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NONLINEAR DYNAMICS AND SYSTEMS THEORY

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PERSONAGE IN SCIENCE

Professor Constantin Corduneanu

to the 84th Birthday Anniversary

A.Yu. Aleksandrov [1], A.A. Martynyuk [2*], N.H. Pavel [3], and S.N. Vassilyev [4]

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[2] *Institute of Mechanics National Academy of Science of Ukraine, Nesterov Str. 3, Kiev, 03057, Ukraine*
[3] *Ohio University, Department of Mathematics, Athens, Ohio, 45701, USA*
[4] *V.A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences, Profsoyuznaya Str. 65, Moscow, 117997, Russia*

The paper contains the biographical sketch and reviews scientific achievements of Constantin Corduneanu, the outstanding researcher in Oscillations, Stability and Control Theory of the 20th century.

1 Brief Outline of C. Corduneanu's Life

Constantin Corduneanu was born on July 26th, 1928, in the City of Iasi, Province of Moldova, Romania, from the parents Costache and Aglaia Corduneanu. At that time, his parents were teachers in the village of Potangeni, Movileni commune in the District of Iasi.

At the age of 12, in 1940, he had to move to the City of Iasi for getting his secondary education. He decided to participate in the fierce competition for a place at the Military Lyceum of Iasi, and he was admitted there, as the 10th, from a number of 400 competitors. Four years later, in 1944, when the capacity exam had to be taken for promotion to the second stage of the secondary education, he was classified the 1st among his peers, with special mention for good answers in Mathematics. In 1945 he was transferred from office to the Nicolae Filipescu National Military College in Predeal (in the Carpathian Mts). There he finished his secondary education in 1947.

C. Corduneanu participated in what is nowadays called "Mathematical Olympiad", in the years 1946 and 1947, winning a prize in each case, the first in 1947. That success convinced him to become a mathematician, and in the Fall of 1947 he registered as a student at the Faculty of Science, Division Mathematics, with the University of Iasi.

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His association with the University of Iasi had lasted until the year 1977, period in which he held positions of Assistant, Lecturer, Associate Professor, Professor, Dean of Mathematics, Vice-Rector for Research and Graduate Studies, as well as some research positions with the Mathematical Institute of the Romanian Academy. C. Corduneanu also served, on different occasions, at the Iasi Polytechnic Institute and for three years at the newly created institution which is known today as the University of Suceava (where he also served as Rector during the period 1966–1967).

In 1977, C. Corduneanu decided to expatriate from Romania, and to reside in the United States of America. In January 1978, after teaching some courses at the International Centre for Theoretical Physics (UNESCO) in Trieste, Italy, he came from Italy to the USA, teaching the Spring Semester of 1978 at the University of Rhode Island, which he had visited before for two academic years and where he was familiar with the place and colleagues. Next academic year, 1978–1979, he was a Visiting Professor at the University of Tennessee in Knoxville. Meantime, the University of Texas at Arlington created a new professorial position, which C. Corduneanu occupied by competition in the Fall of 1979. Ever since, he has been associated with this school, currently holding the title of Emeritus Professor of Mathematics (retired in September 1996, after 47 years in higher education in Romania and the USA).

Besides his usual duties as a Professor, C. Corduneanu had many other activities, such as participating in various national or international conferences (more than 100), paying short visits and talking about his research work in over 60 universities or institutes, in all continents with the exception of Australia, and in over 20 countries (including Russia, Ukraine, Germany, England, France, Italy, China, Japan, Hungary, Poland, Portugal and Chile). He has published during the last 60 years about 200 research papers, including 6 books in a total of 15 editions (Romanian Academy, Academic Press in NY, Springer Verlag, Cambridge University Press, the Taylor and Francis Publishing House in London, John Wiley & Sons in NY, Allyn & Bacon in Boston). He has organized and participated in several conferences, in Romania and in the USA, including the Centennial Volterra Conference on Integral Equations and Applications, 1996, at the University of Texas at Arlington, attended by specialists from many countries.

During the last 45 years, he has been associated with at least 10 mathematical journals from Romania, the USA, South Korea, Israel and Ukraine.

2 Basic Trends of His Scientific Work

2.1 Global Problems in the Theory of Ordinary Differential Equations

This type of problems kept his attention at the beginning of his career, including the doctoral thesis defended in 1956 at the University of Iasi, the committee being composed by Academicians Miron Nicolescu, at that time president of the Romanian Academy, Grigore Moisil and Nicolae Teodorescu from Bucharest, a former student of J. Hadamard at Sorbonne. C. Corduneanu continued research work in this field for several years, studying global existence, stability problems, oscillation theory, with special regard to the almost periodic behavior of solutions to various classes of nonlinear equations.

2.2 Qualitative Theory of Differential Equations, with Special Regard to Stability Theory

The work in this category is mainly directed to ordinary differential equations and equations with causal operators. In [11], he has made one of the first steps in applying the so-called comparison method, and proving in a single theorem all basic results on Liapunov stability, based on using simultaneously the Chaplyguine-Wazewski approach to differential inequalities, and the Liapunov's function in general form. This method has been widely applied by the School of Academician V.M. Matrosov, Russia; and in Ukraine by Academician A.A. Martynyuk and his followers. The result published in [11], has been included in several monographs and treatises, by authors like V. Lakshmikantham and S. Leela, W. Hahn, T. Yoshizawa, A. Halanay, G. Sansone and R. Conti and others.

2.3 Theory of Integral Equations

In this domain he has contributed to generalizing the method due to Massera and Schaffer, from differential equations to integral equations. The book [J] contains the basic results he had obtained until 1987, which became one of most often quoted references in the literature. Also, the book [E] contains qualitative results with application to the stability of systems of automatic control.

2.4 Equations with Causal Operators

This category is aimed at presenting, as much as possible, a unified theory of equations with causal operators (according to Volterra–Tonelli–Tychonoff), that can cover the classical types of ordinary differential equations, equations with delay, integrodifferential equations with Volterra type integral, some discrete equations of evolution. In this regard he has published the book [K] covering research conducted by his group of students, as well as his own or joint projects (Mehran Mahdavi from Tehran and Yizeng Li from Shanghai). A second volume dedicated to this type of equations and their connection with the classical types of equations is now in preparation.

2.5 Fourier Analysis (Generalized)

For over a half century, a vide range of problems have been investigated in this field. The books [A], [B], [I] and [M] are concerned with this subject. The papers [47]–[49] are dealing with recent developments in this field.

3 Teaching Activities

Aug 1996 – Present	Emeritus Professor, University of Texas at Arlington;
1979 – 1996	Professor, University of Texas at Arlington;
1978 – 1979	Visiting Professor, University of Tennessee;
Spring 1978	Visiting Professor, University of Rhode Island;
1968 – 1977	Professor, University of Iasi;
1973 – 1974	Visiting Professor, University of Rhode Island;
1967 – 1968	Visiting Professor, University of Rhode Island;
1962 – 1967	Associate Professor, University of Iasi;
1955 – 1962	Lecturer, University of Iasi;

1950 – 1955 Assistant, University of Iasi;
1949 – 1950 Teaching Assistant, University of Iasi.

4 Administrative

1998 – Present Emeritus President, American Romanian Academy;
1995 – 1998 President, American Romanian Academy of Arts & Sciences;
1982 – 1995 Counselor and member of the Executive Committee, American Romanian Academy of Arts and Sciences;
1972 – 1977 Vice Rector, University of Iasi, 1972–1977 (on leave, 1973–1974). In charge of research and graduate studies;
1968 – 1972 Dean of the Mathematics Faculty, University of Iasi;
1966 – 1967 Rector (President) of the Teachers Training College in Suceava (today the Stefan cel Mare University of Suceava);
1964 – 1967 Head (Chairman) of the Mathematical Division at the Teachers Training College in Suceava.

5 Memberships

American Mathematical Society, Society for Industrial and Applied Mathematics, Mathematical Association of America, American Romanian Academy of Arts and Sciences, Romanian Academy (Bucharest), Phi Beta Delta (International Scholars), International Federation of Nonlinear Analysts.

6 Editorial Activity

Editor:
1981 - Present Libertas Mathematica, the Mathematical Journal of the American Romanian Academy of Arts and Sciences.

Associate Editor:
2001 – Present Nonlinear Dynamics and Systems Theory (Kiev, Ukraine);
2001 – Present Nonlinear Functional Analysis and Applications (Korea);
1996 – Present Annals of Ovidius Univ. (Constantza, Romania);
1995 – Present Functional Differential Equations (Israel);
1994 – Present Communications on Applied Nonlinear Analysis (U.S.A.);
1979 – 1995 Journal of Integral Equations and Applications (U.S.A.);
1988 – 1992 Differential and Integral Equation (U.S.A.);
1977 – 1985 Nonlinear Analysis - Theory, Methods and Applications (U.K.);
1973 – 1978 Revue Roumaine de Math. Pures Appl. (Romania);
1969 – 1977 and 1996 – Present Analele Stiintifice Univ. Iasi (Romania);
1967 – 1975 Mathematical Systems Theory (Germany).

7 Awards

2010 Honorary Doctor, University of Ekaterinburg, Russia;
2003 Doctor Honoris Causa, Stefan cel Mare Univ., Suceava, Romania;
2003 Best Paper Award, CASYS'03, Liege, Belgium;

2002 "V. Pogor" Prize of the Municipality of Iasi;
2001 Medal of Merit in Mathematics from the Union of Czech Mathematicians;
1999 Doctor Honoris Causa, Transylvania University, Brasov, Romania;
1994 Doctor Honoris Causa, University of Iasi, Romania;
1994 Doctor Honoris Causa, Ovidius University, Constantza, Romania;
1991 Distinguished Research Award, University of Texas at Arlington;
1974 Elected Correspondent Member of the Romanian Academy of Sciences in Bucharest, Division of Mathematical Sciences;
1963 The Research Award of the Romanian Academy of Sciences, for research work in "Stability Theory of Automatic Control Systems";
1961 The Research Award of the Department of Education in Bucharest, for research conducted in connection with "Comparison Method in Stability Theory".

8 Invited Lectures (Colloquium Programs, Exchange Programs)

1. Belgium: The University of Louvain (1971, 1976).

2. Canada: The University of Montreal (1973); McGill University (1987); Montreal Polytechnic (1989); University of Victoria (1993); Univ. of Waterloo (1994).

3. Czechoslovakia: The Mathematical Institutes of the Academies of Sciences, and the Universities in Prague, Brunno and Bratislava (1962, 1966, 1971).

4. Morocco: The University of Marrakech (1994, 1995).

5. United Kingdom: The Universities of Warwick, Durham and Sussex (1971, 1973); The University of Wales (1989); The University of Dundee (1992); Univ. of Strathclyde (1994).

6. Italy: The Universities in Milano, Florence, Perugia, Naples, and Politecnico in Torino (1965–1993).

7. Japan: Okayama University of Science (2004).

8. West Germany: Technical University in Aachen (1986).

9. Chile: The University of Osorno (2002).

10. U.S.A.: Arizona State, Brown, Case Western Reserve, Cornell, Drexel, Florida State, Southern Methodist, Texas Christian, and Wichita State Universities; the Universities of Rhode Island, Florida at Gainesville, Georgia at Athens, Colorado at Boulder, Colorado at Colorado Springs, Tennessee at Knoxville, Maryland at College Park, South Florida, Arizona at Tucson, Southern California, Wisconsin at Madison, Texas at Arlington, Dallas at Irving, New Mexico at Albuquerque, California at Los Angeles, Utah at Salt Lake City, Miami at Coral Gables; Bishop College in Dallas, Pomona Colleges, Rensselaer Polytechnic Institute, Georgia Institute of Technology, Virginia Polytechnic Institute and State University; Ohio University, University of Pittsburg, University of Houston (Downtown); Howard University, Washington, D. C.; Virginia State University, Petersburg (1968 – Present).

9 List of Monographs and Books by C. Corduneanu

[A] *Functii aproape periodice.* Editura Academiei, Bucharest, 1961.

[B] *Almost Periodic Functions.* John Wiley & Sons, New York, 1968 (translation of [A], enlarged: with N. Gheorghiu and V. Barbu).

[C] *Principles of Differential and Integral Equations.* Allyn & Bacon, Inc., Boston, 1971.

[D] *Differential and Integral Equations.* Univ. of Iasi Press, 1971. [Romanian]

[E] *Integral Equations and Stability of Feedback Systems.* Academic Press, Inc., New York, 1973.

[F] *Differential and Integral Equations.* Univ. of Iasi Press, 1977. (with an Appendix by N. Pavel). [Romanian]

[G] *Principles of Differential and Integral Equations.* 2nd Ed., Chelsea Publ. Co., The Bronx, New York, 1977.

[H] *Principles of Differential and Integral Equations.* Stereotype edition of [G]. (This edition is currently distributed by the American Math. Society and Oxford Univ. Press).

[I] *Almost Periodic Functions.* Chelsea Publ. Co., The Bronx, New York, 1989. The second English Edition, enlarged. This edition is currently distributed by the American Math. Society and Oxford Univ. Press.

[J] *Integral Equations and Applications.* Cambridge Univ. Press, 1991.

[K] *Functional Equations with Causal Operators.* Taylor and Francis, London, 2002; (Kindle edition, 2007, distributed by amazon.com).

[L] *Integral Equations and Applications.* A paperback edition at Cambridge University Press, 2008.

[M] *Almost Periodic Oscillations and Waves.* Springer Verlag, 2009.

[N] *Special Topics in Functional Equations.* (In preparation; jointly with Y. Li and M. Mahdavi).

10 List of Corduneanu's Selected Papers

[1] Approximation and stability of solutions of hyperbolic equations with characteristic data. *Comm. Acad R.P.R.* **V** (1955) 21–26. [Romanian]

[2] On a boundary value problem for second order nonlinear differential equations. *Analele Stiintifice Univ. Iasi, N.S.* **1** (1955) 11–16. [Romanian]

[3] Differential systems with bounded solutions. *Comptes Rendus Acad. Sci. Paris* **245** (1957) 21–24. [French]

[4] Differential equations in Banach spaces; Theorems of existence and continuability. *Rendiconti Accad. Naz. Lincei* **XXIII** (1957) 226–230. [Italian]

[5] On the existence of bounded solutions for nonlinear differential systems. *Annales Polonici Math.* **V** (1958) 103–106. [French]

[6] On conditional stability under constantly acting disturbances. *Acta Scientiarum Math. Szeged* **XIX** (1958) 229–237. [French]

[7] On boundary value problems for differential systems. *Rendiconti Mat. Napoli* **XXV** (1958) 98–106 [Italian]

[8] On asymptotic stability. I. *Analele Stiintifice Univ. Iasi* **V** (1959) 37–40. [French]

[9] On asymptotic stability. II. *Revue Roumaine Math.* **V** (1960) 209–213. [French]

[10] On the existence of bounded solutions to some classes of nonlinear differential systems. *Doklady Akad. Nauk SSSR* **131** (1960) 735–737. [Russian]

[11] Application of differential inequalities to stability theory. *Analele Stiintifice Univ. Iasi* **VI** (1960) 47–58. [Russian]

[12] On some nonlinear differential systems. *Ibidem* 257–260. [French]

[13] Global existence theorems for differential systems with delayed argument. *Studii Cercetari Mat. Iasi* **XII** (1961) 249–258. [Romanian] (Russian version in the *Proceedings of ICNO Symp.* Kiev, 1961).

[14] An integral equation from the theory of automatic control. *Comptes Rendus Acad. Sci. Paris* **256** (1963) 3564–3567. [French]

[15] On partial stability. *Revue Roumaine Math.* **IX** (1964) 229–236. [French]

[16] Some problems concerning stability theory. *Abhandl. Deutsch. Akad. Wissensch. zu Berlin (Math-Physik Klasse)* (1) (1965) 143–156. (Invited paper at the Equadiff Conf.) [French]

[17] Global problems in the theory of Volterra integral equations. *Annali Mat. Pura Appl.* **67** (1965) 349–363. [French]

[18] Some qualitative problems in the theory of integro-differential equations. *Colloquium Mathematicum* **18** (1967) 77–87. (Invited paper at the Balaton Conference 1956). [French]

[19] Some perturbation problems in the theory of integral equations. *Mathematical Systems Theory* **I** (1967) 143–153.

[20] On certain Volterra functional equations. *Funk. Ekvacioj* **9** (1966) 119–127. [French]

[21] Stability of linear time-varying systems. *Math. Systems Theory* **3** (1969) 151–155.

[22] Periodic and almost periodic solutions of some convolution equations. *Trudy Fifth Int. Conf. Nonlinear Osc., Kiev* **III** (1970) 311–320.

[23] Stability problems for some classes of feedback systems. *In the volume "Eq. Diff. Fonct. non lineaires", Herman, Paris* (1973) 398–405.

[24] On partial stability for delay systems. *Annales Polonici Math.* **XXIX** (1974-1975) 357–362.

[25] Functional equations with infinite delay. *Bolletino Unione Mat. Italiana* **11** (suppl.) (1975) 173–181.

[26] The stability of some feedback systems with delay. *J. Math. An. Appl.* **51** (1975) 377–393 (jointly with N. Luca).

[27] Recent contributions to the theory of differential systems with infinite delay. *Libertas Mathematica. I* (1981) 91–116.

[28] Equations with unbounded delay; A survey. *Nonlinear Analysis, TMA* **4** (1980) 831–877 (jointly with V. Lakshmikantham).

[29] Bounded and almost periodic solutions of certain nonlinear elliptic equations. *Tohoku Math. J.* **32** (1980) 265–278.

[30] Almost periodic discrete processes. *Libertas Mathematica. II* (1982) 159–169.

[31] Two qualitative inequalities. *J. Differential Equations* **61** (1985) 16–25.

[32] Bielecki's method in the theory of integral equations. *Annales Univ. Mariae-Curie Skladowska, Lublin* **38** (2) (1984) 23–40.

[33] A singular perturbation approach to abstract Volterra equations. In: *"Nonlinear Analysis and Applications"*. M. Dekker (1987) 133–138.

[34] Perturbation of linear abstract Volterra equations. *J. Integral Equations and Appl.* **2** (1990) 393–401.

[35] LQ-Optimal control problems for systems with abstract Volterra operators. *Tekhn. Kibernetika* (1) (1993) 132–136. [Russian] (English version in *Libertas Mathematica*)

[36] Discrete qualitative inequalities and applications. *Nonlinear Analysis, TMA* **25** (1995) 933–939.

[37] Neutral functional differential equations with abstract Volterra operators. In: *"Advances in Nonlinear Dynamics"*. Gordon & Breach (A. Martynyuk, Ed.) **5** (1997).

[38] Abstract Volterra Equations (a survey). *Mathematical and Computer Modeling* **32** (2000) 1503–1528.

[39] Existence of solutions for neutral functional differential equations with causal operators. *Journal Differential Equations* **168** (2000) 93–101.

[40] Absolute stability for neutral differential systems. *European J. of Control* (2002) 209–212.

[41] Second order functional equations of neutral type. *Dynamic Systems and Applications* **14** (2005) 83–89.

[42] A modified LQ-Optimal control problem for causal functional differential equations. *Nonlinear Dynamics and Systems Theory* **4** (2004) 139–144.

[43] A duality principle in the theory of dynamical systems. *Nonlinear Dynamics and Systems Theory* **5** (2005) 135–140 (jointly with Y. Li).

[44] Almost periodicity in functional equations. In: *"Progress in Nonlinear Differential Equations and their Applications", Birkhauser (V. Staicu, Ed.)* **75**, 2007.

[45] Boundedness of solutions for a second order differential equations with causal operators. *Nonlinear Studies* **18** (2011) 135–139.

[46] A scale of almost periodic functions spaces. *Differential and Integral Equations* **24** (2011) 1–24.

[47] Formal trigonometric series, almost periodicity and oscillatory functions (to appear).

[48] Elements of an axiomatic construction of the theory of almost periodic functions (to appear). [French]

[49] Searching for generalized Fourier exponents associated with series of oscillatory functions (to appear).

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Time Scales Ostrowski and Grüss Type Inequalities Involving Three Functions

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Abstract: In this paper, we present time scales versions of Ostrowski and Grüss type inequalities containing three functions. We assume that the second derivatives of these functions are bounded. Our results are new also for the discrete case.

Keywords: *Ostrowski–Grüss inequality; Ostrowski-like inequality; Montgomery identity; time scales.*

Mathematics Subject Classification (2010): 34N05, 26D15, 26D10.

1 Introduction

Motivated by a recent paper by B. G. Pachpatte [18], our purpose is to obtain time scales versions of some Ostrowski and Grüss type inequalities including three functions, whose second derivatives are bounded. In detail, we will prove time scales analogues of the following three theorems presented in [18].

Theorem 1.1 [See [18, Theorem 1]] *Let $f, g, h : [a, b] \to \mathbb{R}$ be twice differentiable functions on (a, b) such that $f'', g'', h'' : (a, b) \to \mathbb{R}$ are bounded, i.e.,*

$$\|f''\|_\infty := \sup_{t\in(a,b)} |f''(t)| < \infty, \quad \|g''\|_\infty < \infty, \quad \|h''\|_\infty < \infty.$$

Moreover, let

$$A[f, g, h] := gh \int_a^b f(s)\mathrm{d}s + fh \int_a^b g(s)\mathrm{d}s + fg \int_a^b h(s)\mathrm{d}s$$

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and

$$B[f,g,h] := |gh|\,\|f''\|_\infty + |fh|\,\|g''\|_\infty + |fg|\,\|h''\|_\infty .$$

Then, for all $t \in [a,b]$, we have

$$\left| f(t)g(t)h(t) - \frac{1}{3(b-a)} A[f,g,h](t) - \frac{1}{3}\left(t - \frac{a+b}{2}\right)(fgh)'(t) \right| \\ \leq \frac{1}{6}\left\{ \left(t - \frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12} \right\} B[f,g,h](t).$$

Theorem 1.2 [See [18, Theorem 2]] *In addition to the notation and assumptions of Theorem 1.1, let*

$$L[f,g,h] := gh\frac{f(a)+f(b)}{2} + fh\frac{g(a)+g(b)}{2} + fg\frac{h(a)+h(b)}{2}.$$

Then, for all $t \in [a,b]$, we have

$$\left| f(t)g(t)h(t) - \frac{2}{3(b-a)} A[f,g,h](t) - \frac{1}{3}\left(t - \frac{a+b}{2}\right)(fgh)'(t) + \frac{1}{3}L[f,g,h](t) \right| \\ \leq \frac{1}{3(b-a)} B[f,g,h](t) \int_a^b \left| p(t,s)\left(s - \frac{a+b}{2}\right) \right| \mathrm{d}s,$$

where $p(t,s) = s-a$ for $a \leq s < t$ and $p(t,s) = s-b$ for $t \leq s \leq b$.

Theorem 1.3 [See [18, Theorem 3]] *In addition to the notation and assumptions of Theorem 1.1, let*

$$M[f,g,h] := gh\frac{f(b)-f(a)}{b-a} + fh\frac{g(b)-g(a)}{b-a} + fg\frac{h(b)-h(a)}{b-a}.$$

Then, for all $t \in [a,b]$, we have

$$\left| f(t)g(t)h(t) - \frac{1}{3(b-a)} A[f,g,h](t) - \frac{1}{3}\left(t - \frac{a+b}{2}\right) M[f,g,h](t) \right| \\ \leq \frac{1}{3(b-a)^2} B[f,g,h](t) \int_a^b \int_a^b |p(t,\tau)p(\tau,s)| \,\mathrm{d}s\mathrm{d}\tau,$$

where p is defined as in Theorem 1.2.

Our time scales versions of Theorems 1.1–1.3 will contain Theorems 1.1–1.3 as special cases when the time scale is equal to the set of all real numbers, and they will yield new discrete inequalities when the time scale is equal to the set of all integer numbers. Special cases of our results are contained in [2–5, 12, 15, 20] for the general time scales case, in [8–10, 16] for the continuous case and in [1, 17] for the discrete case. One can also use our results for any other arbitrary time scale to obtain new inequalities, e.g., for the quantum case. For further recent results on time scales calculus published in *Nonlinear Dynamics and Systems Theory*, we refer to [11, 13, 14, 19].

The set up of this paper is as follows. In the next section, we give some necessary details of the time scales calculus. Section 3 contains some auxiliary results as well as the assumptions and notation used in this paper. Finally, in Sections 4–6, we prove time scales analogues of Theorems 1.1–1.3. Each result is followed by several examples and remarks. We would like to point out here that our results are new also for the discrete case.

2 Preliminaries

Now we briefly introduce some necessary time scales elements and refer the reader to the books [6, 7] for further details.

Definition 2.1 A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ are called the forward and backward jump operators, respectively. A point $t \in \mathbb{T}$ is said to be *right-dense*, *right-scattered*, *left-dense*, and *left-scattered* provided $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, and $\rho(t) < t$, respectively. The set $\mathbb{T}^\kappa$ is defined to be equal to the set $\mathbb{T}$ without its left-scattered maximum (if it exists). A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* and we write $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ if it is continuous at all right-dense points and its left-sided limits exist and are finite at all left-dense points, and f is called *delta differentiable* at $t \in \mathbb{T}^\kappa$, with *delta derivative* $f^\Delta(t) \in \mathbb{R}$, provided given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]\right| \le \varepsilon\,|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

If f is differentiable such that f^Δ is rd-continuous, then we write $f \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. The set $\mathrm{C}^2_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ is defined similarly. A function $F : \mathbb{T} \to \mathbb{R}$ is called a *delta antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. Then the *delta integral* of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \text{where} \quad a, b \in \mathbb{T}.$$

Example 2.1 If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $f^\Delta(t) = f'(t)$ for all $t \in \mathbb{R}$ and

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t \quad \text{for all} \quad a, b \in \mathbb{R},$$

and if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and $f^\Delta(t) = f(t+1) - f(t)$ for all $t \in \mathbb{Z}$ and

$$\int_0^n f(t)\Delta t = \sum_{t=0}^{n-1} f(t) \quad \text{for all} \quad n \in \mathbb{N}.$$

Some results about integrals, that will be used in this paper, are contained in [6, Section 1.4] and collected as follows.

Theorem 2.1 *If a function is rd-continuous, then it possesses a delta antiderivative. For $f, g \in \mathrm{C}_{\mathrm{rd}}([a, b], \mathbb{R})$ and $a, b, c \in \mathbb{T}$, we have*

$$\int_a^b [f(t) + g(t)]\,\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t,$$

$$\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t,$$

$$\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t,$$

$$\left|\int_a^b f(t)\Delta t\right| \le \int_a^b |f(t)|\,\Delta t,$$

and, if additionally $f, g \in \mathrm{C}^1_{\mathrm{rd}}([a, b], \mathbb{R})$,

$$\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g(t)\Delta t.$$

We also need the time scales monomials (see [6, Section 1.6]) defined as follows.

Definition 2.2 Define for all $t, s \in \mathbb{T}$

$$g_2(t, s) := \int_s^t (\sigma(\tau) - s)\Delta\tau, \quad h_2(t, s) := \int_s^t (\tau - s)\Delta\tau,$$
$$g_3(t, s) := \int_s^t g_2(\sigma(\tau), s)\Delta\tau, \quad h_3(t, s) := \int_s^t h_2(\tau, s)\Delta\tau.$$

It is known that $g_2(t, s), g_3(t, s), h_2(t, s), h_3(t, s)$ are nonnegative for $t \geq s$ and that $g_2(t, s) = h_2(s, t)$ and $g_3(t, s) = -h_3(s, t)$. Moreover, the following formulas are used in this paper.

Lemma 2.1 *The time scales monomials satisfy the following formulas:*

$$g_2(t, a) - g_2(t, b) = g_2(b, a) + (t - b)(b - a), \tag{1}$$
$$g_2(a, b) + g_2(b, a) = (b - a)^2, \tag{2}$$
$$g_3(t, a) - g_3(t, b) = g_3(b, a) + (t - b)g_2(b, a) + (b - a)g_2(t, b). \tag{3}$$

Proof. The function F defined by $F(t) := g_2(t, a) - g_2(t, b) - g_2(b, a) - (t - b)(b - a)$ satisfies $F^\Delta(t) = \sigma(t) - a - (\sigma(t) - b) - (b - a) = 0$ and $F(b) = 0$. Hence $F = 0$ and so (1) holds. Next, (2) follows by letting $t = a$ in (1). Moreover, the function G defined by $G(t) := g_3(t, a) - g_3(t, b) - g_3(b, a) - (t - b)g_2(b, a) - (b - a)g_2(t, b)$ satisfies $G^\Delta(t) = g_2(\sigma(t), a) - g_2(\sigma(t), b) - g_2(b, a) - (b - a)(\sigma(t) - b) = F(\sigma(t)) = 0$ and $G(b) = 0$. Hence $G = 0$ and so (3) holds.

3 Auxiliary Results and Assumptions

Throughout this paper we assume that $\mathbb{T}$ is a time scale and that $a, b \in \mathbb{T}$ such that $a < b$. Moreover, when writing $[a, b]$, we mean the time scales interval $[a, b] \cap \mathbb{T}$. The following two Montgomery-type results are used in the proofs of our three main results.

Theorem 3.1 *Suppose* $f \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. *Let* $t \in [a, b]$ *and* $u_1, u_2 \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. *If*

$$u(\sigma(s)) = \begin{cases} u_1(\sigma(s)) & \textit{for} \quad a \leq s < t, \\ u_2(\sigma(s)) & \textit{for} \quad t \leq s \leq b, \end{cases} \tag{4}$$

then

$$\begin{aligned} \int_a^b u(\sigma(s))f^\Delta(s)\Delta s = (u_1(t) - u_2(t))f(t) - u_1(a)f(a) + u_2(b)f(b) \\ - \int_a^t u_1^\Delta(s)f(s)\Delta s - \int_t^b u_2^\Delta(s)f(s)\Delta s. \end{aligned} \tag{5}$$

Proof. We use Theorem 2.1 to split the integral into two parts, each of which is evaluated by applying the integration of parts formula, i.e.,

$$\begin{aligned}\int_a^b u(\sigma(s))f^\Delta(s)\Delta s &= \int_a^t u_1(\sigma(s))f^\Delta(s)\Delta s + \int_t^b u_2(\sigma(s))f^\Delta(s)\Delta s\\ &= u_1(t)f(t) - u_1(a)f(a) - \int_a^t u_1^\Delta(s)f(s)\Delta s\\ &\quad + u_2(b)f(b) - u_2(t)f(t) - \int_t^b u_2^\Delta(s)f(s)\Delta s,\end{aligned}$$

from which (5) follows.

Theorem 3.2 *Suppose $f \in \mathrm{C}^2_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. Let $t \in [a,b]$ and $u_i, v_i \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ be such that $u_i^\Delta(s) = v_i(\sigma(s))$ for all $s \in [a,b]$, where $i \in \{1,2\}$. If u satisfies (4), then*

$$\begin{aligned}\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s &= (u_1(t)-u_2(t))f^\Delta(t) - (v_1(t)-v_2(t))f(t)\\ &\quad - u_1(a)f^\Delta(a) + v_1(a)f(a) + u_2(b)f^\Delta(b) - v_2(b)f(b)\\ &\quad + \int_a^t v_1^\Delta(s)f(s)\Delta s + \int_t^b v_2^\Delta(s)f(s)\Delta s.\end{aligned} \tag{6}$$

Proof. Using (5) with f^Δ replaced by $f^{\Delta\Delta}$ and subsequently applying integration by parts twice, we obtain

$$\begin{aligned}\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s &= (u_1(t)-u_2(t))f^\Delta(t) - u_1(a)f^\Delta(a) + u_2(b)f^\Delta(b)\\ &\quad - \int_a^t u_1^\Delta(s)f^\Delta(s)\Delta s - \int_t^b u_2^\Delta(s)f^\Delta(s)\Delta s\\ &= (u_1(t)-u_2(t))f^\Delta(t) - u_1(a)f^\Delta(a) + u_2(b)f^\Delta(b)\\ &\quad - \int_a^t v_1(\sigma(s))f^\Delta(s)\Delta s - \int_t^b v_2(\sigma(s))f^\Delta(s)\Delta s\\ &= (u_1(t)-u_2(t))f^\Delta(t) - u_1(a)f^\Delta(a) + u_2(b)f^\Delta(b)\\ &\quad - \left\{v_1(t)f(t) - v_1(a)f(a) - \int_a^t v_1^\Delta(s)f(s)\Delta s\right\}\\ &\quad - \left\{v_2(b)f(b) - v_2(t)f(t) - \int_t^b v_2^\Delta(s)f(s)\Delta s\right\},\end{aligned}$$

from which (6) follows.

