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Existence and Uniqueness Conditions for a Class of $(k+4j)$-Point n-th Order Boundary Value Problems

P. Eloe [1*], J. Henderson [2] and R.A. Khan [3]

[1] *Department of Mathematics, University of Dayton, Dayton, OH 45469-2316 USA*
[2] *Department of Mathematics, Baylor University, Waco, Texas 76798-7328 USA*
[3] *University of Malakand, Chakadra Dir (L), Khybar Phuktoonkwa, Pakistan*

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Abstract: For the nth order nonlinear differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),$$

we consider uniqueness implies uniqueness and existence results for solutions satisfying certain $(k+4j)-$point boundary conditions, $1 \le j \le n-1$ and $1 \le k \le n-2j$. We define $(j;k;j)-$point unique solvability in analogy to $k-$point disconjugacy and we show that $(j;n-2j;j)-$point unique solvability implies $(j;k;j)-$point unique solvability for $1 \le k \le n-2j$. This result is in analogy to $n-$point disconjugacy implies $k-$point disconjugacy, $2 \le k \le n-1$.

Keywords: *boundary value problem; uniqueness; existence; unique solvability; nonlinear interpolation.*

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1 Introduction

In this paper, we are concerned with uniqueness and existence of solutions for a class of boundary value problems for nth order ordinary differential equation, $n \ge 3$,

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \quad a < x < b, \tag{1}$$

subject to $n-2j$ conjugate boundary conditions and $2j$ nonlocal boundary conditions, where $j \ge 1$. In particular, given $1 \le k \le n-2j$, positive integers $m_1, \ldots, m_k$ such that $m_1 + \cdots + m_k = n-2j$, points $a < t_1 < \ldots < t_{2j} < x_1 < x_2 < \ldots < x_k < s_1 < \ldots < s_{2j} < b$

* Corresponding author: `mailto:peloe1@udayton.edu`

real values $y_i, 1 \le i \le j$, $y_{il}, 1 \le i \le m_l, 1 \le l \le k$, and real values $y_{n-(i-1)}$, $1 \le i \le j$, we are concerned with uniqueness implies uniqueness and existence questions for solutions of (1) satisfying the conjugate and nonlocal boundary conditions of the type

$$\begin{aligned} &a_i y(t_{2i-1}) - b_i y(t_{2i}) = y_i,\ 1 \le i \le j,\ j \text{ nonlocal conditions},\\ &y^{(i-1)}(x_l) = y_{il},\ 1 \le i \le m_l,\ 1 \le l \le k,\ k\text{-point},\ n-2 \text{ conjugate conditions},\\ &c_i y(s_{2i-l}) - d_i y(s_{2i}) = y_{n-(i-1)},\ 1 \le i \le j,\ \ j \text{ nonlocal conditions}, \end{aligned} \tag{2}$$

where a_i, b_i, c_i, d_i, $1 \le i \le j$ are positive real numbers. We shall refer to the boundary conditions, (2), as $(j;k;j)-$point boundary conditions. The $(0;k;0)-$point boundary conditions are referred to as *conjugate* type boundary conditions [18].

Questions of the types with which we deal in this paper have been considered for solutions of (1) satisfying α-point conjugate boundary conditions; in particular, for boundary value problems for (1) satisfying, for $2 \le \alpha \le n$, conjugate boundary conditions of the form,

$$y^{(i-1)}(t_l) = r_{il},\ 1 \le i \le p_l,\ 1 \le l \le \alpha, \tag{3}$$

where $p_1, \ldots, p_\alpha$ are positive integers such that $p_1 + \cdots + p_\alpha = n$, $a < t_1 < \cdots < t_\alpha < b$, and $r_{ij} \in \mathbb{R}, 1 \le i \le p_j, 1 \le j \le \alpha$. These questions have involved: **(i)** whether uniqueness of solutions of (1), (3), for $\alpha = n$, implies uniqueness of solutions of (1), (3), for $2 \le \alpha \le n-1$, and **(ii)** whether uniqueness of solutions of (1), (3), for $\alpha = n$, implies existence of solutions of (1), (3), for $2 \le \alpha \le n$. Of course, a main reason for considering question (i) would be in resolving question (ii).

Hypothesis 1.1 *With respect to equation (1), we assume throughout that*

(A) $f(t, s_1, \ldots, s_n) : (a,b) \times \mathbb{R}^n \to \mathbb{R}$ *is continuous;*

(B) *Solutions of initial problems for (1) are unique and extend to* (a,b).

Given Hypothesis 1.1, Jackson [18] established that indeed (i) is true. In independent works, Hartman [7,8] and Klaasen [21] provided a positive answer to question (ii).

Several other papers have been devoted to uniqueness questions of these types as well as uniqueness implies existence questions for boundary value problems. These works have dealt not only with ordinary differential equations [2,4,9,10,19,22,23], but also with boundary value problems for finite difference equations [11]– [13], and recently with dynamic equations on time scales [6,17]. Some questions of these types have also received recent attention for nonlocal boundary value problems for (1), for the cases of $n = 2,3,4$; see [1,5,15,16]. Recently, [3,20] the case of nonlocal conditions for equations of arbitrary order n have been addressed.

Referring to the methods employed in the papers cited above as shooting methods, the authors shoot from one boundary point with one boundary condition. The contribution in this article is that we shoot from two boundary points, to the left from x_1 and to the right from x_k. New arguments for uniqueness of solutions implies existence of solutions are given to allow for multiple shooting.

2 Uniqueness of Solutions

In the first result of this section, we shall obtain continuous dependence of solutions of (1) on boundary conditions.

