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On Synchronization, Anti-synchronization and Hybrid Synchronization of 3D Discrete Generalized Hénon Map

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Abstract: Suitable stabilization conditions obtained for continuous chaotic systems are generalized, in this paper, to discrete-time chaotic systems. The proposed approach, leading to these conditions for complete synchronization, anti-synchronization and hybrid synchronization phenomena studies, is based on the use of state feedback and aggregation techniques for stability and stabilizability studies associated with the Benrejeb arrow form matrix for system description. The results, easy to use, are successfully applied for two identical 3D generalized Hénon maps.

Keywords: *hyperchaotic discrete-time systems; stability; Benrejeb arrow form matrix; complete synchronization; anti-synchronization; hybrid synchronization.*

Mathematics Subject Classification (2010): 34C28, 93C55.

1 Introduction

Chaos synchronization has received a significant attention due to its potential applications [12, 27] in various fields, for instance, application to control theory, secure communication, chemical reaction and encoding message [13, 24]. There exist many types of synchronization, such as Complete Synchronization (CS) [27], Anti-Synchronization (AS) [19], Hybrid Synchronization (HS) [21, 22], Phase Synchronization [29], Lag Synchronization [30], Generalized Synchronization [31], Projective Synchronization [25] and Q-S Synchronization [32].

Given the two following chaotic systems:
the master one:

$$x_m(k+1) = F(x_m(k)), \tag{1}$$

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the slave one:

$$y_s(k+1) = G(y_s(k)), \tag{2}$$

$x_m(k) = [x_{m1}(k) \ \ldots \ x_{mn}(k)]^T$ is the n-dimensional state vector of the master system and $y_s(k) = [y_{s1}(k) \ \ldots \ y_{sn}(k)]^T$ is the n-dimensional state vector of the slave system. $F : R^n \to R^n$ and $G : R^n \to R^n$ are vector functions in n-dimensional space. If the following conditions are satisfied:

$$\lim_{k\to\infty} \|y_i(k) - \alpha_i x_i(k)\| = 0, \ \forall i = 1, \ldots, n, \tag{3}$$

then the complete synchronization is achieved when all the values of α_i are equal to 1, the anti-synchronization when all α_i are equal to -1 and the hybrid synchronization [21,22] when some pairs of the state variables achieve (AS) and the other pairs of state variables, simultaneously, achieve (CS).

In this paper, the proposed approach is based on establishing a new state feedback stabilizing conditions for nonlinear discrete-time hierarchical systems, which constitute an extension of previous results on synchronization studies of continuous chaotic processes [17, 19]. This approach is based on the Borne and Gentina practical criterion for stability study [7-9] (Appendix) associated with the Benrejeb arrow form matrix for system description [3-6,10,11,16,18].

In fact, the main purpose in this work is to design an adaptive state feedback controller guaranteeing the asymptotic stability followed by the complete synchronization, the anti-synchronization and the hybrid synchronization of the nonlinear discrete-time error of two identical hyperchaotic systems.

The paper is organized as follows. After a brief description of the third order generalized Hénon map in Section 2, sufficient conditions leading to conclude on the asymptotic stability of dynamic nonlinear discrete-time processes characterized, in the state space, by a thin arrow form matrix [11], are given in Section 3. In Section 4, the design of a complete synchronous state feedback stabilizing controller of two identical Hénon maps is proposed. The case of anti-synchronization of two identical Hénon maps is also considered in Section 5, and hybrid synchronization between two identical Hénon maps in Section 6.

2 3D Generalized Hénon Map

In this section, two identical hyperchaotic discrete-time Hénon map master and slave systems are described as follows [2, 15, 23, 26]:

the master one:

$$\begin{cases} x_{m1}(k+1) = \mu - x_{m2}^2(k) - bx_{m3}(k), \\ x_{m2}(k+1) = x_{m1}(k), \\ x_{m3}(k+1) = x_{m2}(k). \end{cases} \tag{4}$$

the slave one:

$$\begin{cases} x_{s1}(k+1) = \mu - x_{s2}^2(k) - bx_{s3}(k) + u_1(k), \\ x_{s2}(k+1) = x_{s1}(k) + u_2(k), \\ x_{s3}(k+1) = x_{s2}(k) + u_3(k), \end{cases} \tag{5}$$