Assumption (H) For the remaining three sections of this paper, we assume that $\mathbb{T}$ is a time scale and that $a, b \in \mathbb{T}$ such that $a < b$. We assume that $f, g, h \in \mathrm{C}^2_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ are such that

$$\left\|f^{\Delta\Delta}\right\|_\infty := \sup_{t\in(a,b)} \left|f^{\Delta\Delta}(t)\right| < \infty, \quad \left\|g^{\Delta\Delta}\right\|_\infty < \infty, \quad \left\|h^{\Delta\Delta}\right\|_\infty < \infty \tag{7}$$

and define

$$
\begin{aligned}
A[f,g,h] &:= gh\int_a^b f(s)\Delta s + fh\int_a^b g(s)\Delta s + fg\int_a^b h(s)\Delta s,\\
B[f,g,h] &:= |gh|\left\|f^{\Delta\Delta}\right\|_\infty + |fh|\left\|g^{\Delta\Delta}\right\|_\infty + |fg|\left\|h^{\Delta\Delta}\right\|_\infty,\\
C[f,g,h] &:= ghf^\Delta + fhg^\Delta + fgh^\Delta,\\
D[f,g,h] &:= \left(\int_a^b g(s)h(s)\Delta s\right)\left(\int_a^b f(s)\Delta s\right) + \left(\int_a^b f(s)h(s)\Delta s\right)\left(\int_a^b g(s)\Delta s\right)\\
&\quad + \left(\int_a^b f(s)g(s)\Delta s\right)\left(\int_a^b h(s)\Delta s\right),\\
L[f,g,h] &:= gh\frac{g_2(b,a)f(a)+h_2(b,a)f(b)}{(b-a)^2} + fh\frac{g_2(b,a)g(a)+h_2(b,a)g(b)}{(b-a)^2}\\
&\quad + fg\frac{g_2(b,a)h(a)+h_2(b,a)h(b)}{(b-a)^2},\\
M[f,g,h] &:= gh\frac{f(b)-f(a)}{b-a} + fh\frac{g(b)-g(a)}{b-a} + fg\frac{h(b)-h(a)}{b-a}.
\end{aligned}
$$

4 Time Scales Version of Theorem 1.1

Theorem 4.1 *Assume* (H). *Then, for all $t\in[a,b]$, we have*

$$
\begin{aligned}
&\left|f(t)g(t)h(t) - \frac{1}{3(b-a)}A[f,g,h](t) - \frac{1}{3}\left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t)\right|\\
&\qquad \le \frac{1}{3}\left(h_2(b,t)+(t-b)\frac{g_2(b,a)}{b-a}+\frac{g_3(b,a)}{b-a}\right)B[f,g,h](t) \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)g(t)h(t)\Delta t - \frac{1}{3(b-a)^2}D[f,g,h]\right.\\
&\qquad\left. -\frac{1}{3(b-a)}\int_a^b\left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t)\Delta t\right|\\
&\quad \le \frac{1}{3(b-a)}\int_a^b\left(h_2(b,t)+(t-b)\frac{g_2(b,a)}{b-a}+\frac{g_3(b,a)}{b-a}\right)B[f,g,h](t)\Delta t. \qquad (9)
\end{aligned}
$$

Proof. Fix $t\in[a,b]$ and define u by (4), where

$$u_1(s)=g_2(s,a),\quad u_2(s)=h_2(b,s).$$

With the notation as in Theorem 3.2, using Definition 2.2, we have

$$v_1(s)=s-a,\quad v_2(s)=s-b,\quad v_1^\Delta(s)=v_2^\Delta(s)=1$$

and $u_1(a)=v_1(a)=u_2(b)=v_2(b)=0$. Moreover, we have

$$u_1(t)-u_2(t)\overset{(1)}{=}(t-b)(b-a)+g_2(b,a),\quad v_1(t)-v_2(t)=b-a.$$

By (6), we therefore obtain

$$\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s = ((t-b)(b-a)+g_2(b,a))f^\Delta(t) - (b-a)f(t) + \int_a^b f(s)\Delta s$$

and thus

$$f(t) = \frac{1}{b-a}\int_a^b f(s)\Delta s + \left(t-b+\frac{g_2(b,a)}{b-a}\right)f^\Delta(t) - \frac{1}{b-a}\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s. \quad (10)$$

Similarly, we get

$$g(t) = \frac{1}{b-a}\int_a^b g(s)\Delta s + \left(t-b+\frac{g_2(b,a)}{b-a}\right)g^\Delta(t) - \frac{1}{b-a}\int_a^b u(\sigma(s))g^{\Delta\Delta}(s)\Delta s \quad (11)$$

and

$$h(t) = \frac{1}{b-a}\int_a^b h(s)\Delta s + \left(t-b+\frac{g_2(b,a)}{b-a}\right)h^\Delta(t) - \frac{1}{b-a}\int_a^b u(\sigma(s))h^{\Delta\Delta}(s)\Delta s. \quad (12)$$

Multiplying (10), (11) and (12) by $g(t)h(t)$, $f(t)h(t)$ and $f(t)g(t)$, respectively, adding the resulting identities and dividing by three, we have

$$\begin{aligned} f(t)g(t)h(t) - \frac{1}{3(b-a)}A[f,g,h](t) - \frac{1}{3}\left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t) \\ = -\frac{1}{3(b-a)}\int_a^b u(\sigma(s))\tilde{B}[f,g,h](t,s)\Delta s, \quad (13)\end{aligned}$$

where

$$\begin{cases} \tilde{B}[f,g,h](t,s) := g(t)h(t)f^{\Delta\Delta}(s) + f(t)h(t)g^{\Delta\Delta}(s) + f(t)g(t)h^{\Delta\Delta}(s) \\ \text{so that} \quad \left|\tilde{B}[f,g,h](t,s)\right| \le B[f,g,h](t). \end{cases} \quad (14)$$

By taking absolute values in (13) and using (7) and

$$\begin{aligned} \int_a^b |u(\sigma(s))|\Delta s &= \int_a^t g_2(\sigma(s),a)\Delta s + \int_t^b h_2(b,\sigma(s))\Delta s \quad (15) \\ &= g_3(t,a) - g_3(t,b) \\ &\overset{(3)}{=} g_3(b,a) + (t-b)g_2(b,a) + (b-a)h_2(b,t), \end{aligned}$$

we obtain (8). Integrating (13) with respect to t from a to b, dividing by $b-a$, noting that

$$\int_a^b A[f,g,h](s)\Delta s = D[f,g,h], \quad (16)$$

taking absolute values and using (7) and (15), we obtain (9).

Example 4.1 If we let $\mathbb{T} = \mathbb{R}$ in Theorem 4.1, then, since $C[f,g,h] = (fgh)'$,

$$b - \frac{g_2(b,a)}{b-a} = b - \frac{(b-a)^2}{2(b-a)} = b - \frac{b-a}{2} = \frac{a+b}{2}$$

and

$$\begin{aligned} h_2(b,t) + (t-b)\frac{g_2(b,a)}{b-a} + \frac{g_3(b,a)}{b-a} &= \frac{1}{2}\left\{(t-b)^2 + (t-b)(b-a) + \frac{(b-a)^2}{3}\right\} \\ &= \frac{1}{2}\left\{\left(t-b+\frac{b-a}{2}\right)^2 - \frac{(b-a)^2}{4} + \frac{(b-a)^2}{3}\right\} \\ &= \frac{1}{2}\left\{\left(t-\frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12}\right\}, \end{aligned}$$

we obtain [18, Theorem 1], in particular, Theorem 1.1.

Example 4.2 If we let $\mathbb{T} = \mathbb{Z}$ and $a = 0$, $b = n \in \mathbb{N}$ in Theorem 4.1, then, since

$$b - \frac{g_2(b,a)}{b-a} = b - \frac{(b-a)(b-a+1)}{2(b-a)} = b - \frac{b-a+1}{2} = \frac{a+b-1}{2} = \frac{n-1}{2}$$

and

$$\begin{aligned} & h_2(b,t) + (t-b)\frac{g_2(b,a)}{b-a} + \frac{g_3(b,a)}{b-a} \\ &= \frac{1}{2}\left\{(b-t)(b-t-1) + (t-b)(b-a+1) + \frac{(b-a+1)(b-a+2)}{3}\right\} \\ &= \frac{1}{2}\left\{\left(t-b+\frac{b-a+2}{2}\right)^2 - \frac{(b-a+2)^2}{4} + \frac{(b-a+1)(b-a+2)}{3}\right\} \\ &= \frac{1}{2}\left\{\left(t+1-\frac{a+b}{2}\right)^2 + \frac{(b-a+2)(b-a-2)}{12}\right\} \\ &= \frac{1}{2}\left\{\left(t+1-\frac{n}{2}\right)^2 + \frac{n^2-4}{12}\right\}, \end{aligned}$$

we obtain

$$\begin{aligned} &\left| f(t)g(t)h(t) - \frac{1}{3n}A[f,g,h](t) - \frac{1}{3}\left(t-\frac{n-1}{2}\right)C[f,g,h](t)\right| \\ &\qquad \le \frac{1}{6}\left\{\left(t+1-\frac{n}{2}\right)^2 + \frac{n^2-4}{12}\right\}B[f,g,h](t) \end{aligned}$$

and

$$\begin{aligned} &\left| \frac{1}{n}\sum_{t=0}^{n-1} f(t)g(t)h(t) - \frac{1}{3n^2}D[f,g,h] - \frac{1}{3n}\sum_{t=0}^{n-1}\left(t-\frac{n-1}{2}\right)C[f,g,h](t)\right| \\ &\qquad \le \frac{1}{6n}\sum_{t=0}^{n-1}\left\{\left(t+1-\frac{n}{2}\right)^2 + \frac{n^2-4}{12}\right\}B[f,g,h](t), \end{aligned}$$

where

$$
\begin{aligned}
A[f,g,h] &= gh\sum_{s=0}^{n-1} f(s) + fh\sum_{s=0}^{n-1} g(s) + fg\sum_{s=0}^{n-1} h(s),\\
B[f,g,h] &= |gh| \max_{1\le s\le n-1}\left|\Delta^2 f(s)\right| + |fh| \max_{1\le s\le n-1}\left|\Delta^2 g(s)\right| + |fg| \max_{1\le s\le n-1}\left|\Delta^2 h(s)\right|,\\
C[f,g,h] &= gh\Delta f + fh\Delta g + fg\Delta h,\\
D[f,g,h] &= \left(\sum_{s=0}^{n-1} g(s)h(s)\right)\left(\sum_{s=0}^{n-1} f(s)\right) + \left(\sum_{s=0}^{n-1} f(s)h(s)\right)\left(\sum_{s=0}^{n-1} g(s)\right)\\
&\quad + \left(\sum_{s=0}^{n-1} f(s)g(s)\right)\left(\sum_{s=0}^{n-1} h(s)\right).
\end{aligned}
$$

These inequalities are new discrete Ostrowski–Grüss type inequalities.

Remark 4.1 If we let $h(t) \equiv 1$ in Theorem 4.1, then (8) becomes

$$
\begin{aligned}
\Bigg| f(t)g(t) - \frac{1}{2(b-a)}\left\{g(t)\int_a^b f(s)\Delta s + f(t)\int_a^b g(s)\Delta s\right\}\\
-\frac{1}{2}\left(t-b+\frac{g_2(b,a)}{b-a}\right)\left\{g(t)f^{\Delta}(t) + f(t)g^{\Delta}(t)\right\}\Bigg|\\
\le \frac{1}{2}\left(h_2(b,t) + (t-b)\frac{g_2(b,a)}{b-a} + \frac{g_3(b,a)}{b-a}\right)\left\{|g(t)|\left\|f^{\Delta\Delta}\right\|_\infty + |f(t)|\left\|g^{\Delta\Delta}\right\|_\infty\right\}
\end{aligned}
$$

and (9) turns into

$$
\begin{aligned}
\Bigg| \frac{1}{b-a}\int_a^b f(t)g(t)\Delta t - \frac{1}{(b-a)^2}\left(\int_a^b f(t)\Delta t\right)\left(\int_a^b g(t)\Delta t\right)\\
-\frac{1}{2(b-a)}\int_a^b\left(t-b+\frac{g_2(b,a)}{b-a}\right)\left\{g(t)f^{\Delta}(t) + f(t)g^{\Delta}(t)\right\}\Delta t\Bigg|\\
\le \frac{1}{2(b-a)}\int_a^b\left(h_2(b,t) + (t-b)\frac{g_2(b,a)}{b-a} + \frac{g_3(b,a)}{b-a}\right)\cdot\\
\cdot\left\{|g(t)|\left\|f^{\Delta\Delta}\right\|_\infty + |f(t)|\left\|g^{\Delta\Delta}\right\|_\infty\right\}\Delta t.
\end{aligned}
$$

If, moreover, we let $g(t) \equiv 1$, then (8) becomes

$$
\begin{aligned}
\left| f(t) - \frac{1}{b-a}\int_a^b f(s)\Delta s - \left(t-b+\frac{g_2(b,a)}{b-a}\right)f^{\Delta}(t)\right|\\
\le \left(h_2(b,t) + (t-b)\frac{g_2(b,a)}{b-a} + \frac{g_3(b,a)}{b-a}\right)\left\|f^{\Delta\Delta}\right\|_\infty.
\end{aligned}
$$

From these inequalities, special cases such as discrete inequalities can be obtained.

5 Time Scales Version of Theorem 1.2

Theorem 5.1 *Assume* (H). *Then, for all* $t \in [a,b]$, *we have*

$$\Bigg| f(t)g(t)h(t) - \frac{2}{3(b-a)}A[f,g,h](t) + \frac{1}{3}L[f,g,h](t) \\ -\frac{1}{3}\left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t)\Bigg| \le \frac{1}{3(b-a)}B[f,g,h](t)I(t) \quad (17)$$

and

$$\Bigg| \frac{1}{b-a}\int_a^b f(t)g(t)h(t)\Delta t - \frac{2}{3(b-a)^2}D[f,g,h] + \frac{1}{3(b-a)}\int_a^b L[f,g,h](t)\Delta t \\ -\frac{1}{3(b-a)}\int_a^b \left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t)\Delta t\Bigg| \\ \le \frac{1}{3(b-a)^2}\int_a^b B[f,g,h](t)I(t)\Delta t, \quad (18)$$

where

$$I(t) := \frac{1}{b-a}\int_a^t |2(b-a)g_2(\sigma(s),a) - (\sigma(s)-a)g_2(b,a)|\,\Delta s \\ + \frac{1}{b-a}\int_t^b |2(b-a)h_2(b,\sigma(s)) - (b-\sigma(s))h_2(b,a)|\,\Delta s.$$

Proof. Fix $t \in [a,b]$ and define u by (4), where

$$u_1(s) = 2(b-a)g_2(s,a) - (s-a)g_2(b,a), \quad u_2(s) = 2(b-a)h_2(b,s) - (b-s)h_2(b,a).$$

With the notation as in Theorem 3.2, using Definition 2.2, we have

$$v_1(s) = 2(b-a)(s-a) - g_2(b,a), \quad v_2(s) = 2(b-a)(s-b) + h_2(b,a), \\ v_1^\Delta(s) = v_2^\Delta(s) = 2(b-a)$$

and $u_1(a) = u_2(b) = 0$, $v_1(a) = -g_2(b,a)$, $v_2(b) = h_2(b,a)$. Moreover, we have

$$\begin{aligned} u_1(t) - u_2(t) &= 2(b-a)(g_2(t,a) - h_2(b,t)) - (t-a)g_2(b,a) + (b-t)h_2(b,a) \\ &\overset{(1),(2)}{=} 2(b-a)(g_2(b,a) + (t-b)(b-a)) \\ &\quad -(t-a)g_2(b,a) + (b-t)\left((b-a)^2 - g_2(b,a)\right) \\ &\overset{(2)}{=} (b-a)g_2(b,a) + (t-b)(b-a)^2, \\ v_1(t) - v_2(t) &= 2(b-a)^2 - g_2(b,a) - h_2(b,a) \\ &\overset{(2)}{=} 2(b-a)^2 - (b-a)^2 = (b-a)^2. \end{aligned}$$

By (6), we therefore obtain

$$\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s = (b-a)\left(g_2(b,a) + (t-b)(b-a)\right)f^\Delta(t) \\ - (b-a)^2 f(t) - g_2(b,a)f(a) - h_2(b,a)f(b) + 2(b-a)\int_a^b f(s)\Delta s$$

and thus

$$f(t) = \frac{2}{b-a}\int_a^b f(s)\Delta s - \frac{g_2(b,a)f(a)+h_2(b,a)f(b)}{(b-a)^2} + \left(t-b+\frac{g_2(b,a)}{b-a}\right)f^{\Delta}(t) - \frac{1}{(b-a)^2}\int_a^b u(\sigma(s))f^{\Delta\Delta}(s)\Delta s. \quad (19)$$

Similarly, we get

$$g(t) = \frac{2}{b-a}\int_a^b g(s)\Delta s - \frac{g_2(b,a)g(a)+h_2(b,a)g(b)}{(b-a)^2} + \left(t-b+\frac{g_2(b,a)}{b-a}\right)g^{\Delta}(t) - \frac{1}{(b-a)^2}\int_a^b u(\sigma(s))g^{\Delta\Delta}(s)\Delta s \quad (20)$$

and

$$h(t) = \frac{2}{b-a}\int_a^b h(s)\Delta s - \frac{g_2(b,a)h(a)+h_2(b,a)h(b)}{(b-a)^2} + \left(t-b+\frac{g_2(b,a)}{b-a}\right)h^{\Delta}(t) - \frac{1}{(b-a)^2}\int_a^b u(\sigma(s))h^{\Delta\Delta}(s)\Delta s. \quad (21)$$

Multiplying (19), (20) and (21) by $g(t)h(t)$, $f(t)h(t)$ and $f(t)g(t)$, respectively, adding the resulting identities and dividing by three, we have

$$f(t)g(t)h(t) - \frac{2}{3(b-a)}A[f,g,h](t) + \frac{1}{3}L[f,g,h](t) - \frac{1}{3}\left(t-b+\frac{g_2(b,a)}{b-a}\right)C[f,g,h](t) = -\frac{1}{3(b-a)^2}\int_a^b u(\sigma(s))\tilde{B}[f,g,h](t,s)\Delta s \quad (22)$$

with $\tilde{B}$ as in (14). By taking absolute values in (22) and using (7) and

$$\frac{1}{b-a}\int_a^b |u(\sigma(s))|\Delta s = I(t), \quad (23)$$

we obtain (17). Integrating (22) with respect to t from a to b, dividing by $b-a$, noting (16), taking absolute values and using (7) and (23), we obtain (18).

Example 5.1 If we let $\mathbb{T}=\mathbb{R}$ in Theorem 5.1, then, since $C[f,g,h]=(fgh)'$,

$$b - \frac{g_2(b,a)}{b-a} = \frac{a+b}{2}$$

and (with p as defined in Theorem 1.2)

$$\begin{aligned} I(t) &= \frac{1}{b-a}\int_a^t \left|(b-a)(s-a)^2 - (s-a)\frac{(b-a)^2}{2}\right|\mathrm{d}s \\ &\quad + \frac{1}{b-a}\int_t^b \left|(b-a)(s-b)^2 - (b-s)\frac{(b-a)^2}{2}\right|\mathrm{d}s \\ &= \int_a^t \left|(s-a)\left(s-\frac{a+b}{2}\right)\right|\mathrm{d}s + \int_t^b \left|(s-b)\left(s-\frac{a+b}{2}\right)\right|\mathrm{d}s \\ &= \int_a^b \left|p(t,s)\left(s-\frac{a+b}{2}\right)\right|\mathrm{d}s, \end{aligned}$$

we obtain [18, Theorem 2], in particular, Theorem 1.2.

Example 5.2 If we let $\mathbb{T} = \mathbb{Z}$ and $a = 0$, $b = n \in \mathbb{N}$ in Theorem 5.1, then, since

$$b - \frac{g_2(b,a)}{b-a} = \frac{n-1}{2}$$

and

$$\begin{aligned} I(t) &= \frac{1}{b-a}\sum_{s=a}^{t-1}\left|(b-a)(s+1-a)(s+2-a) - (s+1-a)\frac{(b-a)(b-a+1)}{2}\right| \\ &\quad + \frac{1}{b-a}\sum_{s=t}^{b-1}\left|(b-a)(b-s-1)(b-s-2) - (b-s-1)\frac{(b-a)(b-a-1)}{2}\right| \\ &= \sum_{s=a}^{t-1}\left|(s+1-a)\left(s+1-\frac{a+b-1}{2}\right)\right| \\ &\quad + \sum_{s=t}^{b-1}\left|(s+1-b)\left(s+1-\frac{a+b-1}{2}\right)\right| \\ &= \sum_{s=0}^{t-1}\left|(s+1)\left(s+1-\frac{n-1}{2}\right)\right| + \sum_{s=t}^{n-1}\left|(s+1-n)\left(s+1-\frac{n-1}{2}\right)\right|, \end{aligned}$$

we have

$$\begin{aligned} &\left|f(t)g(t)h(t) - \frac{2}{3n}A[f,g,h](t) + \frac{1}{3}L[f,g,h](t) - \frac{1}{3}\left(t-\frac{n-1}{2}\right)C[f,g,h](t)\right| \\ &\qquad \le \frac{1}{3n}B[f,g,h](t)\left\{\sum_{s=1}^{t} s\left|s-\frac{n-1}{2}\right| + \sum_{s=t+1}^{n}(n-s)\left|s-\frac{n-1}{2}\right|\right\} \end{aligned}$$

and

$$\begin{aligned} &\left|\frac{1}{n}\sum_{t=0}^{n-1} f(t)g(t)h(t) - \frac{2}{3n^2}D[f,g,h] \right.\\ &\qquad \left. + \frac{1}{3n}\sum_{t=0}^{n-1}L[f,g,h](t) - \frac{1}{3n}\sum_{t=0}^{n-1}\left(t-\frac{n-1}{2}\right)C[f,g,h](t)\right| \\ &\qquad \le \frac{1}{3n^2}\sum_{t=0}^{n-1}B[f,g,h](t)\left\{\sum_{s=1}^{t} s\left|s-\frac{n-1}{2}\right| + \sum_{s=t+1}^{n}(n-s)\left|s-\frac{n-1}{2}\right|\right\}, \end{aligned}$$

where in addition to A, B, C, D defined in Example 4.2,

$$\begin{aligned} L[f,g,h] &= gh\frac{(n+1)f(a)+(n-1)f(b)}{2n} + fh\frac{(n+1)g(a)+(n-1)g(b)}{2n} \\ &\quad + fg\frac{(n+1)h(a)+(n-1)h(b)}{2n}. \end{aligned}$$

These inequalities are new discrete Ostrowski–Grüss type inequalities.

Remark 5.1 If we let $h(t) \equiv 1$ in Theorem 5.1, then (17) becomes

$$\begin{aligned}
&\Bigg| f(t)g(t) - \frac{1}{b-a}\left\{ g(t)\int_a^b f(s)\Delta s + f(t)\int_a^b g(s)\Delta s \right\} \\
&\qquad + g(t)\frac{g_2(b,a)f(a)+h_2(b,a)f(b)}{2(b-a)^2} + f(t)\frac{g_2(b,a)g(a)+h_2(b,a)g(b)}{2(b-a)^2} \\
&\qquad\qquad - \frac{1}{2}\left(t - b + \frac{g_2(b,a)}{b-a} \right)\left\{ g(t)f^{\Delta}(t) + f(t)g^{\Delta}(t) \right\} \Bigg| \\
&\qquad\qquad\qquad \leq \frac{1}{2(b-a)}\left\{ |g(t)| \left\| f^{\Delta\Delta} \right\|_\infty + |f(t)| \left\| g^{\Delta\Delta} \right\|_\infty \right\} I(t),
\end{aligned}$$

(observe (2) when calculating L) and (18) turns into

$$\begin{aligned}
&\Bigg| \frac{1}{b-a}\int_a^b f(t)g(t)\Delta t - \frac{2}{(b-a)^2}\left(\int_a^b f(t)\Delta t \right)\left(\int_a^b g(t)\Delta t \right) \\
&\quad + \frac{1}{b-a}\int_a^b \left\{ g(t)\frac{g_2(b,a)f(a)+h_2(b,a)f(b)}{2(b-a)^2} + f(t)\frac{g_2(b,a)g(a)+h_2(b,a)g(b)}{2(b-a)^2} \right\} \Delta t \\
&\qquad - \frac{1}{2(b-a)}\int_a^b \left(t - b + \frac{g_2(b,a)}{b-a} \right)\left\{ g(t)f^{\Delta}(t) + f(t)g^{\Delta}(t) \right\} \Delta t \Bigg| \\
&\qquad\qquad \leq \frac{1}{2(b-a)^2}\int_a^b \left\{ |g(t)| \left\| f^{\Delta\Delta} \right\|_\infty + |f(t)| \left\| g^{\Delta\Delta} \right\|_\infty \right\} I(t)\Delta t.
\end{aligned}$$

If, moreover, we let $g(t) \equiv 1$, then (17) becomes

$$\begin{aligned}
&\Bigg| f(t) - \frac{2}{b-a}\int_a^b f(s)\Delta s + \frac{g_2(b,a)f(a)+h_2(b,a)f(b)}{(b-a)^2} \\
&\qquad - \left(t - b + \frac{g_2(b,a)}{b-a} \right) f^{\Delta}(t) \Bigg| \leq \frac{1}{b-a}\left\| f^{\Delta\Delta} \right\|_\infty I(t).
\end{aligned}$$

From these inequalities, special cases such as discrete inequalities can be obtained.

6 Time Scales Version of Theorem 1.3

Theorem 6.1 *Assume* (H). *Then, for all* $t \in [a,b]$, *we have*

$$\begin{aligned}
&\left| f(t)g(t)h(t) - \frac{1}{3(b-a)}A[f,g,h](t) - \frac{1}{3}\left(t - b + \frac{g_2(b,a)}{b-a} \right) M[f,g,h](t) \right| \\
&\qquad \leq \frac{1}{3(b-a)^2}B[f,g,h](t)H(t) \qquad (24)
\end{aligned}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)h(t)\Delta t - \frac{1}{3(b-a)^2}D[f,g,h](t)\right.$$
$$\left.-\frac{1}{3(b-a)}\int_a^b \left(t-b+\frac{g_2(b,a)}{b-a}\right) M[f,g,h](t)\Delta t\right| \le \frac{1}{3(b-a)^3}\int_a^b B[f,g,h](t)H(t)\Delta t, \tag{25}$$

where

$$H(t) := \int_a^b \int_a^b |p(t,\tau)p(\tau,s)|\,\Delta s\Delta\tau$$

and

$$p(t,s) := \begin{cases} \sigma(s)-a & \textit{for} \quad a \le s < t, \\ \sigma(s)-b & \textit{for} \quad t \le s \le b. \end{cases}$$

Proof. Fix $t \in [a,b]$. We use Theorem 3.1 three times to obtain

$$\begin{aligned}
&\int_a^b \int_a^b p(t,\tau)p(\tau,s)f^{\Delta\Delta}(s)\Delta s\Delta\tau = \int_a^b p(t,\tau)\left\{\int_a^b p(\tau,s)f^{\Delta\Delta}(s)\Delta s\right\}\Delta\tau \\
&= \int_a^b p(t,\tau)\left\{(b-a)f^{\Delta}(\tau) - \int_a^b f^{\Delta}(s)\Delta s\right\}\Delta\tau \\
&= (b-a)\int_a^b p(t,s)f^{\Delta}(s)\Delta s + (f(a)-f(b))\int_a^b p(t,s)\Delta s \\
&= (b-a)\left\{(b-a)f(t) - \int_a^b f(s)\Delta s\right\} + (f(a)-f(b))\left\{(b-a)t - \int_a^b s\Delta s\right\} \\
&= (b-a)^2 f(t) - (b-a)\int_a^b f(s)\Delta s + (f(a)-f(b))\int_b^a (s-t)\Delta s \\
&= (b-a)^2 f(t) - (b-a)\int_a^b f(s)\Delta s + (g_2(t,a)-h_2(b,t))(f(a)-f(b))
\end{aligned}$$

and thus (by using (1))

$$f(t) = \frac{1}{b-a}\int_a^b f(s)\Delta s + \left(t-b+\frac{g_2(b,a)}{b-a}\right)\frac{f(b)-f(a)}{b-a} + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)p(\tau,s)f^{\Delta\Delta}(s)\Delta s\Delta\tau. \tag{26}$$

Similarly, we get

$$g(t) = \frac{1}{b-a}\int_a^b g(s)\Delta s + \left(t-b+\frac{g_2(b,a)}{b-a}\right)\frac{g(b)-g(a)}{b-a} + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)p(\tau,s)g^{\Delta\Delta}(s)\Delta s\Delta\tau. \tag{27}$$

and

$$h(t) = \frac{1}{b-a}\int_a^b h(s)\Delta s + \left(t - b + \frac{g_2(b,a)}{b-a}\right)\frac{h(b)-h(a)}{b-a} + \frac{1}{(b-a)^2}\int_a^b\int_a^b p(t,\tau)p(\tau,s)h^{\Delta\Delta}(s)\Delta s\Delta\tau. \quad (28)$$

Multiplying (26), (27) and (28) by $g(t)h(t)$, $f(t)h(t)$ and $f(t)g(t)$, respectively, adding the resulting identities and dividing by three, we have

$$f(t)g(t)h(t) - \frac{1}{3(b-a)}A[f,g,h](t) - \frac{1}{3}\left(t - b + \frac{g_2(b,a)}{b-a}\right)M[f,g,h](t) = \frac{1}{3(b-a)^2}\int_a^b\int_a^b p(t,\tau)p(\tau,s)\tilde{B}[f,g,h](t,s)\Delta s\Delta\tau \quad (29)$$

with $\tilde{B}$ as in (14). By taking absolute values in (29) and using (7) and the definition of H, we obtain (24). Integrating (29) with respect to t from a to b, dividing by $b-a$, noting (16), taking absolute values and using (7) and the definition of H, we obtain (25).

Example 6.1 If we let $\mathbb{T} = \mathbb{R}$ in Theorem 6.1, then, by the same calculations as in Example 4.1, we obtain [18, Theorem 3], in particular, Theorem 1.3.

Example 6.2 If we let $\mathbb{T} = \mathbb{Z}$ and $a = 0$, $b = n \in \mathbb{N}$ in Theorem 6.1, then, by the same calculations as in Example 4.2, we obtain

$$\left|f(t)g(t)h(t) - \frac{1}{3n}A[f,g,h](t) - \frac{1}{3}\left(t - \frac{n-1}{2}\right)M[f,g,h](t)\right| \le \frac{1}{3n^2}B[f,g,h](t)H(t)$$

and

$$\left|\frac{1}{n}\sum_{t=0}^{n-1} f(t)g(t)h(t) - \frac{1}{3n^2}D[f,g,h] - \frac{1}{3n}\sum_{t=0}^{n-1}\left(t - \frac{n-1}{2}\right)M[f,g,h](t)\right| \le \frac{1}{3n^3}\sum_{t=0}^{n-1} B[f,g,h](t)H(t),$$

where in addition to A, B, D defined in Example 4.2,

$$M[f,g,h] = gh\frac{f(b)-f(a)}{b-a} + fh\frac{g(b)-g(a)}{b-a} + fg\frac{h(b)-h(a)}{b-a},$$

$$H(t) = \sum_{\tau=0}^{n-1}\sum_{s=0}^{n-1}\left|p(t,\tau)p(\tau,s)\right|,$$

$$p(t,s) = \begin{cases} s+1, & \text{if} \quad 0 \le s < t, \\ s+1-n, & \text{if} \quad t \le s \le n. \end{cases}$$

These inequalities are new discrete Ostrowski–Grüss type inequalities.

Remark 6.1 If we let $h(t) \equiv 1$ in Theorem 6.1, then (24) becomes

$$\begin{aligned}\Bigg| f(t)g(t) - \frac{1}{2(b-a)} &\left\{ g(t) \int_a^b f(s)\Delta s + f(t) \int_a^b g(s)\Delta s \right\} \\ &-\frac{1}{2}\left(t - b + \frac{g_2(b,a)}{(b-a)}\right)\left\{ g(t)\frac{f(b)-f(a)}{b-a} + f(t)\frac{g(b)-g(a)}{b-a} \right\}\Bigg| \\ &\qquad \le \frac{1}{2(b-a)^2}\left\{ |g(t)| \left\| f^{\Delta\Delta} \right\|_\infty + |f(t)| \left\| g^{\Delta\Delta} \right\|_\infty \right\} H(t)\end{aligned}$$

and (25) turns into

$$\begin{aligned}\Bigg| \frac{1}{b-a} \int_a^b f(t)g(t)\Delta t &- \frac{1}{(b-a)^2}\left(\int_a^b f(t)\Delta t\right)\left(\int_a^b g(t)\Delta t\right) \\ &-\frac{1}{2(b-a)} \int_a^b \left(t - b + \frac{g_2(b,a)}{b-a}\right)\left\{ g(t)\frac{f(b)-f(a)}{b-a} + f(t)\frac{g(b)-g(a)}{b-a} \right\}\Delta t\Bigg| \\ &\qquad \le \frac{1}{2(b-a)^3}\int_a^b \left\{ |g(t)| \left\| f^{\Delta\Delta} \right\|_\infty + |f(t)| \left\| g^{\Delta\Delta} \right\|_\infty \right\} H(t)\Delta t.\end{aligned}$$

If, moreover, we let $g(t) \equiv 1$, then (24) becomes

$$\begin{aligned}\left| f(t) - \frac{1}{b-a}\int_a^b f(s)\Delta s - \left(t - b + \frac{g_2(b,a)}{(b-a)}\right)\frac{f(b)-f(a)}{b-a} \right| \\ \le \frac{1}{(b-a)^2}\left\| f^{\Delta\Delta} \right\|_\infty H(t).\end{aligned}$$

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On the Existence of a Common Lyapunov Function for a Family of Nonlinear Mechanical Systems with One Degree of Freedom

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Abstract: Certain classes of essentially nonlinear switched mechanical systems with one degree of freedom are investigated. The conditions are obtained under which, for the families of subsystems corresponding to switched systems, there exist common Lyapunov functions of the prescribed form. The fulfilment of these conditions provides the asymptotic stability of equilibrium positions of switched systems for any switching law.

Keywords: *switched mechanical systems; nonlinear forces; asymptotic stability; common Lyapunov function.*

Mathematics Subject Classification (2010): 34A38, 34D20, 70H14.

1 Introduction

Stability analysis and synthesis of switched systems are fundamental and challenging research problems, see, for example, [4, 7, 11]. In some cases it is required to design a control system in such a way that it remains stable for any admissible switching law [7, 11]. These cases are natural, when switching signal is either unknown, or too complicated to be explicitly taken into account.

A general approach to the above problem is based on the computation of a common Lyapunov function (CLF) for a family of subsystems corresponding to the switched system. This approach has been effectively used in many papers, see [4, 7–9, 11]. However, the conditions of the existence of a CLF are not completely investigated even for the case of families of linear time-invariant systems [7–9].

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This problem is especially complicated for mechanical systems with switching force fields. Motion of mechanical systems is described usually by differential equations of the second order, that results in the appearance of some special properties. In [2], it was mentioned that the known conditions of the existence of CLFs obtained for systems of general form might be ineffective or even nonapplicable for switched mechanical systems. The specific character of mechanical systems leads to the necessity of the separate investigation of such systems as a special subclass of hybrid systems. This subclass possesses certain theoretical features and is of undoubted practical interest [3–5, 11].