Theorem 2.1 *Assume that for some $1 \leq k \leq n-2j$, and positive integers $m_1, \dots, m_k$ such that $m_1 + \dots + m_k = n - 2j$, solutions of the corresponding boundary value problem (1), (2) are unique, when they exist. Given a solution $y(x)$ of (1), an interval $[c, d]$, points $c < x_1 < \dots < x_k < \dots < x_{k+4j} < d$ and an $\epsilon > 0$, there exists $\delta(\epsilon, [c, d]) > 0$ such that, if $|x_i - \xi_i| < \delta$, $1 \leq i \leq k + 4j$, and $c < \xi_1 < \dots < \xi_k < \dots < \xi_{k+4j} < d$, and if*

$$|a_i y(x_{2i-1}) - b_i y(x_{2i}) - z_i| < \delta, \; i = 1, 2, \dots, j,$$
$$|y^{(i-1)}(x_{2j+l}) - z_{il}| < \delta, \; 1 \leq i \leq m_l, \; 1 \leq l \leq k, \; \textit{and}$$
$$|c_i y(x_{k+2j+2i-l}) - d_i y(x_{k+2j+2i}) - z_{n-(i-1)}| < \delta, \; i = 1, 2, \dots, j,$$

then there exists a solution $z(x)$ of (1) satisfying

$$a_i z(\xi_{2i-l}) - b_i z(\xi_{2i})) = z_i, \; 1 \leq i \leq j,$$
$$z^{(i-1)}(\xi_l) = z_{il}, \quad 1 \leq i \leq m_l, 1 \leq l \leq k,$$
$$c_i z(\xi_{k+2j+2i-l}) - d_i z(\xi_{k+2j+2i}) = z_{n-(i-1)}, \; 1 \leq i \leq j,$$

and $|y^{(i-1)}(x) - z^{(i-1)}(x)| < \epsilon$ on $[c, d]$, $1 \leq i \leq n$.

Proof Fix a point $p_0 \in (c, d)$ and define the set

$$G = \{(s_1, \dots, s_{k+4j}, c_1, \dots, c_n) \mid c < s_1 < \dots < s_{k+4j} < d, \; c_1, \dots, c_n \in \mathbb{R}\}.$$

G is an open subset of $\mathbb{R}^{k+4j+n}$. Let $u(x)$ be a solution of the initial value problem for (1) satisfying the initial conditions $u^{(i-1)}(p_0) = c_i$, $1 \leq i \leq n$. Define a mapping $\phi : G \to \mathbb{R}^{k+4j+n}$ by

$$\phi(s_1, \dots, s_{k+4j}, c_1, \dots, c_n) = \big(s_1, \dots, s_{k+4j}, a_1 u(s_l) - b_1 u(s_2), ..., a_j u(s_{2j-l}) - b_j u(s_{2j}),$$
$$u(s_{2j+1}), \dots, u^{(m_1-1)}(s_{2j+1}), \dots, u(s_{2j+k}), \dots, u^{(m_k-1)}(s_{2j+k}),$$
$$c_1 u(s_{k+2j+l}) - d_1 u(s_{k+2j+2}), ..., c_j u(s_{k+4j-l}) - d_j u(s_{k+4j})\big).$$

The continuity of ϕ follows from Condition (B) in Hypothesis 1.1. Moreover, the uniqueness assumption on solutions of (1), (2), for the given k and $m_1, \dots, m_k$, implies that ϕ is one-one. Hence, from the Brouwer theorem on invariance of domain [25], it follows that $\phi(G)$ is an open subset of $\mathbb{R}^{k+4j+n}$, and that ϕ is a homeomorphism from G to $\phi(G)$. The conclusion of the theorem follows directly from the continuity of ϕ^{-1} and the fact that $\phi(G)$ is open.

We now establish that for $k = n-2j$, uniqueness of solutions of the $(j; n-2j; j)-$point BVP (1), (2), implies uniqueness of solutions of the $(j-i; n-2j+i, j)-$point BVP (1), (2), for $i = 1, 2, \dots, j$.

Theorem 2.2 *Let $j \geq 1$. Assume that for $k = n - 2j$, solutions of the $(j; n - 2j; j)-$point BVP (1), (2) are unique, when they exist. Then, for each $i = 1, 2, \dots, j$, solutions of the $(j - i; n - 2j + i, j)-$point BVP (1), (2) are unique, when they exist.*

Proof Assume uniqueness of solutions of the $(j; n-2j; j)-$point BVP (1), (2). Firstly, we show that solutions of the $(j-1; n-2j+1, j)-$point BVP (1), (2) are unique. Assume the conclusion is not true and there exist points $a < t_1 < \dots < t_{2j-2} < x_1 < \dots <$

$x_{n-2j+1} < s_1 < \cdots < s_{2j} < b$ for which there exist distinct solutions $y(x)$ and $z(x)$ of the $(j-1; n-2j+1, j)-$point BVP such that

$$\begin{aligned}
&a_i y(t_{2i-1}) - b_i y(t_{2i}) = a_i z(t_{2i-1}) - b_i z(t_{2i}),\ i = 1, 2, \ldots, j-1,\\
&y(x_1) = z(x_1),\\
&y(x_l) = z(x_l),\ 2 \le l \le n-2j+1,\\
&c_i y(s_{2i-1}) - d_i y(s_{2i}) = c_i z(s_{2i-1}) - d_i z(s_{2i}),\ i = 1, 2, \ldots, j.
\end{aligned}$$

Defining $w = y - z$, we obtain

$$\begin{aligned}
&a_i w(t_{2i-1}) - b_i w(t_{2i}) = 0,\ i = 1, 2, \ldots, j-1,\\
&w(x_1) = 0,\\
&w(x_l) = 0,\ 2 \le l \le n-2j+1,\\
&c_i w(s_{2i-1}) - d_i w(s_{2i}) = 0,\ i = 1, 2, \ldots, j.
\end{aligned}$$

If there exists some $p_1 \in (t_{2j-2}, x_1)$ such that $w(p_1) = 0$, then we have

$$a_j w(p_1) - b_j w(x_1) = 0,\ a_j,\ b_j \in \mathbb{R}.$$

This implies that $y(x)$ and $z(x)$ are distinct solutions of the $(j; n-2j; j)-$point BVP at the points $t_1, \ldots, t_{2j-2}, p_1, x_1, \ldots, x_{n-2j}, s_1, \ldots, s_{2j}$, which is a contradiction. Hence, $w(t) \neq 0$ on (t_{2j-2}, x_1). Let $w(t) > 0$ on (t_{2j-2}, x_1). The case $w(t) < 0$ on (t_{2j-2}, x_1) is dealt with similarly. Then,

$$\max\{w(t) : t \in [t_{2j-2}, x_1]\} = w(\tau_1) > 0.$$

Define

$$v(t) = \begin{cases} a_j w(t) - b_j w(\tau_1), & \text{if} \quad a_j \ge b_j,\\ b_j w(t) - a_j w(\tau_1), & \text{if} \quad a_j \le b_j. \end{cases}$$

Then, $v(\tau_1) > 0$ and $v(x_1) < 0$. By the mean value theorem, there exists $p' \in (\tau_1, x_1)$ such that $v(p') = 0$ which implies that $a_j w(p') - b_j w(\tau_1) = 0$. Hence, there are distinct solutions of the $(j; n-2j; j)-$point BVP at the points

$$t_1, \ldots, t_{2j-2}, \tau_1, p_1', x_2, \ldots, x_{n-2j}, s_1, \ldots, s_{2j},$$

which is again a contradiction. Hence, solutions of the $(j-1; n-2j+1, j)-$point BVP (1), (2) are unique.

