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Instability for Nonlinear Differential Equations of Fifth Order Subject to Delay

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Abstract: This paper studies the instability of zero solution of a certain fifth order nonlinear delay differential equation. Sufficient conditions for the instability of zero solution of the equation considered are obtained by the Lyapunov-Krasovskii functional approach.

Keywords: *instability; Lyapunov–Krasovskii functional; delay differential equation; fifth order.*

Mathematics Subject Classification (2010): 34K20.

1 Introduction

It is well known that in applied sciences some practical problems concerning physics, mechanics and the engineering technique fields associate with differential equations of higher order (Chlouverakis and Sprott [1] and Linz [9]) . Therefore, the investigation of qualitative behaviors of solutions of nonlinear differential equations of higher order has a great importance in theory and applications of differential equations. In particular, by now, several authors have contributed to the theoretical study of instability of solutions of some fifth order nonlinear differential equations without delay (Ezeilo [3–5], Li and Duan [7], Li and Yu [8], Sadek [11], Sun and Hou [12], Tiryaki [13], Tunç [14–16], Tunç and Erdoğan [21], Tunç and Karta [22], Tunç and Şevli [23]). Throughout all of the mentioned papers, based on Krasovskii's properties (Krasovskii [6]), the Lyapunov's second (or direct) method has been used as a basic tool to prove the results established on the instability of solutions, since differential equations studied cannot be solved explicitly. This method, invented by the Russian mathematician Lyapunov in 1892, proves to be

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extremely effective and useful and is still far of being obsolete. On the other hand, it should be noted that the instability of solutions of some certain fifth order nonlinear delay differential equations has been discussed by Tunç [17, 19, 20].

Besides, in 1978, Ezeilo [3] established an instability result for the fifth order nonlinear differential equation without delay

$$x^{(5)} + a_1 x^{(4)} + a_2 x''' + a_3 x'' + a_4 x' + f(x) = 0. \tag{1}$$

In this paper, instead of (1), we consider the fifth order nonlinear delay differential equation

$$x^{(5)} + a_1 x^{(4)} + a_2 x''' + a_3 x'' + a_4 x' + f(x(t-r)) = 0, \tag{2}$$

where a_1, a_2, a_3 and a_4 are some real constants, r is a positive real constant, the primes in (2) denote differentiation with respect to t, $t \in \Re^+ = [0, \infty)$; f is a differentiable function on $\Re$ with $f(0) = 0$. It is assumed that the existence and uniqueness of the solutions of (2) are guaranteed (see [2], pp. 14,15).

We write (2) in system form as follows

$$\begin{aligned} &x' = y, \quad y' = z, \quad z' = w, \quad w' = u, \\ &u' = -a_1 u - a_2 w - a_3 z - a_4 y - f(x) + \int_{t-r}^{t} f'(x(s))y(s)ds. \end{aligned} \tag{3}$$

In all what follows, $x(t),\ y(t),\ z(t),\ w(t)$ and $u(t)$ are abbreviated as $x,\ y,\ z,\ w$ and u, respectively.

The motivation for this paper comes from the above mentioned papers and Martynyuk et. al [10] and Tunç [18]. Our aim is to convey the results established in Ezeilo [3] to Eq. (3).

Consider the linear constant coefficient fifth order differential equation

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + a_3 \ddot{x} + a_4 \dot{x} + a_5 x = 0, \tag{4}$$

where a_1, a_2, a_3, a_4 and a_5 are some real constants. It is well-known from the qualitative behavior of solutions of linear differential equations that the trivial solution of (4) is unstable if and only if, the associated auxiliary equation

$$\psi(\lambda) \equiv \lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0 \tag{5}$$

has at least one root with a positive real part. The existence of such a root naturally depends on (though not always all of) the coefficients a_1, a_2, a_3, a_4 and a_5. For example, if $a_1 < 0$, then it follows from a consideration of the fact that the sum of the roots of (5) equals to $(-a_1)$ and that at the least one root of (5) has a positive real part for arbitrary values of a_2, a_3, a_4 and an analogue consideration, combined with the fact that the product of the roots (5) equals to $(-a_5)$ will verify that at least one root of (5) has a positive real part if

$$a_1 = 0 \text{ and } a_5 \neq 0 \tag{6}$$

for arbitrary a_2, a_3 and a_4. The condition $a_1 = 0$ here in (6) is, however, superfluous when

$$a_5 < 0; \tag{7}$$

for then $\psi(0) = a_5 < 0$ and $\psi(R) > 0$ if $R > 0$ is sufficiently large; thus showing that there is a positive real root of (5) subject to (7) and for arbitrary a_1, a_2, a_3 and a_4.