$x_m(k) = [x_{m1}(k)\ x_{m2}(k)\ x_{m3}(k)]^T$ is the state vector of master system, $x_s(k) = [x_{s1}(k)\ x_{s2}(k)\ x_{s3}(k)]^T$ is the state vector of slave system and $u(k) = [u_1(k)\ u_2(k)\ u_3(k)]^T$ the control vector to be designed later for achieving synchronization, anti-synchronization or hybrid synchronization. The hyperchaotic attractor of system (4) characterized by: $b = 0.1$ and $\mu = 1.76$, with the initial values $x_m(0) = (1, 0.1, 0)$ [15], is shown in Figure 2.1.

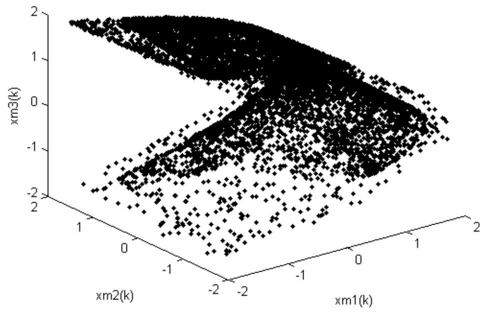

Figure 2.1: Hyperchaotic attractor of system (4) in $x_m(k)$ hyperplane.

The simulation results of the two identical Hénon maps hyperchaotic systems, shown in Figure 2.2, illustrate the systems (4) and (5) responses when the control is turned off and for initial states $x_m(0) = (1, 0.1, 0)$, $x_s(0) = (-0.5, 0, 0.3)$[15]. Then, the states are not synchronized.

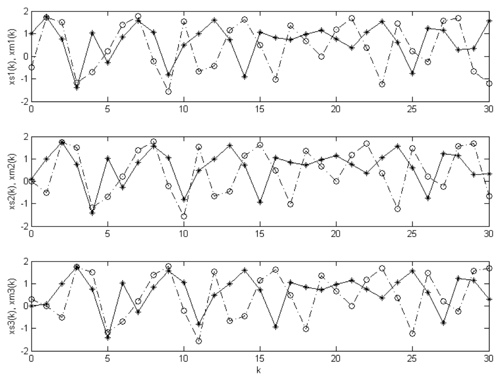

Figure 2.2: Responses of the master (—) and slave (−−) systems.

3 Sufficient Conditions of Asymptotic Stability of Error Dynamics for Chaotic Discrete-time System

Let us consider the following error vector:

$$e_i(k) = x_{si}(k) - \alpha_i x_{mi}(k),\ i = 1, 2, 3,\ \alpha_i \in \{1, -1\}, \tag{6}$$

described in state space by:

$$e(k + 1) = A(k, x(k))e(k) + Bu(k). \tag{7}$$

When system (7) is stabilized by the feedback law $u(k)$, the error converges to zero such as:

$$\lim_{k\to\infty} \|e_i(k)\| = 0,\ i = 1, 2, 3. \tag{8}$$

Then, systems (4) and (5) achieve (CS), (AS) or (HS) according to the values of $\{\alpha_i\}$. To reach this goal, the control law $u(k)$ is chosen such as [6,20]:

$$u(k) = -K(k, x(k))e(k), \tag{9}$$

thus:

$$e(k+1) = A_a(k, x(k))e(k) \tag{10}$$

with

$$A_a(k, x(k)) = A(k, x(k)) - BK(k, x(k)) \tag{11}$$

and the Borne and Gentina criterion [7-9], associated with the particular canonical Benrejeb arrow form matrix $A_a(k, x(k))$ [3-6,10,11,16,18], is used for the formulation of the following theorem [6,16,18].