Now, using the uniqueness of solutions of the $(j-1; n-2j+1, j)-$point BVP, by the same process, we can show uniqueness of solutions of the $(j-2; n-2j+2, j)-$point BVP (1), (2). Continuing in the same fashion, we obtain uniqueness of solution of the $(j-i; n-2j+i, j)-$point BVP for each $i = 1, 2, \ldots, j$.

Corollary 2.1 *Let $j \ge 1$. Assume that for $k = n-2j$, solutions of the $(j; n-2j; j)-$point BVP (1), (2) are unique, when they exist. Then, solutions of the $(0; n-j; j)-$point BVP (1), (2) are unique, when they exist.*

Theorem 2.3 *Let $j \ge 1$. Assume that for $k = n-2j$, solutions of the $(j; n-2j; j)-$point BVP (1), (2) are unique, when they exist. Then, for each $i = 1, 2, \ldots, j$, solutions of the $(j; n-2j+i, j-i)-$point BVP (1), (2) are unique, when they exist.*

Proof Assume uniqueness of solutions of the $(j; n-2j; j)-$point BVP (1), (2). Firstly, we show that solutions of the $(j; n-2j+1, j-1)-$point BVP (1), (2) are unique. Assume the conclusion is not true and there exist points

$$a < t_1 < \cdots < t_2j < x_1 < \cdots < x_{n-2j+1} < s_1 < \cdots < s_{2j-2} < b$$

for which there exist distinct solutions $y(x)$ and $z(x)$ the $(j; n-2j+1, j-1)-$point BVP such that

$$\begin{aligned}
&a_i y(t_{2i-1}) - b_i y(t_{2i}) = a_i z(t_{2i-1}) - b_i z(t_{2i}),\ i = 1, 2, \ldots, j,\\
&y(x_l) = z(x_l),\ 1 \le l \le n-2j,\\
&y(x_{n-2j+1}) = z(x_{n-2j+1}),\\
&c_i y(s_{2i-1}) - d_i y(s_{2i}) = c_i z(s_{2i-1}) - d_i z(s_{2i}),\ i = 1, 2, \ldots, j-1.
\end{aligned}$$

Defining $w = y - z$, then we obtain

$$\begin{aligned}
&a_i w(t_{2i-1}) - b_i w(t_{2i}) = 0,\ i = 1, 2, \ldots, j,\\
&w(x_l) = 0,\ 1 \le l \le n-2j,\\
&w(x_{n-2j+1}) = 0,\\
&c_i w(s_{2i-1}) - d_i w(s_{2i}) = 0,\ i = 1, 2, \ldots, j-1.
\end{aligned}$$

If there exists some $q_1 \in (x_{n-2j+1}, s_1)$ such that $w(q_1) = 0$, then we have

$$c_0 w(x_{n-2j+1}) - d_0 w(q_1) = 0,\ c_0,\ d_0 \in \mathbb{R}.$$

This implies that $y(x)$ and $z(x)$ are distinct solutions of the $(j; n-2j; j)-$point BVP at the points

$$t_1, \ldots, t_{2j}, x_1, \ldots, x_{n-2j}, x_{n-2j+1}, q_1, s_1, \ldots, s_{2j-2},$$

which is a contradiction. Hence, $w(t) \neq 0$ on (x_{n-2j+1}, s_1). Let $w(t) > 0$ on (x_{n-2j+1}, s_1). The case $w(t) < 0$ on (x_{n-2j+1}, s_1) can be dealt with similarly. Then,

$$\max\{w(t) : t \in [x_{n-2j+1}, s_1]\} = w(\tau) > 0.$$

Define

$$v(t) = \begin{cases} c_0 w(t) - d_0 w(\tau), & \text{if} \quad c_0 \ge d_0,\\ d_0 w(t) - c_0 w(\tau), & \text{if} \quad c_0 \le d_0. \end{cases}$$

Then, $v(\tau) > 0$ and $v(x_{n-2j+1}) < 0$. By the mean value theorem, there exists $q' \in (x_n - 2j + 1, \tau)$ such that $v(q') = 0$ which implies that $c_0 w(q') - d_0 w(\tau) = 0$. Hence, there are distinct solutions of the $(j; n-2j; j)$-point BVP at the points

$$t_1, \ldots, t_{2j}, x_1, \ldots, x_{n-2j}, q', \tau, s_1, \ldots, s_{2j-2},$$

which is again a contradiction. Hence, solutions of the $(j; n-2j+1, j-1)-$point BVP (1), (2) are unique.

Now, using the uniqueness of solutions of the $(j; n-2j+1, j-1)-$point BVP, by the same process, we can show uniqueness of solutions of the $(j; n-2j+2, j-2)-$point BVP (1), (2). Continuing in the same fashion, we obtain uniqueness of solution of the $(j; n-2j+i, j-i)-$point BVP for each $i = 1, 2, \ldots, j$.

Corollary 2.2 *Let $j \ge 1$. Assume that for $k = n-2j$, solutions of the $(j; n-2j; j)-$point BVP (1), (2) are unique, when they exist. Then, solutions of the $(j; n-j; 0)-$point BVP (1), (2) are unique, when they exist.*

Corollary 2.3 *Let $j \geq 1$. Assume that for $k = n - 2j$, solutions of the $(j; n - 2j; j)-$point BVP (1), (2) are unique, when they exist. Then, solutions of the n-point conjugate BVP (1), (3) (that is, the $(0; n; 0)-$point BVP), are unique, when they exist.*

In view of the uniqueness implies existence results due to Hartman [7,8] and Klassen [21] as discussed in regard to question (ii), we have an immediate corollary concerning existence of solutions for k-point conjugate boundary value problems for (1).