In the present paper, certain types of switched nonlinear mechanical systems with one degree of freedom are studied. The conditions of the existence of CLFs for families of subsystems corresponding to switched systems are obtained. The fulfilment of these conditions provides that the equilibrium positions of the considered systems are asymptotically stable for arbitrary switching law.

2 Statement of the Problem

First, consider the linear switched mechanical system with one degree of freedom

$$\ddot{x} + a_\sigma \dot{x} + b_\sigma x = 0. \tag{1}$$

Here scalar variable $x(t)$ is the state of the system; $\sigma = \sigma(t)$ is the piecewise constant function defining the switching law, $\sigma(t) : [0, +\infty) \to Q = \{1, \dots, N\}$. In the present paper, we assume that on every bounded time interval the switching function has a finite number of discontinuities, which are called switching instants of time, and takes a constant value on every interval between two consecutive switching instants. This kind of switching law is called admissible one.

Thus, at each time instant, the behaviour of (1) is described by one of the subsystems

$$\ddot{x} + a_s \dot{x} + b_s x = 0, \quad s = 1, \dots, N, \tag{2}$$

where a_s and b_s are constant coefficients.

Let the inequalities $a_s > 0$, $b_s > 0$, $s = 1, \dots, N$, be fulfilled. Then, for every subsystem from the family (2), the equilibrium position $x = \dot{x} = 0$ is asymptotically stable. In spite of this fact, it is well known [4, 7] that there exist parameters a_s and b_s values and switching laws under which the equilibrium position $x = \dot{x} = 0$ of the corresponding switched system (1) is unstable. It is worthy of note that instability can take place even in the case where family (2) consists of two subsystems ($N = 2$), and switching occurs only in the positional forces ($a_1 = a_2 = \text{const} > 0$).

In the present paper, we consider the nonlinear switched system

$$\ddot{x} + a_\sigma \dot{x} + b_\sigma x^\mu = 0 \tag{3}$$

and the corresponding family of subsystems

$$\ddot{x} + a_s \dot{x} + b_s x^\mu = 0, \quad s = 1, \dots, N. \tag{4}$$

Here the switching function $\sigma(t)$ possesses the same properties as in (1); a_s and b_s are positive constants; μ is a rational number with odd numerator and denominator, $\mu > 1$. Thus, subsystems from the family (4) are subjected to linear dissipative forces and essentially nonlinear potential forces. It is known [10] that the equilibrium position $x = \dot{x} = 0$ of each subsystem is asymptotically stable.

We will look for the conditions providing the asymptotic stability of the equilibrium position $x = \dot{x} = 0$ of (3) for any admissible switching law. To solve the problem, we consider the Lyapunov function of a special form and determine the region of the parameters a_s and b_s values under which CLF of the prescribed form can be constructed for the family of subsystems (4).

Furthermore, we extend the obtained results to the case of switched mechanical system with nonlinear dissipative and potential forces.

3 Conditions of the Existence of a CLF

Consider the Lyapunov function

$$V(x,\dot{x}) = \frac{\dot{x}^2}{2} + c\frac{x^{\mu+1}}{\mu+1} + \gamma x^{\beta}\dot{x}. \tag{5}$$

Here c and γ are positive constants, and β is a rational number with odd numerator and denominator, $\beta \geq 1$.

Differentiating $V(x,\dot{x})$ with respect to the sth subsystem from family (4), we obtain

$$\dot{V} = -a_s\dot{x}^2 - \gamma b_s x^{2\mu} + (c-b_s)x^{\mu}\dot{x} - a_s\gamma x^{\beta}\dot{x} + \gamma\beta x^{\beta-1}\dot{x}^2 \equiv W_s(x,\dot{x}).$$

By the use of generalized homogeneous functions properties [12], one gets the following necessary condition of the negative definiteness of functions $W_1(x,\dot{x}),\dots,W_N(x,\dot{x})$:

$$\beta = \mu. \tag{6}$$

For such value of the parameter β, the Lyapunov function (5) is positive definite for any $c > 0$ and $\gamma > 0$, and functions $W_1(x,\dot{x}),\dots,W_N(x,\dot{x})$ are negative definite if and only if the quadratic forms

$$\omega_s(y_1,y_2) = -a_s y_2^2 - \gamma b_s y_1^2 + (c - b_s - a_s\gamma)y_1y_2, \quad s = 1,\dots,N, \tag{7}$$

possess the same property.

Applying the Sylvester criterion, we obtain $4a_sb_s\gamma > (c - b_s - a_s\gamma)^2$, $s = 1,\dots,N$. Hence, the inequalities

$$\left(\sqrt{a_s\gamma} - \sqrt{b_s}\right)^2 < c < \left(\sqrt{a_s\gamma} + \sqrt{b_s}\right)^2, \quad s = 1,\dots,N,$$

should be valid. It means that, for the existence of the required value of the parameter c, it is necessary and sufficient the fulfilment of the conditions

$$\left(\sqrt{a_s\gamma} - \sqrt{b_s}\right)^2 < \left(\sqrt{a_j\gamma} + \sqrt{b_j}\right)^2, \quad s,j = 1,\dots,N. \tag{8}$$

Conditions (8) can be rewritten in the form

$$\sqrt{\gamma}(\sqrt{a_s} + \sqrt{a_j}) > \sqrt{b_s} - \sqrt{b_j}, \qquad \sqrt{\gamma}(\sqrt{a_s} - \sqrt{a_j}) < \sqrt{b_s} + \sqrt{b_j}, \qquad s,j = 1,\dots,N.$$

Denote

$$A = \max_{s,j=1,\dots,N} \frac{\sqrt{b_s} - \sqrt{b_j}}{\sqrt{a_s} + \sqrt{a_j}},$$

$B = +\infty$ if $a_s = a_j$ for all $s, j = 1, \dots, N$, and

$$B = \min_{s,j:\ a_s > a_j} \frac{\sqrt{b_s} + \sqrt{b_j}}{\sqrt{a_s} - \sqrt{a_j}}$$

otherwise.

Finally, we arrive at

Theorem 3.1 *Family* (4) *admits a CLF of the form* (5) *satisfying the assumptions of the Lyapunov asymptotic stability theorem if and only if the inequality*

$$A < B \tag{9}$$

holds.

Remark 3.1 Theorem 3.1 gives us the constructive algorithm for finding the CLF for family (4). If inequality (9) is fulfilled, then the value of parameter β is determined by formula (6), while $\gamma \in (A, B)$, and, for the value of γ chosen from this interval, $c \in (\underline{c}(\gamma), \overline{c}(\gamma))$, where

$$\underline{c}(\gamma) = \max_{s=1,\dots,N} \left(\sqrt{a_s\gamma} - \sqrt{b_s}\right)^2, \qquad \overline{c}(\gamma) = \min_{s=1,\dots,N} \left(\sqrt{a_s\gamma} + \sqrt{b_s}\right)^2.$$

Although we have obtained the necessary and sufficient conditions of the existence of a CLF for family (4), however only for the Lyapunov function of the special form (5). Nevertheless, these conditions permit us to deduce the following interesting and important conclusions about stability of the equilibrium position $x = \dot{x} = 0$ of switched system (3).

Corollary 3.1 *Let the switching take place in the velocity forces only* ($b_s = b = \text{const} > 0$, $s = 1, \dots, N$)*. Then the equilibrium position* $x = \dot{x} = 0$ *of system* (3) *is asymptotically stable for any admissible switching law.*

Corollary 3.2 *Let the switching take place in the potential forces only* ($a_s = a = \text{const} > 0$, $s = 1, \dots, N$)*. Then the equilibrium position* $x = \dot{x} = 0$ *of system* (3) *is asymptotically stable for any admissible switching law.*

Corollary 3.3 *Let family* (4) *consist of two subsystems* ($N = 2$)*, and the switching take place both in the velocity forces and in the potential forces* ($a_1 \neq a_2$, $b_1 \neq b_2$)*. Then the equilibrium position* $x = \dot{x} = 0$ *of system* (3) *is asymptotically stable for any admissible switching law.*

Remark 3.2 As it was mentioned in Section 2, the statements of Corollaries 3.2 and 3.3 are not true for the linear case ($\mu = 1$). Thus, in comparison with linear systems, nonlinear ones are "more stable" with respect to the switching of parameters values.

4 Systems with Nonlinear Dissipative and Potential Forces

Consider now the switched system

$$\ddot{x} + a_\sigma x^\nu \dot{x} + b_\sigma x^\mu = 0. \tag{10}$$

The corresponding family of subsystems is described as follows

$$\ddot{x} + a_s x^\nu \dot{x} + b_s x^\mu = 0, \quad s = 1, \dots, N. \tag{11}$$

Here a_s and b_s are positive constants; μ is a rational number with odd numerator and denominator, $\mu > 1$; ν is a positive rational number with even numerator and odd denominator. In this case, considered subsystems are subjected to essentially nonlinear dissipative and potential forces. Equations of such type are called the Lienard ones [6, 10]. It is known [10] that the equilibrium position $x = \dot{x} = 0$ of each subsystem from (11) is asymptotically stable.

To obtain the conditions providing the asymptotic stability of the equilibrium position of (10) for any admissible switching law, construct a CLF for the family (11) in the form

$$V(x, \dot{x}) = \frac{\dot{x}^2}{2} + c\frac{x^{\mu+1}}{\mu+1} + \gamma x^\beta \dot{x} + \varepsilon x \dot{x}^\lambda, \tag{12}$$

where $c > 0$, $\gamma > 0$, $\varepsilon < 0$, while β and λ are rational numbers with odd numerators and denominators, $\beta \geq 1$, $\lambda \geq 1$.

Differentiating $V(x, \dot{x})$ with respect to the sth subsystem from (11), one gets

$$\dot{V} = \varepsilon \dot{x}^{\lambda+1} - a_s x^\nu \dot{x}^2 - \gamma b_s x^{\mu+\beta} + (c - b_s) x^\mu \dot{x} - a_s \gamma x^{\beta+\nu} \dot{x}$$

$$+\gamma\beta x^{\beta-1}\dot{x}^2 - \varepsilon\lambda a_s x^{\nu+1}\dot{x}^\lambda - \varepsilon\lambda b_s x^{\mu+1}\dot{x}^{\lambda-1} \equiv W_s(x, \dot{x}).$$

By the use of generalized homogeneous functions properties [12] and Lemma 2 from [1], it is easy to obtain the following necessary conditions of the negative definiteness of functions $W_1(x, \dot{x}), \dots, W_N(x, \dot{x})$:

(i) if $\mu > 2\nu + 1$, then

$$\beta = \mu - \nu; \tag{13}$$

(ii) if $\mu \leq 2\nu + 1$, then $\lambda = 1 + 2(\beta - 1)/(\mu + 1)$.

It is worthy of note that, in the case where $\mu = 2\nu + 1$, systems

$$\dot{x} = y, \quad \dot{y} = -a_s x^\nu y - b_s x^\mu, \qquad s = 1, \dots, N,$$

corresponding to equations from (11) are generalized homogeneous.

In what follows, we consider the only case where $\mu > 2\nu + 1$. Under the condition (13), we have

$$W_s(x, \dot{x}) = x^\nu \left(-a_s \dot{x}^2 - \gamma b_s x^{2(\mu-\nu)} + (c - b_s - a_s\gamma) x^{\mu-\nu} \dot{x}\right) + \varepsilon \dot{x}^{\lambda+1}$$

$$+\gamma\beta x^{\mu-\nu-1}\dot{x}^2 - \varepsilon\lambda a_s x^{\nu+1}\dot{x}^\lambda - \varepsilon\lambda b_s x^{\mu+1}\dot{x}^{\lambda-1}, \quad s = 1, \dots, N.$$

Let

$$\lambda > \frac{2\mu - 2\nu - 1}{\mu - \nu}. \tag{14}$$

Then the Lyapunov function (12) is positive definite, and for the negative definiteness of functions $W_1(x, \dot{x}), \dots, W_N(x, \dot{x})$ it is sufficient the negative definiteness of quadratic forms (7).

With the numbers A and B defined in a similar way as in Section 3, we claim the following result

Theorem 4.1 *Let $\mu > 2\nu + 1$. If inequality* (9) *holds, then for family* (11) *there exists a CLF of the form* (12) *satisfying the assumptions of the Lyapunov asymptotic stability theorem.*

Remark 4.1 In contrast to Theorem 3.1, the conditions of Theorem 4.1 are only sufficient ones for the existence of a CLF of the given form for the considered family.

Remark 4.2 Under the conditions of Theorem 4.1, we obtain the following constructive algorithm for the finding of a CLF for family (11). The Lyapunov function can be chosen in the form (12), where β is defined by the formula (13), λ satisfies inequality (14), ε is an arbitrary negative number, while the values of parameters γ and c are defined in a similar way as in Remark 3.1.

Corollary 4.1 *Let $\mu > 2\nu+1$. If the switching takes place in the velocity forces only ($b_s = b = \mathrm{const} > 0$, $s = 1, \dots, N$), then the equilibrium position $x = \dot{x} = 0$ of system* (10) *is asymptotically stable for any admissible switching law.*

Corollary 4.2 *Let $\mu > 2\nu + 1$. If the switching takes place in the potential forces only ($a_s = a = \mathrm{const} > 0$, $s = 1, \dots, N$), then the equilibrium position $x = \dot{x} = 0$ of system* (10) *is asymptotically stable for any admissible switching law.*

Corollary 4.3 *Let $\mu > 2\nu+1$. If family* (11) *consists of two subsystems ($N = 2$), and the switching takes place both in the velocity forces and in the potential forces ($a_1 \neq a_2$, $b_1 \neq b_2$), then the equilibrium position $x = \dot{x} = 0$ of system* (10) *is asymptotically stable for any admissible switching law.*

5 Conclusion

In the present paper, for certain classes of families of nonlinear mechanical systems with one degree of freedom the conditions of the existence of CLFs of the given form are obtained. The fulfilment of these conditions provides the asymptotic stability of equilibrium positions of corresponding switched systems for any switching law. It is proved that, for considered families of essentially nonlinear systems, we can guarantee the existence of CLFs under weaker assumptions than for linear ones. Thus, in comparison with linear systems, nonlinear ones are "more stable" with respect to the switching of parameters values. Theorems 3.1 and 4.1 can be used for the design of stabilizing controls for mechanical systems. A challenging direction for further research is the extention of the obtained results to the switched nonlinear mechanical systems with several degrees of freedom.

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Homoclinic Orbits for a Class of Second Order Hamiltonian Systems

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Abstract: A new result for existence of homoclinic orbits is obtained for the second order Hamiltonian systems $\ddot{x}(t)+V'(t,x(t)) = f(t)$, where $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, $V \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, $V(t,x) = -K(t,x) + W(t,x)$ is $T-$periodic in t, $T > 0$ and $f : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a continuous bounded function, under an assumption weaker than the so-called Ambrosetti–Rabinowitz-type condition.

Keywords: *homoclinic orbits; Hamiltonian systems; critical point; diagonal method.*

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1 Introduction

In this paper we are concerned with the study of the existence of homoclinic solutions for second order time-dependent Hamiltonian systems of the type

$$\ddot{x}(t) + V'(t,x(t)) = f(t), \tag{HS}$$

where $x = (x_1, ..., x_N)$, $V \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, $V'(t,x) = \frac{\partial V}{\partial x}(t,x)$ and $f : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a continuous function. Here, as usual, we say that a solution x of (HS) is homoclinic $(to\ 0)$ if $x(t) \to 0$ as $t \to \pm\infty$. In addition x is called nontrivial if $x \not\equiv 0$.

The existence of homoclinic solutions for (HS) has been extensively investigated in many papers via the critical point theory, see [8, 11]. These results were obtained under the fact that the potential V is of the type

$$V(t,x) = -\frac{1}{2}L(t)x.x + W(t,x),$$

where $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix-valued function and $W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$.

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Recently, in [2], Izydorek and Janczewska have studied the existence of such solutions when the potential V is of the form

$$V(t,x) = -K(t,x) + W(t,x),$$

where $K, W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. Precisely, they established the following result.

Theorem 1.1 *Assume that V and f satisfy the conditions*
(V_1) $V(t,x) = -K(t,x) + W(t,x)$, *where* $K, W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *are* C^1*-maps,* $T-$*periodic with respect to* t, $T > 0$,
(V_2) *there are constants* $b_1, b_2 > 0$ *such that* $b_1 |x|^2 \leq K(t,x) \leq b_2 |x|^2$ *for all* $(t,x) \in \mathbb{R} \times \mathbb{R}^N$,
(V_3) *for all* $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, $K(t,x) \leq K'(t,x).x \leq 2K(t,x)$,
(V_4) $W'(t,x) = o(|x|)$, *as* $|x| \to 0$ *uniformly with respect to* t,
(V_5) *there is a constant* $\mu > 2$ *such that* $0 < \mu W(t,x) \leq W'(t,x).x$ *for every* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^N \setminus \{0\}$,
(V_6) $f : \mathbb{R} \to \mathbb{R}^N$ *is a bounded continuous function,*
(V_7) $\bar{b}_1 = min\{1, 2b_1\} > 2M$ *and* $\left(\int_{\mathbb{R}} |f(t)|^2 dt\right)^{1/2} \leq \frac{\beta}{2C}$, *where* $0 < \beta < \bar{b}_1 - 2M$, $M = \sup\left\{W(t,x)\ t \in [0,T], x \in \mathbb{R}^N, |x| = 1\right\}$ *and* C *is a positive Sobolev constant defined in [2]. Then the system* (HS) *possesses a nontrivial homoclinic solution.*

Here and in the following $x.y$ denotes the inner product of $x, y \in \mathbb{R}^N$ and $|.|$ denotes the associated norm.

The so-called Ambrosetti–Rabinowitz-type condition (V_5) appears frequently in the studying of existence of homoclinic solutions for (HS). The goal of this work is to prove that Theorem 1.1 still holds if (V_5) is replaced by a weaker condition. The motivation for the paper comes mainly from a paper by An [14], in which he dealt with the existence of periodic solutions for (HS) with a condition weaker than (V_5).

Definition 1.1 A vector field v defined on $\mathbb{R}^N$ is called positive if $v(x).x > 0$ for all $x \in \mathbb{R}^N \setminus \{0\}$. We call v a normalized positive vector field if v is positive, linear and satisfies the following condition:

$$v(x).x = x.x,\ \forall\ x \in \mathbb{R}^N. \tag{v_1}$$

Consider the following assumptions:
(V_3') there exists normalized positive vector field v such that for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$

$$K(t,x) \leq K'(t,x).v(x) \leq 2K(t,x),$$

(V_5') there exists constant $\mu > 2$ such that for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N \setminus \{0\}$

$$0 < \mu W(t,x) \leq W'(t,x).v(x).$$

The main result of this paper is as follows.

Theorem 1.2 *Assume that V and f satisfy* (V_1), (V_2), (V_3'), (V_4), (V_5'), (V_6), (V_7) *and the following assumption:*

$$W(t,x) \leq M |x|^{\mu},\ \forall\ t \in \mathbb{R},\ \forall |x| \leq 1. \tag{V_8}$$

Then the system (HS) *possesses a nontrivial homoclinic solution.*

It is obvious that if $v(x) = x$, then (V_3') becomes (V_3) and (V_5') becomes (V_5). Consider the following examples.

Example 1.1 Let $\theta(x)$ be the argument of $x = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ defined by

$$\theta(x) = \begin{cases} arctan(\frac{\xi_2}{\xi_1}), \ if \ \xi_1 > 0, \xi_2 \geq 0, \\ \frac{\pi}{2}, \ if \ \xi_1 = 0, \xi_2 > 0, \\ arctan(\frac{\xi_2}{\xi_1}) + \pi, \ if \ \xi_1 < 0, \\ \frac{3\pi}{2}, \ if \ \xi_1 = 0, \xi_2 < 0, \\ arctan(\frac{\xi_2}{\xi_1}) + 2\pi, \ if \ \xi_1 > 0, \xi_2 < 0. \end{cases}$$

Define a function $K \in C^1(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ as follows:

$$K(t,x) = \begin{cases} \frac{|x|^2}{exp(2\sin 4(\ln|x| + \theta(x)))}, \ if x \neq 0, \\ 0, \ if \ x = 0. \end{cases}$$

Define a normalized positive vector field v by $v(x) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} x$. An easy computation shows that K satisfies (V_2) and (V_3').

Example 1.2 For any $\mu > 2$, define a function $W \in C^1(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ as follows:

$$W(t,x) = \begin{cases} \frac{|x|^\mu}{exp(\mu(2 + \sin 4(\ln|x| + \theta(x))))}, \ if \ x \neq 0, \\ 0, \ if \ x = 0. \end{cases}$$

A direct computation (see [14]) shows that W satisfies (V_4), (V_5') and (V_8). Moreover, W does not satisfy (V_5).

In order to obtain homoclinic solution of (HS), we consider a sequence of systems of differential equations:

$$\ddot{x}(t) + V'(t, x(t)) = f_k(t), \qquad (HS_k)$$

where $f_k : \mathbb{R} \to \mathbb{R}^N$ is a $2kT$-periodic extension of f to the interval $[-kT, kT[, \ k \in \mathbb{N}$. We will prove the existence of a homoclinic solution of (HS) as the limit of the $2kT$-periodic solution of (HS_k) as in [2,8].

2 Preliminaries

For each $k \in \mathbb{N}$, let $E_k = W^{1,2}_{2kT}(\mathbb{R}, \mathbb{R}^N)$ denote the Hilbert space of $2kT$-periodic functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the norm

$$\|x\|_{E_k} = \left(\int_{-kT}^{kT} (|\dot{x}(t)|^2 + |x(t)|^2) dt \right)^{1/2},$$

and let $L^2_{2kT}(\mathbb{R}, \mathbb{R}^N)$ denote the Hilbert space of $2kT$-periodic functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the norm

$$\|x\|_{L^2_{2kT}} = \left(\int_{-kT}^{kT} |x(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

Furthermore, let $L^{\infty}_{2kT}(\mathbb{R},\mathbb{R}^N)$ be the space of $2kT$-periodic essentially bounded measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the norm

$$\|x\|_{L^{\infty}_{2kT}} = ess\sup\left\{|x(t)| : t\in[-kT,kT]\right\}.$$

Let $\phi_k : E_k \to \mathbb{R}$ be defined by

$$\phi_k(x) = \int_{-kT}^{kT}\left[\frac{1}{2}|\dot{x}(t)|^2 + K(t,x(t)) - W(t,x(t)) + f_k(t).x(t)\right]dt. \tag{2.1}$$

It is well known that $\phi_k \in C^1(E_k,\mathbb{R})$ and for all $x, y \in E_k$

$$\phi_k'(x)y = \int_{-kT}^{kT}\left[\dot{x}(t).\dot{y}(t) + K'(t,x(t)).y(t) - W'(t,x(t)).y(t) + f_k(t).y(t)\right]dt. \tag{2.2}$$

Moreover, the critical points of ϕ_k in E_k are exactly the classical $2kT$-periodic solution of (HS_k) (see [6,9]). We will obtain a critical point of ϕ_k by using the following Mountain Pass Theorem.

Theorem 2.1 *[8] Let E be a real Banach space and $\phi\in C^1(E,\mathbb{R})$ satisfying the Palais-Smale condition. If ϕ satisfies the following conditions:*
(i) $\phi(0)=0$,
(ii) there exist constants $\rho,\alpha>0$ such that $\phi_{/\partial B_\rho(0)}\geq\alpha$,
(iii) there exist $e\in E\backslash\overline{B}_\rho(0)$ such that $\phi(e)\leq 0$.
Then ϕ possesses a critical value $c\geq\alpha$ given by $c=\inf_{g\in\Gamma}\max_{s\in[0,1]}\phi(g(s))$, where
$\Gamma = \{g\in C([0,1],E) : g(0)=0, g(1)=e\}$.

Lemma 2.1 *[2] Let $x:\mathbb{R}\to\mathbb{R}^N$ be a continuous mapping such that $\dot{x}\in L^2_{loc}(\mathbb{R},\mathbb{R}^N)$. For every $t\in\mathbb{R}$ the following inequality holds:*

$$|x(t)| \leq \sqrt{2}\left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}}(|\dot{x}(s)|^2+|x(s)|^2)ds\right)^{1/2},$$

where $L^2_{loc}(\mathbb{R},\mathbb{R}^N)$ denotes the space of locally square integrable functions from $\mathbb{R}$ into $\mathbb{R}^N$.

Lemma 2.2 *[14] Denote by φ_s the flow of the linear vector field v with property (v_1), then*

$$|\varphi_s x| = e^s|x|, \forall s\in\mathbb{R}, \forall x\in\mathbb{R}^N.$$

Lemma 2.3 *There exist $a_1, a_2>0$ such that*

$$W(t,x)\geq a_1|x|^{\mu} - a_2, \forall t\in\mathbb{R}, \forall x\in\mathbb{R}^N. \tag{2.3}$$

Proof. Denote by S^{N-1} the unit sphere in $\mathbb{R}^N$. For any $x\in\mathbb{R}^N\backslash\{0\}$, since

$$\frac{d}{ds}(|\varphi_s x|^2) = 2\varphi_s x.v(\varphi_s(x)) > 0,$$

$(|\varphi_s x|^2)$ is increasing in s. Hence, there exist $s\in\mathbb{R}$ and $\xi\in S^{N-1}$ such that $x=\varphi_s\xi$ (see[13] for details). Since $|x|=|\varphi_s\xi|=e^s$, by (V_5') we have

$$\frac{d}{ds}[W(t,\varphi_s\xi)] = W'(t,\varphi_s\xi).v(\varphi_s\xi)\geq\mu W(t,\varphi_s\xi)>0, \forall s,t\in\mathbb{R}. \tag{2.4}$$

Let $R>0$, integrating (2.4) over $[\ln R, s]$ we obtain

$$\int_{\ln R}^{s} \frac{\frac{d}{dl}[W(t,\varphi_l \xi)]}{W(t,\varphi_l \xi)} dl \geq \mu s - \mu \ln R.$$

By (V_5') the quantity $a_1 = \inf_{t\in\mathbb{R},|x|=R}(W(t,x))R^{-\mu}$ is strictly positive and

$$W(t,x) \geq a_1 |x|^{\mu}, \forall |x| \geq R, \forall t \in \mathbb{R}.$$

Let $a_2 = \sup_{t\in\mathbb{R},|x|\leq R} W(t,x)$, then (2.3) holds. $\square$ Let v be the normalized positive vector field in (V_3') and (V_5') of Theorem 1.2. Then v is an invertible linear operator from $\mathbb{R}^N$ to $\mathbb{R}^N$. Let $a = \frac{1}{\|v^{-1}\|}, b = \|v\|$, where $\|v\|$ and $\|v^{-1}\|$ are operator norms. For any $x \in \mathbb{R}^N$, one has

$$a|x| \leq |v(x)| \leq b|x|. \tag{2.5}$$

Define a vector field $\tilde{v}$ on E_k by

$$(\tilde{v}(x))(t) = v(x(t)). \tag{2.6}$$

Using condition (v_1) and a direct computation we have the following Lemma.

Lemma 2.4 *For any $x \in E_k$, there hold*

$$\int_{-kT}^{kT} |\dot{x}(t)|^2 dt = \int_{-kT}^{kT} \dot{x}(t).\, \dot{\overbrace{\tilde{v}(x)}}(t) dt. \tag{2.7}$$

$$a\|x\|_{E_k} \leq \|\tilde{v}(x)\|_{E_k} \leq b\|x\|_{E_k}. \tag{2.8}$$

Lemma 2.5 *Let $Y : [0,+\infty[\to [0,+\infty[$ be given as follows*

$$Y(s) = \begin{cases} \max_{t\in[0,T],0<|x|\leq s} \frac{W'(t,x).v(x)}{|x|^2}, & s>0, \\ 0, & s=0. \end{cases}$$

Then Y is continuous, nondecreasing, $Y(s)>0$ for $s>0$ and $Y(s)\to+\infty$ as $s\to+\infty$.

It is easy to prove this lemma by applying $(V_4), (V_5'), (V_8)$, (2.3) and (2.5).

Remark 2.1 Assumptions $(V_4), (V_5'), (V_8)$ and (2.5) imply that $W(t,x) = o(|x|^2)$ as $x \to 0$ uniformly for $t \in [0,T]$ and $W(t,0)=0$, $W'(t,0)=0$. Moreover, from (V_2) and (V_3') we conclude that $K(t,0)=0$, $K'(t,0)=0$.

3 Proof of Theorem 1.2

Let $\gamma_k : E_k \to [0,+\infty[$ be given by

$$\gamma_k(x) = \left(\int_{-kT}^{kT} \left[|\dot{x}(t)|^2 + 2K(t,x(t))\right] dt\right)^{1/2}. \tag{3.1}$$

Let $\bar{b}_2 = \max\{1, 2b_2\}$, by (V_2) we have

$$\bar{b}_1 \|x\|_{E_k}^2 \leq \gamma_k^2(x) \leq \bar{b}_2 \|x\|_{E_k}^2. \tag{3.2}$$

By (2.1) and (3.1) we have:

$$\phi_k(x) = \frac{1}{2}\gamma_k^2(x) - \int_{-kT}^{kT} W(t, x(t))dt + \int_{-kT}^{kT} f_k(t).x(t)dt. \tag{3.3}$$

Moreover, using (V_3'), (2.6) and (2.7) we obtain

$$\begin{aligned}\phi_k'(x).\tilde{v}(x) &\le \int_{-kT}^{kT} \left(|\dot{x}(t)|^2 + 2K(t, x(t))\right) dt \\ &\quad - \int_{-kT}^{kT} W'(t, x(t)).v(x(t))dt + \int_{-kT}^{kT} f_k(t).v(x(t))dt \\ &= \gamma_k^2(x) - \int_{-kT}^{kT} W'(t, x(t)).v(x(t))dt + \int_{-kT}^{kT} f_k(t).v(x(t))dt. \end{aligned}\tag{3.4}$$

Lemma 3.1 *Assume that V and f satisfy (V_1), (V_2), (V_3'), (V_4), (V_5'), and $(V_6)-(V_8)$. Then for every $k \in \mathbb{N}$ the system (HS_k) possesses a $2kT$-periodic solution $x_k \in E_k$.*

Proof. It is clear that $\phi_k(0) = 0$. We show that ϕ_k satisfies the Palais-Smale condition. Assume that $(x_j)_{j\in\mathbb{N}} \subset E_k$ is a sequence such that $(\phi_k(x_j))_{j\in\mathbb{N}}$ is bounded and $\phi_k'(x_j) \to 0$ as $j \to +\infty$. Then there exists a constant $C_k > 0$ such that

$$|\phi_k(x_j)| \le C_k, \quad \|\phi_k'(x_j)\|_{E_k^*} \le C_k, \tag{3.5}$$

for every $j \in \mathbb{N}$. By (3.3) and (V_5') we have

$$\gamma_k^2(x_j) \le 2\phi_k(x_j) + \frac{2}{\mu}\int_{-kT}^{kT} W'(t, x(t)).v(x(t))dt - 2\int_{-kT}^{kT} f_k(t).x_j(t)dt. \tag{3.6}$$

From (3.4) and (3.6) we obtain

$$(1-\frac{2}{\mu})\gamma_k^2(x_j) \le 2\phi_k(x_j) - \frac{2}{\mu}\phi_k'(x_j)\tilde{v}(x_j) - 2\int_{-kT}^{kT} f_k(t).x_j(t)dt + \frac{2}{\mu}\int_{-kT}^{kT} f_k(t).v(x_j(t))dt. \tag{3.7}$$

By (2.8), (3.2) and (3.7) we have

$$\begin{aligned}(1-\frac{2}{\mu})\bar{b}_1 \|x_j\|_{E_k}^2 &\le 2\phi_k(x_j) + \frac{2}{\mu}\|\phi_k'(x_j)\|_{E_k^*} b\|x_j\|_{E_k} + 2\left(\int_{-kT}^{kT} |f_k(t)|^2 dt\right)^{\frac{1}{2}} \|x_j\|_{E_k} \\ &\quad + \frac{2}{\mu}\left(\int_{-kT}^{kT} |f_k(t)|^2 dt\right)^{\frac{1}{2}} b\|x_j\|_{E_k}. \end{aligned}\tag{3.8}$$

From (3.5), (3.8) and (V_7) we obtain

$$(1-\frac{2}{\mu})\bar{b}_1 \|x_j\|_{E_k}^2 - \frac{2C_k}{\mu} b\|x_j\|_{E_k} - (2+\frac{2b}{\mu})\frac{\beta}{2C}\|x_j\|_{E_k} - 2C_k \le 0. \tag{3.9}$$

Since $\mu > 2$, (3.9) shows that $(x_j)_{j\in\mathbb{N}}$ is bounded in E_k. In a similar way to Proposition 4.3 in [6], we can prove that $(x_j)_{j\in\mathbb{N}}$ has a convergent subsequence in E_k. Hence, ϕ_k satisfies the Palais-Smale condition.