A root with a positive real part also exists for certain equations (5) with a_5 positive and sufficiently large. To see this easily we refer to the well-known Routh-Hurwitz criteria which stipulate that each root of (5) has a negative real part. Namely, a necessary and sufficient condition for the negativity of the real parts of all the roots of the polynomial equation (5) is the positivity of all the principal diagonals of the minors of the Hurwitz matrix:

$$H_5 = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & 0 & a_5 & a_4 & a_3 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix}.$$

It should be also noted that the principal diagonal of the Hurwitz matrix H_5 exhibits the coefficients of the polynomial equation (5) in the order of their numbers from a_1 to a_5 . The fourth order minor, say Δ_4, concerned here is given by the determinant

$$\Delta_4 = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & a_5 & a_4 \end{vmatrix},$$

that is, on multiplying out:

$$\Delta_4 = -a_5^2 + a_5(2a_1a_4 + a_2a_3 - a_1a_2^2) + a_4(a_1a_2a_3 - a_3^2 - a_1^2a_4). \tag{8}$$

It is thus clear, in particular, that if $\Delta_4 < 0$, as would indeed be the case from (8), if

$$a_5 \geq R_0 > 0 \tag{9}$$

with $R_0 = R_0(a_1, a_2, a_3, a_4)$ sufficiently large, then at the least one root of (5) has a non-negative real part subject to (9).

Let $r \geq 0$ be given, and let $C = C([-r, 0], \ \Re^n)$ with $\|\phi\| = \max\limits_{-r \leq s \leq 0} |\phi(s)|\,, \ \ \phi \in C.$

For $H > 0$ define $C_H \subset C$ by $C_H = \{\phi \in C : \|\phi\| < H\}$.

If $x : [-r, \ a) \to \Re^n$ is continuous, $0 < A \leq \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t + s), -r \leq s \leq 0, t \geq 0.$$

Let G be an open subset of C and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), x_t = x(t + \theta), -r \leq \theta \leq 0, t \geq 0,$$

where $F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from the conditions on F that each initial value problem

$$\dot{x} = F(x_t), x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A)$, $0 < A \leq \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition 1.1 The zero solution $x = 0$ of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.

2 Main Results

Our first main result is given by the following theorem.

Theorem 2.1 *In addition to the assumptions imposed to the function f in Eq. (2), we assume that there exist constants a_1, a_3, δ (> 0), δ_5 and $\bar{\delta}_5$ such that the following conditions hold:*

$$a_1 > 0, f(0) = 0, f(x) \neq 0,\ (x \neq 0), \bar{\delta}_5 \geq f'(x) > \delta_5 \geq 0 \text{ for all } x,$$

where

$$\delta_5 > \begin{cases} 0, & \text{if} \quad a_3 \leq 0, \\ a_3^2 a_1^{-1}, & \text{if} \quad a_3 > 0. \end{cases}$$

Then the trivial solution $x = 0$ of Eq. (2) is unstable provided

$$r < 2\min\{1, \frac{\delta_5 - \delta a_3}{(1+\delta)\delta_5}, \frac{\delta a_1 - a_3}{\bar{\delta}_5}\}.$$

Remark 2.1 The kernel of the proof of Theorem 2.1 will be to show that, under the conditions sated in Theorem 2.1, there exists a continuous Lyapunov functional $V_0 = V_0(x_t, y_t, z_t, w_t, u_t)$ which has the following three properties:

(P_1) in every neighborhood of $(0,0,0,0,0)$, there exists a point $(\xi, \eta, \zeta, \mu, \rho)$ such that $V_0(\xi, \eta, \zeta, \mu, \rho) > 0$,

(P_2) the time derivative $\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t)$ along solution paths of the corresponding equivalent differential system for Theorem 2.1 is positive semi-definite,

(P_3) the only solution $(x, y, z, w, u) = (x(t), y(t), z(t), w(t), u(t))$ of (3) which satisfies $\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) = 0$ is the trivial solution $(0,0,0,0,0)$.