Corollary 2.4 *Let $j \geq 1$. Assume that for $k = n - 2j$, solutions of the $(j; n - 2j; j)-$point BVP (1), (2) are unique, when they exist. Then, solutions of the l-point conjugate BVP (1), (3) (that is, the $(0; l; 0)-$point BVP), for $2 \leq l \leq n$, are unique, when they exist.*

We now establish that uniqueness of solutions of (1), (2), when $k = n - 2j$, implies uniqueness of solutions of (1), (2), when $1 \leq k \leq n - 2j - 1$.

Theorem 2.4 *Assume that for $k = n - 2j$, solutions of the $(j; n - 2j; j)-$point BVP (1), (2) are unique, when they exist. Then, for each $1 \leq k \leq n - 2j - 1$, solutions of the $(j; k; j)-$point BVP (1), (2) are unique, when they exist.*

Proof Assume that solutions of the $(j; n - 2j; j)-$point BVP (1), (2) are unique. Assume that, for some $1 \leq k \leq n - 2j - 1$, some $(j; k; j)-$point BVP (1), (2) has distinct solutions. Let

$$h = \max\{k = 1, \ldots, n - 2j - 1 \mid (j; k; j) - \text{point BVP has distinct solutions}\}.$$

Then, there are positive integers, $m_1, \ldots, m_h$, such that $m_1 + \cdots + m_h = n - 2j$, and points $a < t_1 < \cdots < t_{2j} < x_1 < \cdots < x_h < s_1 < \cdots < s_{2j} < b$, for which there exist distinct solutions $y(x)$ and $z(x)$ of the $(j; h; j)-$point boundary value problem (1), (2), for these $m_1, \ldots, m_h$; that is,

$$\begin{aligned}
&a_i y(t_{2i-1}) - b_i y(t_{2i}) = a_i z(t_{2i-1}) - b_i z(t_{2i}),\ i = 1, 2, \ldots, j,\\
&y^{(i-1)}(x_l) = z^{(i-1)}(x_l),\ 1 \leq i \leq m_l,\ 1 \leq l \leq h,\\
&c_i y(s_{2i-1}) - d_i y(s_{2i}) = c_i z(s_{2i-1}) - d_i z(s_{2i}),\ i = 1, 2, \ldots, j.
\end{aligned}$$

Since $h \leq n - 2j - 1$, so some $m_l \geq 2$. Let

$$m_{l_0} = \max\{m_l \mid 1 \leq l \leq h\};$$

then $m_{l_0} \geq 2$. Since, x_l is a zero of $y - z$ of exact multiplicity $m_l, 1 \leq l \leq h$ and y and z are distinct solutions of (1), we may assume, with no loss of generality, that

$$y^{(m_{l_0})}(x_{l_0}) > z^{(m_{l_0})}(x_{l_0}).$$

Now fix $a < \tau < x_1$. By the maximality of h, solutions of the $(j; h + 1; j)-$problems (1), (2) at the points $t_1, \ldots, t_{2j}, \tau, x_1, \ldots, x_h, s_1, \ldots, s_{2j}$ are unique. Hence, it follows from Theorem 2.1 that, for each $\epsilon > 0$, there is a $\delta > 0$ and there is a solution $z_\delta(x)$ of the $(j; h + 1; j)-$point problem (1), (2), (corresponding to $k = h + 1$), satisfying at the points $t_1, \ldots, t_{2j}, \tau, x_1, \ldots, x_h, s_1, \ldots, s_{2j}$,

$$\begin{aligned}
&a_i z_\delta(t_{2i-1}) - b_i z_\delta(t_{2i}) = a_i z(t_{2i-1}) - b_i z(t_{2i}) = a_i y(t_{2i-1}) - b_i y(t_{2i}),\ i = 1, 2, \ldots, j,\\
&z_\delta(\tau) = z(\tau),\\
&z_\delta^{(i-1)}(x_l) = z^{(i-1)}(x_l) = y^{(i-1)}(x_l), \quad 1 \leq i \leq m_l, \quad 1 \leq l \leq h, \quad l \neq l_0,\\
&z_\delta^{(i-1)}(x_{l_0}) = z^{(i-1)}(x_{l_0}) = y^{(i-1)}(x_{l_0}), \quad 1 \leq i \leq m_{l_0} - 2, \quad (\text{if } m_{l_0} > 2),\\
&z_\delta^{(m_{l_0}-2)}(x_{l_0}) = z^{(m_{l_0}-2)}(x_{l_0}) + \delta = y^{(m_{l_0}-2)}(x_{l_0}) + \delta,\\
&c_i z_\delta(s_{2i-1}) - d_i z_\delta(s_{2i}) = c_i z(s_{2i-1}) - d_i z(s_{2i}) = c_i y(s_{2i-1}) - d_i y(s_{2i}),\ i = 1, 2, \ldots, j,
\end{aligned}$$

and $|z_\delta(x) - z(x)| < \epsilon$ on $[t_1, s_{2j}]$. For $\epsilon > 0$, sufficiently small, there exist points $x_{l_0-1} < \rho_1 < x_{l_0} < \rho_2 < x_{l_0+1}$ such that

$$
\begin{aligned}
&a_i z_\delta(t_{2i-1}) - b_i z_\delta(t_{2i}) = a_j y(t_{2i-1}) - b_i y(t_{2i}),\ i = 1, 2, \ldots, j,\\
&z_\delta^{(i-1)}(x_l) = y^{(i-1)}(x_l), \quad 1 \le i \le m_l, \quad 1 \le l \le l_0 - 1,\\
&z_\delta(\rho_1) = y(\rho_1),\\
&z_\delta^{(i-1)}(x_{l_0}) = y^{(i-1)}(x_{l_0}), \quad 1 \le i \le m_{l_0} - 2, \quad (\text{if } m_{l_0} > 2),\\
&z_\delta(\rho_2) = y(\rho_2),\\
&z_\delta^{(i-1)}(x_l) = y^{(i-1)}(x_l), \quad 1 \le i \le m_l, \quad l_0 + 1 \le l \le h,\\
&c_i z_\delta(s_{2i-1}) - d_i z_\delta(s_{2i}) = c_i y(s_{2i-1}) - d_i y(s_{2i}),\ i = 1, 2, \ldots, j.
\end{aligned}
$$

Thus, $z_\delta(x)$ and $y(x)$ are distinct solutions of the $(j; h+1; j)-$point boundary value problem at the points $t_1, \ldots, t_{2j}, x_1, \ldots, x_{l_0-1}, \rho_1, \rho_2, x_{l_0+1}, \ldots, x_h, s_1, \ldots, s_{2j}$, which is a contradiction because of the maximality of h. The proof is complete.