Now, let us show that there exist constants $\rho, \alpha > 0$ independent of k such that ϕ_k satisfies the assumption (ii) of Theorem 2.1 with these constants. Let $x \in E_k$ such that $0 < \|x\|_{L^\infty_{2kT}} \leq 1$. By (V_8) we have

$$\int_{-kT}^{kT} W(t, x(t))dt \leq M \int_{-kT}^{kT} |x(t)|^2 \, dt \leq M \|x\|^2_{E_k}. \tag{3.10}$$

From (3.2), (3.10) and (V_7) we have

$$\begin{aligned}\phi_k(x) &\geq \frac{1}{2}\bar{b}_1 \|x\|^2_{E_k} - M \|x\|^2_{E_k} - \|f_k\|_{L^2_{2kT}} \|x\|_{L^2_{2kT}} \\ &\geq \frac{1}{2}\bar{b}_1 \|x\|^2_{E_k} - M \|x\|^2_{E_k} - \frac{\beta}{2C} \|x\|_{L^2_{2kT}} \\ &\geq \frac{1}{2}(\bar{b}_1 - \beta - 2M) \|x\|^2_{E_k} + \frac{\beta}{2} \|x\|^2_{E_k} - \frac{\beta}{2C} \|x\|_{E_k}. \end{aligned} \tag{3.11}$$

Note that (V_7) implies $\bar{b}_1 - \beta - 2M > 0$. Set

$$\rho = \frac{1}{C}, \ \alpha = \frac{\bar{b}_1 - \beta - 2M}{2C^2}.$$

(3.11) shows that $\|x\|_{E_k} = \rho$ implies that $\phi_k(x) \geq \alpha$ for $k \in \mathbb{N}$. Finally, it remains to show that ϕ_k satisfies assumption (iii) of Theorem 2.1. By the use of (3.2), (3.3) and (2.3), for every $r \in \mathbb{R}\setminus\{0\}$ and $x \in E_k \setminus \{0\}$, the following inequality holds:

$$\phi_k(rx) \leq \frac{\bar{b}_2 r^2}{2} \|x\|^2_{E_k} - a_1 |r|^\mu \int_{-kT}^{kT} |x(t)|^\mu \, dt + |r| \, \|f_k\|_{L^2_{2kT}} \|x\|_{L^2_{2kT}} + 2kTa_2. \tag{3.12}$$

Take $X \in E_1$ such that $X(\pm T) = 0$. Since $\mu > 2$ and $a_1 > 0$, (3.12) implies that there exists $r_0 \in \mathbb{R}\setminus\{0\}$ such that $\|r_0 X\|_{E_1} > \rho$ and $\phi_1(r_0 X) < 0$. Set $e_1(t) = r_0 X(t)$ and

$$e_k(t) = \begin{cases} e_1(t), \ |t| \leq T \\ 0, \ T < |t| \leq kT \end{cases} \tag{3.13}$$

for $k > 0$. Then $e_k \in E_k, \|e_k\|_{E_k} = \|e_1\|_{E_1} > \rho$ and $\phi_k(e_k) = \phi_1(e_1) < 0$ for every $k \in \mathbb{N}$. By Theorem 2.1, ϕ_k possesses a critical value $c_k \geq \alpha$ given by

$$c_k = \inf_{g \in \Gamma_k} \max_{s \in [0,1]} \phi_k(g(s)), \tag{3.14}$$

where $\Gamma_k = \{g \in C([0,1], E_k) : g(0) = 0, g(1) = e_k\}$. Hence, for every $k \in \mathbb{N}$, there exists $x_k \in E_k$ such that

$$\phi_k(x_k) = c_k, \ \phi'_k(x_k) = 0. \tag{3.15}$$

The function x_k is a desired classical $2kT-$periodic solution of (HS_k) for $k \in \mathbb{N}$. Since $c_k > 0$, x_k is a nontrivial solution even if $f_k(t) = 0$. □

Lemma 3.2 *Let $x_k \in E_k$ be a solution of system (HS_k) satisfying* (3.15). *Then there exists a positive constant M_1 independent of k such that*

$$\|x_k\|_{L^\infty_{2kT}} \leq M_1, \ \forall k \in \mathbb{N}. \tag{3.16}$$

Proof. For $k \in \mathbb{N}$, let $g_k : [0,1] \to E_k$ be a curve given by $g_k(s) = se_k$, where e_k is defined by (3.13). Then $g_k \in \Gamma_k$ and $\phi_k(g_k(s)) = \phi_1(g_1(s))$ for all $k \in \mathbb{N}$ and $s \in [0,1]$. Therefore, by (3.14)

$$c_k \leq \max_{s\in[0,1]} \phi_1(g_1(s)) \equiv M_0,\ \forall k \in \mathbb{N}, \tag{3.17}$$

where M_0 is independent of k. Since $\phi_k'(x_k) = 0$, we get from $(2.7), (3.3), (V_3')$ and (V_5')

$$c_k = \phi_k(x_k) - \frac{1}{2}\phi_k'(x_k).\tilde{v}(x_k)$$

$$\geq (\frac{\mu}{2} - 1)\int_{-kT}^{kT} W(t, x_k(t))dt + \int_{-kT}^{kT} f_k(t).x_k(t)dt - \frac{1}{2}\int_{-kT}^{kT} f_k(t).v(x_k(t))dt,$$

and hence

$$\int_{-kT}^{kT} W(t, x_k(t))dt \leq \frac{2}{\mu-2}c_k - \frac{2}{\mu-2}\int_{-kT}^{kT} f_k(t).x_k(t)dt + \frac{1}{\mu-2}\int_{-kT}^{kT} f_k(t).v(x_k(t))dt. \tag{3.18}$$

Combining (3.18) with $(2.8), (3.2), (3.17)$ and (V_7) we obtain

$$\frac{\bar{b}_1}{2}\left\|x_k\right\|_{E_k}^2 \leq \frac{\mu M_0}{\mu-2} + \frac{\beta(\mu+b)}{2C(\mu-2)}\left\|x_k\right\|_{E_k}. \tag{3.19}$$

Since $\bar{b}_1 > 0$ and all coefficients of (3.19) are independent of k, we see that there exist $M_1' > 0$ independent of k such that

$$\left\|x_k\right\|_{E_k} \leq M_1',\ \forall k \in \mathbb{N}, \tag{3.20}$$

which, together with [2, Proposition 1.1] impliy that (3.16) holds. □

Let $C^p_{loc}(\mathbb{R}, \mathbb{R}^N)$, where $p \in \mathbb{N}$, denotes the space of C^p functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the topology of almost uniformly convergence of functions and all derivatives up to the order p.

Lemma 3.3 *Let $x_k \in E_k$ be a solution of system (HS_k) satisfying* (3.16). *Then there exists a subsequence (x_{k_m}) of $(x_k)_{k\in\mathbb{N}}$ convergent to a certain $x_0 \in C^1(\mathbb{R}, \mathbb{R}^N)$ in $C^1_{loc}(\mathbb{R}, \mathbb{R}^N)$.*

Proof. By (3.16), we know that $(x_k)_{k\in\mathbb{N}}$ is a uniformly bounded sequence. Next, we will show that $(\dot{x}_k)_{k\in\mathbb{N}}$ and $(\ddot{x}_k)_{k\in\mathbb{N}}$ are also uniformly bounded sequences. Since x_k satisfies (HS_k), if $t \in [-kT, kT[$ we have

$$|\ddot{x}_k(t)| \leq |f_k(t)| + |V'(t, x_k(t))| = |f(t)| + |V'(t, x_k(t))|$$

$$\leq \sup_{t\in\mathbb{R}} |f(t)| + \sup_{(t,x)\in[0,T]\times[-M_1,M_1]} |V'(t, x(t))|,\ t \in [-kT, kT[. \tag{3.21}$$

From $(3.16), (3.21), (V_1)$ and (V_6) there is $M_2 > 0$ independent of k such that

$$\left\|\ddot{x}_k\right\|_{L^\infty_{2kT}} \leq M_2,\ \forall k \in \mathbb{N}. \tag{3.22}$$

Let $i = -k, -k+1, ..., k-1$. By the continuity of $\dot{x}_k(t)$, we can choose $t_{k_i} \in [iT, (i+1)T]$, such that

$$\dot{x}_k(t_{k_i}) = \frac{1}{T}\int_{iT}^{(i+1)T} \dot{x}_k(s)ds = \frac{1}{T}\left(x_k((i+1)T) - x_k(iT)\right),$$

it follows that for $t \in [iT, (i+1)T],\ i = -k, -k+1, ..., k-1$

$$|\dot{x}_k(t)| = \left| \int_{t_{k_i}}^{t} \ddot{x}_k(s)ds + \dot{x}_k(t_{k_i}) \right| \leq \int_{iT}^{(i+1)T} |\ddot{x}_k(s)|\, ds + |\dot{x}_k(t_{k_i})|$$

$$\leq M_2 T + T^{-1} |x_k((i+1)T) - x_k(iT)| \leq M_2 T + 2M_1 T^{-1} \equiv M_3.$$

Consequently,

$$\|\dot{x}_k\|_{L^\infty_{2kT}} \leq M_3,\ \forall k \in \mathbb{N}. \tag{3.23}$$

The task is now to show that $(x_k)_{k\in\mathbb{N}}$ and $(\dot{x}_k)_{k\in\mathbb{N}}$ are equicontinuous. Of course, it suffices to prove that both sequences satisfy the Lipschitz condition with some constants independent of k. Let $k \in \mathbb{N}$ and $t, t_0 \in \mathbb{R}$, we have by (3.23)

$$|x_k(t) - x_k(t_0)| = \left| \int_{t_0}^{t} \dot{x}_k(s)ds \right| \leq \left| \int_{t_0}^{t} |\dot{x}_k(s)|\, ds \right| \leq M_3 |t - t_0|.$$

Analogously, we have by (3.22) $|\dot{x}_k(t) - \dot{x}_k(t_0)| \leq M_2 |t - t_0|$. For each $k \in \mathbb{N}$, set $C^1_k = C^1([-kT, kT], \mathbb{R}^N)$ with the norm defined as follows:

$$\|x\|_{C^1_k} = \max_{t\in[-kT,kT]} (|\dot{x}(t)| + |x(t)|),\ x \in C^1_k.$$

Now, we will show that $(x_k)_{k\in\mathbb{N}}$ possesses a convergent subsequence (x_{k_m}) in $C^1_{loc}(\mathbb{R}, \mathbb{R}^N)$. First, let $(x_k)_{k\in\mathbb{N}}$ be restricted to $[-T, T]$. It is clear that (x_k) and $(\dot{x}_k)$ are uniformly bounded and equicontinuous. By Arzela-Ascoli theorem, there exist a subsequence (x^1_k) of $(x_k)_{k\in\mathbb{N}\setminus\{1\}}$, $x^1 \in C([-T, T], \mathbb{R}^N)$ and $y^1 \in C([-T, T], \mathbb{R}^N)$ such that

$$\left\|x^1_k - x^1\right\|_{C([-T,T],\mathbb{R}^N)} \to 0,\ \left\|\dot{x}^1_k - y^1\right\|_{C([-T,T],\mathbb{R}^N)} \to 0,\ as\ k \to +\infty. \tag{3.24}$$

Note that for $t \in [-T, T]$

$$x^1_k(t) = x^1_k(-T) + \int_{-T}^{t} \dot{x}^1_k(s)ds,\ k \in \mathbb{N}. \tag{3.25}$$

Let $k \to \infty$ in (3.25) and using (3.24) we obtain

$$x^1(t) = x^1(-T) + \int_{-T}^{t} y^1(s)ds,\ for\ t \in [-T, T] \tag{3.26}$$

which shows that $y^1(t) = \dot{x}^1(t)$ for $t \in [-T, T]$ and $x^1 \in C^1_1$. Moreover, it follows from (3.24) that

$$\left\|x^1_k - x^1\right\|_{C^1_1} \to 0,\quad as\ k \to +\infty.$$

Secondly, let (x^1_k) be restricted to $[-2T, 2T]$. It is clear that (x^1_k) and $(\dot{x}^1_k)$ are uniformly bounded and equicontinuous. Similarly as above, by Arzela-Ascoli theorem, there exist a subsequence (x^2_k) of (x^1_k) satisfying $x_2 \notin (x^2_k)$ and $x^2 \in C^1_2$ such that

$$\left\|x^2_k - x^2\right\|_{C^1_2} \to 0,\ as\ k \to +\infty.$$

By repeating this procedure for all $k \in \mathbb{N}$, there exist $(x^m_k) \subset (x^{m-1}_k)$, $x_m \notin (x^m_k)$ and $x^m \in C^1_m$ such that

$$\|x^m_k - x^m\|_{C^1_m} \to 0,\ as\ k \to +\infty,\ m = 1, 2, ... \tag{3.27}$$

Moreover, we have

$$\left\|x^{m+1}-x^m\right\|_{C^1_m}\leq\left\|x^{m+1}_k-x^{m+1}\right\|_{C^1_m}+\left\|x^m_k-x^m\right\|_{C^1_m}+\left\|x^{m+1}_k-x^m_k\right\|_{C^1_m}\to 0$$

as $k\to+\infty$, which leads to

$$x^{m+1}(t)=x^m(t),\ for\ t\in[-mT,mT],\ m=1,2,\ldots\,. \tag{3.28}$$

Let

$$x_0(t)=x^m(t),\ for\ t\in[-mT,mT],\ m=1,2,\ldots\,. \tag{3.29}$$

Then $x_0\in C^1(\mathbb{R},\mathbb{R}^N)$ and $x^m\to x_0$ as $m\to+\infty$ in $C^1_{loc}(\mathbb{R},\mathbb{R}^N)$. Now take a diagonal sequence (x_{k_m}) consisting of $x^1_1,x^2_2,x^3_3,\ldots$ (see [4]). For any $m\in\mathbb{N}$, $(x^i_i)^\infty_{i=m}$ is a subsequence of $(x^m_k)_{k\in\mathbb{N}}$, so it follows from (3.27) and (3.29) that

$$\left\|x^i_i-x_0\right\|_{C^1_m}=\left\|x^i_i-x^m\right\|_{C^1_m}\to 0,\ as\ i\to+\infty,\ m=1,2,\ldots$$

That is

$$x_{k_m}\to x_0,\ as\ m\to+\infty\ in\ C^1_{loc}(\mathbb{R},\mathbb{R}^N). \tag{3.30}$$

Lemma 3.4 *The function x_0 defined in Lemma 3.3 is the desired homoclinic solution of (HS).*

Proof. Firstly we will show that x_0 satisfies (HS). For every $k\in\mathbb{N}$, and $t\in\mathbb{R}$ we have by Lemma 3.1:

$$\ddot{x}_{k_m}(t)=f_{k_m}(t)-V'(t,x_{k_m}(t)). \tag{3.31}$$

Take $l_1,l_2\in\mathbb{R}$ such that $l_1<l_2$. There exists $m_0\in\mathbb{N}$ such that for all $m>m_0$

$$\ddot{x}_{k_m}(t)=f(t)-V'(t,x_{k_m}(t)),\ \forall t\in[l_1,l_2]. \tag{3.32}$$

Integrating (3.32) from l_1 to $t\in[l_1,l_2]$, we have

$$\dot{x}_{k_m}(t)-\dot{x}_{k_m}(l_1)=\int_{l_1}^t[f(s)-V'(s,x_{k_m}(s))]ds. \tag{3.33}$$

Since (3.30) shows that $x_{k_m}\to x_0$ uniformly on $[l_1,l_2]$ and $\dot{x}_{k_m}\to\dot{x}_0$ uniformly on $[l_1,l_2]$ as $m\to+\infty$, then by taking $m\to+\infty$ in (3.33), we get

$$\dot{x}_0(t)-\dot{x}_0(l_1)=\int_{l_1}^t[f(s)-V'(s,x_0(s))]ds,\ for\ t\in[l_1,l_2]. \tag{3.34}$$

Since l_1 and l_2 are arbitrary, (3.34) shows that x_0 is a solution of (HS). Secondly, we prove that $x_0(t)\to 0,\ as\ t\to\pm\infty$. We have, from (3.20)

$$\int_{-kT}^{kT}(|\dot{x}_k(t)|^2+|x_k(t)|^2)dt\leq M_1'^2,\ \forall k\in\mathbb{N}. \tag{3.35}$$

For every $l\in\mathbb{N}$, there exists $m_1\in\mathbb{N}$ such that for $m>m_1$

$$\int_{-lT}^{lT}(|\dot{x}_{k_m}(t)|^2+|x_{k_m}(t)|^2)dt\leq M_1'^2. \tag{3.36}$$

Let $m \to +\infty$ in (3.36) and use (3.30), it follows that for each $l \in \mathbb{N}$,

$$\int_{-lT}^{lT} (|\dot{x}_0(t)|^2 + |x_0(t)|^2)dt \le M_1'^2. \tag{3.37}$$

Letting $l \to +\infty$ in (3.37), we obtain

$$\int_{-\infty}^{+\infty} (|\dot{x}_0(t)|^2 + |x_0(t)|^2)dt \le M_1'^2, \tag{3.38}$$

and so

$$\int_{|t|\ge r} (|\dot{x}_0(t)|^2 + |x_0(t)|^2)dt \to 0, \ as\ t \to \pm\infty. \tag{3.39}$$

Combining (3.39) with Lemma 2.3 we obtain our claim.

Now, we show that $\dot{x}_0(t) \to 0, \ as\ t \to \pm\infty$. To do this, observe that by Lemma 2.3

$$|\dot{x}_0(t)|^2 \le 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|x_0(s)|^2 + |\dot{x}_0(s)|^2)ds + 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{x}_0(s)|^2\, ds. \tag{3.40}$$

From (3.39) and (3.40) it suffices to prove that

$$\int_{r}^{r+1} |\ddot{x}_0(s)|^2\, ds \to 0, \ as\ r \to \pm\infty. \tag{3.41}$$

By (HS) we obtain

$$\int_{r}^{r+1} |\ddot{x}_0(s)|^2\, ds = \int_{r}^{r+1} (|V'(s, x_0(s))|^2 + |f(s)|^2)ds - 2\int_{r}^{r+1} V'(s, x_0(s)).f(s)ds.$$

Since $V'(t, 0) = 0$ for all $t \in \mathbb{R}$, $x_0 \to 0, \ as\ t \to \pm\infty$ and $\int_r^{r+1} |f(s)|^2\, ds \to 0$, $as\ r \to \pm\infty$, then (3.41) follows.

Finally, we will show that if $f \equiv 0$ then $x_0 \not\equiv 0$. For this purpose we will use the properties of Y given by (2.9). The definition of Y implies that

$$\int_{-kT}^{kT} W'(t, x_k(t)).v(x_k(t))dt \le Y(\|x_k\|_{L^\infty_{2kT}}) \|x_k\|^2_{E_k}. \tag{3.42}$$

Since $\phi_k'(x_k).v(x_k) = 0$, then (3.4) gives

$$\int_{-kT}^{kT} W'(t, x_k(t)).v(x_k(t))dt = \int_{-kT}^{kT} |\dot{x}_k(t)|^2\, dt + \int_{-kT}^{kT} K'(t, x_k(t)).v(x_k(t))dt. \tag{3.43}$$

Substituting (3.43) into (3.42), and applying (V_3') and (V_2) we obtain

$$Y(\|x_k\|_{L^\infty_{2kT}}) \ge \min\{1, b_1\} \|x_k\|^2_{E_k},$$

and hence

$$Y(\|x_k\|_{L^\infty_{2kT}}) \ge \min\{1, b_1\} > 0. \tag{3.44}$$

If $\|x_{k_m}\|_{L^\infty_{2k_mT}} \to 0$, as $m \to +\infty$, we would have $Y(0) \ge \min\{1, b_1\} > 0$, a contradiction. Passing to a subsequence of $(x_{k_m})_{m\in\mathbb{N}}$ if necessary, there is $\eta > 0$ such that

$$\|x_{k_m}\|_{L^\infty_{2k_mT}} \ge \eta. \tag{3.45}$$

Moreover, for all $j \in \mathbb{N}$, $t \mapsto x_{k_m,j}(t) = x_{k_m}(t + jT)$ is also a $2k_mT$-periodic solution of (HS_{k_m}). Hence, if the maximum of $|x_{k_m}|$ occurs in $h_{k_m} \in [-k_mT, k_mT]$ then, the maximum of $|x_{k_m,j}|$ occurs in $s_{k_m,j} = h_{k_m} - jT$. Then there exists a $j_{k_m} \in \mathbb{Z}$ such that $s_{k_m,j_{k_m}} \in [-T, T]$. Consequently,

$$\left\|x_{k_m,j_{k_m}}\right\|_{L^\infty_{2k_mT}} = \max_{t\in[-T,T]} \left|x_{k_m,j_{k_m}}(t)\right|.$$

Suppose, contrary to our claim, that $x_0 = 0$. Then, by Lemma 3.3,

$$\left\|x_{k_m,j_{k_m}}\right\|_{L^\infty_{2k_mT}} = \max_{t\in[-T,T]} \left|x_{k_m,j_{k_m}}(t)\right| \to 0,$$

which contradicts (3.45). □

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A Decentralized Stabilization Approach of a Class of Nonlinear Polynomial Interconnected Systems Application for a Large Scale Power System

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Abstract: This paper presents a new approach dealing with the decentralized control of non linear interconnected systems. The key of this work is, on one hand, the description of the nonlinear systems using the Kronecker product notations which allow important manipulations, and on the other hand the use of the Lyapunov's direct method of stability analysis, associated with a quadratic function. The proposed approach is then applied to an industrial process: a three-machine-based interconnected power system, to improve its decentralized stabilization.

Keywords: *nonlinear systems; interconnected systems; decentralized stabilization; Kronecker product; power systems.*

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1 Introduction

In recent years, modern control methods have found their way into decentralized design of interconnected large scale nonlinear systems, leading to a wide variety of new concepts and results ([2]- [4], [18], [23]).

Decentralized control aims mainly to carry out a feedback control for each subsystem using only its local state variables.

The decentralized control law implementation is more feasible and more economical than a centralized control being dependant on the whole state variables for each subsystem local control. This kind of control is very important for the power systems which

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are generally large scale, interconnected and highly nonlinear systems. Centralized control for the large scale power system is usually impractical: first, because it requires an intensive exchange of information between many sub-systems that are geographically located in different and, generally distant areas; and second for lack of computing capacity. Consequently, a decentralized nonlinear controller, for which the development is based only on local information and measurements, is often preferable in power industry applications. A wide variety of properties for the decentralized control of power systems are extensively studied in the literature and different design approaches are proposed accordingly ([2], [8], [11], [21], [22], [24]).

It is essential to verify that the collection of these decentralized local controls should obviously guarantee the stability of the global interconnected system.

Analysis of decentralized stability properties of large scale systems has been the motivation of many works over the past twenty years ([3], [5], [9], [16]- [18]).

Power system stability has been recognized as an important problem for secure system operation. Many major blackouts caused by power system instability have illustrated the importance of this phenomenon. Historically, transient instability has been the dominant stability problem for most systems and also the focus of much of the power industry's attention related to system instability ([8], [11]- [15]). It is mainly interested in the maintenance of synchronism between generators following a severe disturbance
In this context, we propose in this work a new decentralized control for the stability of a class of non linear interconnected continuous systems based on polynomial modeling. The description of these systems using Kronecker product [19] and the use of a quadratic Lyapunov function have allowed the definition of sufficient conditions for the global asymptotic stability of the system equilibrium.

This paper is organized as follows: The next part exposes a brief summary of the main mathematical background that has supported this work. The third part will first present the studied systems, then expose the approach outcome of this work. The fourth and final part aims to show the applicability of the proposed design tool, on the basis of an illustrative example of a three-machine-based interconnected power system, followed by the concluding section.

2 Mathematical Notations and Properties

The dimensions of the matrices used in this section are the following:

$$A(p \times q), \quad B(r \times s), \quad C(q \times g), \quad D(s \times h), \quad E(n \times p), \quad P(n \times n), \quad X(n \times 1) \in \mathbb{R}^n,$$
$$Y(m \times 1) \in \mathbb{R}^m, \quad Z(q \times 1) \in \mathbb{R}^q.$$

Throughout the paper, the following notations are used: I_n is the identity matrix of order n, $\mathbb{O}_{n\times m}$ is the $(n \times m)$ null matrix and A^T is the transpose matrix of A.

2.1 Kronecker product

The Kronecker product of A and B denoted by $A \otimes B$ is the $(pr \times qs)$ matrix defined by:

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{pmatrix}.$$

2.2 Kronecker power of vectors

The Kronecker power of order i, $X^{[i]}$, of the vector X is defined by

$$\begin{cases} X^{[0]} = 1, \\ X^{[i]} = X^{[i-1]} \otimes X = X \otimes X^{[i-1]}, X^{[i]} \in \mathbb{R}^{n^i}, \text{for } i \geq 1. \end{cases} \tag{1}$$

2.3 Permutation matrix

Let e_i^n denote the i^{th} vector of the canonic basis of $\mathbb{R}^n$, the permutation matrix denoted by $U_{n\times m}$ is defined by [19]:

$$U_{n\times m} = \sum_{i=1}^{n}\sum_{k=1}^{m}(e_i^n.e_k^{m^T}) \otimes (e_k^m.e_i^{n^T}). \tag{2}$$

This matrix is square $(nm \times nm)$ and has precisely a single "1" in each row and in each column. The main useful properties of this matrix are the following:

$$U_{n\times m}^{-1} = U_{n\times m}^T = U_{m\times n}, \tag{3}$$

$$U_{n\times 1} = U_{1\times n} = U_n. \tag{4}$$

This matrix ensures the following relations

$$B \otimes A = U_{r\times p}(A \otimes B)U_{q\times s}, \tag{5}$$

$$X \otimes Y = U_{n\times m}(Y \otimes X), \tag{6}$$

$$X^{[k]} = U_{n^i\times n^{k-i}}X^{[k]}, \quad \forall i \leq k. \tag{7}$$

2.4 Vec-function

The function Vec of a matrix was defined in [19] as follows:

$$A = [A_1 \quad A_2 \quad \dots \quad A_q], \quad vec(A) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_q \end{pmatrix}, \tag{8}$$

where $\forall i \in \{1, \dots, q\}$, A_i is a vector of $\mathbb{R}^p$. We recall the following useful rules of this function, given in [19]:

$$Vec(E.A.C) = (C^T \otimes E)Vec(A), \tag{9}$$

$$Vec(A^T) = U_{p\times q}Vec(A). \tag{10}$$

2.5 Mat-function

An important matrix-valued linear function of a vector, denoted by $Mat_{(n,m)}(.)$ was defined in [20] as follows. If V is a vector of dimension $p = n.m$ then $M = Mat_{(n,m)}(V)$ is the $(n \times m)$ matrix verifying: $V = Vec(M)$. We recall the following useful lemma for this function, given in [20].

Lemma 2.1 *Consider the matrix A with $p = n$ and $q = n^k$ $(k \in \mathbb{N})$, and let i and j be two integers verifying $i + j = k + 1$ and $i \geq 1$. Then*

$$Mat_{(n^i,n^j)}(Vec(PA)) = U_{n^{i-1}\times n}(P \otimes I_{n^{i-1}}).M_{i-1,j}(A) \tag{11}$$

with

$$M_{i-1,j}(A) = \begin{pmatrix} Mat_{(n^{i-1},n^j)}(A^{1^T}) \\ Mat_{(n^{i-1},n^j)}(A^{2^T}) \\ \vdots \\ Mat_{(n^{i-1},n^j)}(A^{n^T}) \end{pmatrix},$$

where A^i denotes the i^{th} row of the matrix A. i.e.,

$$A = \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^n \end{pmatrix}.$$

3 The Proposed Decentralized Stabilization Approach

3.1 Description of the studied systems

We consider the class of nonlinear systems, formed by the interconnection of n subsystems, and for which the r order polynomial development is composed only with the odd Kronecker power of vectors, i.e., $r = 2s - 1, \quad s \in \mathbb{N}$:

$$\begin{array}{l} \dot{X}_i = f_i(X_i) + B_iU_i + g_i(X_1, X_2, \ldots, X_n), \\ i = 1, 2, \ldots, n \end{array} \tag{12}$$

with

$$f_i(X_i) = \sum_{k=0}^{s-1} A_{i,2k+1}X_i^{[2k+1]} \tag{13}$$

and

$$g_i(X_1,\ldots,X_n) = \sum_{\substack{s_1,\ldots,s_n \\ \sum_i s_i \leq r}}^{r} G_{s_1,\ldots,s_n} X_1^{[s_1]} \otimes \ldots \otimes X_i^{[s_i]} \otimes \ldots \otimes X_n^{[s_n]}, \tag{14}$$

where $X_i \in \mathbb{R}^{n_i}$ is the state vector of the i^{th} subsystem, B_i is the control matrix of the i^{th} subsystem, U_i is the control of the i^{th} subsystem, $A_{i,2k+1} \in \mathbb{R}^{n_i \times n_i^{2k+1}}$, $G_{s_1,\ldots,s_n}$ are matrices with appropriate dimensions.

The overall interconnected system is described by the following compact form:

$$\begin{aligned} \dot{\mathcal{X}} &= \mathcal{A}_1\mathcal{X} + \mathcal{A}_3\mathcal{X}^{[3]} + \mathcal{A}_5\mathcal{X}^{[5]} + \ldots + \mathcal{A}_r\mathcal{X}^{[r]} + \mathcal{B}U \\ &= \sum_{j=0}^{s-1} \mathcal{A}_{2j+1}\mathcal{X}^{[2j+1]} + \mathcal{B}U, \quad r = 2s-1, \quad s \in \mathbb{N}, \end{aligned} \tag{15}$$

where $\mathcal{X} = [X_1^T, X_2^T, \ldots, X_n^T]^T, \mathcal{X} \in \mathbb{R}^N$, $N = \sum_{i=1}^{n} n_i$, $\mathcal{A}_{2j+1} \in \mathbb{R}^{N \times N^{2j+1}}$,
$\mathcal{B} = diag(B_1, B_2, \ldots, B_n)$, $U = (U_1^T, U_2^T, \ldots, U_n^T)^T$.

3.2 Nonlinear decentralized control stabilization

We expose in this section our approach of a decentralized control synthesis of the interconnected global system (15). The decentralized control laws of the n subsystems are taken in the following form:

$$\begin{array}{l} U_i = -(K_{i1}X_i + K_{i3}X_i^{[3]} + K_{i5}X_i^{[5]} + \ldots + K_{ir}X_i^{[r]}), \\ i = 1, \ldots, n, \end{array} \tag{16}$$

which leads to the following global control law

$$\begin{array}{rcl} U & = & \begin{pmatrix} U_1 & \ldots & U_n \end{pmatrix}^T \\ & = & -(K_1\mathcal{X} + K_3\mathcal{X}^{[3]} + K_5\mathcal{X}^{[5]} + \ldots + K_r\mathcal{X}^{[r]}) \\ & = & -\sum_{j=0}^{s-1} K_{2j+1}\mathcal{X}^{[2j+1]}, \qquad r = 2s-1, \end{array} \tag{17}$$

where $K_1 = \mathrm{diag}(K_{i1})$, $i = 1, \ldots, n$ and matrices K_{2j+1}, $j = 1, \ldots, s-1$ are expressed from $K_{i,2j+1}$.

Let $Q_i(n_i \times n_i)$, $i = 1, \ldots, n$ be symmetric positive definite matrices, and α be a positive real. And let P_i $(i = 1, \ldots, n)$ be the symmetric positive definite matrices solution of the following Riccati equations

$$A_{i1}^T P_i + P_{i1}A_{i1} - P_i(B_iR_i^{-1}B_i^T)P_i + Q_i + 2\alpha P_i = 0, \tag{18}$$

where A_{i1} is the characteristic matrix of the i^{th} subsystem. And let the gains $K_{i,2j+1}$ $(i = 1, \ldots, n$ and $j = 1, \ldots, s-1)$ be given by

$$\begin{cases} K_{i1} = R_i^{-1}B_i^T P_i, \\ M_{j,j+1}(K_{i,2j+1}) = (R_i^{-1}B_i^T P_i) \otimes I_{n_i^j}. \end{cases} \tag{19}$$

We have then the following theorem.

Theorem 3.1 *The decentralized control law (16) (or (17)) is globally and asymptotically stabilizable for system (15) if there exist $(n_i \times n_i)$ positive definite matrices Q_i, $i = 1, \ldots, n$ and $\alpha \in \mathbb{R}$ such that matrices F_1, F_3, F_{2s-1} defined by*

$$F_1 = Q + P\mathcal{B}R^{-1}\mathcal{B}^T P + 2\alpha P - (PH + H^T P) \tag{20}$$

with $Q = diag(Q_i)$, $P = diag(P_i)$, $R^{-1} = diag(R_i^{-1})$, H is the interconnection linear part, and for $j \geq 1$,

$$F_{2j+1} = (P\mathcal{B}R^{-1}\mathcal{B}^T P) \otimes I_{N^j} - (P \otimes I_{N^j})M_{j,j+1}(\mathcal{A}_{2j+1}) \tag{21}$$

are semi-positive definite.

