge (|z - y| < \frac{1}{4}), \tag{24}$$

$$\forall (\eta, s, z) \in S_r \quad |\omega(\eta, s) - f_p(s, z, v_{\xi_r,t_r,y_r})| < \delta(\eta), \tag{25}$$

(this group of inequalities characterizes the size of S_r and the properties of the feedback controller to be constructed), second

$$E \subset \bigcup_{r=1}^{\infty} S_r; \ \text{ and } \ E_q \subset \bigcup_{r=1}^{r_q} S_r, \text{ for all } q \in \mathbb{N}, \tag{26}$$

$$S_r \bigcap E_1 \neq \emptyset, \qquad \text{if } 1 \le r \le r_1; \quad \text{and} \quad S_r \bigcap \left(\left(\overline{B_{R_{q+1}}(0)} \backslash B_{R_q}(0) \right) \times I \times K_{q+1} \right) \neq \emptyset,$$
$$\text{if } r_q + 1 \le r \le r_{q+1}, \tag{27}$$

$$S_r \bigcap \left(\bigcup_{j=1}^{r_q} S_j \right) = \emptyset, \quad \text{if } r \ge r_{q+1} + 1, \ \ q \in \mathbb{N}. \tag{28}$$

(this group of inclusions and inequalities characterizes the local finiteness of the countable covering $\{S_r\}_{r=1}^\infty$ of E), and third

(A$_1$) $\bigcup\limits_{r=1}^{r_q} S_r = \bigcup\limits_{l=1}^{l_q} \Sigma_l$ for all $q \in \mathbb{N}$ (which implies that $\bigcup\limits_{l=1}^{\infty} \Sigma_l = \bigcup\limits_{r=1}^{\infty} S_r$);
(A$_2$) $\Sigma_{l'} \bigcap \Sigma_{l''} = \emptyset$ for all $l' \neq l''$;
(A$_3$) for each $r \in \mathbb{N}$, there is a finite set of indices $P(r) \subset \mathbb{N}$ such that $S_r = \bigcup\limits_{l \in P(r)} \Sigma_l$.

This group of conditions characterizes the relationship between the original countable covering $\{S_r\}_{r=1}^\infty$ of E and its derivative covering $\{\Sigma_l\}_{l=1}^\infty \subset \mathcal{F}$, of E by mutually disjoint sets Σ_l, obtained by using the properties of semiring $\mathcal{F}$ [12, vol. 2, p. 18].

From (A$_1$), (A$_2$), (A$_3$), it follows that for every $l \in \mathbb{N}$ there exists $r(l) \in \mathbb{N}$ such that $\Sigma_l \subset S_{r(l)}$, and such that, if $1 \le l \le l_1$, then $1 \le r(l) \le r_1$, and if $l_q + 1 \le l \le l_{q+1}$ ($q \in \mathbb{N}$), then $r_q + 1 \le r(l) \le r_{q+1}$. Since $\Sigma_l \subset S_{r(l)}$, we obtain from (24), (26), (27):

$$\left(B_{\frac{1}{2}}(\xi) \times I \times \mathbb{R}^k \right) \bigcap \Sigma_l = \emptyset, \quad \text{whenever} \quad l \notin \Omega(\xi), \ \ l \in \mathbb{N}, \ \ \xi \in \mathbb{R}^k, \tag{29}$$

where $\Omega(\xi)$ is the finite number of indices given by

$$\Omega(\xi) := \begin{cases} \{l\}_{l=1}^{l_3}, & \text{if } \xi \in \Xi_1 \cup \Xi_2; \\ \{l\}_{l=l_{q-1}+1}^{l_{q+2}}, & \text{if } \ \xi \in \Xi_{q+1}, \ q \ge 2. \end{cases} \tag{30}$$

Define

$$v(\xi,t,y)=v_{\xi_{r(l)},t_{r(l)},y_{r(l)}},\ \ \text{whenever } (\xi,t,y)\in\Sigma_l,\ l\in\mathbb{N}. \tag{31}$$

Then, from (25), (31), and from the inclusion $\Sigma_l\subset S_{r(l)}$, we obtain:

$$|\omega(\eta,s)-f_p(s,z,v(\eta,s,z))|<\delta(\eta) \text{ for all } (\eta,s,z)\in\bigcup_{l=1}^{\infty}\Sigma_l \tag{32}$$

Lemma 5.1 *1) For every $\xi\in\mathbb{R}^k$, there are a unique $z(\xi,\cdot)\in C(I;\mathbb{R}^k)$ such that*

$$z(\xi,t_0)=\xi, \tag{33}$$

a unique finite sequence of indices $\{\nu_j(\xi)\}_{j=1}^{N(\xi)}=\{\nu_j\}_{j=1}^{N(\xi)}\subset\Omega(\xi)$ such that $N(\xi)\le|\Omega(\xi)|$, and $\nu_\mu\neq\nu_j$ whenever $\mu\neq j$, and a unique finite sequence $t_0=\tau_1^*(\xi)<\tau_2^*(\xi)<\ldots<\tau_{N(\xi)}^*(\xi)<\tau_{N(\xi)+1}^*(\xi)=T$ such that:

1.a) $\dot z(\xi,t)$ is defined and continuous at each t in $I\setminus\{\tau_1^*(\xi),...,\tau_{N(\xi)}^*(\xi)\}$, and

$$(\xi,t,z(\xi,t))\in E \text{ and } |\omega(\xi,t)-f_p(t,z(\xi,t),v(\xi,t,z(\xi,t)))|<\delta(\xi),\ \ t\in I \tag{34}$$