In view of Theorem 2.2 and Theorem 2.4, we have the following corollaries.

Corollary 2.5 *Let $j \ge 1$. Assume that for $k = n - 2j$, solutions of the $(j; n - 2j; j)-$point BVP (1), (2) are unique, when they exist. Then, for $1 \le k \le n - 2j$ and $1 \le i \le j$, solutions of the $(j; k + i; j - i)-$point BVP are unique, when they exist.*

Corollary 2.6 *Let $j \ge 1$. Assume that for $k = n - 2j$, solutions of the $(j; n - 2j; j)-$point BVP (1), (2) are unique, when they exist. Then, for $1 \le k \le n - 2j$ and $1 \le i \le j$, solutions of the $(j - i; k + i; j)-$point BVP are unique, when they exist.*

3 Existence of Solutions

Now we deal with uniqueness implies existence for these problems. For such existence results, continuous dependence as in Theorem 2.1 plays a role. In addition, we shall make use of a Schrader [24] precompactness result on bounded sequences of solutions of (1) which is stated as follows:

Theorem 3.1 *Assume the uniqueness of solutions for (1), (3), when $\ell = n$. If $\{y_\nu(x)\}$ is a sequence of solutions of (1) which is uniformly bounded on a nondegenerate compact subinterval $[c, d] \subset (a, b)$, then there is a subsequence $\{y_{\nu_l}(x)\}$ such that $\{y_{k_l}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b), for each $i = 0, \ldots, n - 1$.*

We have as a corollary a precompactness condition in terms of (1), (2), when $k = n - 2j$.

Corollary 3.1 *Assume that for $k = n - 2j$, solutions of the $(j; n - 2j; j)-$point BVP (1), (2), are unique. If $\{y_\nu(x)\}$ is a sequence of solutions of (1) which is uniformly bounded on a nondegenerate compact subinterval $[c, d] \subset (a, b)$, then there is a subsequence $\{y_{\nu_l}(x)\}$ such that $\{y_{k_l}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b), for each $i = 0, \ldots, n - 1$.*

We now present our uniqueness implies existence result for the $(j; k; j)-$point boundary value problems.

Theorem 3.2 *Let $j \geq 1$. Assume that solutions of (1), (2), when $k = n-2j$, are unique. Then, for each $1 \leq k \leq n-2j$, positive integers $m_1, \dots, m_k$ such that $m_1 + \cdots + m_k = n-2j$, points $a < t_1 < \cdots < t_{2j} < x_1 < \cdots < x_k < s_1 < \cdots < s_{2j} < b$, real values y_i, $1 \leq i \leq j$, $y_{il}, 1 \leq i \leq m_l, 1 \leq l \leq k$ and $y_{n-i}, 0 \leq i \leq j-1$, there exists a unique solution of (1), (2).*

Proof Let $1 \leq k \leq n-2j$, positive integers $m_1, \dots, m_k$ such that $m_1 + \cdots + m_k = n-2j$, points $a < t_1 < \cdots < t_{2j} < x_1 < \cdots < x_k < s_1 < \cdots < s_{2j} < b$, real values y_i, $1 \leq i \leq j$, $y_{il}, 1 \leq i \leq m_l, 1 \leq l \leq k$ and $y_{n-i}, 0 \leq i \leq j-1$, be given.

Assume that for $k = n-2j$, solutions of the $(j; n-2j; j)-$point BVP, (1), (2), are unique. For $1 \leq k \leq n-2j$, in view of Corollary 2.4, solutions of the $(0; l; 0)-$point BVP (l-point conjugate BVP) for $2 \leq l \leq n$, are also unique. Let $z(x)$ be the unique solution of (1) satisfying the $(k+2j+2)$-point conjugate boundary conditions (3) at the points $t_1, p_1, t_2, \dots, t_j, x_1, \dots, x_k, s_1, \dots, s_{j+1}$ if $m_1 > 1$, $m_k > 1$ (or alternatively, if $m_1 = 1, m_k = 1$, $z(x)$ satisfies the $(k+2j)$-point conjugate boundary conditions and if one of $m_1 = 1, m_k = 1$ hold, then $z(x)$ satisfies the $(k+2j+1)$-point conjugate boundary conditions), that is,

$$\begin{array}{l}
z(t_1) = \frac{y_1}{a_1},\ z(p_1) = 0, \\
z(t_i) = \frac{y_i}{a_i},\ 2 \leq i \leq j, \\
z^{(i-1)}(x_1) = y_{i1}, \quad 1 \leq i \leq m_1 - 1, \\
z^{(i-1)}(x_l) = y_{il}, \quad 1 \leq i \leq m_l, \quad 2 \leq l \leq k-1, \\
z^{(i-1)}(x_k) = y_{ik}, \quad 1 \leq i \leq m_k - 1, \\
z(s_i) = \frac{y_{n-(i-1)}}{c_i},\ 1 \leq i \leq j-1, \\
z(s_j) = \frac{y_{n-(j-1)}}{c_j},\ z(s_{j+1}) = 0.
\end{array}$$

From the first and the last lines, we obtain

$$a_1 z(t_1) - b_1 z(p_1) = y_1,\ c_j z(s_j) - d_j z(s_{j+1}) = y_{n-(j-1)}.$$

Now, define the set

$$\begin{aligned}
S = \{ (u^{(m_1-1)}(x_1), u^{(m_k-1)}(x_k)) \mid u & \text{ is a solution of (1) satisfying} \\
& a_1 u(t_1) - b_1 u(p_1) = y_1,\ u(t_i) = \frac{y_i}{a_i},\ 2 \leq i \leq j, \\
& u^{(i-1)}(x_1) = y_{i1},\ 1 \leq i \leq m_1 - 1, \\
& u^{(i-1)}(x_l) = y_{il},\ 1 \leq i \leq m_l,\ 2 \leq l \leq k-1, \\
& u^{(i-1)}(x_k) = y_{ik},\ 1 \leq i \leq m_k - 1, \\
& u(s_i) = \frac{y_{n-(i-1)}}{c_i},\ 1 \leq i \leq j-1,\ c_j u(s_j) - d_j u(s_{j+1}) = y_{n-(j-1)} \}.
\end{aligned}$$

Clearly, $(z^{(m_1-1)}(x_1), z^{(m_k-1)}(x_k)) \in S$, and so S is a nonempty subset of $\mathbb{R}^2$.