Proof. Consider the Lyapunov functional $V_0 = V_0(x_t, y_t, z_t, w_t, u_t)$ defined by

$$\begin{aligned} V_0 &= \tfrac{1}{2}\{-\delta a_4 x^2 + (a_4 + \delta a_2)y^2 + (a_2 - \delta)z^2 - w^2\} + \delta yw + \delta a_1 yz \\ &\quad -\delta xu - \delta a_1 xw - \delta a_2 xz - \delta a_3 xy + zu + a_1 zw + yf(x) \\ &\quad -\lambda \int_{-r}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds, \end{aligned} \tag{10}$$

where δ is a fixed positive constant, as is possible in view of the condition $\delta_5 > a_3^2 a_1^{-1}$ such that $a_3 a_1^{-1} < \delta < \delta_5 a_3^{-1}$, and s is a real variable such that the integral $\int_{-r}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds$ is non-negative, and λ is a positive constant which will be determined later in the proof.

It is clear from (10) that

$$V_0(-\varepsilon^2, 0, 0, 0, \varepsilon) = \delta(\varepsilon^3 - \frac{1}{2}a_4\varepsilon^4) > 0$$

for all sufficiently small ε. Hence, in every neighborhood of the origin, $(0,0,0,0,0)$, there exists a point $(-\varepsilon^2, 0, 0, 0, \varepsilon)$ such that $V_0(-\varepsilon^2, 0, 0, 0, \varepsilon) > 0$, which shows that the property (P_1) holds for V_0.

By an elementary differentiation, time derivative of the functional $V_0(x_t, y_t, z_t, w_t, u_t)$ in (10) along the solutions of (3) yields

$$\begin{aligned}\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) &= \delta x f(x) + \{f'(x) - \delta a_3\}y^2 + (\delta a_1 - a_3)z^2 + a_1 w^2 \\ &-\delta x \int_{t-r}^{t} f'(x(s))y(s)ds + z \int_{t-r}^{t} f'(x(s))y(s)ds \\ &-\lambda r y^2 + \lambda \int_{t-r}^{t} y^2(s)ds.\end{aligned}$$

The assumptions $f(0) = 0$, $\bar{\delta}_5 \geq f'(x) > \delta_5 \geq 0$ and the estimate $2|mn| \leq m^2 + n^2$ imply

$$\delta x f(x) \geq (\delta\delta_5)x^2,$$

$$-\delta x \int_{t-r}^{t} f'(x(s))y(s)ds \geq -\delta|x| \int_{t-r}^{t} f'(x(s))\ |y(s)|\, ds \geq -\tfrac{1}{2}(\delta\bar{\delta}_5 r)x^2 - \tfrac{1}{2}(\delta\bar{\delta}_5) \int_{t-r}^{t} y^2(s)ds$$

$$z \int_{t-r}^{t} f'(x(s))y(s)ds \geq -|z| \int_{t-r}^{t} f'(x(s))\ |y(s)|\, ds \geq -\tfrac{1}{2}\bar{\delta}_5 r z^2 - \tfrac{1}{2}\bar{\delta}_5 \int_{t-r}^{t} y^2(s)ds$$

so that

$$\begin{aligned}\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) &= (\delta\delta_5 - \tfrac{1}{2}\delta\delta_5 r)x^2 + \{\delta_5 - \delta a_3 - \lambda r\}y^2 \\ &+(\delta a_1 - a_3 - \tfrac{1}{2}\bar{\delta}_5 r)z^2 + a_1 w^2 \\ &+2^{-1}\{2\lambda - (1+\delta)\bar{\delta}_5\} \int_{t-r}^{t} y^2(s)ds.\end{aligned}$$