Theorem 3.1 *The error process, described by (7) is stabilized by the control law defined by (9), if the matrix $A_a(k, x(k))$, defined by (11), is in the arrow form such that:*

i. the nonlinear elements are isolated in one row of the matrix $A_a(k, x(k))$;

ii. the diagonal elements, $a_{a_{ii}}(k, x(k))$, of the matrix $A_a(k, x(k))$ are such that:

$$1 - |a_{a_{ii}}(k, x(k))| > 0,\ \forall i = 1, \ldots, n-1, \tag{12}$$

iii. there exist $\varepsilon > 0$ such that:

$$1 - |a_{a_{nn}}(k, x(k))| - \sum_{i=1}^{n-1} (|a_{a_{in}}(k, x(k)) a_{a_{ni}}(k, x(k))| \times (1 - |a_{a_{ii}}(k, x(k))|)^{-1}) > \varepsilon. \tag{13}$$

Proof The overvaluing system $M(A_a(k, x(k)))$, associated with the vectorial norm $p(z(k))$ is defined (Appendix), in this case, by the following equation

$$z(k+1) = M(A_a(k, x(k)))z(k). \tag{14}$$

The process, described by (7), is stabilized by the control law (9), if the matrix $(I - M(A_a(k, x(k))))$ is an M matrix [28] or if, by application of the stability criterion of Borne and Gentina [7-9], we have

$$\begin{cases} 1 - |a_{a_{ii}}(k, x(k))| > 0,\ \forall i = 1, \ldots, n-1, \\ det(I - M(A_a(k, x(k)))) > \varepsilon \end{cases} \tag{15}$$

The computation of the first member of the last inequality leads to the following expression

$$\begin{aligned} & det(I - M(A_a(k, x(k)))) = \\ & (1 - |a_{a_{nn}}(k, x(k))| - \sum_{i=1}^{n-1} (|a_{a_{in}}(k, x(k)) a_{a_{ni}}(k, x(k))| \times (1 - |a_{a_{ii}}(k, x(k))|)^{-1})) \\ & \times \prod_{j=1}^{n-1} (1 - |a_{a_{ii}}(k, x(k))|) \end{aligned} \tag{16}$$

and achieves easily the proof of the theorem.

4 Synchronization of Two Identical 3D Generalized Hénon Maps

In this section, we propose a systematic procedure to synchronize two identical third-order generalized Hénon maps. This approach determines a control vector $u(k)$ which makes the slave system achieve synchronism with the master system [1,14].

4.1 Problem statement of synchronization of two identical Hénon maps

Let us consider the synchronization error between systems (4) and (5) described by

$$e_i(k) = x_{si}(k) - x_{mi}(k),\ \forall i = 1, 2, 3, \tag{17}$$

$$\begin{cases} e_1(k+1) = -(x_{s2}(k) + x_{m2}(k))e_2(k) - 0.1e_3(k) + u_1(k), \\ e_2(k+1) = e_1(k) + u_2(k), \\ e_3(k+1) = e_2(k) + u_3(k), \end{cases} \tag{18}$$

or, in state space, by

$$e(k+1) = A_s(x(k))e(k) + Bu(k) \tag{19}$$

with

$$A_s(x(k)) = \begin{bmatrix} 0 & -(x_{s2}(k) + x_{m2}(k)) & -0.1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{20}$$

and

$$B = I_{3\times 3}. \tag{21}$$

Figure 4.1 shows the error states between systems (4) and (5) when the control is turned off. It is obvious that the error grows chaotically with time.

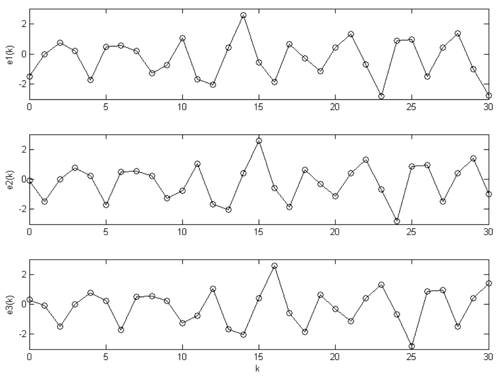

Figure 4.1: Error dynamics for 3D generalized Hénon maps for deactivated controller.