Next, choose $(\rho_0, \sigma_0) \in S$. Then, there is a solution $u_0(x)$ of (1) satisfying

$$
\begin{aligned}
&a_1u_0(t_1)-b_1u_0(p_1)=y_1,\ u_0(t_i)=\tfrac{y_i}{a_i},\ 2\le i\le j,\\
&u_0^{(i-1)}(x_1)=y_{i1},\quad 1\le i\le m_1-1,\\
&u_0^{(m_1-1)}(x_1)=\rho_0,\\
&u_0^{(i-1)}(x_l)=y_{il},\quad 1\le i\le m_l,\quad 2\le l\le k-1,\\
&u_0^{(i-1)}(x_k)=y_{ik},\ 1\le i\le m_k-1,\\
&u_0^{(m_k-1)}(x_k)=\sigma_0,\\
&u_0(s_i)=\tfrac{y_{n-(i-1)}}{c_i},\ 1\le i\le j-1,\ c_ju_0(s_j)-d_ju_0(s_{j+1})=y_{n-(j-1)}.
\end{aligned}
$$

By the uniqueness of solutions of the $(1;k+2j-2;1)-$point BVP by Corollary 2.6, and in view of Theorem 2.1, there exists a $\delta>0$ such that, for each $|\rho-\rho_0|<\delta$, $|\sigma-\sigma_0|<\delta$, there is a solution $u_{\rho\sigma}(x)$ of (1) satisfying

$$
\begin{aligned}
&a_1u_{\rho\sigma}(t_1)-b_1u_{\rho\sigma}(p_1)=y_1,\ u_{\rho\sigma}(t_i)=\tfrac{y_i}{a_i},\ 2\le i\le j,\\
&u_{\rho\sigma}^{(i-1)}(x_1)=y_{i1},\quad 1\le i\le m_1-1,\\
&u_{\rho\sigma}^{(m_1-1)}(x_1)=\rho,\\
&u_{\rho\sigma}^{(i-1)}(x_l)=y_{il},\quad 1\le i\le m_l,\quad 2\le l\le k-1,\\
&u_{\rho\sigma}^{(i-1)}(x_k)=y_{ik},\ 1\le i\le m_k-1,\\
&u_{\rho\sigma}^{(m_k-1)}(x_k)=\sigma,\\
&u_{\rho\sigma}(s_i)=\tfrac{y_{n-(i-1)}}{c_i},\ 1\le i\le j-1,\ c_ju_{\rho\sigma}(s_j)-d_ju_{\rho\sigma}(s_{j+1})=y_{n-(j-1)}
\end{aligned}
$$

and $|u_{\rho\sigma}-u_0|<\delta$ on $[t_1,s_{j+1}]$, which implies that $(u_{\rho\sigma}^{(m_1-1)}(x_1),u_{\rho\sigma}^{(m_k-1)}(x_k))\in S$, that is, $(\rho,\sigma)\in S$. Hence, $\{(\rho,\sigma)|:|\rho-\rho_0|<\delta,|\sigma-\sigma_0|<\delta\}\subset S$. Thus, S is an open, nonempty subset of $\mathbb{R}^2$.

Now, we show that S is also a closed subset of $\mathbb{R}^2$. To do this, assume that S is not closed and assume there exists $r_0=(p_0,q_0)\in\overline{S}\setminus S$. Let $\{r_n\}=\{(p_n,q_n)\}\subset S$ such that

$$\lim_{n\to\infty}r_n=\lim_{n\to\infty}(p_n,q_n)=(p_0,q_0)=r_0.$$

We can assume that each sequence $\{p_n\},\{q_n\}$ is monotone. For the sake of this argument, we shall assume that each of $\{p_n\}$ and $\{q_n\}$ is monotone nondecreasing; the arguments for the other three cases, $\{p_n\}$ nondecreasing and $\{q_n\}$ nonincreasing, $\{p_n\}$ nonincreasing and $\{q_n\}$ nondecreasing, and each of $\{p_n\}$, $\{q_n\}$ nonincreasing are analogous.

So assume $p_n<p_{n+1}\le p_0$, $q_n<q_{n+1}\le q_0$ and assume one of the inequalities, $p_{n+1}\le p_0$, $q_{n+1}\le q_0$, is strict. By the definition of S, for each term $r_n, n\in\mathbb{N}$, there exists a unique solution $u_n(x)$ of (1) satisfying

$$
\begin{aligned}
&a_1u_n(t_1)-b_1u_n(p_1)=y_1,\ u_n(t_i)=\tfrac{y_i}{a_i},\ 2\le i\le j,\\
&u_n^{(i-1)}(x_1)=y_{i1},\quad 1\le i\le m_1-1,\\
&u_n^{(m_1-1)}(x_1)=p_n,\\
&u_n^{(i-1)}(x_l)=y_{il},\quad 1\le i\le m_l,\quad 2\le l\le k-1,\\
&u_n^{(i-1)}(x_k)=y_{ik},\ 1\le i\le m_k-1,\\
&u_n^{(m_k-1)}(x_k)=q_n,\\
&u_n(s_i)=\tfrac{y_{n-(i-1)}}{c_i},\ 1\le i\le j-1,\ c_ju_n(s_j)-d_ju_n(s_{j+1})=y_{n-(j-1)}.
\end{aligned}
$$

Set $w_n = u_n - u_{n+1}$. Then

$$\begin{aligned}
&a_1 w_n(t_1) - b_1 w_n(p_1) = 0,\ w_n(t_i) = 0,\ 2 \le i \le j,\\
&w_n^{(i-1)}(x_1) = 0, \quad 1 \le i \le m_1 - 1,\\
&w_n^{(m_1-1)}(x_1) = p_n - p_{n+1} \le 0,\\
&w_n^{(i-1)}(x_l) = 0, \quad 1 \le i \le m_l, \quad 2 \le l \le k-1,\\
&w_n^{(i-1)}(x_k) = 0,\ 1 \le i \le m_k - 1,\\
&w_n^{(m_k-1)}(x_k) = q_n - q_{n+1} \le 0,\\
&w_n(s_i) = 0,\ 1 \le i \le j-1,\ c_j w_n(s_j) - d_j w_n(s_{j+1}) = 0.
\end{aligned}$$