1.b) for each $j=1,\ldots,N(\xi)$, we have:

$$(\xi,t,z(\xi,t))\in\Sigma_{\nu_j}\quad \text{for all } \ t\in]\tau_j^*(\xi),\tau_{j+1}^*(\xi)[, \tag{35}$$

$$\dot z(\xi,t)=\varphi(t,z(\xi,t),v(\xi,t,z(\xi,t)))+\int_{t_0}^{t}\psi(t,s,z(\xi,s),v(\xi,s,z(\xi,s)))ds$$

$$\text{for all } t\in]\tau_j^*(\xi),\tau_{j+1}^*(\xi)[, \tag{36}$$

$$\tau_{j+1}^*(\xi)=\Theta_{\nu_j}(\xi,z(\xi,\tau_j^*(\xi))),\quad \text{and}\quad \tau_j^*(\xi)=\vartheta_{\nu_j}(\xi,z(\xi,\tau_{j+1}^*(\xi))) \tag{37}$$

2) Given any $\xi\in\mathbb{R}^k$, and any $l\in\mathbb{N}$, define $t\mapsto s_l(\xi,t)$ and $t\mapsto t_l(\xi,t)$ by

$$s_l(\xi,t)=t-\vartheta_l(\xi,z(\xi,t)),\ \ t_l(\xi,t)=t-\Theta_l(\xi,z(\xi,t))\quad \text{for all } t\in I. \tag{38}$$

Then, for every $\xi\in\mathbb{R}^k$, and every $l\in\mathbb{N}$, first,

$$\frac{3(t-\tau)}{2}\ge s_l(\xi,t)-s_l(\xi,\tau)\ge\frac{t-\tau}{2} \text{ whenever } t>\tau,\ l\in\mathbb{N}, \tag{39}$$

$$\frac{3(t-\tau)}{2}\ge t_l(\xi,t)-t_l(\xi,\tau)\ge\frac{t-\tau}{2} \text{ whenever } t>\tau,\ l\in\mathbb{N}, \tag{40}$$

for all $t\in I$ and $\tau\in I$, and, second, there are unique $s_l^*(\xi)\in I$ and $t_l^*(\xi)\in I$ such that $s_l(\xi,s_l^*(\xi))=0$ and $t_l(\xi,t_l^*(\xi))=0$. Moreover, $t_0=s_{\nu_1}^*(\xi)$; $\tau_i^*(\xi)=t_{\nu_{i-1}}^*(\xi)=s_{\nu_i}^*(\xi)$ for every $i=2,...,N(\xi)$; and $T=t_{\nu_{N(\xi)}}^*(\xi)$.

The proof of the current Lemma 5.1, which is omitted, is by induction on $i\in\{1,\ldots,N(\xi)\}$ and is similar to that of Lemma 6.1 from [16]. The only difference is that the induction argument starts with the initial instant $t_0=\tau_1^*(\xi)$ whereas in [16] it starts with $T=\tau_1^*(\xi)$ down to t_0. Having proved Lemma 5.1 one combines it with the implicit function theorem and proves Lemma 5.2 (again by induction on $i\in\{1,\ldots,N(\xi)\}$).

Lemma 5.2 *For all* $i \in \{1, ..., N(\xi)\}$, *functions* $\eta \mapsto s^*_{\nu_i}(\eta)$, $\eta \mapsto t^*_{\nu_i}(\eta)$, $\eta \mapsto z(\eta, s^*_{\nu_i}(\eta))$, *and* $\eta \mapsto z(\eta, t^*_{\nu_i}(\eta))$ *defined in the previous Lemma 5.1 are continuous at every* $\xi \in \mathbb{R}^k$

The only differences of the proof of the current Lemma 5.2. in comparison with that of Lemma 6.2 from [16] are as follows: first one should use the implicit function theorem for the continuous monothone functions instead of C^1 - case (due to nonsmoothness), and second one needs to invert the time again in comparison with [16].

Finally, define the desired family of controls $\{u(\xi, \cdot)\}_{\xi \in \mathbb{R}^k}$ by

$$u(\xi, t) = v(\xi, t, z(\xi, t)) \quad \text{whenever} \quad t \in I, \;\; \xi \in \mathbb{R}^k. \tag{41}$$

From Lemmas 5.1 and 5.2 it immediately follows that the family $\{u(\xi, \cdot)\}_{\xi \in \mathbb{R}^k}$ defined by (41) satisfies all the Conditions 1),2),3) of Proposition 5.1. The proof of Proposition 5.1 is complete. This completes the proof of Theorem 4.1 and respectively those of Theorems 3.1 and 3.2.

6 Conclusion

The problem of global controllability of triangular integro-differential Volterra equations with noninvertible input-output links and with nonsmooth (Lipschitz continuous) dynamics has been solved. In addition we proved the global controllability of large scale interconnections of such systems when the cross-terms are bounded and Lipschitz continuous. The main distinctions of the current work in comparison with the thechniques used in preceeding works [15, 16] are as follows. First, in contrast to [15, 16], since the dynamics is not differentiable (but satisfies the local Lipschitz condition only) we cannot refer to the properties of the Frechet derivative of the input-state map $u(\cdot) \mapsto x(\cdot)$ that was essential in [15, 16] and cannot consider the linearized control system around a trajectory (which characterizes this Frechet derivative). Second, in contrast to [15] the input-output links $x_{i+1}(\cdot) \mapsto x_i(\cdot)$ are not invertible, which is why each virtual control needeed at each step of the backstepping procedure cannot be obtained as in [15] by solving the corresponding Volterra equations. To handle the second problem, we update some auxilary construction from [16] to the case of Volterra nonsmooth systems and to handle the first one we develop a backstepping design which is different from that from [15, 16]. All the arguments that are similar to those from [15, 16] are omitted and only essential changes are highlighted.

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