4.2 Chaos synchronization via feedback control law

From the control theory viewpoint, the synchronization of system (18) is equivalent to the stabilization of system (19) by the feedback control law $u(k)$. To achieve this goal, let consider $u(k)$ introduced in (9) such that:

$$K(x(k)) = \begin{bmatrix} k_{11}(x(k)) & k_{12}(x(k)) & k_{13}(x(k)) \\ k_{21}(x(k)) & k_{22}(x(k)) & k_{23}(x(k)) \\ k_{31}(x(k)) & k_{32}(x(k)) & k_{33}(x(k)) \end{bmatrix}. \tag{22}$$

Then, the error system becomes:

$$e(k+1) = A_{sc}(x(k))e(k) \tag{23}$$

with

$$A_{sc}(x(k)) = A_s(x(k)) - BK(x(k)). \tag{24}$$

$A_{sc}(x(k))$ can be rewritten as

$$A_{sc}(x(k)) = \begin{bmatrix} -k_{11}(x(k)) & -(x_{s2}(k)+x_{m2}(k)) - k_{12}(x(k)) & -0.1 - k_{13}(x(k)) \\ 1 - k_{21}(x(k)) & -k_{22}(x(k)) & -k_{23}(x(k)) \\ -k_{31}(x(k)) & 1 - k_{32}(x(k)) & -k_{33}(x(k)) \end{bmatrix}. \tag{25}$$

A circular permutation on the components of state vector and the choice of correction parameters k_{23} and k_{32} constant as follows

$$\begin{cases} 1 - k_{32} = 0, \\ k_{23} = 0, \end{cases} \tag{26}$$

make the matrix $A_{sa}(x(k))$ in Benrejeb arrow form:

$$A_{sa}(x(k)) = \begin{bmatrix} -k_{11}(x(k)) & -(x_{s2}(k)+x_{m2}(k)) - k_{12}(x(k)) & -0.1 - k_{13}(x(k)) \\ 1 - k_{21}(x(k)) & -k_{22}(x(k)) & 0 \\ -k_{31}(x(k)) & 0 & -k_{33}(x(k)) \end{bmatrix}. \tag{27}$$

The system characterized by (27) is asymptotically stable, if the control gains $k_{ij}(x(k))$, $i, j = 1, 2, 3$, are chosen so that the following constraints are satisfied:

i. the nonlinear elements are isolated in one row of the matrix $A_{sa}(x(k))$;

ii. the diagonal elements of the matrix $A_{sa}(x(k))$ are such that:

$$\begin{cases} 1 - |k_{33}(x(k))| > 0, \\ 1 - |k_{22}(x(k))| > 0, \end{cases} \tag{28}$$

iii. there exist $\varepsilon > 0$ such that:

$$\begin{aligned} &1 - |k_{11}(x(k))| - \frac{|k_{31}(x(k))(0.1 + k_{13}(x(k)))|}{1 - |k_{33}(x(k))|} \\ &- \frac{|(k_{12}(x(k)) + x_{s2}(k) + x_{m2}(k))(1 - k_{21}(x(k)))|}{1 - |k_{22}(x(k))|} > \varepsilon. \end{aligned} \tag{29}$$

Then, instantaneous gains $k_{ij}(x(k))$, $\forall i, j = 1, 2, 3$, satisfying inequalities (28) and (29) such as:

$$K(x(k)) = \begin{bmatrix} 0.05 & 0.5 - x_{s2}(k) - x_{m2}(k) & 0.1 \\ 0.5 & 0.5 & 0 \\ 0.2 & 1 & 0.8 \end{bmatrix} \tag{30}$$

guaranty the synchronization, between systems (4) and (5), as shown in Figures 4.2 and 4.3.

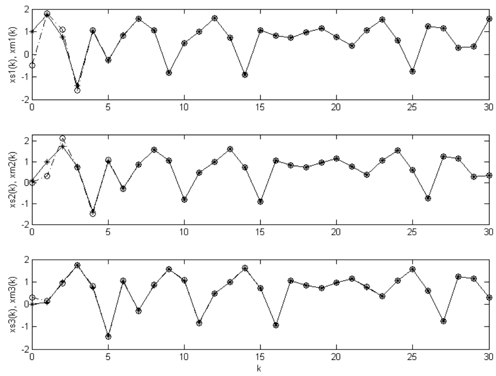

Figure 4.2: Time responses of spatiotemporal chaos synchronization master (—) and slave (−−) outputs.