Proof. The proof of the above theorem is based on Lyapunov direct method. Let V be the Lyapunov function defined by the following quadratic form:

$$V = \mathcal{X}^T P \mathcal{X}, \tag{22}$$

where $P = \text{diag}(P_i)$ is an $(n \times n)$ definite symmetric matrix. The global asymptotic stability of the equilibrium state $\mathcal{X} = 0$ of system (15) is ensured when the time derivative $\dot{V}(\mathcal{X})$ of $V(\mathcal{X})$ is negative definite for all $\mathcal{X} \in \mathbb{R}^n$. One has

$$\dot{V} = \dot{\mathcal{X}}^T P \mathcal{X} + \mathcal{X}^T P \dot{\mathcal{X}}. \tag{23}$$

Using (15), expression (23) leads to

$$\begin{aligned} \dot{V} &= 2\sum_{j=0}^{s-1} (Vec(P\mathcal{A}_{2j+1} - P\mathcal{B}K_{2j+1}))^T \mathcal{X}^{[2j+2]} \\ &= 2\sum_{j=0}^{s-1} \mathcal{X}^{[j+1]^T} Mat_{(n^{j-1},n^j)}(Vec(P\mathcal{A}_{2j+1} - P\mathcal{B}K_{2j+1}))\mathcal{X}^{[j+1]}. \end{aligned} \tag{24}$$

Using Lemma 1, we get

$$Mat_{(n^{j+1},n^{j+1})}(Vec(P\mathcal{A}_{2j+1} - P\mathcal{B}K_{2j+1})) = U_{n^j,n}(P \otimes I_{n^j})M_{n^j,n^{j+1}}(\mathcal{A}_{2j+1} - \mathcal{B}K_{2j+1}). \tag{25}$$

The use of (25) and the following expression

$$\forall i, j \in \mathbb{N}; \quad U_{n^i \times n^j} \mathcal{X}^{[i+j]} = \mathcal{X}^{[i+j]} \tag{26}$$

yield

$$\begin{aligned} \dot{V} &= 2\sum_{j=0}^{s-1} \mathcal{X}^{[j+1]^T} Mat_{(n^{j-1},n^j)}(Vec(P\mathcal{A}_{2j+1} - P\mathcal{B}K_{2j+1}))\mathcal{X}^{[j+1]} \\ &= 2\sum_{j=0}^{s-1} \mathcal{X}^{[j+1]^T} U_{n^j,n}(P \otimes I_{n^j})M_{n^j,n^{j+1}}(\mathcal{A}_{2j+1} - \mathcal{B}K_{2j+1})\mathcal{X}^{[j+1]} \\ &= 2\sum_{j=0}^{s-1} \mathcal{X}^{[j+1]^T} (P \otimes I_{n^j})M_{n^j,n^{j+1}}(\mathcal{A}_{2j+1} - \mathcal{B}K_{2j+1})\mathcal{X}^{[j+1]} \\ &= 2\sum_{j=0}^{s-1} \mathcal{X}^{[j+1]^T} (P \otimes I_{n^j})(M_{n^j,n^{j+1}}(\mathcal{A}_{2j+1}) - M_{n^j,n^{j+1}}(\mathcal{B}K_{2j+1}))\mathcal{X}^{[j+1]} \\ &= 2\sum_{j=0}^{s-1} \mathcal{X}^{[j+1]^T} (P \otimes I_{n^j})(M_{n^j,n^{j+1}}(\mathcal{A}_{2j+1}) - \mathcal{B}R^{-1}B^T P \otimes I_{n^{j-1}})\mathcal{X}^{[j+1]}. \end{aligned} \tag{27}$$

Then we obtain the following expression:

$$\dot{V} = -\mathbb{X}^T \mathbb{M} \mathbb{X} \tag{28}$$

with

$$\mathbb{X} = \begin{pmatrix} \mathcal{X} \\ \vdots \\ \mathcal{X}^{[j+1]} \end{pmatrix}^T, \quad \mathbb{M} = \begin{pmatrix} F_1 & & \mathbb{O} \\ & \ddots & \\ \mathbb{O} & & 2(P \otimes I_{n^j})F_{2j+1}. \end{pmatrix}. \tag{29}$$

To ensure the asymptotic stability of system (15) with the control law (17), $\dot{V}$ should be negative definite, then the matrix $\mathbb{M}$ should be positive definite, which is equivalent to F_1 of expression (20) is positive definite and F_{2j+1}, for $j \geq 1$, of expression (21) are semi-positive definite.

3.2.1 Second version of decentralized stabilizability conditions

We consider system model (15) of the global interconnected system, and let

$$\mathcal{A}_{2j+1} = \mathcal{A}^1_{2j+1} + \mathcal{A}^2_{2j+1}, \tag{30}$$

where $\mathcal{A}^1_{2j+1}$ expressed from $A_{i,2k+1}$ (matrices of separated subsystems) and $\mathcal{A}^2_{2j+1}$ expressed from $G^{k,s}_{ij}$ (corresponding to interconnections).

If there exist symmetric positive definite matrices $Q_{i,j+1}(n_i^{j+1} \times n_i^{j+1})$, $j = 1, \dots, s-1$, such that the matrices P_i, solutions of Riccati equations (18), will be solutions of the following equations, for $i = 1, \dots, n$:

$$(P_i \otimes I_{n_i^j}) M_{j,j+1}(A_{i,2j+1}) + M^T_{j,j+1}(A_{i,2j+1})(P_i \otimes I_{n_i^j}) - (P_i B_i R_i^{-1} B_i^T P_i) \otimes I_{n_i^j} + Q_{i,j+1} = 0. \tag{31}$$

Each of isolated decoupled subsystems, in which all the interactions are assumed to be zero, can be stabilized with control vector U_i of (16), where the gains $K_{i,2j+1}$, $i = 1, \dots, n$ and $j = 1, \dots, s-1$ are given by

$$\begin{cases} K_{i1} = R_i^{-1} B_i^T P_i, \\ M_{j,j+1}(K_{i,2j+1}) = (R_i^{-1} B_i^T P_i) \otimes I_{n_i^j}. \end{cases} \tag{32}$$

Now the presence of interconnections will influence the stability, and it is necessary to obtain sufficient conditions to guarantee the stability of the overall system. This is given by the following theorem.

Theorem 3.2 *The decentralized control law (16) (or (17)) is globally and asymptotically stabilizable for system (15) if there exist $(n_i \times n_i)$ positive definite matrices Q_i, $i = 1, \dots, n$, $\alpha \in \mathbb{R}$, and $(n_i^{j+1} \times n_i^{j+1})$ positive definite matrices $Q_{i,j+1}$, $j \geq 1$, such that matrix F_1, defined by*

$$F_1 = Q_1 + P\mathcal{B}R^{-1}\mathcal{B}^T P + 2\alpha P - (PA_1^2 + A_1^{2^T} P), \tag{33}$$

where $Q_1 = diag(Q_i)$, $P = diag(P_i)$, $R^{-1} = diag(R_i^{-1})$, A_1^2 defined in (30) is positive definite, and for $j \geq 1$

$$F_{2j+1} = Q_{j+1} + (P\mathcal{B}R^{-1}\mathcal{B}^T P) \otimes I_{n^j} - [M^T_{j,j+1}(\mathcal{A}^2_{2j+1})(P \otimes I_{n^j}) + (P \otimes I_{n^j}) M_{j,j+1}(\mathcal{A}^2_{2j+1})], \tag{34}$$

where $Q_{j+1} = diag(Q_{i,j+1})$ are semi positive definite.

Proof. Let V be the Lyapunov function defined by the following quadratic form

$$V = \mathcal{X}^T P \mathcal{X}. \tag{35}$$

The development of $\dot{V}$ leads to

$$\begin{aligned} \dot{V} = {} & \mathcal{X}^T (P\mathcal{A}_1 + \mathcal{A}_1^T P - PBK_1 - K_1^T B^T P)\mathcal{X} + 2\sum_{j=1}^{s-1} \mathcal{X}^{[j+1]^T} (\otimes I_{n^j})(M_{j,j+1}(\mathcal{A}_{2j+1}) \\ & -(\mathcal{B}R^{-1}B^T P) \otimes I_{n^j})\mathcal{X}^{[j+1]}. \end{aligned} \tag{36}$$

Then using (18) and (31) in (36), we get

$$\begin{aligned} \dot{V} = {} & -\mathcal{X}^T (Q_1 + P\mathcal{B}R^{-1}\mathcal{B}^T P + 2\alpha P - (PA_1^2 + A_1^{2^T} P))\mathcal{X} - \sum_{j=1}^{s-1} \mathcal{X}^{[j+1]^T} \{Q_{j+1} \\ & + (P\mathcal{B}R^{-1}\mathcal{B}^T P) \otimes I_{n^j} - [M^T_{j,j+1}(\mathcal{A}^2_{2j+1})(P \otimes I_{n^j}) + (P \otimes I_{n^j}) M_{j,j+1}(\mathcal{A}^2_{2j+1})]\} \mathcal{X}^{[j+1]}. \end{aligned} \tag{37}$$

The expression (37) is then equivalent to

$$\dot{V} = -\mathbb{X}^T \mathbb{M} \mathbb{X} \tag{38}$$

with

$$\mathbb{X} = \begin{pmatrix} \mathcal{X} \\ \vdots \\ \mathcal{X}^{[j+1]} \end{pmatrix}^T, \quad \mathbb{M} = \begin{pmatrix} F_1 & & \mathbb{O} \\ & \ddots & \\ \mathbb{O} & & F_{2j+1} \end{pmatrix}. \tag{39}$$

To ensure the asymptotic stability of system (15) with the control law (17), $\dot{V}$ should be negative definite, then the matrix $\mathbb{M}$ should be positive definite, which is equivalent to F_1 is positive definite and F_{2j+1}, $j \geq 1$ is semi-positive definite.

4 Application of the Proposed Control to a Multimachine Power System

We propose in this part to show that it is possible to apply the proposed decentralized control method to an industrial process. It consists in studying the stability by decentralized control of a power system composed of three interconnected machines, (Figure 1), characterized by the parameters indicated in Table 1.

4.1 Multimachine power system modelisation

A three machine power system controlled by the steam valve opening, can be described with the interconnection of three subsystems as follows [21]:

$$\dot{X}_i(t) = A_i X_i(t) + B_i U_i(t) + \sum_{j=1, j\neq i}^{3} p_{ij} G_{ij} g_{ij}(X_i, X_j); \quad i = 1, \cdots, 3, \tag{40}$$

where $X_i(t)$ is the state vector defined by $X_i(t)^T$=[$\Delta\delta_i(t) \quad \omega_i(t) \quad \Delta P_{m_i}(t) \quad \Delta X_{e_i}(t)$], $\Delta\delta_i(t) = \delta_i(t) - \delta_i 0$, $\Delta P_{m_i}(t) = P_{m_i}(t) - P_{m_i 0}$, $\Delta X_{e_i}(t) = X_{e_i}(t) - X_{e_i 0}$, $U_i(t)$ is the control, $U_i(t) = \Delta X_{ei}(t)$,

$$A_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\dfrac{D_i}{2H_i} & 0 & -\dfrac{\omega_0}{2H_i} & 0 \\ 0 & 0 & -\dfrac{1}{T_{m_i}} & \dfrac{K_{m_i}}{T_{m_i}} \\ -\dfrac{K_{e_i}}{T_{e_i} R_i \omega_0} & 0 & 0 & -\dfrac{1}{T_{e_i}} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{T_{e_i}} \end{bmatrix}, G_{ij} = \begin{bmatrix} 0 \\ -\dfrac{\omega_0 E'_{qi} E'_{qj} B_{ij}}{2H_i} \\ 0 \\ 0 \end{bmatrix},$$

$g_{ij}(x_i, x_j) = \sin(\delta_i(t) - \delta_j(t)) - \sin(\delta_{i0} - \delta_{j0})$, where:

p_{ij} a constant of either 1 or 0 (if 0, then j^{th} machine has no connection with i^{th} one);
δ_i the rotor angle for i^{th} machine, in *radian*;
ω_i the relative speed for i^{th} machine, in $radian/second$;
P_{m_i} the mechanical power for i^{th} machine, in *pu*;
X_{e_i} the steam valve opening for i^{th} machine, in *pu*;
H_i the inertia constant for i^{th} machine, in *second*;
D_i the damping coefficient for i^{th} machine, in *pu*;
T_{m_i} the time constant for i^{th} machine's turbine, in *second*;
K_{m_i} the gain of i^{th} machine turbine;
T_{e_i} the time constant of i^{th} machine's speed governor, in *second*;
T_{e_i} the gain of i^{th} machine's speed governor;
R_i the regulation constant of i^{th} machine, in *pu*;
B_{ij} the nodal susceptance between i^{th} and j^{th} machines, in *pu*;
ω_0 the synchronous machine speed, in $radian/second$;
E'_{qi} the internal transient voltage for i^{th} machine, in *pu*, which is a constant;
E'_{qj} the internal transient voltage for i^{th} machine, in *pu*, which is a constant;
x_{di} the direct axis reactance of the i^{th} generator, in *pu*;
x'_{di} the direct axis transient reactance of the i^{th} generator, in *pu*;
x_{Ti} the transformer reactance;
x_{adi} the mutual reactance between the excitation coil and the stator coil, in *p.u.*;
T'_{d0i} the direct axis transient short-circuit time constant, in *second*;
x_{ij} the transmission line reactance between the i^{th} and the j^{th} generators, in *pu*;

δ_{i0}, P_{m_i0} and X_{e_i0} are the initial values of $\delta_i(t)$, $P_{m_i}(t)$ and $X_{e_i}(t)$.

	Machine 1	Machine 2	Machine 3
$x_d(pu)$	1.863	2.36	2.36
$x'_d(pu)$	0.257	0.319	0.319
$x_T(pu)$	0.129	0.11	0.11
$x_{ad}(pu)$	1.712	0.712	0.712
$T'_{d0}(pu)$	6.9	7.96	7.96
$H(s)$	4	5.1	5.1
$D(pu)$	5	3	3
$T_m(s)$	0.35	0.35	0.35
$T_e(s)$	0.1	0.1	0.1
R	0.05	0.05	0.05
K_m	1	1	1
K_e	1	1	1

$x_{12}(pu)$	0.55
$x_{13}(pu)$	0.53
$x_{23}(pu)$	0.6
$\omega_0(rad/s)$	314.159

Table 1: Three-machine-based system parameters.

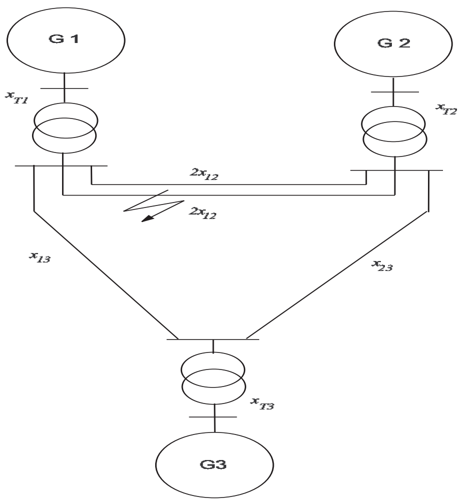

Figure 1: Three-machine example system.

4.1.1 Polynomial model

The nonlinear analytic model (40) can be represented with a third order truncated polynomial form, which is considered to be sufficient for the studied power system modeling:

$$\dot{X}_i = A_{i,1}X_i + A_{i,3}X_i^{[3]} + B_iU_i + \sum_{\substack{j=1\\ j\neq i}}^{3}\sum_{k=1}^{3}\sum_{s=1}^{k} p_{ij}G_{ij}^{k,s}X_i^{[k-s]}\otimes X_j^{[s]}, \quad i = 1,2,3. \tag{41}$$

The global interconnected system is then modelled with the following polynomial form

$$\dot{\mathcal{X}} = \mathcal{A}_1\mathcal{X} + \mathcal{A}_3\mathcal{X}^{[3]} + \mathcal{B}U, \tag{42}$$

where $\mathcal{X} = [X_1^T, X_2^T, X_3^T]^T$,

$$\begin{cases}\mathcal{A}_1 = \mathrm{diag}(A_1, A_2, A_3), \mathcal{A}_1(2,1) = -54.98,\\ \mathcal{A}_1(2,5) = 27.49, \mathcal{A}_1(2,9) = 27.49, \mathcal{A}_1(6,5) = -46.2,\\ \mathcal{A}_1(6,1) = 23.1, \mathcal{A}_1(6,9) = 23.1, \mathcal{A}_1(10,9) = -50.59,\\ \mathcal{A}_1(10,1) = 23.1, \mathcal{A}_1(10,5) = 27.49,\end{cases}$$

$$\begin{cases}\mathcal{A}_3(2,1)=9.16, \mathcal{A}_3(6,769)=-13.745, \mathcal{A}_3(8,1537)=-13.745,\\ \mathcal{A}_3(2,65)=-13.745, \mathcal{A}_3(6,833)=7.7, \mathcal{A}_3(8,1601)=-13.745,\\ \mathcal{A}_3(2,129)=-13.745, \mathcal{A}_3(6,897)=-11.55, \mathcal{A}_3(8,1665)=8.43,\\ \mathcal{A}_3(2,257)=13.745, \mathcal{A}_3(6,577)=13.745, \mathcal{A}_3(8,1153)=13.745,\\ \mathcal{A}_3(2,513)=13.745, \mathcal{A}_3(6,1089)=11.55, \mathcal{A}_3(8,1409)=13.745,\\ \mathcal{A}_3(2,833)=-4.58, \mathcal{A}_3(6,1)=-3.85, \mathcal{A}_3(8,1)=-3.85,\\ \mathcal{A}_3(2,1664)=-4.58, \mathcal{A}_3(6,1665)=-13.745, \mathcal{A}_3(8,833)=-4.58,\\ \mathcal{A}_3(i,j) = 0 \text{ for the other values of i and j } 1 \le i \le 12,\\ \text{ et } 1 \le j \le 1728.\end{cases}$$

We want to compute the decentralized control laws given by (16) and (19) for $i = 1,2,3$. For α, R_i and Q_i, $i = 1,2,3$, given by $\alpha = 0$, $R_i = 2$, $Q_i = \mathrm{diag}\{0.001, 0.001, 0.01, 0.01\}$, we obtain:

$$K_{11} = [55.90 \quad 24.48 \quad 349.25 \quad 103.87], \; K_{21} = K_{31} = [55.90 \quad 33.06 \quad 359.28 \quad 106.62].$$

We can easily verify that matrices F_1 and F_3 given in Theorem 1 are positive defined, which guarantees the stability of system (42) by the decentralized control law.

Firstly, we want to know the behavior of the proposed power system in free operating conditions. The curves of Figure 2 show the strongly transient evolution of the power system state variables, when it is simulated under these conditions towards a perturbation on the first machine rotor angle.

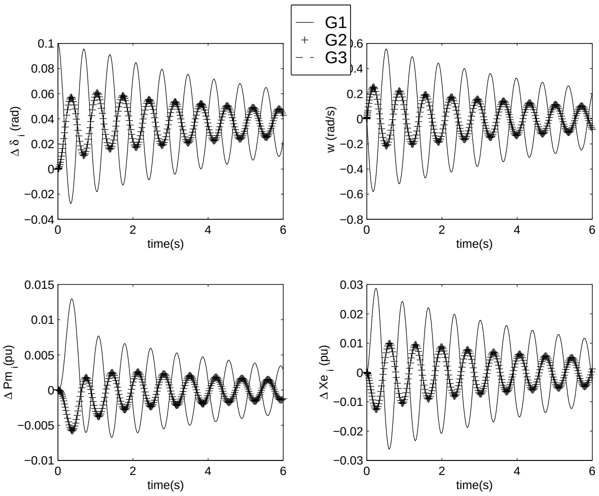

Figure 2: State variable evolution in free operating conditions, toward a perturbation on $\delta 1$.

Now, to test the performances of the established decentralized control law, we carry on the simulation of the controlled power system towards some perturbations occurred on state variables. Figure 3 shows the case when a perturbation is occurred on the rotor angle of the first machine. Figure 4 illustrates the corresponding control signal evolution. Regarding to Figure 5 and Figure 6 they show, respectively, the evolution of the three-machine state variables when a perturbation occurs on the relative speed of the second machine, and the corresponding control.

From the simulation results shown in these figures, it can be seen that the nonlinear decentralized control is able to damp the oscillations of the system and to enhance transient stability of the multimachine power system and this despite different fault locations that occur on state variables.

5 Conclusion

In this paper, we have developed and validated a new decentralized control approach of nonlinear interconnected polynomial systems. The studied systems are described by a polynomial model with odd Kronecker power of state vectors.

The nonlinear decentralized control law, which is also described by a polynomial form, can guarantee the asymptotic stability of the overall interconnected system when some sufficient conditions are verified.

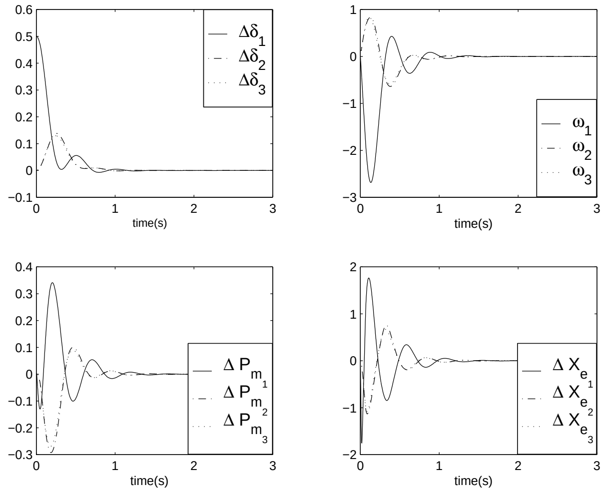

Figure 3: State variable evolution towards a perturbation on δ_1.

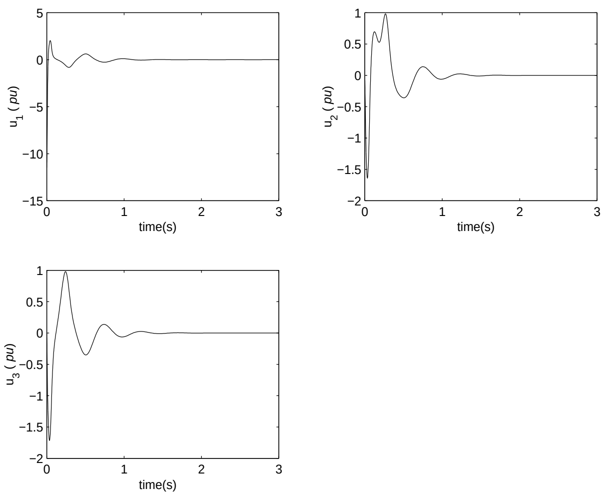

Figure 4: The corresponding control signal evolution.

This new approach is then validated by numerical simulation study on a three-interconnected-machine power system. The proposed study has shown the high performances of the considered control which is able to damp the system oscillations and to enhance the power system transient stability and this despite the high nonlinear interconnections between generators and different perturbations that can occur on the system state variables.

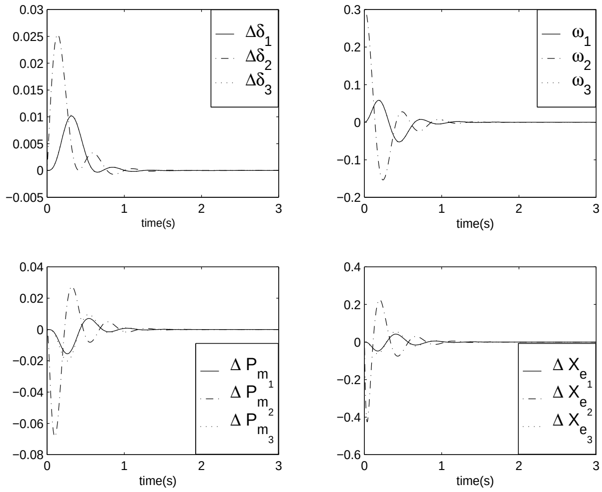

Figure 5: State variable evolution towards a perturbation on ω_2.

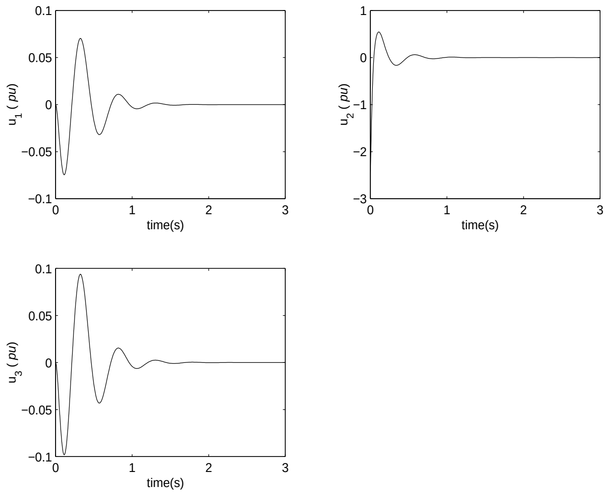

Figure 6: The corresponding control signal evolution.

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Positive Solutions for a Fourth Order Three Point Focal Boundary Value Problem

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Abstract: The authors consider a fourth order three point boundary value problem. Some a priori estimates to positive solutions for the boundary value problem are obtained. Sufficient conditions for the existence and nonexistence of positive solutions for the problem are established.

Keywords: *fixed point theorem; cone; nonlinear boundary value problem; positive solution.*

Mathematics Subject Classification (2010): 34B18.

1 Introduction

In this paper, we consider the fourth order differential equation

$$u''''(t) + g(t)f(u(t)) = 0, \quad 0 \le t \le 1, \tag{1}$$

together with the boundary conditions

$$u(0) = u'(p) = u''(1) = u'''(1) = 0. \tag{2}$$

Throughout this paper, we assume that

(H1) p is a real constant such that $1 - \sqrt{3}/3 \le p \le 1$, $f : [0, \infty) \to [0, \infty)$ and $g : [0, 1] \to [0, \infty)$ are continuous functions, and $g(t) \not\equiv 0$ on $[0, 1]$.

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In this paper, we will study positive solutions of the problem (1)–(2). By a *positive solution*, we mean a solution $u(t)$ to the problem (1)–(2) such that $u(t) > 0$ for $t \in (0,1)$.

The fourth order equation (1), known as the beam equation, has been studied by many authors under various boundary conditions and by different approaches. For example, in 2006, Anderson and Avery [2] considered the fourth order four-point right focal boundary value problem

$$u''''(t) + f(u(t)) = 0, \quad 0 < t < 1, \tag{3}$$

$$u(0) = u'(q) = u''(r) = u'''(1) = 0, \tag{4}$$

under the assumption that $1/2 < q < (1+q)/2 < r < 1$. We note that if we allow $r = 1$, then (4) reduces to (2). In 2005, Yang [9] considered the boundary value problem

$$u''''(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{5}$$

$$u(0) = u'(0) = u''(1) = u'''(1) = 0, \tag{6}$$

and obtained sufficient conditions for the existence and nonexistence of positive solutions to the problem (5)–(6). We note that if we let $p = 0$, then (2) reduces to (6).

For some other results on boundary value problems for the beam equation, we refer the reader to the papers [1, 3–6, 8].

In this paper, we shall first prove some upper and lower estimates to positive solutions of the problem (1)–(2), and then establish some sufficient conditions for the existence and non-existence of positive solutions.

This paper is organized as follows. In Section 2, we give the Green function for the problem (1)–(2), state the Krasnosel'skii's fixed point theorem, and fix some notations. In Section 3, we present some a priori estimates to positive solutions to the problem. In Section 4, we establish some existence and nonexistence results for positive solutions.

2 Preliminaries

The Green function $G : [0,1] \times [0,1] \to [0,\infty)$ for the problem (1)–(2) is

$$\begin{aligned} G(t,s) \quad = \quad & -t[p^2/2 - ps - ((p-s)^2/2)H(p-s)] \\ & -t^2 s/2 + t^3/6 - ((t-s)^3/6)H(t-s). \end{aligned}$$

Here, $H : (-\infty,\infty) \to (-\infty,\infty)$ is the unit step function given by

$$H(t) = \begin{cases} 1, & \text{if } t \ge 0, \\ 0, & \text{if } t < 0. \end{cases}$$

The problem (1)–(2) is then equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)g(s)f(u(s))\,ds, \quad 0 \le t \le 1. \tag{7}$$

It is easy to verify that G is a continuous function. Also, we note that if $0 \le s \le p$, then

$$G(p,s) = s^3/6 \ge 0;$$

if $p \le s \le 1$, then

$$G(p,s) = p^2(3s - 2p)/6 \ge 0.$$

In summary, we have $G(p,0)=0$ and

$$G(p,s)>0, \quad 0<s\le 1.$$

We will need the following simplified version of the Krasnosel'skii fixed point theorem (see [7]) to prove some of our results.

Theorem 2.1 *Let $(X,\|\cdot\|)$ be a Banach space over the reals, and let $P\subset X$ be a cone in X. Let H_1 and H_2 be distinct positive numbers. If $L:P\to P$ is a completely continuous operator such that*

(K1) If $v\in P$ and $\|v\|=H_1$, then $\|Lv\|\le\|v\|$, and

(K2) If $v\in P$ and $\|v\|=H_2$, then $\|Lv\|\ge\|v\|$.

Then L has a fixed point v in P with $\min\{H_1,H_2\}\le\|v\|\le\max\{H_1,H_2\}$.

For the rest of this paper, we let $X=C[0,1]$ be equipped with the norm

$$\|v\|=\max_{t\in[0,1]}|v(t)|, \quad \text{for all} \quad v\in X.$$

Clearly, X is a Banach space. We define

$$Y=\{v\in X\mid v(t)\ge 0 \text{ for } 0\le t\le 1\},$$

and define the operator $T:Y\to X$ by

$$(Tu)(t)=\int_0^1 G(t,s)g(s)f(u(s))\,ds, \quad 0\le t\le 1. \tag{8}$$

It is easy to see that if (H1) holds, then $T:Y\to Y$ is a completely continuous operator. We also define the constants

$$F_0=\limsup_{x\to 0^+}\frac{f(x)}{x}, \quad f_0=\liminf_{x\to 0^+}\frac{f(x)}{x},$$
$$F_\infty=\limsup_{x\to +\infty}\frac{f(x)}{x}, \quad f_\infty=\liminf_{x\to +\infty}\frac{f(x)}{x}.$$

These constants, which are associated with the function f, will be used in Sections 4 and 5.

3 Estimates for Positive Solutions

In this section, we derive some upper and lower estimates for positive solutions of the problem (1)–(2).

Lemma 3.1 *If (H1) holds, then $G(t,s)\le G(p,s)$ for $0\le t,s\le 1$.*

Proof. We take four cases to prove this inequality. If $t\le s\le p$, then

$$G(p,s)-G(t,s)=\frac{(s-t)^3}{6}\ge 0.$$

If $s \le t \le p$ or $s \le p \le t$, then

$$G(p,s) - G(t,s) = 0.$$

If $t \le p \le s$ or $p \le t \le s$, then

$$G(p,s) - G(t,s) = \frac{(t-p)^2}{6}(2s - 2p + s - t) \ge 0.$$

If $p \le s \le t$, then

$$G(p,s) - G(t,s) = \frac{(s-p)^2}{6}(2t - 2p + t - s) \ge 0.$$

The proof is now complete.

We define the function $a : [0,1] \to [0,\infty)$ by

$$a(t) = \frac{3p(2-p)t - 3t^2 + t^3}{p^2(3-2p)}, \quad 0 \le t \le 1.$$

We notice that

$$a(0) = 0, \quad a(1) = \frac{3(\sqrt{3}/3 + 1 - p)(p - (1 - \sqrt{3}/3))}{p^2(3-2p)} \ge 0,$$

and

$$a''(t) = \frac{-6(1-t)}{p^2(3-2p)} \le 0, \quad 0 \le t \le 1.$$

Therefore, $a(t)$ is concave downward on $[0,1]$. Since $a(0) = 0$ and $a(1) \ge 0$, we have

$$a(t) \ge 0, \quad 0 \le t \le 1.$$

It is easy to see that

$$a(t) \ge \min\{t, 1-t\}, \quad 0 \le t \le 1. \tag{9}$$

We leave the verification of (9) to the reader.

Lemma 3.2 *Suppose (H1) holds. Then $G(t,s) \ge a(t)G(p,s)$ for $0 \le t, s \le 1$.*

Proof. We take four cases to prove the lemma.
If $t \le s \le p$, then

$$\begin{aligned} G(t,s) - a(t)G(p,s) &= \frac{t}{6(3-2p)p^2}[s^2(3-s-2p)(s-p)^2 \\ &\quad + s(s-t)(p-s)(2s - 2s^2 + 2p - 2p^2 + s - sp + p - sp) \\ &\quad + (s-t)^2(2p^2 - 2p^3 + p^2 - s^3)] \\ &\ge 0. \end{aligned}$$

If $s \le t \le p$ or $s \le p \le t$, then

$$G(t,s) - a(t)G(p,s) = \frac{s^3(t-p)^2(3-t-2p)}{6(3-2p)p^2} \ge 0.$$

If $t \le p \le s$ or $p \le t \le s$, then

$$G(t,s) - a(t)G(p,s) = \frac{t(t-p)^2(1-s)}{2(3-2p)} \ge 0.$$

If $p \le s \le t$, then

$$\begin{aligned} G(t,s) - a(t)G(p,s) &= \frac{t(p-t)^2(1-s)}{6-4p} + \frac{(s-t)^3}{6} \\ &\ge \frac{1}{6}\left[t(p-t)^2(1-s) + (s-t)^3\right] \\ &\ge \frac{1}{6}\left[t(p-t)^2(1-s) + (p-t)^2(s-t)\right] \\ &= \frac{1}{6}(p-t)^2 s(1-t) \\ &\ge 0. \end{aligned}$$

This completes the proof of the lemma.

Lemma 3.3 *Suppose (H1) holds. Then* $G(t,s) \ge 0$ *for* $0 \le t, s \le 1$.

Proof. The lemma follows easily from Lemma 3.2 and the facts that $a(t) \ge 0$ for $0 \le t \le 1$ and $G(p,s) \ge 0$ for $0 \le s \le 1$.

Lemma 3.4 *Suppose (H1) holds. If* $u \in C^4[0,1]$ *satisfies the boundary conditions* (2)*, and*

$$u''''(t) \le 0 \quad \text{for} \quad 0 \le t \le 1, \tag{10}$$

then $\|u\| = u(p)$, $u(t) \ge 0$, *and*

$$a(t)u(p) \le u(t) \le u(p) \quad \text{for} \quad 0 \le t \le 1. \tag{11}$$

Proof. Suppose $u \in C^4[0,1]$ satisfies (2) and (10). If $0 \le t \le 1$, then

$$u(t) = \int_0^1 G(t,s)(-u''''(s))ds \ge 0,$$

$$u(t) = \int_0^1 G(t,s)(-u''''(s))ds \ge a(t)\int_0^1 G(p,s)(-u''''(s))ds = a(t)u(p),$$

and

$$u(t) = \int_0^1 G(t,s)(-u''''(s))ds \le \int_0^1 G(p,s)(-u''''(s))ds = u(p),$$

which proves the lemma.

The next theorem follows immediately from Lemma 3.4.

Theorem 3.1 *Suppose (H1) holds. If* $u \in C^4[0,1]$ *is a non-negative solution to the problem* (1)–(2)*, then* $u(t)$ *satisfies* (11)*.*

We now define

$$P = \left\{v \in X : v(p) \ge 0,\ a(t)v(p) \le v(t) \le v(p) \text{ on } [0,1]\right\}.$$

Clearly P is a positive cone in X. It is obvious that if $u \in P$, then $u(p) = \|u\|$. We see from Theorem 3.1 that if $u(t)$ is a nonnegative solution to the problem (1)–(2), then $u \in P$. In a similar fashion to Lemma 3.4, we can show that $T(P) \subset P$. To find a positive solution to the problem (1)–(2), we need only to find a fixed point u of T such that $u \in P$ and $u(p) = \|u\| > 0$.