First assume $p_{n+1} < p_0$ and $q_{n+1} < q_0$. By the uniqueness of solutions of the $(1; k + 2j - 2; 1)-$point BVP, there exists $\epsilon_n > 0$ such that

(a) $u_n(x) < u_{n+1}(x)$ on $(x_1 - \epsilon_n, x_1) \cup (x_1, x_2)$, if m_1 is odd,

(b) $u_n(x) > u_{n+1}(x)$ on $(x_1 - \epsilon_n, x_1)$ and $u_n(x) < u_{n+1}(x)$ on (x_1, x_2), if m_1 is even,

(c) $u_n(x) < u_{n+1}(x)$ on $(x_{k-1}, x_k) \cup (x_k, x_k + \epsilon_n)$, if m_k is odd,

(d) $u_n(x) > u_{n+1}(x)$ on (x_{k-1}, x_k) and $u_n(x) < u_{n+1}(x)$ on $(x_k, x_k + \epsilon_n)$, if m_k is even.

For the sake of this argument, we shall assume that m_1 and m_k are odd; the other cases are argued analogously. We also note that either $u_n(x) < u_{n+1}(x)$ on (t_j, x_1) or $u_n(x) < u_{n+1}(x)$ on (x_k, s_1). If neither of these inequalities hold, then there exist $t_j < \hat{t} < x_1$ and $x_k < \hat{s} < s_1$ such that $u_n(\hat{t}) - u_{n+1}(\hat{t}) = 0 = u_n(\hat{s}) - u_{n+1}(\hat{s})$ violating the uniqueness of solutions of $(1; k + 2j; 1)-$point BVPs. For the sake of this argument, let us assume that $u_n(x) < u_{n+1}(x)$ on (t_j, x_1). The sequence $\{r_n\}$ converges to r_0 and $r_0 \notin S$. In view of Corollary 3.1, the sequence $\{u_n(x)\}$ is not uniformly bounded on any compact subset of each of $(t_j, x_1), (x_1, x_2)$, and (x_{k-1}, x_k).

Now, let $w(x)$ be the unique solution of the $(0; k+2j; 0)-$point conjugate BVP (1),(3) satisfying at the points $t_1, p_1, t_2, \dots, t_j, x_1, \dots, x_k, s_1, \dots, s_j$,

$$\begin{aligned}
&w(t_1) = \tfrac{y_1}{a_1}, w(p_1) = 0,\\
&w(t_i) = \tfrac{y_i}{a_i},\ 2 \le i \le j,\\
&w^{(i-1)}(x_1) = y_{i1}, \quad 1 \le i \le m_1 - 1,\ (\text{if } m_1 > 1),\\
&w^{(m_1-1)}(x_1) = p_0,\\
&w^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, \quad 2 \le l \le k-1,\\
&w_n^{(i-1)}(x_k) = y_{ik},\ 1 \le i \le m_k - 1,\ (\text{if } m_k > 1)\\
&w_n^{(m_k-1)}(x_k) = q_0,\\
&w(s_i) = \tfrac{y_{n-(i-1)}}{c_i},\ 1 \le i \le j-1.
\end{aligned}$$

From the monotonicity and unboundedness property of the sequence $\{u_n(x)\}$, it follows that, for some large n_0, there exist a solution u_{n_0} of (1) and points $t_j < \tau_1 < x_1 < \tau_2 < x_2$, $x_{k-1} < \rho_1 < x_k$ such that

$$u_{n_0}(\tau_1) = w(\tau_1),\ u_{n_0}(\tau_2) = w(\tau_2),\ u_{n_0}(\rho_1) = w(\rho_1).$$

In particular,

$$\begin{aligned}
&a_1 u_{n_0}(t_1) - b_1 u_{n_0}(p_1) = y_1 = aw(t_1) - b_1 w(p_1),\\
&u_{n_0}(t_i) = \tfrac{y_i}{a_i} = w(t_i),\ 2 \le i \le j,\\
&u_{n_0}(\tau_1) = w(\tau_1),\\
&u_{n_0}^{(i-1)}(x_1) = y_{i1} = w^{(i-1)}(x_1), \quad 1 \le i \le m_1 - 1,\\
&u_{n_0}(\tau_2) = w(\tau_2),\\
&u_{n_0}^{(i-1)}(x_l) = y_{il} = w^{(i-1)}(x_l), \quad 1 \le i \le m_l, \quad 2 \le l \le k-1,\\
&u_{n_0}(\rho_1) = w(\rho_1),\\
&u_{n_0}^{(i-1)}(x_k) = y_{ik} = w^{(i-1)}(x_k), \quad 1 \le i \le m_k - 1,\\
&u_{n_0}(s_i) = \tfrac{y_{n-(i-1)}}{c_i} = w(s_i),\ 1 \le i \le j-1.
\end{aligned}$$

Thus, $u_{n_0}(x)$ and $w(x)$ are distinct solutions of the same $(1; k+2j+1; 0)-$point (or if $m_1 = 1$ and $m_k = 1$, the same $(1; k+2j+2; 0)-$point) BVP which contradicts Corollary 2.5.

If $q_{n+1} = q_0$, (and keeping with the assumptions that m_1, m_k odd) then

$$u_n(x) < u_{n+1}(x), \quad t_j < x < x_2.$$

Now w is already constructed and as before, find $u_{n_0}, t_j < \tau_1 < x_1 < \tau_2 < x_2$, such that

$$u_{n_0}(\tau_1) = w(\tau_1), \quad u_{n_0}(\tau_2) = w(\tau_2).$$

Then,

$$\begin{aligned}
&a_1 u_{n_0}(t_1) - b_1 u_{n_0}(p_1) = y_1 = aw(t_1) - b_1 w(p_1),\\
&u_{n_0}(t_i) = \tfrac{y_i}{a_i} = w(t_i),\ 2 \le i \le j,\\
&u_{n_0}(\tau_1) = w(\tau_1),\\
&u_{n_0}^{(i-1)}(x_1) = y_{i1} = w^{(i-1)}(x_1), \quad 1 \le i \le m_1 - 1,\\
&u_{n_0}(\tau_2) = w(\tau_2),\\
&u_{n_0}^{(i-1)}(x_l) = y_{il} = w^{(i-1)}(x_l), \quad 1 \le i \le m_l, \quad 2 \le l \le k,\\
&u_{n_0}(s_i) = \tfrac{y_{n-(i-1)}}{c_i} = w(s_i),\ 1 \le i \le j-1,
\end{aligned}$$

and Corollary 2.5 is contradicted.