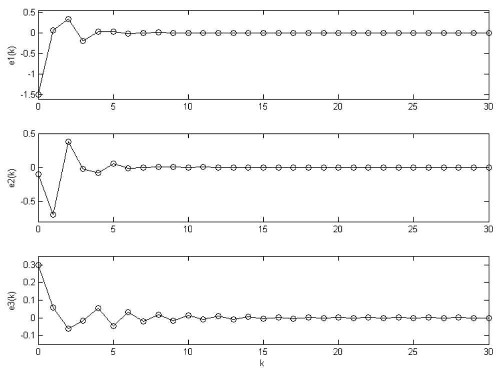

Figure 4.3: Error dynamics of the the 3D generalized Hénon maps for activated controller.

Figure 4.3 shows that $e_1(k)$ converges to zero after 4 iterations and $e_2(k)\; and\; e_3(k)$ after 5 iterations.

5 Anti-synchronization of Two Identical 3D Generalized Hénon Maps

In this section, the objective is to design a controller such that the controlled third order generalized Hénon map (5) is anti-synchronous with the third order generalized Hénon map (4), i.e., to make the sum of the oscillating signals converge to zero, when $k \to \infty$.

5.1 Problem statement of anti-synchronization of two identical Hénon maps

Let us consider, in the present case, the error vector as

$$e(k+1) = x_{si}(k) + x_{mi}(k),\ \forall i = 1,2,3 \tag{31}$$

and the error system as

$$\begin{cases} e_1(k+1) = -x_{m2}^2(k) - x_{s2}^2(k) - 0.1e_3(k) + 3.52 + u_1(k), \\ e_2(k+1) = e_1(k) + u_2(k), \\ e_3(k+1) = e_2(k) + u_3(k). \end{cases} \tag{32}$$

The previous equations (32) can be rewritten under the following matrix description

$$e(k+1) = A_{As}(x(k))e(k) + Bu(k) + C_{As}(x(k)) \tag{33}$$

with

$$A_{As}(x(k)) = \begin{bmatrix} 0 & x_{s2}(k) - x_{m2}(k)) & -0.1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{34}$$

and

$$C_{As}(x(k)) = \begin{bmatrix} 3.52 - 2x_{s2}^2(k) \\ 0 \\ 0 \end{bmatrix}, \tag{35}$$

$B = I_{3\times3}$. Figure 5.1 shows the states error between systems (4) and (5) when the control is turned off. It is obvious that the error grow chaotically with time.

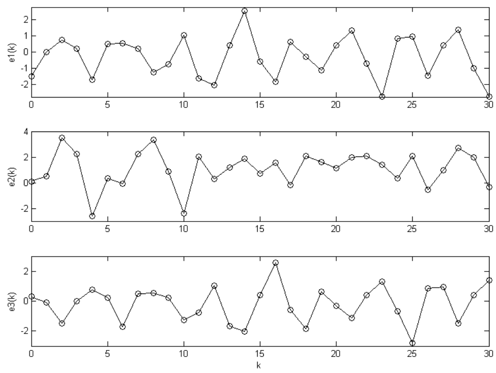

Figure 5.1: Error dynamics of the 3D generalized Hénon maps for deactivated controller.

5.2 Anti-synchronization using state feedback control law

To achieve the property of anti-synchronization between the identical Hénon maps (4) and (5) and by referring to the hypothesis mentioned in the theorem announced in Section 3, let us define the active control functions as follows:

$$u_i(k) = -f_i(x(k)) - \sum_{j=1}^{3} k_{ij}(x(k))e_j(k),\ \forall i = 1, 2, 3, \tag{36}$$

$$u(k) = -\begin{bmatrix} 3.52 - 2x_{s2}^2(k) \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} k_{11}(x(k)) & k_{12}(x(k)) & k_{13}(x(k)) \\ k_{21}(x(k)) & k_{22}(x(k)) & k_{23}(x(k)) \\ k_{31}(x(k)) & k_{32}(x(k)) & k_{33}(x(k)) \end{bmatrix} e(k). \tag{37}$$