4 Existence and Nonexistence Results

First, we define some important constants:

$$A = \int_0^1 G(p,s)g(s)a(s)\,ds \quad \text{and} \quad B = \int_0^1 G(p,s)g(s)\,ds.$$

The next two theorems provide sufficient conditions for the existence of at least one positive solution for the problem (1)–(2).

Theorem 4.1 *Suppose that (H1) holds. If $BF_0 < 1 < Af_\infty$, then the problem* (1)–(2) *has at least one positive solution.*

Proof. First, we choose $\varepsilon > 0$ such that $(F_0 + \varepsilon)B \le 1$. By the definition of F_0, there exists $H_1 > 0$ such that $f(x) \le (F_0 + \varepsilon)x$ for $0 < x \le H_1$. Now for each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}
\|Tu\| = (Tu)(p) &= \int_0^1 G(p,s)g(s)f(u(s))\,ds \\
&\le \int_0^1 G(p,s)g(s)(F_0+\varepsilon)u(s)\,ds \\
&\le (F_0+\varepsilon)\|u\| \int_0^1 G(p,s)g(s)\,ds \\
&= (F_0+\varepsilon)\|u\|B \le \|u\|.
\end{aligned}$$

Hence, condition (K1) in Theorem 2.1 is satisfied.

Next we choose $\delta > 0$ and $\tau \in (0, 1/4)$ such that

$$\int_\tau^{1-\tau} G(p,s)g(s)a(s)\,ds \cdot (f_\infty - \delta) \ge 1.$$

There exists $H_3 > 2H_1$ such that $f(x) \ge (f_\infty - \delta)x$ for $x \ge H_3$. Let $H_2 = H_3/\tau$. If $u \in P$ and $\|u\| = H_2$, then for each $t \in [\tau, 1-\tau]$, we have

$$u(t) \ge H_2 a(t) \ge H_2 \min\{t, 1-t\} \ge H_2\tau = H_3.$$

Therefore, for each $u \in P$ with $\|u\| = H_2$, we have

$$\begin{aligned}
\|Tu\| = (Tu)(p) &= \int_0^1 G(p,s)g(s)f(u(s))\,ds \\
&\ge \int_\tau^{1-\tau} G(p,s)g(s)f(u(s))\,ds \\
&\ge \int_\tau^{1-\tau} G(p,s)g(s)(f_\infty-\delta)u(s)\,ds \\
&\ge \int_\tau^{1-\tau} G(p,s)g(s)a(s)\,ds \cdot (f_\infty-\delta)\|u\| \ge \|u\|.
\end{aligned}$$

Thus, condition (K2) of Theorem 2.1 is satisfied. By Theorem 2.1, T has a fixed point u such that $\min\{H_1, H_2\} = H_1 \le \|u\| \le \max\{H_1, H_2\} = H_2$. This completes the proof of the theorem.

The proof of the following companion result is very similar to that of Theorem 4.1 and is therefore omitted.

Theorem 4.2 *Suppose that (H1) holds. If $BF_\infty < 1 < Af_0$, then the problem* (1)–(2) *has at least one positive solution.*

The next two theorems provide sufficient conditions for the nonexistence of positive solutions to the problem (1)–(2).

Theorem 4.3 *Suppose (H1) holds. If $Bf(x) < x$ for all $x > 0$, then the problem* (1)–(2) *has no positive solutions.*

Proof. Assume to the contrary that $u(t)$ is a positive solution of the problem (1)–(2). Then $u \in P$, $u(t) > 0$ for $0 < t < 1$, and

$$\begin{aligned} u(p) &= \int_0^1 G(p,s)g(s)f(u(s))\,ds \\ &< B^{-1}\int_0^1 G(p,s)g(s)u(s)\,ds \\ &\le B^{-1}u(p)\int_0^1 G(p,s)g(s)\,ds \\ &= B^{-1}u(p)B = u(p), \end{aligned}$$

which is a contradiction.

The proof of our next theorem is similar to the one above.

Theorem 4.4 *Suppose (H1) holds. If $Af(x) > x$ for all $x > 0$, then the problem* (1)–(2) *has no positive solutions.*

We conclude this paper with an example.

Example 4.1 Consider the fourth order boundary value problem

$$u''''(t) = \lambda(1+t)u(t)(1+3u(t))/(1+u(t)), \quad 0 \le t \le 1, \tag{12}$$

$$u(0) = u'(3/4) = u''(1) = u'''(1) = 0. \tag{13}$$

Here $\lambda > 0$ is a parameter. In this example, $p = 3/4$, $g(t) = 1+t$, and

$$f(u) = \lambda u(1+3u)/(1+u).$$

It is easy to see that $f_0 = F_0 = \lambda$, $f_\infty = F_\infty = 3\lambda$, and

$$\lambda x < f(x) < 3\lambda x \quad \text{for} \quad x > 0.$$

Calculations indicate that

$$A = 142837/2064384, \quad B = 363/5120.$$

By Theorem 4.1, if

$$4.8176 \approx 1/(3A) < \lambda < 1/B \approx 14.1046,$$

then the problem (12)–(13) has at least one positive solution. From Theorems 4.3 and 4.4 we see that if

$$\lambda \le 1/(3B) \approx 4.7015 \quad \text{or} \quad \lambda \ge 1/A \approx 14.4528,$$

then the problem (12)–(13) has no positive solutions.

This example shows that our existence and nonexistence results can be quite sharp.

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Existence, Uniqueness and Asymptotic Stability of Solutions to Non-Autonomous Semi-Linear Differential Equations with Deviated Arguments

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Abstract: We consider a non-autonomous semi-linear differential equation of parabolic type with a deviated argument in an arbitrary Banach space. Using the Sobolevskiĭ-Tanabe theory of parabolic equations, we prove the existence and uniqueness of a solution. We also discuss the asymptotic stability of a solution. As an application, we give an example to illustrate the main results.

Keywords: *analytic semigroup, parabolic equation, differential equation with a deviated argument, Banach fixed point theorem.*

Mathematics Subject Classification (2010): 34G10, 34G20, 34K30, 35K90, 47N20.

1 Introduction

The purpose of this article is to study the following differential equation in a Banach space $(X, \|\cdot\|)$:

$$\left.\begin{aligned} \frac{du}{dt} + A(t)u(t) &= f(t, u(t), u(h(u(t), t))),\ t > 0; \\ u(0) &= u_0,\ u_0 \in X. \end{aligned}\right\} \tag{1}$$

We assume that for each $t \geq 0$, $-A(t)$ generates an analytic semigroup of bounded linear operators on X, $f\colon [0, \infty) \times X \times X \to X$ and $h\colon X \times [0, \infty) \to [0, \infty)$. The nonlinear continuous functions f and h satisfy suitable growth conditions in their arguments stated in Section 2.

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Differential equations with deviated arguments model certain real world systems in the theory of automatic control, the study of problems related with combustion in rocket motion, the theory of self-oscillating systems, problems of long-term planning in economics, biological systems, and many other systems in the areas of science and technology [3]. Recently, many authors have studied the existence, uniqueness and continuous dependence of a solution of the differential equation of the type (1) (see e.g. Gal [6, 7]; Grimm [8]; Jankowski [12]; Oberg [16]). More details of differential equation with deviated arguments can be found in Bahuguna and Muslim [1], Dubey [2], El'sgol'ts and Norkin [3], Gal [6,7], Grimm [8], Jankowski [12], Kwaspisz [14] and Pandey et. al [17,18].

Oberg [16] has studied the following problem in $\mathbb{R}^n$:

$$\left.\begin{aligned} \frac{du(t)}{dt} &= f(t, u(t), u(h(t, u(t)))),\ t > 0; \\ u(0) &= u_0, u_0 \in \mathbb{R}^n, \end{aligned}\right\} \tag{2}$$

where $u : \mathbb{R}_+ \to \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$. The existence theorem for a solution to Problem (2) has been obtained by the Banach fixed point theorem, when f and h are continuous and uniformly locally Lipschitz on all of their variables.

The following problem with a deviated argument in a Banach space $(X, \|\cdot\|)$ has been studied by Gal [6],

$$\left.\begin{aligned} \frac{du}{dt} - Au(t) &= f(t, u(t), u(h(u(t), t))), \quad t > 0; \\ u(0) &= u_0,\ u_0 \in X, \end{aligned}\right\} \tag{3}$$

where $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators on X. The existence and uniqueness of a solution of (3) has been established under the following conditions on the functions f and h:

(a) $f : [0, \infty) \times X_\alpha \times X_{\alpha-1} \to X$ satisfies

$$\|f(t, x, x') - f(s, y, y')\| \leq L_f(|t - s|^{\theta_1} + \|x - y\|_\alpha + \|x' - y'\|_{\alpha-1})$$

for all $x, y \in X_\alpha,\ x', y' \in X_{\alpha-1},\ s, t \in [0, \infty)$, for some constants $L_f > 0$ and $\theta_1 \in (0, 1]$.

(b) $h : X_\alpha \times [0, \infty) \to [0, \infty)$ satisfies

$$|h(x, t) - h(y, s)| \leq L_h(\|x - y\|_\alpha + |t - s|^{\theta_2})$$

for all $x, y \in X_\alpha,\ s, t \in [0, \infty)$, for some constants $L_h > 0$ and $\theta_2 \in (0, 1]$.

For $0 < \alpha \leq 1$, $\|x\|_\alpha = \|(-A)^\alpha x\|$, denotes the norm on X_α, the domain of $(-A)^\alpha$.

The main objective is to establish the existence, uniqueness and asymptotic stability of a solution to Problem (1) generalizing some results of Gal [6]. In addition, we establish a stability theorem.

The article is organized as follows. We provide preliminaries, assumptions and lemmas needed for proving the main results in Section 2. We prove the local and global existence, and stability of a solution in Section 3. An example is considered to illustrate the application of the main results.

2 Preliminaries and Assumptions

In this section, we give basic assumptions, preliminaries and lemmas necessary to prove the main results. The material presented here can be found in more details by Friedman [4], Henry [9], Krien [13], Ladas and Lakshmikantham [15], Sobolevskiǐ [19] and Tanabe [20].

Let $(X, \|\cdot\|)$ be a complex Banach space. Let $T \in [0, \infty)$ and $\{A(t) : 0 \le t \le T\}$ be a family of closed linear operators on the Banach space X. We will use the following assumptions [4].

(A1) The domain $D(A)$ of $A(t)$ is dense in X and independent of t.

(A2) For each $t \in [0, T]$, the resolvent $R(\lambda; A(t))$ exists for all $\mathrm{Re}\lambda \le 0$ and there is a constant $C > 0$ (independent of t and λ) such that

$$\|R(\lambda; A(t))\| \le \frac{C}{|\lambda| + 1}, \ \mathrm{Re}\lambda \le 0, \ t \in [0, T].$$

(A3) For each fixed $s \in [0, T]$, there are constants $C > 0$ and $\rho \in (0, 1]$, such that

$$\|[A(t) - A(\tau)]A^{-1}(s)\| \le C|t - \tau|^{\rho}$$

for any $t, \tau \in [0, T]$. Here C and ρ are independent of t, τ and s.

The assumption **(A2)** implies that for each $s \in [0, T]$, $-A(s)$ generates a strongly continuous analytic semigroup $\{e^{-tA(s)} : t \ge 0\}$ in $B(X)$, where $B(X)$ denotes the Banach algebra of all bounded linear operators on X. Then there exist positive constants C and d such that

$$\|e^{-tA(s)}\| \le Ce^{-dt}, \quad t \ge 0; \tag{4}$$

$$\|A(s)e^{-tA(s)}\| \le \frac{Ce^{-dt}}{t}, \quad t > 0, \tag{5}$$

for all $s \in [0, T]$ [4].

The assumptions **(A1)**, **(A2)** and **(A3)** imply the existence of a unique fundamental solution $\{U(t, s) : 0 \le s \le t \le T\}$ to the homogeneous Cauchy problem that possesses the following properties [4]:

(i) $U(t, s) \in B(X)$ and $U(t, s)$ is strongly continuous in t, s for all $0 \le s \le t \le T$.

(ii) $U(t, s)x \in D(A)$ for each $x \in X$, for all $0 \le s \le t \le T$.

(iii) $U(t, r)U(r, s) = U(t, s)$ for all $0 \le s \le r \le t \le T$.

(iv) the derivative $\partial U(t, s)/\partial t$ exists in the strong operator topology and belongs to $B(X)$ for all $0 \le s < t \le T$, and strongly continuous in t, where $s < t \le T$.

(v) $\dfrac{\partial U(t, s)}{\partial t} + A(t)U(t, s) = 0$ and $U(s, s) = I$ for all $0 \le s < t \le T$.

For $\alpha > 0$, we define negative fractional powers $A(t)^{-\alpha}$ [4][cf. inequality 4] by

$$A(t)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-\tau A(t)} \tau^{\alpha - 1} d\tau.$$

Then $A(t)^{-\alpha}$ is bijective and bounded linear operator on X. We define the positive fractional powers of $A(t)$ by $A(t)^{\alpha} \equiv [A(t)^{-\alpha}]^{-1}$. Then $A(t)^{\alpha}$ is a closed linear operator with the domain $D(A(t)^{\alpha})$ dense in X and $D(A(t)^{\alpha}) \subset D(A(t)^{\beta})$ if $\alpha > \beta$. For $0 < \alpha \leq 1$, let $X_{\alpha} = D(A(0)^{\alpha})$ and equip this space with the graph norm

$$\|x\|_{\alpha} = \|A(0)^{\alpha}x\|.$$

Then X_{α} is a Banach space endowed with the norm $\|\cdot\|_{\alpha}$. If $0 < \alpha \leq 1$, the embedding $X_1 \hookrightarrow X_{\alpha} \hookrightarrow X$ are dense and continuous. For each $\alpha > 0$, define $X_{-\alpha} = (X_{\alpha})^*$, the dual space of X_{α}, and endow with the natural norm

$$\|x\|_{-\alpha} = \|A(0)^{-\alpha}x\|.$$

Also the assumption **(A3)** implies that there exists a constant $C > 0$ such that

$$\|A(t)A(s)^{-1}\| \leq C$$

for all $0 \leq s, t \leq T$. Hence, for each t, the functional $y \rightarrow \|A(t)y\|$ defines an equivalent norm on $D(A) \equiv D(A(0))$ and the mapping $t \rightarrow A(t)$ from $[0, T]$ into $\mathcal{L}(X_1, X)$ is uniformly Hölder continuous [10].

Let f and h be two continuous functions. For $0 < \alpha \leq 1$, let W_{α} and $W_{\alpha-1}$ be open sets in X_{α} and $X_{\alpha-1}$, respectively. For each $u' \in W_{\alpha}$ and $u'' \in W_{\alpha-1}$, there are balls such that $B_{\alpha}(u', r') \subset W_{\alpha}$ and $B_{\alpha-1}(u'', r'') \subset W_{\alpha-1}$, for some positive numbers r' and r''. We will use the following assumptions:

(A4) (a) There exist constants $L_f \equiv L_f(t, u', u'', r', r'') > 0$ and $0 < \theta_1 \leq 1$, such that the nonlinear map $f : [0, T] \times W_{\alpha} \times W_{\alpha-1} \rightarrow X$ satisfies the following condition

$$\|f(t, x, x') - f(s, y, y')\| \leq L_f(|t - s|^{\theta_1} + \|x - y\|_{\alpha} + \|x' - y'\|_{\alpha-1}) \tag{6}$$

for all $x, y \in B_{\alpha}$, $x', y' \in B_{\alpha-1}$ and for all $s, t \in [0, T]$.
(b) There exist constants $L_h \equiv L_h(t, u', r') > 0$ and $0 < \theta_2 \leq 1$ such that $h(\cdot, 0) = 0$, $h : W_{\alpha} \times [0, T] \rightarrow [0, T]$ satisfies the following condition

$$|h(x, t) - h(y, s)| \leq L_h(\|x - y\|_{\alpha} + |t - s|^{\theta_2}) \tag{7}$$

for all $x, y \in B_{\alpha}$ and for all $s, t \in [0, T]$.

For $t_0 \geq 0$ and $0 < \beta \leq 1$, let $C^{\beta}([t_0, T]; X)$ denote the space uniformly Hölder continuous on $[t_0, T]$ with exponent β. Then $C^{\beta}([t_0, T]; X)$ is a Banach space endowed with the norm

$$\|h\|_{C^{\beta}([t_0,T];X)} = \sup_{t_0 \leq t \leq T} \|h(t)\| + \sup_{t,s \in [t_0,T], t \neq s} \frac{\|h(t) - h(s)\|}{|t - s|^{\beta}}.$$

Now we consider the following inhomogeneous Cauchy problem

$$\frac{du}{dt} + A(t)u = f(t), \quad u(t_0) = u_0. \tag{8}$$

Theorem 2.1 *[4, Theorem II. 3.1] Suppose that the assumptions **(A1)**–**(A3)** hold. If $f \in C^{\beta}([t_0, T]; X)$, then the unique solution of (8) is given by*

$$u(t) = U(t, t_0)u_0 + \int_{t_0}^{t} U(t, s)f(s)ds, \quad t_0 \leq t \leq T.$$

Indeed, $u : [t_0, T] \rightarrow X$ is strongly continuously differentiable on $(t_0, T]$.

The following lemmas will be used in the subsequent sections.

Lemma 2.1 *[5, Lemma 1.1] For $h \in C^\beta([t_0,T];X)$, we define $Q : C^\beta([t_0,T];X) \to C([t_0,T];X_1)$ by*

$$Qh(t) = \int_{t_0}^{t} U(t,s)h(s)ds, \ t_0 \le t \le T.$$

Then Q is a bounded mapping and $\|Qh\|_{C([t_0,T];X_1)} \le C\|h\|_{C^\beta([t_0,T];X)}$ for some $C > 0$.

We have the following corollary from Lemma 2.1.

Corollary 2.1 *For $y \in X_1$, we define*

$$H(y;h) = U(t,0)y + \int_0^t U(t,s)h(s)ds, \ 0 \le t \le T.$$

Then H is a bounded linear mapping from $X_1 \times C^\beta([t_0,T];X)$ into $C([t_0,T];X_1)$.

Lemma 2.2 *[10, Lemma 2] Let $0 < \alpha \le 1$ and $f \in C([t_0,T];X_\alpha)$. We define*

$$v(t) = \int_{t_0}^{t} U(t,s)f(s)ds, \ t_0 \le t \le T.$$

Then $v \in C([t_0,T];X_1) \cap C^1((t_0,T];X)$ and $v'(t) + A(t)v(t) = f(t), \ t_0 < t \le T$.

3 Main Results

In this section, we establish the main results. Let $I = [0,\delta]$ for some positive number δ to be specified later. Let $\mathcal{C}_\alpha$, $0 \le \alpha \le 1$ denote the space of all X_α-valued continuous functions on I, endowed with the sup-norm, $\sup_{t\in I} \|\psi(t)\|_\alpha$, $\psi \in C(I;X_\alpha)$. Let

$$Y_\alpha = \mathcal{C}_{L_\alpha}(I;X_{\alpha-1}) = \{\psi \in \mathcal{C}_\alpha : \|\psi(t) - \psi(s)\|_{\alpha-1} \le L_\alpha|t-s|, \text{ for all } t,s \in I\},$$

where L_α is a positive constant to be specified later. It is clear that Y_α is a Banach space under the sup-norm of $\mathcal{C}_\alpha$.

Definition 3.1 A continuous function $u : I \to X$ said to be a solution of Problem (1) if the following are satisfied:

(i) $u(\cdot) \in \mathcal{C}_{L_\alpha}(I;X_{\alpha-1}) \cap C^1((0,\delta);X) \cap C(I;X)$;

(ii) $u(t) \in W_\alpha$, for all $t \in (0,\delta)$;

(iii) $\dfrac{du}{dt} + A(t)u(t) = f(t,u(t),u(h(u(t),t)))$ for all $t \in (0,\delta)$;

(iv) $u(0) = u_0$.

For $0 < \alpha < \beta \le 1$, let $u_0 \in X_\alpha$. Let $r > 0$ be chosen small enough such that the assumption **(A4)** holds for the closed balls $B_\alpha \equiv B_\alpha(u_0,r)$ and $B_{\alpha-1} \equiv B_{\alpha-1}(u_0,r)$. Let $K > 0$ and $0 < \eta < \beta - \alpha$ be fixed constants. Let

$$\mathcal{S}_\alpha = \{y \in \mathcal{C}_\alpha \cap Y_\alpha : y(0) = u_0, \sup_{t\in I} \|y(t) - u_0\|_\alpha \le r, \|y(t) - y(s)\|_\alpha \le K|t-s|^\eta \ \forall\, t,s \in I\}.$$

Then $\mathcal{S}_\alpha$ is a non-empty closed and bounded subset of $\mathcal{C}_\alpha$.

3.1 Local existence of solution

Now we prove the following theorem of the local existence of a solution to Problem (1). The proof is based on the ideas of Friedman [4] and Gal [6].

Theorem 3.1 *Let $u_0 \in X_\beta$, where $0 < \alpha < \beta \leq 1$. If the assumptions **(A1)-(A4)** hold, then there exist a positive number $\delta \equiv \delta(\alpha, u_0)$ and a unique solution $u(t)$ to Problem (1) on the interval $[0, \delta]$ such that $u \in \mathcal{S}_\alpha \cap C^1((0,\delta); X)$.*

Proof. Let $v \in \mathcal{S}_\alpha$. We define $f_v(t) = f(t, v(t), v(h(v(t), t)))$. Then the assumption **(A4)** implies that f_v is Hölder continuous on I of exponent $\gamma = \min\{\theta_1, \theta_2, \eta\}$. We consider the following problem:

$$\left.\begin{array}{rcl} \dfrac{du}{dt} + A(t)u(t) &=& f_v(t), \quad t \in I; \\ u(0) &=& u_0. \end{array}\right\} \tag{9}$$

Then by Theorem 2.1, there exists a unique solution u_v of (9) which is given by

$$u_v(t) = U(t,0)u_0 + \int_0^t U(t,s) f_v(s) ds, \quad t \in I.$$

We define a map F by

$$Fv(t) = U(t,0)u_0 + \int_0^t U(t,s) f_v(s) ds, \qquad \text{for each } t \in I.$$

We will claim that F maps from $\mathcal{S}_\alpha$ into itself, for sufficiently small $\delta > 0$. Indeed, if $t_1, t_2 \in I$ with $t_2 > t_1$, then we have

$$\begin{aligned} \|Fv(t_2) - Fv(t_1)\|_{\alpha-1} \leq{}& \|[U(t_2,0) - U(t_1,0)]u_0\|_{\alpha-1} \\ &+ \left\| \int_0^{t_2} U(t_2,s) f_v(s) ds - \int_0^{t_1} U(t_1,s) f_v(s) ds \right\|_{\alpha-1}. \end{aligned} \tag{10}$$

We will use the bounded inclusion $X \subset X_{\alpha-1}$ to estimate each of the terms on the right hand side of (10). The first term on the right hand side of (10) is estimated as follows [4, see Lemma II. 14.1],

$$\|(U(t_2,0) - U(t_1,0))u_0\|_{\alpha-1} \leq C_1 \|u_0\|_\alpha (t_2 - t_1), \tag{11}$$

where C_1 is some positive constant. We have the following estimate for the second term on the right hand side of (10) [4, Lemma II. 14.4],

$$\begin{aligned} &\left\| \int_0^{t_2} U(t_2,s) f_v(s) ds - \int_0^{t_1} U(t_1,s) f_v(s) ds \right\|_{\alpha-1} \\ &\quad \leq C_2 N_1 (t_2 - t_1)(|\log(t_2 - t_1)| + 1), \end{aligned} \tag{12}$$

where $N_1 = \sup\limits_{s \in [0,T]} \|f_v(s)\|$ and C_2 is some positive constant.

Using the estimates (11) and (12), we get from the inequality (10),

$$\|Fv(t_2) - Fv(t_1)\|_{\alpha-1} \leq L_\alpha |t_2 - t_1|,$$

where $L_\alpha = \max\{C_1\|u_0\|_\alpha, C_2N_1(|\log(t_2-t_1)|+1)\}$ that depends on C_1, C_2, N_1, δ.

Next our aim is to show that $\sup_{t\in I}\|F(y)(t)-u_0\|_\alpha \le r$, for sufficiently small $\delta > 0$. Since $u_0 \in X_\alpha$, we can choose sufficiently small $\delta_1 > 0$ such that [4, Lemma II.14.1],

$$\|U(t,0)u_0 - u_0\|_\alpha \le \frac{r}{3}, \quad \text{for all } t \in [0,\delta_1]. \tag{13}$$

We choose $\delta_2 > 0$ such that

$$\left(\frac{C(\alpha)}{1-\alpha}L_f[(1+L_\alpha L_h)r + \delta_2^{\theta_2}] + \frac{C(\alpha)K_1}{1-\alpha}\right)\delta_2^{1-\alpha} \le \frac{2r}{3}.$$

Let $K_1 := \sup_{0\le t\le T}\|f(t,u_0,u_0)\|$. For $v \in \mathcal{S}_\alpha$ and $t \in [0,\delta_2]$, it follows from the assumption **(A4)** [19, cf. inequality (1.65), p. 23], (6), (7) and $h(u_0,0) = 0$ that

$$\begin{aligned}
&\left\|\int_0^t U(t,s)f_v(s)ds\right\|_\alpha \\
&\le C(\alpha)L_f\int_0^t (t-s)^{-\alpha}[\|v(s)-u_0\|_\alpha + \|v([h(v(s),s)]) - u_0\|_{\alpha-1}]ds \\
&\quad + C(\alpha)K_1\int_0^t (t-s)^{-\alpha}ds \\
&\le C(\alpha)L_f\int_0^t (t-s)^{-\alpha}[\|v(s)-u_0\|_\alpha + L_\alpha|h((v(s),s)) - h(u(0),0)|]ds \\
&\quad + C(\alpha)K_1\int_0^t (t-s)^{-\alpha}ds \\
&\le C(\alpha)L_f\int_0^t (t-s)^{-\alpha}[\|v(s)-u_0\|_\alpha + L_\alpha|h((v(s),s)) - h(u(0),0)|]ds \\
&\quad + \frac{C(\alpha)K_1\delta^{1-\alpha}}{1-\alpha} \\
&\le C(\alpha)L_f\int_0^t (t-s)^{-\alpha}[r + L_\alpha L_h(\|v(s)-u_0\|_\alpha + s^{\theta_2})]ds + \frac{C(\alpha)K_1\delta_2^{1-\alpha}}{1-\alpha} \\
&\le C(\alpha)L_f[(1+L_\alpha L_h)r + \delta_2^{\theta_2}]\int_0^t (t-s)^{-\alpha}ds + \frac{C(\alpha)K_1\delta_2^{1-\alpha}}{1-\alpha} \\
&\le \left(\frac{C(\alpha)}{1-\alpha}L_f[(1+L_\alpha L_h)r + \delta_2^{\theta_2}] + \frac{C(\alpha)K_1}{1-\alpha}\right)\delta_2^{1-\alpha}.
\end{aligned} \tag{14}$$

Combining (13) and (14), we obtain $\sup_{t\in I}\|Fv(t)-u_0\|_\alpha \le r$, where $\delta_3 = \min\{\delta_1,\delta_2\}$ [6, cf. p. 977].

Next we show that $\|Fv(t+h) - Fv(t)\|_\alpha \le Kh^\eta$ for some constant $K > 0$ and $0 < \eta < 1$. If $0 \le \alpha < \beta \le 1$ and $0 \le t \le t+h \le \delta$, then we have

$$\begin{aligned}
\|Fv(t+h) - Fv(t)\|_\alpha \le &\|[U(t+h,0) - U(t,0)]u_0\|_\alpha \\
&+ \left\|\int_0^{t+h} U(t+h,s)f_v(s)ds - \int_0^t U(t,s)f_v(s)ds\right\|_\alpha.
\end{aligned}$$

Using [4, Lemma II.14.1 and Lemma II.14.4], we get the following estimates

$$\|[U(t+h,0)-U(t,0)]u_0\|_\alpha \leq C(\alpha,u_0)h^{\beta-\alpha}; \tag{15}$$

$$\left\|\int_0^{t+h} U(t+h,s)f_v(s)ds - \int_0^t U(t,s)f_v(s)ds\right\|_\alpha \leq C(\alpha)N_1h^{1-\alpha}(1+|\log h|). \tag{16}$$

From (15) and (16), it is clear that

$$\|Fv(t+h)-Fv(t)\|_\alpha \leq h^\eta[C(\alpha,u_0)\delta^{\beta-\alpha-\eta}+C(\alpha)N_1\delta^\nu h^{1-\alpha-\eta-\nu}(|\log h|+1)]$$

for any $\nu > 0$ and $\nu < 1-\alpha-\eta$. Hence, for sufficiently small $\delta > 0$, we have

$$\|Fv(t+h)-Fv(t)\|_\alpha \leq Kh^\eta$$

for some $K > 0$. Thus F maps $\mathcal{S}_\alpha$ into itself.

Finally, we show that F is a contraction map. We choose $\delta_4 > 0$ such that

$$\frac{C(\alpha)}{1-\alpha}L_f(2+L_\alpha L_h)\delta_4^{1-\alpha} < \frac{1}{2}.$$

Let $v_1, v_2 \in S_\alpha$ and $t \in [0,\delta_4]$. Then we have [19, cf. inequality (1.65), page 23],

$$\begin{aligned}\|Fv_1(t)-Fv_2(t)\|_\alpha &\leq C(\alpha)L_f\int_0^t(t-s)^{-\alpha}(\|v_1(s)-v_2(s)\|_\alpha \\ &\qquad + \|v_1([h(v_1(s),s)])-v_2([h(v_2(s),s)])\|_{\alpha-1})ds \\ &\leq C(\alpha)L_f(2+L_\alpha L_h)\int_0^t(t-s)^{-\alpha}\|v_1(s)-v_2(s)\|_\alpha ds \\ &\leq \frac{C(\alpha)}{1-\alpha}L_f(2+L_\alpha L_h)\delta_4^{1-\alpha}\sup_{t\in I}\|v_1(t)-v_2(t)\|_\alpha. \end{aligned} \tag{17}$$

Then, from (17), it is clear that F is a contraction map. Since $\mathcal{S}_\alpha$ is a complete metric space, by the Banach fixed-point theorem, there exists $u \in \mathcal{S}_\alpha$ such that $Fu = u$. From Lemma 2.1 and Theorem 2.1, it follows that $u \in C^1((0,\delta);X)$. Thus u is a solution to Problem (1) on $[0,\delta]$, where $\delta = \min\{\delta_3,\delta_4\}$.

3.2 Global existence of solution

In this section, we prove the global existence of a solution to Problem (1).

Theorem 3.2 *Assume that **(A1)–(A4)** hold. Suppose that there are positive constants k_1 and k_2 such that*

$$\begin{aligned}\|f(t,x,y)\| &\leq k_1(1+\|x\|_\alpha+\|y\|_{\alpha-1}) \text{ for } \quad 0<\alpha<1, &(18)\\ |h(z,t)| &\leq k_2(1+\|z\|_\alpha) &(19)\end{aligned}$$

for all t, where $0 \leq t \leq T$, $x, z \in X_\alpha$ and $y \in X_{\alpha-1}$, then the initial value problem (1) has a unique solution that exists for all $t \in [0,T]$, for each $u_0 \in W_\beta$, where $0 < \alpha < \beta \leq 1$.

Proof. Let $\delta > 0$ be sufficiently small such that $u(t)$, $t \in (0, \delta]$, be the local solution of (1) obtained in Theorem 3.1. So for the global existence of a solution to problem (1), it is enough to show that $\|u(t)\|_\alpha$ is bounded as $t \uparrow \delta$ and this bound is independent of t.

Now using (6), (7), (18) and (19), we get, for $u(.) \in X_1$,

$$\begin{aligned}\|u(t)\|_\alpha &\leq \|U(t,0)u_0\|_\alpha + \left\| \int_0^t U(t,s) f(s, u(s), u(h(u(s), s))) ds \right\|_\alpha \\ &\leq \|A(0)^\alpha A(t)^{-\beta} A(t)^\beta U(t,0) A(0)^{-\beta} A(0)^\beta u_0\| \\ &\quad + k_1 \int_0^t (t-s)^{-\alpha} [(1 + \|u(s)\|_\alpha + L_\alpha |h(u(s), s) - h(u_0, 0)| + \|u_0\|_{\alpha-1}] ds.\end{aligned} \tag{20}$$

Using [4, inequality (II.14.12) and (II.14.14)] in (20), we get

$$\|u(t)\|_\alpha \leq (C' + D)\|u_0\|_\alpha + k_1[1 + (1 + L_\alpha k_2)] \int_0^t (t-s)^{-\alpha}(1 + \|u(s)\|_\alpha) ds,$$

where $D = \sup_{t \in [0,T]} K k_1 \int_0^t (t-s)^{-\alpha} ds$, K is the constant in the bounded inclusion $X \subset X_{\alpha-1}$ and C' is some positive constant. Applying the Gronwall lemma, we get that $\|u(t)\|_\alpha$ is bounded as $t \uparrow \delta$.

Remark 3.1 In the case when $A(t)$ is a self adjoint positive definite operator in a Hilbert space X, Theorem 3.1 and Theorem 3.2 can be strengthened. Assumptions **(A1), (A2)** and **(A3)** imply that, for $0 \leq \alpha \leq 1$ and for all $s, t \in [0,T]$ [13, p. 185],

$$\|A(t)^\alpha A(s)^{-\alpha}\| \leq C\|A(t)A(s)^{-1}\|^\alpha \leq C', \tag{21}$$

where $C, C' > 0$. Then we can prove Theorem 3.1 and Theorem 3.2 with a less regularity assumption on u_0.

3.3 Existence of solution with regularity

In this section, we give a theorem with more regularity on f and u_0. We denote $D(A(0))$ by X_1. We equipped this space X_1 with the graph norm

$$\|x\|_1 := (\|x\|^2 + \|A(0)x\|^2)^{\frac{1}{2}},$$

that is equivalent to the usual norm $\|A(0)x\|$ for $x \in D(A(0))$.