Let $\lambda = \frac{(1+\delta)\bar{\delta}_5}{2}$. Hence

$$\begin{aligned}\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) &= (\delta\delta_5 - 2^{-1}\delta\delta_5 r)x^2 + \{\delta_5 - \delta a_3 - 2^{-1}(1+\delta)\delta_5 r\}y^2 \\ &+(\delta a_1 - a_3 - 2^{-1}\bar{\delta}_5 r)z^2 + a_1 w^2 > 0\end{aligned}$$

provided $r < 2\min\{1, \frac{\delta_5 - \delta a_3}{(1+\delta)\delta_5}, \frac{\delta a_1 - a_3}{\delta_5}\}$, which verifies that the property (P_2) holds for V_0.

On the other hand, $\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) = 0$ if and only if $x = y = z = w = 0$, which implies that $x = y = z = w = u = 0$. Furthermore, by $f(x) \neq 0$ for all $x \neq 0$, it follows that $\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) = 0$ if and only if $x = y = z = w = u = 0$. Thus, the property (P_3) holds for V_0. By the above discussion, we conclude that the zero solution of Eq. (2) is unstable. The proof of Theorem 2.1 is completed. □

Our second main result is given by the following theorem.

Theorem 2.2 *In addition to the assumptions imposed to the function f in Eq. (2), we assume that there exist constants a_1, a_3, δ (> 0), $\bar{\delta}'_5$ and δ'_5 such that the following conditions hold:*

$$a_1 < 0, f(0) = 0, f(x) \neq 0,\ (x \neq 0), -\bar{\delta}'_5 \leq f'(x) < -\delta'_5 \ \textit{for all } x,$$

where

$$\delta_5' = \begin{cases} 0, & if \;\; a_3 \geq 0, \\ a_3^2 \, |a_1|^{-1}, & if \;\; a_3 < 0. \end{cases}$$

Then the trivial solution $x = 0$ of Eq. (2) is unstable provided

$$r < 2 \min\big\{1, \frac{\delta_5 - \delta a_3}{(1+\delta)\delta_5}, \frac{\delta a_1 - a_3}{\bar{\delta}_5}\big\}.$$

Proof. Consider the Lyapunov functional $V_1 = V_1(x_t, y_t, z_t, w_t, u_t)$ defined by

$$\begin{aligned} V_1 \;\; &= \tfrac{1}{2}\{\delta a_4 x^2 - (a_4 + \delta a_2) y^2 - (a_2 - \delta) z^2 + w^2\} - \delta y w - \delta a_1 y z \\ &+ \delta x u + \delta a_1 x w + \delta a_2 x z + \delta a_3 x y - z u - a_1 z w - y f(x) \\ &- \lambda \int_{-r}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds. \end{aligned}$$

Now, the constant δ is fixed as follows $|a_3| \;\; |a_1|^{-1} < \delta < \delta_5' \, |a_3|^{-1}$.

It is clear from V_1 that

$$V_1(\varepsilon^2, 0, 0, 0, \varepsilon) = \delta(\varepsilon^3 + \frac{1}{2} a_4 \varepsilon^4) > 0$$

for all sufficiently small ε, so that V_1 has the property (P_1).

Calculating the time derivative of V_1 along solutions of (3), we obtain

$$\begin{aligned} \tfrac{d}{dt} V_1(x_t, y_t, z_t, w_t, u_t) \;\; &= -\delta x f(x) - \{f'(x) - \delta a_3\} y^2 - (\delta a_1 - a_3) z^2 - a_1 w^2 \\ &+ \delta x \int_{t-r}^{t} f'(x(s)) y(s) ds - z \int_{t-r}^{t} f'(x(s)) y(s) ds \\ &- \lambda r y^2 + \lambda \int_{t-r}^{t} y^2(s) ds. \end{aligned}$$