Hence, the error system (32) becomes:

$$e(k+1) = A_{Asc}(x(k))e(k) \tag{38}$$

with

$$A_{Asc}(x(k)) = A_{As}(x(k)) - BK(x(k)). \tag{39}$$

$A_{Asc}(x(k))$ can be rewritten as

$$A_{Asc}(x(k)) = \begin{bmatrix} -k_{11}(x(k)) & x_{s2}(k) - x_{m2}(k) - k_{12}(x(k)) & -0.1 - k_{13}(x(k)) \\ 1 - k_{21}(x(k)) & -k_{22}(x(k)) & -k_{23}(x(k)) \\ -k_{31}(x(k)) & 1 - k_{32}(x(k)) & -k_{33}(x(k)) \end{bmatrix}. \tag{40}$$

Proceeding as before, we make the appropriate choice of the instantaneous control gains k_{23} and k_{32} as shown in equalities (26). Then, with the gains $k_{ij}(x(k))$, $\forall i, j = 1, 2, 3$, satisfying inequalities (12) and (13) of the theorem announced in Section 3 such as

$$K(x(k)) = \begin{bmatrix} 0.05 & 0.5 - x_{m2}(k) + x_{s2}(k) & 0.1 \\ 0.5 & 0.5 & 0 \\ 0.2 & 1 & 0.8 \end{bmatrix} \tag{41}$$

the system (31) converges and hence, the anti-synchronization of (4) and (5) is realized as shown in Figures 5.2, 5.3 and 5.4.

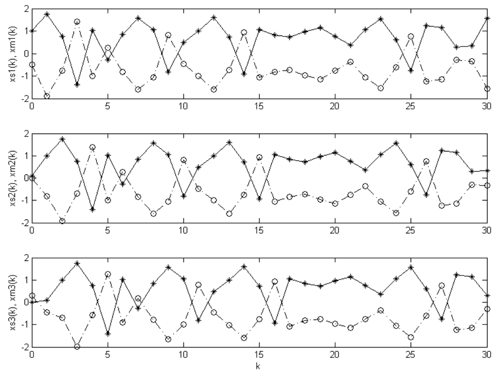

Figure 5.2: Time responses of spatiotemporal chaos anti-synchronization of master (—) and slave (−−) outputs.

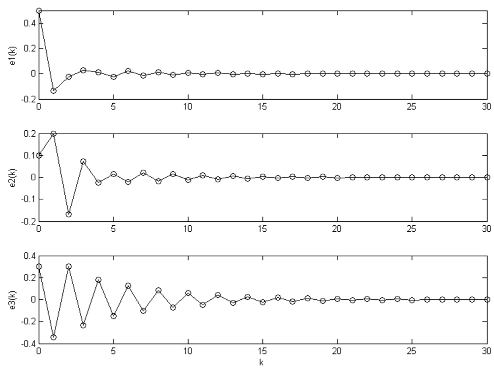

Figure 5.3: Error dynamics of the 3D generalized Hénon for activated controller.

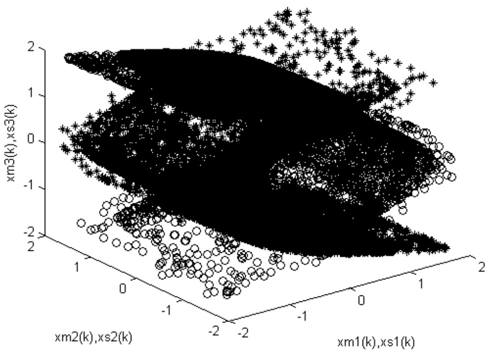

Figure 5.4: Hyperchaotic attractor of system (4)(o) and (5)($*$).

Figure 5.3 shows the time response of the anti-synchronization errors, one can observe that $e_1(k)$ $e_2(k)$ *and* $e_3(k)$ converges to zero respectively in 2, 4 and 11 iterations. Figure 5.4 depicts the projection of the anti-synchronized attractors onto the $x_{mi}(k)$ *and* $x_{si}(k)$, $\forall\, i = 1, 2, 3$ hyperplane, where the state vectors of the master and slave systems evolve in the opposite directions.

6 Hybrid Synchronization of Two Identical 3D Generalized Hénon Maps

In this section, we focus on the problem of hybrid synchronization process of two identical chaotic Hénon maps.