Let f and h be two continuous functions. Let W_1 and W be open sets in X_1 and X, respectively. For each $u \in W_1$ and $u' \in W$, there are balls such that $B_1(u, r) \subset W_1$ and $B(u', r') \subset W$. We will make use of the following stronger assumptions:

$(\mathbf{A4})'$ (a) There exist constants $L_f \equiv L_f(t, u, u', r, r') > 0$ and $0 < \theta_1 \leq 1$, such that the nonlinear map $f : [0,T] \times W_1 \times W \to X_\alpha$ satisfies:

$$\|f(t, x, x') - f(s, y, y')\|_\alpha \leq L_f(|t-s|^{\theta_1} + \|x - y\|_1 + \|x' - y'\|) \tag{22}$$

for all $x, y \in B_1$, $x', y' \in B$, for all $s, t \in [0,T]$ and $\alpha \in (0,1)$.

(b) There exist constants $L_h \equiv L_h(t, u', r') > 0$ and $0 < \theta_2 \leq 1$, such that $h(\cdot, 0) = 0$, $h : W_1 \times [0,T] \to [0,T]$ satisfies:

$$|h(x,t) - h(y,s)| \leq L_h(\|x - y\|_1 + |t-s|^{\theta_2}) \tag{23}$$

for all $x, y \in B_1$ and for all $s, t \in [0,T]$.

Then we have the following theorem.

Theorem 3.3 *Let $u_0 \in W_1$. Suppose that the assumptions **(A1)-(A3)** and $(\mathbf{A4})'$ hold. Then there exist a positive number $\delta \equiv \delta(u_0)$ and a unique solution $u(t)$ of Problem (1) on the interval $[0,\delta]$ such that $\in C_L(I;X) \cap C^1((0,\delta);X) \cap C(I;X)$, where*

$$C_L(I;X) = \{\psi \in C(I;X_1) : \|\psi(t) - \psi(s)\| \leq L|t-s|, \text{ for all } t,s \in I\},$$

for some $L > 0$. Further, we assume that there are positive constants k_1 and k_2 such that

$$\begin{aligned} \|f(t,x,y)\|_\alpha &\leq k_1(1 + \|x\|_1 + \|y\|) \ \text{ for } \ 0 < \alpha < 1, && (24)\\ |h(z,t)| &\leq k_2(1 + \|z\|_1), && (25) \end{aligned}$$

for all t, $x,z \in X_1$ and $y \in X$, where $0 \leq t \leq T$. Then the unique solution of (1) exists for all $t \geq 0$.

Proof. We denote the interval $[0,\delta]$ by I. For each $v \in C(I,B_1)$, we define a map F by

$$Fv(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s,v(s),v(h(v(s),s)))ds \quad \text{ for each } t \in I.$$

By Lemma 2.2, the map F from $C(I,B_1)$ into $C(I;X_1)$ is well defined. The proof of this Theorem can be obtained by the same argument as in the proof of Theorem 3.1 and Theorem 3.2. Thus, we omit the details of the proof.

3.4 Asymptotic stability of solution

In this section, we discuss the asymptotic stability of a solution to Problem (1) in X. The proof is based on the ideas of Friedman [4] and Webb [21].

Theorem 3.4 *Suppose that the assumptions **(A1)-(A4)** hold, $u_0 \in X_\beta$, where $0 < \alpha < \beta \leq 1$ and there exists a continuous solution $u \in X_\alpha$. Suppose there exist a continuous function $\epsilon : [0,\infty) \to [0,\infty)$ and a constant $k_3 > 0$ such that*

$$\|f(t,u(t),u(h(u(t),t)))\| \leq k_3(\epsilon(t) + \|u(t)\|_\alpha + \|u(t)\|_{\alpha-1}) \ \text{ for } \ 0 < \alpha < 1, t \geq 0. \quad (26)$$

Then

(i) if $\epsilon(t)$ is bounded on $[0,\infty)$, then $\|u(t)\|_\alpha$ is bounded on $[0,\infty)$;

(ii) if $\epsilon(t) = \mathrm{O}(e^{\sigma t})$ for some $-1 < \sigma < 0$, then $\|u(t)\|_\alpha = \mathrm{O}(e^{\sigma t})$;

(iii) if $\epsilon(t) = \mathrm{o}(1)$, then $\|u(t)\|_\alpha = \mathrm{o}(1)$.

Proof. It is known [4, p. 176] that there exists $0 < \theta < d$, such that

$$\|A^\gamma(t)U(t,0)\| \leq \frac{C}{t^\gamma}e^{-\theta t} \text{ if } \ t > 0, \quad (27)$$

for any $0 \leq \gamma \leq 1$.

Now, for $t > 0$, put $\varphi(t) = e^{\theta t}\|u(t)\|_\alpha$. Using (27) to the solution of Problem (1), we obtain

$$\begin{aligned}\varphi(t) &\le Ct^{-\alpha}\|u_0\| + C\int_0^t e^{\theta s}(t-s)^{-\alpha}k_3[\epsilon(s) + \|u(s)\|_\alpha + \|u(s)\|_{\alpha-1}]ds \\ &\le Ct^{-\alpha}\|u_0\| + Ck_3\int_0^t e^{\theta s}(t-s)^{-\alpha}\epsilon(s)ds + Ck_3(1+K)\int_0^t (t-s)^{-\alpha}\varphi(s)ds \\ &\le \left\{C_0t^{-\alpha}\|u_0\| + C_0\int_0^t e^{\theta s}(t-s)^{-\alpha}\epsilon(s)ds\right\} + C_0\int_0^t (t-s)^{-\alpha}\varphi(s)ds, \qquad (28)\end{aligned}$$

where $C_0 = \max\{C, Ck_3, Ck_3(1+K)\}$. We denote

$$\chi(t) = C_0t^{-\alpha}\|u_0\| + C_0\int_0^t e^{\theta s}(t-s)^{-\alpha}\epsilon(s)ds.$$

Then it is clear that

$$\chi(t) \le C_0t^{-\alpha}\|u_0\| + \tilde{C}e^{\theta t}\sup_{0\le s<\infty}\epsilon(s), \qquad (29)$$

for some constant $\tilde{C} > 0$. We get from (28) by the method of iteration that [21],

$$\varphi(t) \le \chi(t) + \int_0^t \left[\sum_0^\infty \frac{(t-s)^{j-1-j\alpha}[\Gamma(1-\alpha)]^j}{\Gamma(j-j\alpha)}\right]\chi(s)ds.$$

We note that the series in the bracket is bounded by $B_1(t-s)^{-\alpha}\exp[B_2(t-s)^{1-\alpha}]$ for some constants $B_1, B_2 > 0$. Thus it follows that, for $t \ge 1$ and for any $\lambda > 0$,

$$\varphi(t) \le B_3e^{\lambda t}\|u_0\| + B_4e^{\theta t}\sup_{0\le s<\infty}\epsilon(s), \qquad (30)$$

where B_3 and B_4 are some positive constants. Thus, for any $0 < \theta_0 < \theta$, we get

$$\|u(t)\|_\alpha \le B_3e^{-\theta_0 t}\|u_0\| + B_4\sup_{0\le s<\infty}\epsilon(s). \qquad (31)$$

The proof follows from the inequality (31).

4 Example

Consider the following differential equation with deviated argument [6, 10]:

$$\left.\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(k(t,x)\frac{\partial}{\partial x}u(x)) &= \widetilde{H}(x, u(t,x)) + \widetilde{G}(t, x, u(t,x)); \\ u(t,0) &= u(t,1), \quad t > 0; \\ u(0,x) &= u_0(x), \quad x \in (0,1).\end{aligned}\right\} \qquad (32)$$

Here, $\widetilde{H}(x, u(t,x)) = \int_0^x K(x,y)u(\widetilde{g}(t)|u(t,y)|, y)dy$ for all $(t,x) \in (0,\infty)\times(0,1)$. Assume that $\widetilde{g} : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in t with $\widetilde{g}(0) = 0$ and $K \in C^1([0,1]\times[0,1];\mathbb{R})$. The function $\widetilde{G} : \mathbb{R}_+ \times [0,1]\times\mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u, uniformly in x [6].

We assume that k is positive function with continuous partial derivative k_x such that, for all $0 \le t < \infty$ and $x \in (0,1)$,

(i) $0 < k_0 \le k(t,x) < k_0^{'}$,

(ii) $|k_x(t,x)| \le k_1$,

(iii) $|k(t,x) - k(s,x)| \le C|t-s|^{\epsilon}$,

(iv) $|k_x(t,x) - k_x(s,x)| \le C|t-s|^{\epsilon}$,

for some ϵ with $0 < \epsilon \le 1$, some constants k_0, $k_0^{'}$, and $C > 0$.

Let $X = L^2((0,1);\mathbb{R})$. We define $X_1 = D(A(0)) = H^2(0,1) \cap H_0^1(0,1)$ and $A(t)u(t) = -\dfrac{\partial}{\partial x}(k(t,x)\dfrac{\partial}{\partial x}u(x))$. Then $X_{1/2} = D((A(0))^{1/2}) = H_0^1(0,1)$. Then the family $\{A(t) : t > 0\}$ satisfies the assumptions **(A1)**, **(A2)** and **(A3)** on each bounded interval $[0,T]$ [10].

For $x \in (0,1)$, we define $f : \mathbb{R}_+ \times H^2(0,1) \times L^2(0,1) \to H_0^1(0,1)$ by

$$f(t,\phi,\psi) = \widetilde{H}(x,\psi) + \widetilde{G}(t,x,\phi),$$

where $\widetilde{H}(x,\psi(x,t)) = \displaystyle\int_0^x K(x,y)\psi(y,t)dy$ and $\widetilde{G} : \mathbb{R}_+ \times [0,1] \times H^2(0,1) \to H_0^1(0,1)$ satisfies $\|\widetilde{G}(t,x,u)\|_{H_0^1(0,1)} \le C(1 + \|u\|_{H^2(0,1)})$, for some $C > 0$. Then it can be shown that f satisfies the condition (22)(see Gal [6]) and $h : H^2(0,1) \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by $h(\phi(x,t),t) = \widetilde{g}(t)|\phi(x,t)|$ satisfies (23) (see Gal [6]). Thus, we can apply the results of previous sections to study the existence, uniqueness and asymptotic stability of solution of (32).

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Boundary Stabilization of a Plate in Contact with a Fluid

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Abstract: This paper presents a solution to the boundary stabilization of a vibrating plate under fluid loading. The fluid is considered to be compressible, barotropic and inviscid. A linear control law is constructed to suppress the plate vibration. The control forces and moments consist of feedbacks of the velocity and normal derivative of the velocity at the boundaries of the plate. The novel features of the proposed method are that (1) it asymptotically stabilizes vibrations of a plate in contact with fluid (the fluid has a free surface) via boundary control and without truncation of the model; and (2) the stabilization of both plate vibrations and fluid motions are simultaneously achieved by using only a linear feedback from the plate boundaries.

Keywords: *semigroups of operators; LaSalle invariant set theorem; asymptotic stabilization; Kirchhoff plate; compressible Newtonain barotropic fluid.*

Mathematics Subject Classification (2010): 35M12, 35Q30.

1 Introduction

The vibration of a plate in contact with fluids has been thoroughly analyzed by many authors [1–3]. Such problems appear frequently in practice, for example when studying the veins, pulmonary passages and urinary systems which can be modeled as shells conveying fluid, aero-elastic instabilities around flexible aircraft, container conveying the fluids and dams [1–5].

One of the most challenging practical difficulties which is present in many of the fluid-structure applications is the vibration of the structures. This may be due to relatively low rigidity and small structural damping and a little excitation may lead long vibration decay time. Vibration is the most destructing source for the flexible structures.

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Therefore, vibration of flexible structures is capable for disturbance, discomfort, damage and destruction. In particular, many researchers have studied the problem of vibration suppression (stabilization) of plates (without and with being in contact with a fluid) since the plate is a necessary element in many applications such as aircraft's skin and flexible structures. In particular, it is widely used in fluid-structure systems [1, 2, 4, 5]. Therefore, an important question in the research of experimentalists and applied mathematicians in the field of flexible structures is the control and stability of vibrating plate under arbitrary loading (such as fluid loading) [6, 8–11]. That is, if the equilibrium state is slightly disturbed, do the perturbations grow or decay? Therefore, suppressing the vibration of such plates (under heavy fluid loading) takes attention of control researchers that investigate in this field.

Boundary stabilization methods are efficient methods to exclude the problems of both in-domain measurement and actuation. The boundary actuators designed for the nondiscretized PDE models are often simple compensators which ensure closed-loop stability for an infinite number of modes.

For some references in boundary stabilization methods, see [12]. Several researchers have proposed boundary actuators for a variety of flexible systems [9, 10, 12–15]. Some researches have been concerned with the fluid-structure stabilization problem, [3, 16]. In these studies, the fluid doesn't have free surface; however, in fact, in most of fluid-structure problems such as dams, large containers, the fluid has at least a free surface. Therefore, in this work we study the stabilization problem of vibrating plate in contact with a fluid having free surface; also we present the simulation results which verify our mathematical results. The fluid is considered to be barotropic compressible Newtonian fluid whereas the plate is taken to be Kirchhoff plate. We use the semigroup techniques to demonstrate the well-posedness of the system. Then benefitting from the Lyapunov stability method and the LaSalle's invariant set theorem, we prove the asymptotic stability of the closed loop system. The main objective of this paper is to use boundary control method for stabilizing the plate vibration in contact with a fluid having free surface via boundary actuators at the plate boundary. It should be noted that the Lyapunov methods are extended to various applications [17, 18]. The presented method uses control actuators at the boundaries of structure.

This article is arranged as follows. In Section 2, the dynamics of a plate and surrounding fluid are presented. Section 3 is devoted to well-posedness and boundary stabilization proof of the fluid-structure problem. Section 4 presents the simulation results. Section 5 is devoted to the conclusion.

2 Governing Equations of Motion

2.1 Fluid domain

The governing equations for the Newtonian barotropic fluid with low velocity can be simplified from the Navier–Stockes equation to the wave equation [19]. The related equations are listed below

$$\begin{cases} c^2\Delta\phi = \phi_{,tt} & \text{in} \quad \Theta, \\ \rho_0\phi_{,t} = -p(x,y,0,t) & \text{in} \quad \Omega, \\ \rho_0\phi_{,tt} + \rho_0 g\phi_{,n} + p_{e,t} = 0 & \text{in} \quad \Omega_2, \\ \phi_{,n} = 0 & \text{in} \quad \Omega_3, \end{cases} \tag{1}$$

where Ω, Ω_2 and Ω_3 are defined as follows:

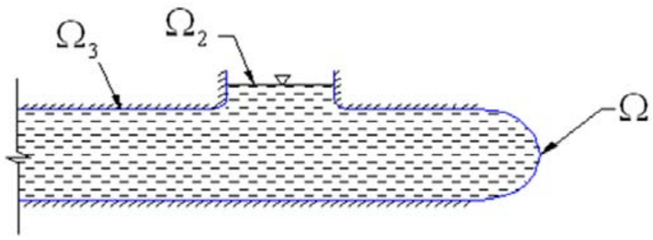

Figure 1: Different boundaries of the fluid-structure system.

1) The wet surface or the fluid structure interface (see Figure 1).

This is the most essential part of the fluid boundary. The motion of the structure and the normal component of the fluid motion coincide, that is [19]:

$$\mathbf{v_f}.\mathbf{n} = \mathbf{v}_s.\mathbf{n}, \tag{2}$$

where $\mathbf{v_f}$ is the fluid velocity and $\mathbf{v}_s$ is the structure velocity.

In this boundary the following equation can be attained [19]:

$$\rho_0\phi_{,t} = -p(x, y, 0, t). \tag{3}$$

2) A free surface with prescribed external pressure, where we allow the linearized (gravitational) waves Ω_2 (see Figure 2) [19]:

$$\rho_0\phi_{,tt} + \rho_0 g\phi_{,n} + p_{e,t} = 0. \tag{4}$$

3) Fixed surface with prescribed external pressure, Ω_3, see Figure1, [19]:

$$\phi_{,n} = 0, \tag{5}$$

where $\phi(x, y, z, t)$ is the velocity potential. This means that $\mathbf{v} = \nabla\phi$ and c is the sound speed in the fluid.

2.2 Structure Domain

The governing equation of a Kirchhoff's plate with external pressure $p(x, y, 0, t)$ can be written as follows [8]:

$$\left\{\begin{array}{lll} D\nabla^4 w + \rho h w_{,tt} = p & \text{in} & \Omega, \\ w = \partial w/\partial n = 0 & \text{in} & \Gamma_0, \\ V^{(n)} + \partial M^{(ns)}/\partial s = U_1\ ,\ M^{(n)} = U_2 & \text{in} & \Gamma_1, \end{array}\right\} \tag{6}$$

$\forall(x, y, t) \in \Omega\times[0, \infty)$; where $w(x, y, t)$ represents the transverse displacement, $p(x, y, 0, t)$ is the external transverse force distribution (hydrodynamic pressure due to fluid loading) on the plate, h is the thickness of the plate, E is the Young's modulus of elasticity, ν is the Poisson's ratio and $D = Eh/(12(1 - \nu^2))$ is the flexural rigidity. It should be noted that Ω is a bounded simple region and $\mathbf{n} = (n_1, n_2)$ is the unit outward normal vector to the boundaries of the plate. M_{11}, M_{12}, M_{22} and V_1, V_2 are defined in the Appendix.

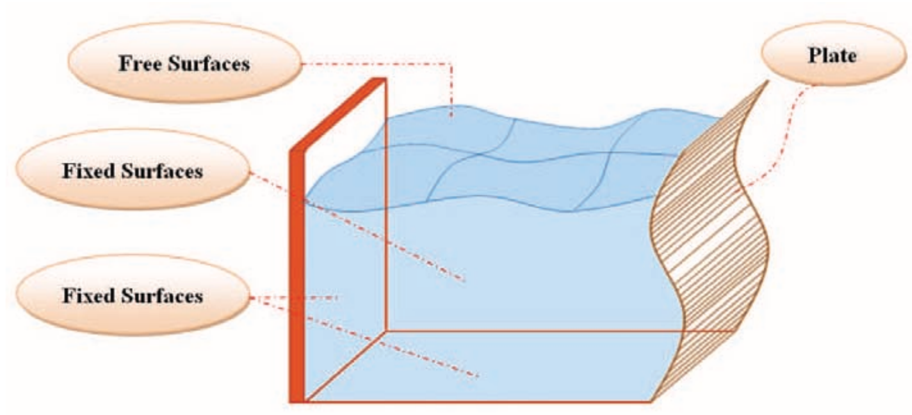

Figure 2: Schematic view of the fluid-structure problem.

3 Stabilization of Plate Under Heavy Fluid Loading

In this section, we consider the stabilization problem of the vibration of a plate without any boundary attachment. For this purpose, first, the following definitions will be used. The inner product on the space $\mathbf{H} = H^1_{\Omega_3}(\Theta) \times \mathbb{L}^2(\Theta) \times H^2_{\Gamma_0}(\Omega) \times L^2(\Omega)$ will be presented as

$$< X, Y >_H = \int_\Theta [\frac{\rho_0}{2c^2}\tau_1\tau_2 + \frac{\rho_0}{2}\Pi(\kappa_1, \kappa_2)]d\theta + \int_\Omega [\frac{\rho_0}{2g}\tau_1\tau_2 + \frac{\rho h}{2}\zeta_1\zeta_2 + \Lambda(\eta_1, \eta_2)]d\Omega, \quad (7)$$

where $X, Y \in \mathbf{H}, X = (\kappa_1, \tau_1, \eta_1, \zeta_1), Y = (\kappa_2, \tau_2, \eta_2, \zeta_2), H^2_{\Omega_3}(\Theta) = \{\kappa_1 : \kappa_1 \in H^2(\Theta) : \partial\kappa_1/\partial n = 0|_{\Omega_1}\}$ and $H^2_{\Gamma_0}(\Omega) = \{\xi_1 : \xi_1 \in H^2(\Omega) : \xi_1 = 0|_{\Gamma_0}, \partial\xi_1/\partial n|_{\Gamma_0} = 0\}$; also the following relations hold

$$\left\{ \begin{array}{lcl} \Pi(\kappa_1, \kappa_2) & = & \kappa_{1,x}\kappa_{2,x} + \kappa_{1,y}\kappa_{2,y} + \kappa_{1,z}\kappa_{2,z}, \\ \Lambda(\eta_1, \eta_2) & = & (1/2)\Delta\eta_1\Delta\eta_2. \end{array} \right. \quad (8)$$

It should be noticed that $\Pi(\kappa, \kappa)$ and $\Lambda(\eta, \eta)$ take the roles of the strain energy of the plate and fluid respectively and therefore must be nonnegative.

The plate governing equations and related boundary conditions are as follows (see [8]):

$$\left\{ \begin{array}{lcll} D\Delta^2 w + \rho h w_{,tt} & = & p, & \\ w = \partial w/\partial n & = & 0 & \text{in} \quad \Gamma_0, \\ V^{(n)} + \partial M^{(ns)}/\partial s & = & U_1 & \text{in} \quad \Gamma_1, \\ M^{(n)} & = & U_2 & \text{in} \quad \Gamma_1. \end{array} \right\} \quad (9)$$

For this problem, our main intention is to show that the system (9) under boundary feedbacks $U_1 = -w_{,t}$ and $U_2 = \partial(w_{,t})/\partial n$ is well-posed and asymptotically stable. Note that $\partial\Omega = \Gamma = \Gamma_1 \cup \Gamma_2$ and

$$\begin{array}{lcl} M^{(n)} & = & M_{11}n_1^2 + M_{22}n_2^2 - 2M_{11}n_1 n_2, \\ M^{(ns)} & = & (M_{11} - M_{22})n_1 n_2 + M_{12}(n_1^2 - n_2^2), \\ V_1 & = & M_{11,x} + M_{12,y}, \\ V_2 & = & M_{12,x} + M_{22,y}, \\ V^{(n)} & = & V_1 n_1 + V_2 n_2, \end{array} \quad (10)$$

where $\overrightarrow{n} = (n_1, n_2)$ is the unit outward vector normal to the boundary. V_1, V_2 stand for transversal forces which lay in the planes being perpendicular to unit vectors in x and y directions. $V^{(n)}, M^{(n)}$, are, respectively, transverse force and bending moment which lay perpendicular to the normal direction. For definitions of the remaining parameters see Appendix. To analyze the system using the notion of the linear operators, we utilize the following notation

$$AX = \begin{bmatrix} \tau_1 \\ c^2 \Delta \kappa_1 \\ \zeta_1 \\ \frac{-D}{\rho h} \Delta^2 \eta_1 + p \end{bmatrix}. \tag{11}$$

The state space representation of the system (9) is

$$\left\{ \begin{array}{lll} \dot{\Xi} = A\Xi, & & \\ w = 0, \partial w/\partial n = 0 & \text{in} & \Gamma_0, \\ V^{(n)} + M_{,s}^{(ns)} = -w_{,t}, \ , \ M^{(n)} = \partial(w_{,t})/\partial n & \text{in} & \Gamma_1, \\ \rho_0 \phi_{,t} = -p & \text{in} & \Omega, \\ \rho_0 \phi_{,tt} + \rho_0 g \phi_{,n} = 0 & \text{in} & \Omega_2, \\ \phi_{,n} = 0 & \text{in} & \Omega_3, \\ \Xi(0) = \Xi_0, & & \end{array} \right\} \tag{12}$$

where $\Xi = (\xi_1, \xi_2, \xi_3, \xi_4), \phi = \xi_1, \phi_{,t} = \xi_2, w = \xi_3$ and $w_{,t} = \xi_4$. At first, it will be shown that the operator A with the following domain is a dissipative operator.

$$\begin{aligned} D(A) \quad = \quad & \{(\xi_1, \xi_2, \xi_3, \xi_4) | \xi_1 \in H^2(\Theta) \cap H^1_{\Omega_3}(\Theta), \xi_2 \in H^1_{\Omega_3}(\Theta), \\ & \xi_3 \in H^2_{\Gamma_0}(\Omega) \cap H^4(\Omega), \xi_4 \in H^2_{\Gamma_0} \text{ such that } \rho_0 \xi_2|_\Omega = -p, \} \end{aligned} \tag{13}$$

where $H^4_{\Gamma_0}(\Omega) = \{\xi_3 : \xi_3 \in H^4(\Omega) : \xi_3 = 0|_{\Gamma_0}, \partial\xi_3/\partial n|_{\Gamma_0} = 0\}$ and $H^2_{\Omega_3}(\Theta) = \{\xi_1 : \xi_1 \in H^2(\Theta), \xi_1 = 0|_{\Omega_3}\}$.

Lemma 3.1 *A is a dissipative operator.*

Proof. We start from the fact that the total mechanical energy of the systems is equal to the following inner product $E(t) =< \Xi, \Xi >$, therefore

$$\dot{E}(t) = 2 < \Xi, \dot{\Xi} >= 2 < \Xi, A\Xi > . \tag{14}$$

With the above premise and referring to the Lemma 5.1 of Appendix, the proof will be complete.

Lemma 3.2 *The resolvent* $(\alpha I - A)^{-1}$ *exists and is compact* $(\forall \alpha > 0)$.

Proof. For this purpose, we utilize the following relation

$$(\alpha I - A)X = X_0, X_0 \in \mathbf{H} \tag{15}$$

it can be seen that

$$< (\alpha I - A)X, X >_{\mathbf{H}} = \alpha \|X\|^2_{\mathbf{H}} + \|\xi_4\|^2_{L^2(\Gamma_1)} + \|\partial\xi_4/\partial n\|^2_{L^2(\Gamma_1)} \geq \alpha \|X\|^2_{\mathbf{H}}, \tag{16}$$

where $\|X\|^2_{\mathbf{H}} =< X, X >$.

Using Lax-Milgram lemma, one can easily prove that the above equation has a unique weak solution (see [20–22]). In particular one can infer that:

$R(\alpha I - A) = H^2(\Theta) \times H^1(\Theta) \times H^4(\Omega) \times H^2(\Omega)$, where $\alpha > 0$.

On the other hand, it is clear that $D(A)$ is dense in $H^2(\Theta) \times L^2(\Theta) \times H^4(\Omega) \times \text{Ł}^2(\Omega)$, hence, according to Lumer-Phillips theorem; it is proved that A generates a C_0-semigroup of contractions (see [24]). Finally one can obtain the following result

$$\|X_0\|_{\mathbf{H}} \geq \alpha \|X\|_{\mathbf{H}}. \tag{17}$$

Using Sobolev embedding theorem (Rellich-Kondrachov compact embedding theorem), since $(\alpha I - A)^{-1} V$ is compactly embedded in $\text{Ł}^2(\Theta) \times \text{Ł}^2(\Theta) \times \text{Ł}^2(\Omega) \times \text{Ł}^2(\Omega)$, therefore the compactness of the above-mentioned resolvent is evident.

Theorem 3.1 *Let in the system (22), the initial condition Ξ_0 belong to $D(A)$. Then the system (22) is well-posed.*

Proof. Based on Lemma 3.1, it is evident that the system (22) is well-posed [24]. Also its strong solution has the following regularity (see [23, 24]).

$$\begin{array}{lcl} \phi(t) & \in & C^0([0,t], H^2(\Theta) \cap H^1_{\Omega_3}(\Theta)) \cap C^1([0,t], H^1_{\Omega_3}(\Theta)) \cap C^2([0,t], L^2(\Theta)), \\ w(t) & \in & C^0([0,t], H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)) \cap C^1([0,t], H^2_{\Gamma_0}(\Omega)) \cap C^2([0,t], L^2(\Theta)). \end{array} \tag{18}$$

Now, we turn our attention to the proof of the asymptotic stability of the closed loop system.

Theorem 3.2 *Using the boundary feedback control laws (19), the states of the system Ξ will eventually tend toward zero,*

$$U_1 = -\xi_4 \text{ and } U_2 = \partial \xi_4 / \partial n. \tag{19}$$

Proof. The mechanical energy of the system as discussed previously, is

$$E(t) = < \Xi, \Xi > . \tag{20}$$

By performing some algebraic operations and using Green's Lemma, the following can be obtained (see Appendix):

$$\dot{E}(t) = -\|\xi_4\|^2_{L^2(\Gamma_1)} - \|\partial \xi_4 / \partial n\|^2_{L^2(\Gamma_1)} \leq 0. \tag{21}$$

At this step, because of the compactness of the resolvent $(\alpha I - A)^{-1}$, one can use LaSalle's invariant set theorem and therefore, it is sufficient to show that the following system has the trivial solution as its unique solution:

$$\left\{ \begin{array}{lll} \dot{\Xi} = A\Xi & \text{in} & \Omega, \\ \xi_4 = \partial \xi_4/\partial n = 0 \textit{ and } M^{(n)} = V^{(n)} = 0 & \text{in} & \Gamma_1, \\ \xi_3 = \partial \xi_3 / \partial n = 0 & \text{in} & \Gamma_0, \\ \rho_0 \phi_{,t} = -p & \text{in} & \Omega, \\ \rho_0 \phi_{,tt} + \rho_0 g \phi_{,n} = 0 & \text{in} & \Omega_2, \\ \phi = 0 & \text{in} & \Omega_3, \\ \Xi(0) = \Xi_0. & & \end{array} \right\} \tag{22}$$

Using the Holmgren uniqueness theorem [25], one can easily show that the above system of equations admits only trivial solution. Then, by regarding the LaSalle's invariant set theorem,

$$\lim_{t \to \infty} E(t) = 0, \tag{23}$$

which yields the desired stability.

4 Simulation Results

In this section, we compare the controlled vibration of the plate in contact to a fluid with the uncontrolled one. We plot displacements of some points of the plate in the controlled and uncontrolled cases. We will see the effect of the boundary actuators.

4.1 Geometric Properties of the Plate and the Acoustic Fluid Models

Acoustic fluid region is a $0.5m \times 0.5m \times 0.5m$ cubic space. All sides of the fluid except one which is in contact with the plate are fixed and; therefore, the normal velocities of the fluid at those faces are zero. One face is in contact with the plate and the other face is a free surface (see Figure 2).

4.2 Mechanical Properties of Plate and Acoustic Fluid

The mechanical properties of the fluid and plate are shown in Table 1 and Table 2, respectively.

Bulk Modulus	Density
225e7	1000 Kg/m^3

Table 1: Material properties of the fluid.

Young Modulus	Poisson's Ratio	Density
200e9 Pa	0.3	1920 Kg/m^3

Table 2: Material properties of the plate.

4.3 Results

We present two sets of results. First, the results of the vibration of middle point of the plate without boundary actuators at the plate boundaries are presented and then the other set is for the vibrations of the same point of the plate in the presence of the boundary actuators. We attach a set of boundary actuators with controller gain $k_f = 3N.s/m$ at two controlled sides of the plate. First, the results for the free vibrations of the plate are presented and subsequently the simulation results for the controlled vibrations of the plate are demonstrated. The displacements of the mentioned points of the plate are illustrated by Figures 3–8.

5 Conclusion

Asymptotic stability of the vibration of plates in contact with a fluid was proved. It is shown that the mechanical energy of the systems would converge asymptotically toward zero. Since the control laws consisted only of the feedback from the shear force and bending moment at the boundary of plate, measurement cost was minimized. Also, the proposed method avoids installation of distributed actuators / sensors which meant observation of vibration data along the plate or in the interior of the fluid is not required and the asymptotical stability of the fluid is accomplished without using any actuation in the fluid domain or its boundary.

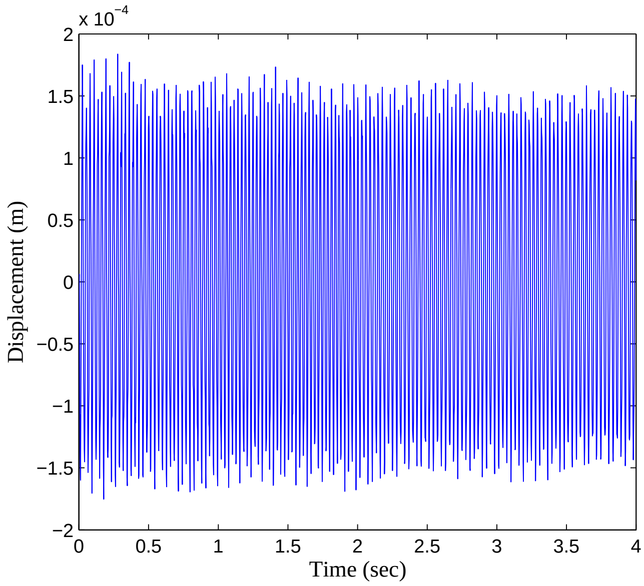

Figure 3: Displacement of the point (0.25, 0) of the plate in contact with the fluid in its free vibration.

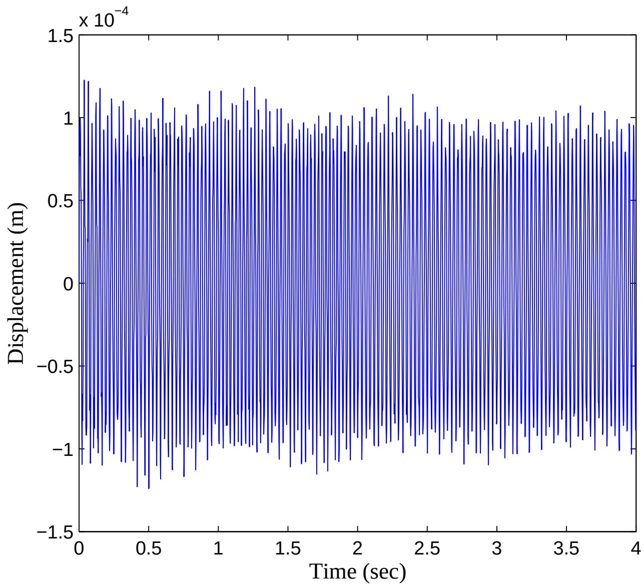

Figure 4: Displacement of point (0.25, 0.25) of the plate in contact with the fluid in its free vibration.