The conclusion then is that S contains all its limit points and is a closed subset of $\mathbb{R}^2$; since S is open and nonempty, $S \equiv \mathbb{R}^2$.

By choosing $(y_{m_1 1}, y_{m_k k}) \in S$, there is a corresponding solution $y(x)$ of (1) such that

$$\begin{aligned}
&a_1 y(t_1) - b_1 y(p_1) = y_1,\\
&y(t_i) = \tfrac{y_i}{a_i},\ 2 \le i \le j,\\
&y^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, \quad 1 \le l \le k,\\
&y(s_i) = \tfrac{y_{n-(i-1)}}{c_i},\ 1 \le i \le j-1,\\
&c_j y(s_j) - d_j y(s_{j+1}) = y_{n-(j-1)},
\end{aligned}$$

which is the desired solution of the $(1; k+2j-2; 1)-$point BVP.

Now, let $z_1(x)$ be the unique solution of the $(1; k+2j-2; 1)-$point BVP satisfying the $(k+2j-2)$-point conjugate boundary conditions (or the $(k+2j)$-point conjugate boundary conditions if $m_1 > 1$ and $m_k > 1$) at the points

$$t_1, p_1, t_2, p_2, t_3, \ldots, t_j, x_1, \ldots, x_k, s_1, \ldots, s_{j-1}, q_1, s_j, s_{j+1},$$

that is,

$$\begin{aligned}
&a_1 z(t_1) - b_1 z(p_1) = y_1,\\
&z_1(t_2) = \tfrac{y_2}{a_2},\ z_1(p_2) = 0,\\
&z_1^{(i-1)}(x_1) = y_{i1}, \quad 1 \le i \le m_1 - 1,\\
&z_1^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, \quad 2 \le l \le k-1,\\
&z_1^{(i-1)}(x_k) = y_{ik}, \quad 1 \le i \le m_k - 1,\\
&z_1(s_i) = \tfrac{y_{n-(i-1)}}{c_i},\ 1 \le i \le j-2,\\
&z_1(s_{j-1}) = \tfrac{y_{n-(j-2)}}{c_{j-1}},\ z_1(q_1) = 0,\\
&c_j z_1(s_j) - d_j z_1(s_{j+1}) = y_{n-(j-1)}.
\end{aligned}$$

We have

$$a_2 z_1(t_2) - b_2 z_1(p_2) = y_2,\ c_{j-1} z_1(s_{j-1}) - d_{j-1} z_1(q_1) = y_{n-(j-2)}.$$

Define the set

$$\begin{aligned}
S_1 = \{(u^{(m_1-1)}(x_1), u^{(m_k-1)}(x_k)) \mid u \ &\text{ is a solution of (1) satisfying}\\
&a_1 u(t_1) - b_1 u(p_1) = y_1,\ a_2 u(t_2) - b_2 u(p_2) = y_2,\\
&u(t_i) = \frac{y_i}{a_i},\ 3 \le i \le j,\\
&u^{(i-1)}(x_1) = y_{i1},\ 1 \le i \le m_1 - 1,\\
&u^{(i-1)}(x_l) = y_{il},\ 1 \le i \le m_l,\ 2 \le l \le k-1,\\
&u^{(i-1)}(x_k) = y_{ik},\ 1 \le i \le m_k - 1,\\
&u(s_i) = \frac{y_{n-(i-1)}}{c_i},\ 1 \le i \le j-2,\\
&c_{j-1} u(s_{j-1}) - d_{j-1} u(q_1) = y_{n-(j-2)},\ c_j u(s_j) - d_j u(s_{j+1}) = y_{n-(j-1)}\}.
\end{aligned}$$

Clearly, $(z_1^{(m_1-1)}(x_1), z_1^{(m_k-1)}(x_k)) \in S_1$, and so S_1 is a nonempty subset of $\mathbb{R}^2$. By the same process as we did previously, we can show that $S_1 = \mathbb{R}^2$. Hence, $(y_{m_1 1}, y_{m_k k}) \in S_1$, which implies that there is a solution $y_1(x)$ of (1) such that

$$\begin{aligned}
&a_1 y(t_1) - b_1 y(p_1) = y_1,\ a_2 y(t_2) - b_2 y(p_2) = y_2,\\
&y(t_i) = \tfrac{y_i}{a_i},\ 3 \le i \le j,\\
&y_1^{(i-1)}(x_l) = y_{il}, \quad 1 \le i \le m_l, \quad 1 \le l \le k,\\
&y(s_i) = \tfrac{y_{n-(i-1)}}{c_i},\ 1 \le i \le j-2,\\
&c_{j-1} y(s_{j-1}) - d_{j-1} y(q_1) = y_{n-(j-2)},\ c_j y(s_j) - d_j y(s_{j+1}) = y_{n-(j-1)},
\end{aligned}$$

which is the desired solution of the $(2; k+2j-4; 2)-$point BVP. Continuing in the same way, we obtain a unique solution of the $(j; k; j)-$point BVP, that is, a solution $y(x)$ of (1) such that at the points $t_1, \ldots, t_{2j}, x_1, \ldots, x_k, s_1, \ldots, s_{2j}$, satisfies

$$\begin{aligned}
&a_i y(t_{2i-1}) - b_i y(t_{2i}) = y_i,\ i = 1, 2, ..., j,\\
&y^{(i-1)}(x_l) = y_{il},\ 1 \le i \le m_l,\ 1 \le l \le k,\\
&c_i y(s_{2i-1}) - d_i y(s_{2i}) = y_{n-(i-1)},\ i = 1, 2, \ldots, j.
\end{aligned}$$

We restate Theorem 3.2 in the terminology introduced in Introduction.

Corollary 3.2 *Assume that $k = n - 2j$, solutions of the $(j; n-2j; j)-$point BVP, are unique. Then, for each $1 \le k \le n-2j$, (1) is $(j; k; j)-$point uniquely solvable.*

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