6.1 Problem statement of hybrid synchronization of two identical Hénon maps

The error vector defined as

$$\begin{cases} e_1(k+1) = x_{s1}(k) - x_{m1}(k), \\ e_2(k+1) = x_{s2}(k) + x_{m2}(k), \\ e_3(k+1) = x_{s3}(k) - x_{m3}(k), \end{cases} \tag{42}$$

leads to the following error system

$$\begin{cases} e_1(k+1) = (x_{m2}(k) - x_{s2}(k))e_2(k) - 0.1e_3(k) + u_1(k), \\ e_2(k+1) = e_1(k) + 2x_{m1}(k) + u_2(k), \\ e_3(k+1) = e_2(k) - 2x_{m2}(k) + u_3(k), \end{cases} \tag{43}$$

The previous equations (43) can be rewritten under the following matrix form

$$e(k+1) = A_{Hs}(x(k))e(k) + Bu(k) + C_{Hs}(x(k)) \tag{44}$$

with

$$A_{Hs}(x(k)) = \begin{bmatrix} 0 & x_{m2}(k) - x_{s2}(k) & -0.1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C_{Hs}(x(k)) = \begin{bmatrix} 0 \\ 2x_{m1}(k) \\ -2x_{m2}(k) \end{bmatrix}, \tag{45}$$

$B = I_{3\times3}$. Figure 6.1 shows the error dynamics when the control is turned off. One can observe that errors grow chaotically with time.

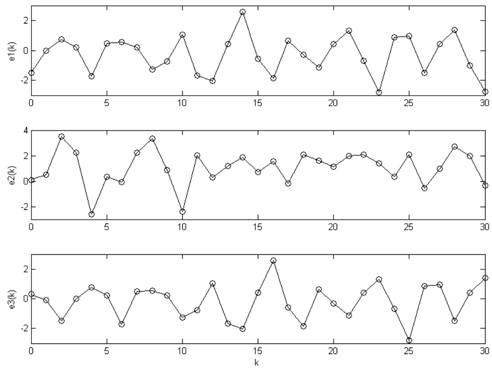

Figure 6.1: Error dynamics of the 3D generalized Hénon maps for deactivated controller.

6.2 Hybrid synchronization via state feedback control law

We seek, as before, for an error system stabilizing control law:

$$u_i(k) = -f\prime_i(x(k)) - \sum_{j=1}^{3} k_{ij}(x(k))e_j(k),\ \forall i = 1, 2, 3, \tag{46}$$

$$u(k) = -\begin{bmatrix} 0 \\ 2x_{m1}(k) \\ -2x_{m2}(k) \end{bmatrix} - \begin{bmatrix} k_{11}(x(k)) & k_{12}(x(k)) & k_{13}(x(k)) \\ k_{21}(x(k)) & k_{22}(x(k)) & k_{23}(x(k)) \\ k_{31}(x(k)) & k_{32}(x(k)) & k_{33}(x(k)) \end{bmatrix} e(k). \tag{47}$$

It cames to the following error dynamical system

$$e(k+1) = A_{Hsc}(x(k))e(k) \tag{48}$$

with:

$$A_{Hsc}(x(k)) = A_{Hs}(x(k)) - BK(x(k)). \tag{49}$$

$A_{Hsc}(x(k))$ can be rewritten as

$$A_{Hsc}(x(k)) = \begin{bmatrix} -k_{11}(x(k)) & x_{m2}(k) - x_{s2}(k) - k_{12}(x(k)) & -0.1 - k_{13}(x(k)) \\ 1 - k_{21}(x(k)) & -k_{22}(x(k)) & -k_{23}(x(k)) \\ -k_{31}(x(k)) & 1 - k_{32}(x(k)) & -k_{33}(x(k)) \end{bmatrix}. \tag{50}$$

These feedback laws stabilize system (43). $e_1(k)$, $e_2(k)$ and $e_3(k)$ converging to zero as time tends to infinity, imply that the hybrid synchronization of the two identical Hénon map systems (4) and (5) is obtained. To achieve this goal, the instantaneous gain matrix $k_{ij}(x(k))$, $\forall i, j = 1, 2, 3$, have to satisfy equalities (26) for system description and inequalities (12) and (13) of the theorem announced in Section 3 for stability study such as

$$K(x(k)) = \begin{bmatrix} 0.05 & 0.5 + x_{m2}(k) - x_{s2}(k) & 0.1 \\ 0.5 & 0.5 & 0 \\ 0.2 & 1 & 0.8 \end{bmatrix}. \tag{51}$$

The guaranteed hybrid synchronization is shown in Figures 6.2 and 6.3.