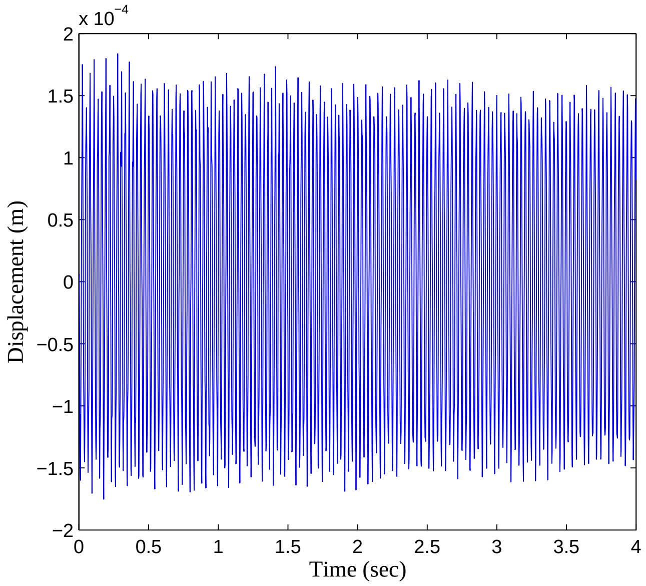

Figure 5: Displacement of point (0.25, 0.5) of the plate in contact with the fluid in its free vibration.

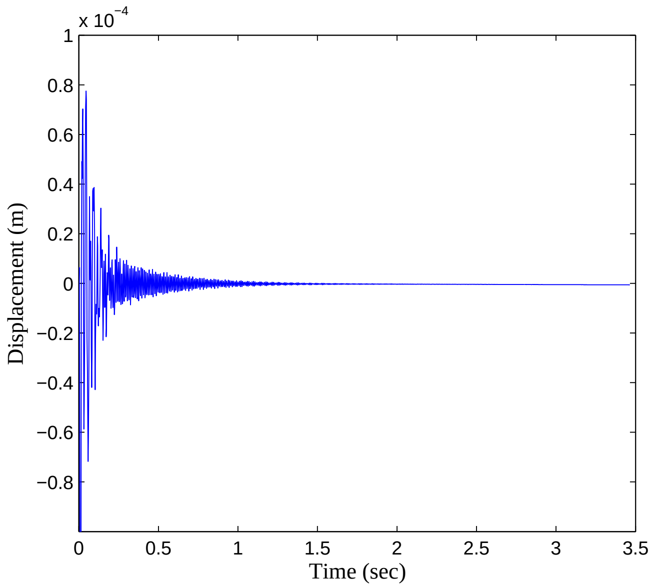

Figure 6: Displacement of point (0.25, 0) of the plate in contact with the fluid in the presence of the boundary actuators.

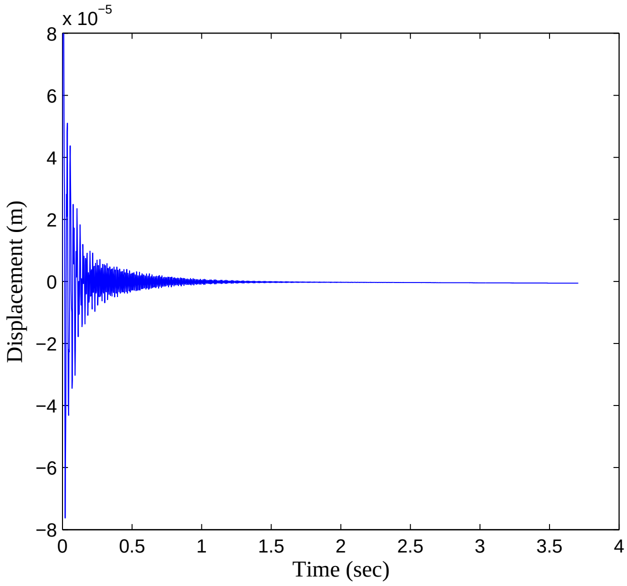

Figure 7: Displacement of point (0.25, 0.25) of the plate in contact with the fluid in the presence of the boundary actuators.

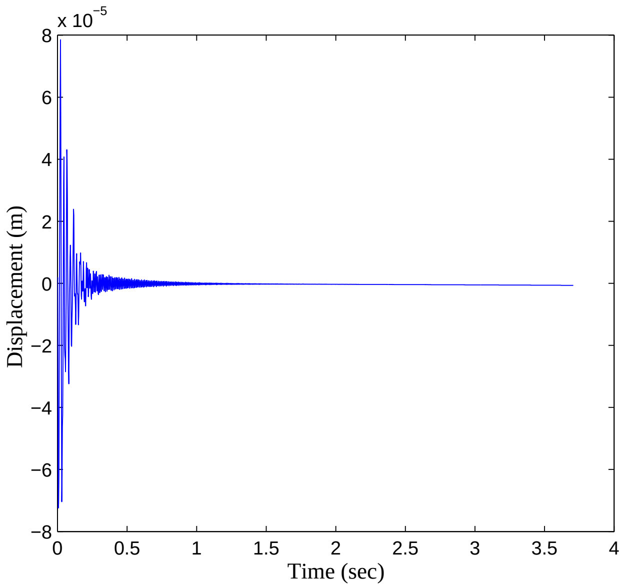

Figure 8: Displacement of point (0.25, 0.5) of the plate in contact with the fluid in the presence of the boundary actuators.

Appendix

In this section it will be shown that the time derivative of the mechanical energy of the system is negative semi-definite and in the sequel we show that the operator A is dissipative.

Lemma 5.1 *For the operator A, with definition (11), one can have*

$$\dot{E}(t) = 2 < \Xi, A\Xi >= -\|\xi_4\|^2_{L^2(\Gamma_1)} - \|\partial\xi_4/\partial n\|^2_{L^2(\Gamma_1)}. \tag{24}$$

Proof. It is clear that $\dot{E}(t) = 2 < \Xi, A\Xi >$. For the rest of the proof, we define some parameters

$$\begin{aligned} M_{11} &= -D(w_{,xx} + \nu w_{,yy}), && (25)\\ M_{22} &= -D(w_{,yy} + \nu w_{,xx}), && (26)\\ M_{12} &= -D(1-\nu)w_{,xy}, && (27)\\ \kappa_{11} &= -w_{,xx}\ , \quad \kappa_{22} = -w_{,yy}\ , \quad \kappa_{12} = -2w_{,xy}, && (28)\\ V_1 &= M_{11,x} + M_{12,y}, && (29)\\ V_2 &= M_{12,x} + M_{22,y}. && (30) \end{aligned}$$

We notice that the governing equation of motion can be rewritten in the following form [8]

$$M_{11,xx} + 2M_{12,xy} + M_{22,yy} = \rho h w_{,tt}. \tag{31}$$

The energy functional takes the following form

$$\begin{aligned} E(t) &= \tfrac{1}{2}\textstyle\int_\Omega [M_{11}\kappa_{11} + M_{22}\kappa_{22} + M_{12}\kappa_{12} + \rho h w^2_{,t}]d\Omega \\ &+ \textstyle\int_\Theta [\frac{\rho_0}{2c^2}\phi^2_{,t} + \frac{\rho_0}{2}|\nabla\phi|^2]d\Theta + \int_{\Omega_2}[\frac{\rho_0}{2g}\phi^2_{,t}]d\Omega. \end{aligned} \tag{32}$$

Therefore, time derivative of $E(t)$ will be

$$\begin{aligned} \dot{E}(t) = \frac{1}{2}\int_\Omega [\dot{M}_{11}\kappa_{11} + \dot{M}_{22}\kappa_{22} + \dot{M}_{12}\kappa_{12} + M_{11}\dot{\kappa}_{11} + M_{22}\dot{\kappa}_{22} + M_{12}\dot{\kappa}_{12} + 2\rho h w_{,t}w_{,tt}]d\Omega \\ + \int_{\Omega_2}[\frac{\rho_0}{g}\phi_{,t}\phi_{,tt}]d\Omega + \int_\Theta [\frac{\rho_0}{c^2}\phi_{,t}\phi_{,tt} + \rho_0(\phi'_{tx}\phi_{,x} + \phi_{,ty}\phi'_{y})]d\Theta \end{aligned} \tag{33}$$

and therefore

$$\begin{aligned} 2\dot{E}(t) &= \textstyle\int_\Omega [M_{11}\dot{\kappa}_{11} + M_{22}\dot{\kappa}_{22} + M_{12}\dot{\kappa}_{12} + (M_{11,xx} + M_{22,yy} + 2M_{12,xy})w_t]d\Omega + \\ &\textstyle\int_\Omega [\dot{M}_{11}\kappa_{11} + \dot{M}_{22}\kappa_{22} + \dot{M}_{12}\kappa_{12} + (M_{11,xx} + M_{22,yy} + 2M_{12,xy})w_{,t}]d\Omega + \\ &\textstyle\int_{\Omega_2}[\frac{\rho_0}{g}\phi_{,t}\phi_{,tt}]d\Omega + \rho_0\int_\Theta [\frac{\partial}{\partial x}((\phi_{,t}\phi_{,tt}) + \frac{\partial}{\partial y}(\phi_{,t}\phi_{,tt})]d\Theta. \end{aligned} \tag{34}$$

Employing the relations for the resultant moments in directions x and y (see (24)– (28)), we get

$$\begin{aligned} 2\dot{E}(t) &= \textstyle\int_\Omega [(M_{11,xx}w_{,t} - M_{11}w_{,xxt}) + (M_{22,yy}w_{,t} - M_{22}w_{,yyt}) + 2(M_{12,xy}w_{,xyt})]d\Omega + \\ &\textstyle\int_\Omega D[w_{,xx}w_{,xxt} + \nu w_{,yyt}w_{,xx}]d\Omega + \int_\Omega D[\nu w_{,yy}w_{,xxt} + w_{,yyt}w_{,yy}]d\Omega + \\ &\textstyle\int_\Omega 2D(1-\nu)w_{,xyt}w_{,xy}d\Omega - \int_\Omega D[w_{,xxxx}w_{,t} + \nu w_{,xxyy}w_{,t}]d\Omega - \\ &\textstyle\int_\Omega 2D(1-\nu)w_{,xxyy}w_{,t}d\Omega - \int_\Omega D[w_{,yyyy}w_{,t} + \nu w_{,xxyy}w_{,t}]d\Omega + \\ &\textstyle\int_\Omega pw_{,t}d\Omega + \int_{\Omega_2}\frac{\rho_0}{g}\phi_{,t}\phi_{,tt}d\Omega + \rho_0\int_\Omega \phi_{,t}\phi_{,n}d\Omega + \\ &\textstyle\rho_0\int_{\Omega_2}\phi_{,t}\phi_{,n}d\Omega + \rho_0\int_{\Omega_3}\phi_{,t}\phi_{,n}d\Omega. \end{aligned} \tag{35}$$

Rearranging the terms and using the boundary conditions for the fluid yield

$$\begin{array}{rcl} 2\dot{E}(t) & = & 2\int_{\Omega}[(M_{11,x}w_{,t}+M_{12,y}w_{,t}-M_{11}w_{,tx}-M_{12}w_{,yt})_{,x}d\Omega+ \\ & & \int_{\Omega}[(M_{22,y}w_{,t}-M_{12,x}w_{,t}-M_{22}w_{,ty}-M_{12}w_{,xt})_{,y}d\Omega+ \\ & & \int_{\Omega}pw_{,t}d\Omega+\int_{\Omega_2}\frac{\rho_0}{g}\phi_{,t}\phi_{,tt}d\Omega+\rho_0\int_{\Omega}\phi_{,t}\phi_{,n}d\Omega+\rho_0\int_{\Omega_2}\phi_{,t}\phi_{,n}d\Omega. \end{array} \tag{36}$$

Applying Green's Lemma and also boundary conditions of the fluid yield

$$\begin{array}{rcl} 2\dot{E}(t) & = & \oint_{\Gamma}(M_{11,x}w_{,t}+M_{12,y}w_{,t}-M_{11}w_{,tx}-M_{12}w_{,yt})n_1d\Gamma+ \\ & & \oint_{\Gamma}[(M_{22,y}w_{,t}-M_{12,x}w_{,t}-M_{22}w_{,ty}-M_{12}w_{,xt})n_2d\Gamma+ \\ & & \int_{\Omega}pw_{,t}d\Omega-\rho_0\int_{\Omega_2}\phi_{,t}\phi_{,n}d\Omega-\int_{\Omega}pw_{,t}d\Omega+\rho_0\int_{\Omega_2}\phi_{,t}\phi_{,n}d\Omega. \end{array} \tag{37}$$

Grouping the terms and noting that

$$\frac{\partial\Delta}{\partial x} = n_1\frac{\partial\Delta}{\partial n}-n_2\frac{\partial\Delta}{\partial s}, \tag{38}$$

$$\frac{\partial\Delta}{\partial y} = n_1\frac{\partial\Delta}{\partial s}-n_2\frac{\partial\Delta}{\partial n}, \tag{39}$$

yield the following result

$$\dot{E}(t)=\oint_{\Gamma}[(V^{(n)}+\frac{\partial M_{ns}}{\partial s})w_{,t}-M^{(n)}(w_{,t})_{,n}]d\Gamma. \tag{40}$$

By applying the assumptions of Theorem 2, and using the related boundary conditions, the following result is attained:

$$\dot{E}(t)=-\|\xi_4\|^2_{L^2(\Gamma_1)}-\|\frac{\partial\xi_4}{\partial n}\|^2_{L^2(\Gamma_1)}. \tag{41}$$

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Instability for Nonlinear Differential Equations of Fifth Order Subject to Delay

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Abstract: This paper studies the instability of zero solution of a certain fifth order nonlinear delay differential equation. Sufficient conditions for the instability of zero solution of the equation considered are obtained by the Lyapunov-Krasovskii functional approach.

Keywords: *instability; Lyapunov–Krasovskii functional; delay differential equation; fifth order.*

Mathematics Subject Classification (2010): 34K20.

1 Introduction

It is well known that in applied sciences some practical problems concerning physics, mechanics and the engineering technique fields associate with differential equations of higher order (Chlouverakis and Sprott [1] and Linz [9]) . Therefore, the investigation of qualitative behaviors of solutions of nonlinear differential equations of higher order has a great importance in theory and applications of differential equations. In particular, by now, several authors have contributed to the theoretical study of instability of solutions of some fifth order nonlinear differential equations without delay (Ezeilo [3–5], Li and Duan [7], Li and Yu [8], Sadek [11], Sun and Hou [12], Tiryaki [13], Tunç [14–16], Tunç and Erdoğan [21], Tunç and Karta [22], Tunç and Şevli [23]). Throughout all of the mentioned papers, based on Krasovskii's properties (Krasovskii [6]), the Lyapunov's second (or direct) method has been used as a basic tool to prove the results established on the instability of solutions, since differential equations studied cannot be solved explicitly. This method, invented by the Russian mathematician Lyapunov in 1892, proves to be

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extremely effective and useful and is still far of being obsolete. On the other hand, it should be noted that the instability of solutions of some certain fifth order nonlinear delay differential equations has been discussed by Tunç [17, 19, 20].

Besides, in 1978, Ezeilo [3] established an instability result for the fifth order nonlinear differential equation without delay

$$x^{(5)} + a_1 x^{(4)} + a_2 x''' + a_3 x'' + a_4 x' + f(x) = 0. \tag{1}$$

In this paper, instead of (1), we consider the fifth order nonlinear delay differential equation

$$x^{(5)} + a_1 x^{(4)} + a_2 x''' + a_3 x'' + a_4 x' + f(x(t-r)) = 0, \tag{2}$$

where a_1, a_2, a_3 and a_4 are some real constants, r is a positive real constant, the primes in (2) denote differentiation with respect to t, $t \in \Re^+ = [0, \infty)$; f is a differentiable function on $\Re$ with $f(0) = 0$. It is assumed that the existence and uniqueness of the solutions of (2) are guaranteed (see [2], pp. 14,15).

We write (2) in system form as follows

$$\begin{aligned} &x' = y, \quad y' = z, \quad z' = w, \quad w' = u, \\ &u' = -a_1 u - a_2 w - a_3 z - a_4 y - f(x) + \int\limits_{t-r}^{t} f'(x(s)) y(s) ds. \end{aligned} \tag{3}$$

In all what follows, $x(t),\ y(t),\ z(t),\ w(t)$ and $u(t)$ are abbreviated as $x,\ \ y,\ \ z,\ \ w$ and u, respectively.

The motivation for this paper comes from the above mentioned papers and Martynyuk et. al [10] and Tunç [18]. Our aim is to convey the results established in Ezeilo [3] to Eq. (3).

Consider the linear constant coefficient fifth order differential equation

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + a_3 \ddot{x} + a_4 \dot{x} + a_5 x = 0, \tag{4}$$

where a_1, a_2, a_3, a_4 and a_5 are some real constants. It is well-known from the qualitative behavior of solutions of linear differential equations that the trivial solution of (4) is unstable if and only if, the associated auxiliary equation

$$\psi(\lambda) \equiv \lambda^5 + a_1 \lambda^4 + a_2 \lambda^3 + a_3 \lambda^2 + a_4 \lambda + a_5 = 0 \tag{5}$$

has at least one root with a positive real part. The existence of such a root naturally depends on (though not always all of) the coefficients a_1, a_2, a_3, a_4 and a_5. For example, if $a_1 < 0$, then it follows from a consideration of the fact that the sum of the roots of (5) equals to $(-a_1)$ and that at the least one root of (5) has a positive real part for arbitrary values of a_2, a_3, a_4 and an analogue consideration, combined with the fact that the product of the roots (5) equals to $(-a_5)$ will verify that at least one root of (5) has a positive real part if

$$a_1 = 0 \text{ and } a_5 \neq 0 \tag{6}$$

for arbitrary a_2, a_3 and a_4. The condition $a_1 = 0$ here in (6) is, however, superfluous when

$$a_5 < 0; \tag{7}$$

for then $\psi(0) = a_5 < 0$ and $\psi(R) > 0$ if $R > 0$ is sufficiently large; thus showing that there is a positive real root of (5) subject to (7) and for arbitrary a_1, a_2, a_3 and a_4.

A root with a positive real part also exists for certain equations (5) with a_5 positive and sufficiently large. To see this easily we refer to the well-known Routh-Hurwitz criteria which stipulate that each root of (5) has a negative real part. Namely, a necessary and sufficient condition for the negativity of the real parts of all the roots of the polynomial equation (5) is the positivity of all the principal diagonals of the minors of the Hurwitz matrix:

$$H_5 = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & 0 & a_5 & a_4 & a_3 \\ 0 & 0 & 0 & 0 & a_5 \end{bmatrix}.$$

It should be also noted that the principal diagonal of the Hurwitz matrix H_5 exhibits the coefficients of the polynomial equation (5) in the order of their numbers from a_1 to a_5 . The fourth order minor, say Δ_4, concerned here is given by the determinant

$$\Delta_4 = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & a_5 & a_4 \end{vmatrix},$$

that is, on multiplying out:

$$\Delta_4 = -a_5^2 + a_5(2a_1a_4 + a_2a_3 - a_1a_2^2) + a_4(a_1a_2a_3 - a_3^2 - a_1^2a_4). \tag{8}$$

It is thus clear, in particular, that if $\Delta_4 < 0$, as would indeed be the case from (8), if

$$a_5 \geq R_0 > 0 \tag{9}$$

with $R_0 = R_0(a_1, a_2, a_3, a_4)$ sufficiently large, then at the least one root of (5) has a non-negative real part subject to (9).

Let $r \geq 0$ be given, and let $C = C([-r, 0],\ \Re^n)$ with $\|\phi\| = \max\limits_{-r \leq s \leq 0} |\phi(s)|\,,\ \ \phi \in C.$

For $H > 0$ define $C_H \subset C$ by $C_H = \{\phi \in C : \|\phi\| < H\}$.

If $x : [-r,\ a) \to \Re^n$ is continuous, $0 < A \leq \infty$, then, for each t in $[0, A)$, x_t in C is defined by

$$x_t(s) = x(t + s), -r \leq s \leq 0, t \geq 0.$$

Let G be an open subset of C and consider the general autonomous delay differential system with finite delay

$$\dot{x} = F(x_t), x_t = x(t + \theta), -r \leq \theta \leq 0, t \geq 0,$$

where $F : G \to \Re^n$ is continuous and maps closed and bounded sets into bounded sets. It follows from the conditions on F that each initial value problem

$$\dot{x} = F(x_t), x_0 = \phi \in G$$

has a unique solution defined on some interval $[0, A)$, $0 < A \leq \infty$. This solution will be denoted by $x(\phi)(.)$ so that $x_0(\phi) = \phi$.

Definition 1.1 The zero solution $x = 0$ of $\dot{x} = F(x_t)$ is stable if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\| < \delta$ implies that $|x(\phi)(t)| < \varepsilon$ for all $t \geq 0$. The zero solution is said to be unstable if it is not stable.

2 Main Results

Our first main result is given by the following theorem.

Theorem 2.1 *In addition to the assumptions imposed to the function f in Eq. (2), we assume that there exist constants a_1, a_3, $\delta\ (>0)$, δ_5 and $\bar{\delta}_5$ such that the following conditions hold:*

$$a_1 > 0, f(0) = 0, f(x) \neq 0,\ (x \neq 0), \bar{\delta}_5 \geq f'(x) > \delta_5 \geq 0 \text{ for all } x,$$

where

$$\delta_5 > \begin{cases} 0, & \text{if} \quad a_3 \leq 0, \\ a_3^2 a_1^{-1}, & \text{if} \quad a_3 > 0. \end{cases}$$

Then the trivial solution $x = 0$ of Eq. (2) is unstable provided

$$r < 2 \min \{1, \frac{\delta_5 - \delta a_3}{(1+\delta)\delta_5}, \frac{\delta a_1 - a_3}{\bar{\delta}_5}\}.$$

Remark 2.1 The kernel of the proof of Theorem 2.1 will be to show that, under the conditions sated in Theorem 2.1, there exists a continuous Lyapunov functional $V_0 = V_0(x_t, y_t, z_t, w_t, u_t)$ which has the following three properties:

(P_1) in every neighborhood of $(0,0,0,0,0)$, there exists a point $(\xi, \eta, \zeta, \mu, \rho)$ such that $V_0(\xi, \eta, \zeta, \mu, \rho) > 0$,

(P_2) the time derivative $\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t)$ along solution paths of the corresponding equivalent differential system for Theorem 2.1 is positive semi-definite,

(P_3) the only solution $(x, y, z, w, u) = (x(t), y(t), z(t), w(t), u(t))$ of (3) which satisfies $\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) = 0$ is the trivial solution $(0,0,0,0,0)$.

Proof. Consider the Lyapunov functional $V_0 = V_0(x_t, y_t, z_t, w_t, u_t)$ defined by

$$\begin{aligned} V_0 \quad &= \tfrac{1}{2}\{-\delta a_4 x^2 + (a_4 + \delta a_2) y^2 + (a_2 - \delta) z^2 - w^2\} + \delta y w + \delta a_1 y z \\ &-\delta x u - \delta a_1 x w - \delta a_2 x z - \delta a_3 x y + z u + a_1 z w + y f(x) \\ &-\lambda \int_{-r}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds, \end{aligned} \tag{10}$$

where δ is a fixed positive constant, as is possible in view of the condition $\delta_5 > a_3^2 a_1^{-1}$ such that $a_3 a_1^{-1} < \delta < \delta_5 a_3^{-1}$, and s is a real variable such that the integral $\int_{-r}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds$ is non-negative, and λ is a positive constant which will be determined later in the proof.

It is clear from (10) that

$$V_0(-\varepsilon^2, 0, 0, 0, \varepsilon) = \delta(\varepsilon^3 - \frac{1}{2} a_4 \varepsilon^4) > 0$$

for all sufficiently small ε. Hence, in every neighborhood of the origin, $(0,0,0,0,0)$, there exists a point $(-\varepsilon^2, 0, 0, 0, \varepsilon)$ such that $V_0(-\varepsilon^2, 0, 0, 0, \varepsilon) > 0$, which shows that the property (P_1) holds for V_0.

By an elementary differentiation, time derivative of the functional $V_0(x_t, y_t, z_t, w_t, u_t)$ in (10) along the solutions of (3) yields

$$\begin{aligned}\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) &= \delta x f(x) + \{f'(x) - \delta a_3\}y^2 + (\delta a_1 - a_3)z^2 + a_1 w^2 \\ &-\delta x \int_{t-r}^{t} f'(x(s))y(s)ds + z \int_{t-r}^{t} f'(x(s))y(s)ds \\ &-\lambda r y^2 + \lambda \int_{t-r}^{t} y^2(s)ds.\end{aligned}$$

The assumptions $f(0) = 0$, $\bar{\delta}_5 \geq f'(x) > \delta_5 \geq 0$ and the estimate $2|mn| \leq m^2 + n^2$ imply

$$\delta x f(x) \geq (\delta\delta_5)x^2,$$

$$-\delta x \int_{t-r}^{t} f'(x(s))y(s)ds \quad \geq -\delta|x| \int_{t-r}^{t} f'(x(s)) \ |y(s)|\, ds \geq -\tfrac{1}{2}(\delta\bar{\delta}_5 r)x^2 - \tfrac{1}{2}(\delta\bar{\delta}_5) \int_{t-r}^{t} y^2(s)ds$$

$$z \int_{t-r}^{t} f'(x(s))y(s)ds \quad \geq -|z| \int_{t-r}^{t} f'(x(s)) \ |y(s)|\, ds \geq -\tfrac{1}{2}\bar{\delta}_5 r z^2 - \tfrac{1}{2}\bar{\delta}_5 \int_{t-r}^{t} y^2(s)ds$$

so that

$$\begin{aligned}\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) &= (\delta\delta_5 - \tfrac{1}{2}\delta\delta_5 r)x^2 + \{\delta_5 - \delta a_3 - \lambda r\}y^2 \\ &+(\delta a_1 - a_3 - \tfrac{1}{2}\bar{\delta}_5 r)z^2 + a_1 w^2 \\ &+2^{-1}\{2\lambda - (1+\delta)\bar{\delta}_5\} \int_{t-r}^{t} y^2(s)ds.\end{aligned}$$

Let $\lambda = \frac{(1+\delta)\bar{\delta}_5}{2}$. Hence

$$\begin{aligned}\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) &= (\delta\delta_5 - 2^{-1}\delta\delta_5 r)x^2 + \{\delta_5 - \delta a_3 - 2^{-1}(1+\delta)\delta_5 r\}y^2 \\ &+(\delta a_1 - a_3 - 2^{-1}\bar{\delta}_5 r)z^2 + a_1 w^2 > 0\end{aligned}$$

provided $r < 2\min\{1, \frac{\delta_5 - \delta a_3}{(1+\delta)\delta_5}, \frac{\delta a_1 - a_3}{\delta_5}\}$, which verifies that the property (P_2) holds for V_0.

On the other hand, $\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) = 0$ if and only if $x = y = z = w = 0$, which implies that $x = y = z = w = u = 0$. Furthermore, by $f(x) \neq 0$ for all $x \neq 0$, it follows that $\frac{d}{dt}V_0(x_t, y_t, z_t, w_t, u_t) = 0$ if and only if $x = y = z = w = u = 0$. Thus, the property (P_3) holds for V_0. By the above discussion, we conclude that the zero solution of Eq. (2) is unstable. The proof of Theorem 2.1 is completed. □

Our second main result is given by the following theorem.

Theorem 2.2 *In addition to the assumptions imposed to the function f in Eq. (2), we assume that there exist constants a_1, a_3, δ (> 0), $\bar{\delta}'_5$ and δ'_5 such that the following conditions hold:*

$$a_1 < 0, f(0) = 0, f(x) \neq 0, \ (x \neq 0), -\bar{\delta}'_5 \leq f'(x) < -\delta'_5 \ \textit{for all } x,$$

where

$$\delta_5' = \begin{cases} 0, & if \;\; a_3 \geq 0, \\ a_3^2 \left| a_1 \right|^{-1}, & if \;\; a_3 < 0. \end{cases}$$

Then the trivial solution $x = 0$ of Eq. (2) is unstable provided

$$r < 2 \min \big\{ 1, \frac{\delta_5 - \delta a_3}{(1+\delta)\delta_5}, \frac{\delta a_1 - a_3}{\bar{\delta}_5} \big\}.$$

Proof. Consider the Lyapunov functional $V_1 = V_1(x_t, y_t, z_t, w_t, u_t)$ defined by

$$\begin{aligned} V_1 \;\; &= \tfrac{1}{2}\{\delta a_4 x^2 - (a_4 + \delta a_2)y^2 - (a_2 - \delta)z^2 + w^2\} - \delta y w - \delta a_1 y z \\ &+ \delta x u + \delta a_1 x w + \delta a_2 x z + \delta a_3 x y - z u - a_1 z w - y f(x) \\ &- \lambda \int_{-r}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds. \end{aligned}$$

Now, the constant δ is fixed as follows $|a_3| \;\; |a_1|^{-1} < \delta < \delta_5' \, |a_3|^{-1}$.

It is clear from V_1 that

$$V_1(\varepsilon^2, 0, 0, 0, \varepsilon) = \delta(\varepsilon^3 + \frac{1}{2} a_4 \varepsilon^4) > 0$$

for all sufficiently small ε, so that V_1 has the property (P_1).

Calculating the time derivative of V_1 along solutions of (3), we obtain

$$\begin{aligned} \tfrac{d}{dt} V_1(x_t, y_t, z_t, w_t, u_t) \;\; &= -\delta x f(x) - \{f'(x) - \delta a_3\} y^2 - (\delta a_1 - a_3) z^2 - a_1 w^2 \\ &+ \delta x \int_{t-r}^{t} f'(x(s)) y(s) ds - z \int_{t-r}^{t} f'(x(s)) y(s) ds \\ &- \lambda r y^2 + \lambda \int_{t-r}^{t} y^2(s) ds. \end{aligned}$$

The assumptions $f(0) = 0$, $-\bar{\delta}_5' \leq f'(x) < -\delta_5'$ and the estimate $2\,|mn| \leq m^2 + n^2$ imply

$$-\delta x f(x) \geq (\delta \delta_5') x^2,$$

$$-\delta x \int_{t-r}^{t} f'(x(s)) y(s) ds \;\; \geq \delta \, |x| \int_{t-r}^{t} f'(x(s)) \;\; |y(s)| \, ds \geq -\tfrac{1}{2}(\delta \bar{\delta}_5' r) x^2 - \tfrac{1}{2}(\delta \bar{\delta}_5') \int_{t-r}^{t} y^2(s) ds$$

and

$$z \int_{t-r}^{t} f'(x(s)) y(s) ds \;\; \geq |z| \int_{t-r}^{t} f'(x(s)) \;\; |y(s)| \, ds \geq -\tfrac{1}{2} \bar{\delta}_5' r z^2 - \tfrac{1}{2} \bar{\delta}_5' \int_{t-r}^{t} y^2(s) ds$$

so that

$$\begin{aligned} \tfrac{d}{dt} V_1(x_t, y_t, z_t, w_t, u_t) \;\; &= \delta(\delta_5' - \tfrac{1}{2}\bar{\delta}_5' r) x^2 + \{\delta_5' - \delta a_3 - \lambda r\} y^2 + (-\delta a_1 + a_3 - \tfrac{1}{2}\bar{\delta}_5' r) z^2 \\ &- a_1 w^2 + 2^{-1}\{2\lambda - (1+\delta)\bar{\delta}_5'\} \int_{t-r}^{t} y^2(s) ds. \end{aligned}$$

Let $\lambda = \frac{(1+\delta)\bar{\delta}_5'}{2}$. Hence

$$\begin{aligned}\frac{d}{dt}V_1(x_t, y_t, z_t, w_t, u_t) &= \delta(\delta_5' - 2^{-1}\bar{\delta}_5' r)x^2 + \{\delta_5' - \delta a_3 - 2^{-1}(1+\delta)\bar{\delta}_5' r\}y^2 \\ &+ (-\delta a_1 + a_3 - 2^{-1}\bar{\delta}_5' r)z^2 - a_1 w^2 > 0\end{aligned}$$

provided $r < 2\min\{\frac{\delta_5'}{\bar{\delta}_5'}, \frac{\delta_5' - \delta a_3}{(1+\delta)\bar{\delta}_5'}, \frac{-\delta a_1 + a_3}{\bar{\delta}_5'}\}$, which verifies that the property (P_2) holds for V_1.

The remaining of the proof is similar to the proof of Theorem 2.1. Therefore, we omit the details. The proof of Theorem 2.2 is now completed. □

Remark 2.2 When we take into account the assumptions established in Tunç ([19, 20]), it can be seen that our assumptions are completely different from that of ([19,20]). That is to say, Theorems 2.1 and 2.2 raise two new results on the instability of solutions of a delay differential equation (2).

Example 2.1 Consider nonlinear differential equation of fifth order with delay

$$x^{(5)} + x^{(4)} + x''' + \frac{1}{2}x'' + x' + 3x(t-r) = 0. \tag{11}$$

We write (11) in system form as follows

$$x' = y, y' = z, z' = w, w' = u, \quad u' = -u - w - \tfrac{1}{2}z - y - 3x + 3\int_{t-r}^{t} y(s)ds.$$

It follows that Eq. (11) is special case of Eq. (2) and

$$a_1 = 1 > 0, a_2 = 1 > 0, a_3 = \frac{1}{2} > 0, a_4 = 1 > 0,$$

$$f(x) = 3x, f(0) = 0, f(x) \neq 0,\ (x \neq 0), f'(x) = 3,$$

$$3 = \bar{\delta}_5 = f'(x) > \delta_5 > 0, \delta_5 > \frac{1}{4} = \frac{a_3^2}{a_1},$$

$$\frac{1}{2} = a_3 a_1^{-1} < \delta < \delta_5 a_3^{-1} = 2\delta_5.$$

In view of the above estimates, we conclude that all the assumptions of Theorem 2.1 hold. Hence, if

$$r < 2\min\{1, \frac{\delta_5 - 2^{-1}\delta}{(1+\delta)\delta_5}, \frac{\delta - 2^{-1}}{3}\},$$

then the zero solution of (11) is unstable.

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