The assumptions $f(0) = 0$, $-\bar{\delta}_5' \leq f'(x) < -\delta_5'$ and the estimate $2\,|mn| \leq m^2 + n^2$ imply

$$-\delta x f(x) \geq (\delta \delta_5') x^2,$$

$$-\delta x \int_{t-r}^{t} f'(x(s)) y(s) ds \;\; \geq \delta \, |x| \int_{t-r}^{t} f'(x(s)) \;\; |y(s)| \, ds \geq -\tfrac{1}{2}(\delta \bar{\delta}_5' r) x^2 - \tfrac{1}{2}(\delta \bar{\delta}_5') \int_{t-r}^{t} y^2(s) ds$$

and

$$z \int_{t-r}^{t} f'(x(s)) y(s) ds \;\; \geq |z| \int_{t-r}^{t} f'(x(s)) \;\; |y(s)| \, ds \geq -\tfrac{1}{2} \bar{\delta}_5' r z^2 - \tfrac{1}{2} \bar{\delta}_5' \int_{t-r}^{t} y^2(s) ds$$

so that

$$\begin{aligned} \tfrac{d}{dt} V_1(x_t, y_t, z_t, w_t, u_t) \;\; &= \delta(\delta_5' - \tfrac{1}{2} \bar{\delta}_5' r) x^2 + \{\delta_5' - \delta a_3 - \lambda r\} y^2 + (-\delta a_1 + a_3 - \tfrac{1}{2} \bar{\delta}_5' r) z^2 \\ &- a_1 w^2 + 2^{-1}\{2\lambda - (1+\delta)\bar{\delta}_5'\} \int_{t-r}^{t} y^2(s) ds. \end{aligned}$$

Let $\lambda = \frac{(1+\delta)\bar{\delta}'_5}{2}$. Hence

$$\frac{d}{dt}V_1(x_t, y_t, z_t, w_t, u_t) = \delta(\delta'_5 - 2^{-1}\bar{\delta}'_5 r)x^2 + \{\delta'_5 - \delta a_3 - 2^{-1}(1+\delta)\bar{\delta}'_5 r\}y^2 + (-\delta a_1 + a_3 - 2^{-1}\bar{\delta}'_5 r)z^2 - a_1 w^2 > 0$$

provided $r < 2\min\{\frac{\delta'_5}{\bar{\delta}'_5}, \frac{\delta'_5 - \delta a_3}{(1+\delta)\bar{\delta}'_5}, \frac{-\delta a_1 + a_3}{\bar{\delta}'_5}\}$, which verifies that the property (P_2) holds for V_1.

The remaining of the proof is similar to the proof of Theorem 2.1. Therefore, we omit the details. The proof of Theorem 2.2 is now completed. □

Remark 2.2 When we take into account the assumptions established in Tunç ([19, 20]), it can be seen that our assumptions are completely different from that of ([19,20]). That is to say, Theorems 2.1 and 2.2 raise two new results on the instability of solutions of a delay differential equation (2).

Example 2.1 Consider nonlinear differential equation of fifth order with delay

$$x^{(5)} + x^{(4)} + x''' + \frac{1}{2}x'' + x' + 3x(t-r) = 0. \tag{11}$$

We write (11) in system form as follows

$$x' = y, y' = z, z' = w, w' = u, \quad u' = -u - w - \tfrac{1}{2}z - y - 3x + 3\int_{t-r}^{t} y(s)ds.$$

It follows that Eq. (11) is special case of Eq. (2) and

$$a_1 = 1 > 0, a_2 = 1 > 0, a_3 = \frac{1}{2} > 0, a_4 = 1 > 0,$$

$$f(x) = 3x, f(0) = 0, f(x) \neq 0,\ (x \neq 0), f'(x) = 3,$$

$$3 = \bar{\delta}_5 = f'(x) > \delta_5 > 0, \delta_5 > \frac{1}{4} = \frac{a_3^2}{a_1},$$

$$\frac{1}{2} = a_3 a_1^{-1} < \delta < \delta_5 a_3^{-1} = 2\delta_5.$$

In view of the above estimates, we conclude that all the assumptions of Theorem 2.1 hold. Hence, if

$$r < 2\min\{1, \frac{\delta_5 - 2^{-1}\delta}{(1+\delta)\delta_5}, \frac{\delta - 2^{-1}}{3}\},$$

then the zero solution of (11) is unstable.

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