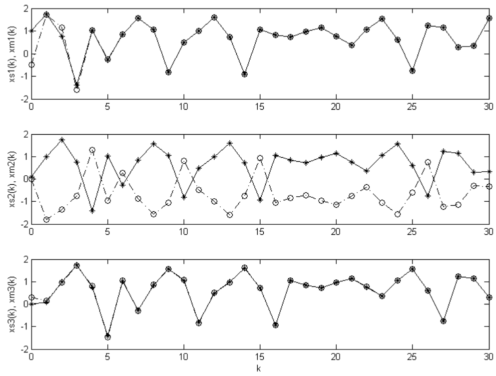

Figure 6.2: Time responses of spatiotemporal chaos hybrid synchronization master (—) and slave (−−) outputs.

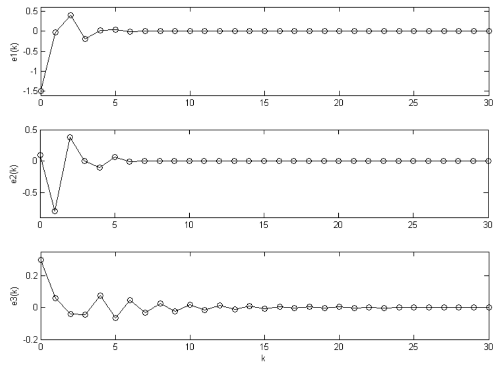

Figure 6.3: Error dynamics of the 3D generalized Hénon map for activated controller.

One can observe, in Figure 6.3 that $e_1(k)$, $e_2(k)$ and $e_3(k)$ converge to zero respectively after 4, 6 and 5 iterations.

7 Conclusion

Stability and stabilisability analysis of discrete-time chaotic systems approaches leading to suitable stabilization conditions is proposed, in this paper, for synchronization studies using the practical stability criterion of Borne and Gentina associated with the particular matrix description, namely the Benrejeb arrow form matrix. Numerical simulations illustrate the efficiency of above stabilization conditions for synchronization studies of two identical Hénon maps. Obtained results can be applied to secure communication and message encoding.

8 Appendix

Borne-Gentina practical stability criterion [7-9]:
Let us consider the nonlinear discrete-time system described in the state space by

$$x(k+1) = A(k, x(k))x(k), \tag{52}$$

where $A(k, x(k))$ is a $n \times n$ matrix, $A(k, x(k)) = \{a_{ij}(k, x(k))\}$ and $x(k) = [x_1(k) \ \ldots \ x_n(k)]^T \in R^n$ is the state vector. Consider the overvaluing matrix $M(A(k, x(k)))$, associated with the vectorial norm $p(z(k)) = [|z_1(k)| \ \ldots \ |z_n(k)|]^T$, $z(k) = [z_1(k) \ \ldots \ z_n(k)]^T$, such that

$$M(A(k, x(k))) : \{a^*_{ij}(k, x(k)) = |a_{ij}(k, x(k))|, \ \forall i, j = 1, \ldots, n\}. \tag{53}$$

If non-constant elements are isolated in only one row of the overvaluing matrix $M(A(k, x(k)))$, asymptotic stability is ensured if all the successive principal minors of the matrix $(I - M(A(k, x(k))))$ are positive.
Thus, the stability conditions of the initial system (53) are the following

$$1 - a^*_{11} \geq \varepsilon > 0, \quad \begin{vmatrix} 1 - a^*_{11} & -a^*_{12} \\ -a^*_{21} & 1 - a^*_{22} \end{vmatrix} \geq \varepsilon > 0, \ldots, \tag{54}$$

$$\begin{vmatrix} 1 - a^*_{11} & -a^*_{12} & \ldots & -a^*_{1n} \\ -a^*_{21} & 1 - a^*_{22} & \ldots & -a^*_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a^*_{n1} & -a^*_{n2} & \ldots & 1 - a^*_{nn} \end{vmatrix} \geq \varepsilon > 0 \ \forall (k, x(k)).$$

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