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Approximate Controllability of Nonlocal Semilinear Time-varying Delay Control Systems

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Abstract: In this work the controllability problem for a class of semilinear control system with nonlocal initial conditions is considered. Under some simple conditions the relation between the reachable set of semilinear system and that of its corresponding linear system is established. In particular, approximate controllability of semilinear abstract control system is proved. Examples are presented to explain the application of the proposed result.

Keywords: *infinite-dimensional spaces; semilinear time-varying delay systems; approximate controllability; nonlocal conditions.*

Mathematics Subject Classification (2010): 93B05.

1 Introduction

Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{C}_t = C([-\tau, t]; X), \tau > 0, 0 \le t \le T < \infty$, be a Banach space of all continuous functions from $[-\tau, t]$ into X endowed with the norm $||\phi||_{\mathcal{C}_t} = \sup_{-\tau \le \eta \le t} ||\phi(\eta)||$. Now, consider the following nonlocal semilinear delay control system

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t) + f(t, x(t), x_{b(t)}) \text{ on } (0, T], \\ h(x) &= \phi \text{ on } [-\tau, 0], \end{aligned} \tag{1}$$

where the state variable $x(\cdot)$ takes values in Banach space X and the control function $u(\cdot)$ belongs to $Y = L^2([0, T]; U)$, the Banach space of admissible control functions with a Banach space U. Standing assumptions on system operators are as follows:

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(H1) $A : X \supset D(A) \to X$ is a linear operator such that it generates a C_0-semigroup on X, denoted by $S(t) : t \geq 0$. Let $M \geq 1$ and $\omega \geq 0$ be such that $||S(t)|| \leq Me^{\omega t}$; $t \geq 0$.

(H2) $b : [0, T] \to [0, T]$ is a map such that it satisfies the property $b(t) \leq t$, $\forall t \in [0, T]$. For a continuous function $x \in \mathcal{C}_T$ and $t \in [0, T]$, $x_{b(t)} \in \mathcal{C}_0$ and is defined by $x_{b(t)}(\theta) = x(b(t) + \theta)$; $\theta \in [-\tau, 0]$.

(H3) $h : \mathcal{C}_0 \to \mathcal{C}_0$, and there exists a function $\chi \in \mathcal{C}_0$ such that $h(\chi) = \phi$.

(H4) Nonlinear map $f : [0, T] \times X \times \mathcal{C}_0 \to X$ is continuous in first variable and satisfies the Lipschitz-like condition in second and third argument, that is, there exists some constant $l > 0$ such that $\|f(t, x(t), y_{b(t)}) - f(t, v(t), w_{b(t)})\| \leq l(\|x - v\|_{\mathcal{C}_T} + \|y - w\|_{\mathcal{C}_T})$ for all $x, y, v, w \in \mathcal{C}_T$ and $t \in [0, T]$.

(H5) $B : U \to X$ is a bounded linear operator.

Semilinear differential equation (1) can be seen as an abstract formulation for many control systems described by partial or functional differential equations. Here, nonlocal condition is generally more practical for the physical measurements as compared to the classical condition. The importance of nonlocal conditions has been discussed in the pioneering work by Byszewski and Lakshmikantham [6,7]. Nonlocal conditions were used by Deng in [10] to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube. It is a well known fact that the problem of controllability of semilinear systems in infinite-dimensional spaces can be converted into solvability problem of a functional operator equation in appropriate Banach spaces, and fixed-point theory has been widely used in the literature to establish this solvability; [2,9,14,15]. These concepts has been extended to infinite-dimensional semilinear delay control systems with local or nonlocal initial conditions, among others, we refer to the papers [5, 17, 19, 21, 23, 24, 26] for local conditions and papers [3, 4, 13, 16] for nonlocal conditions.

The purpose of this paper is to compare the trajectory reachable set of nonlinear system (1) to the trajectory reachable set of its corresponding linear system [$f = 0$ in (1)] and this is motivated by the paper of Naito and Park [19] and Ryu, Park, and Kwun [21]. In particular, approximate controllability of system (1) is shown provided the corresponding linear system is controllable. In the proof of the main controllability result in the next section, we do not require any inequality condition, compactness of $S(t)$, and uniform-boundedness of f. In this respect, this paper relaxes some restrictions made by earlier authors if an another simple condition is satisfied by the system operators. In the last section, theory is illustrated with some examples.

2 The Main Results

Let us first consider the following functional delay differential system:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t), x_{b(t)}), & t \in (0, T], \\ h(x) = \phi, & \text{on } [-\tau, 0]. \end{cases} \tag{2}$$

Definition 2.1 A solution function $x \in \mathcal{C}_T$ of the integral equation

$$x(t) = \begin{cases} \chi(t), & t \in [-\tau, 0], \\ S(t)\chi(0) + \int_0^t S(t-s)f(s, x(s), x_{b(s)})\mathrm{d}s, & t \in [0, T], \end{cases} \tag{3}$$

is called a mild solution of problem (2).

The existence and uniqueness of the mild solution of (2) is discussed in the following theorem, and the proof is motivated by the work of Bahuguna and his coworkers, see [1,11].

Theorem 2.1 *If assumptions (H1)-(H4) are satisfied, then there exists a mild solution of (2) on* $[0, T]$ *for some* $T > 0$*. Moreover, the mild solution is unique if and only if* χ *is unique.*

Proof. We choose a $T > 0$ such that $2lTMe^{\omega T} < 1$. Define a map $\mathcal{F}$ from $\mathcal{C}_T$ into itself by

$$(\mathcal{F}x)(t) = \begin{cases} \chi(t), & t \in [-\tau, 0], \\ S(t)\chi(0) + \int_0^t S(t-s)f(s, x(s), x_{b(s)})\mathrm{d}s, & t \in [0, T]. \end{cases} \tag{4}$$

It is clear that $\mathcal{F}$ is well defined and assumption $(H3)$ ensures a fixed point of $\mathcal{F}$ on $t \in [-\tau, 0]$. Now we show that $\mathcal{F}$ is a contraction for the case when $t \in [0, T]$. For this purpose, consider any $x, y \in \mathcal{C}_T$, then we have

$$\begin{aligned} ||(\mathcal{F}x)(t) - (\mathcal{F}y)(t)||_X &\leq ||\int_0^t S(t-s)(f(s, x(s), x_{b(s)}) - f(s, y(s), y_{b(s)}))\mathrm{d}s||_X \\ &\leq 2lTMe^{\omega T}||x - y||_{\mathcal{C}_T}. \end{aligned} \tag{5}$$

Since $2lTMe^{\omega T} < 1$, $\mathcal{F}$ is a contraction on $\mathcal{C}_T$ and hence by Banach Contraction Principle $\mathcal{F}$ has a unique fixed point. Obviously, the uniqueness of χ in $(H3)$ reveals the uniqueness of the mild solution. □

From the above result, a mild solution of the control system (1) can be written as follows

$$x(t) = \begin{cases} \chi(t), & t \in [-\tau, 0], \\ S(t)\chi(0) + \int_0^t S(t-s)[Bu(s) + f(s,\, x(s),\, x_{b(s)})]\mathrm{d}s, & t \in [0, T]. \end{cases} \tag{6}$$

Note that, mild solution (6) depends on control functions $u(\cdot)$. The solution of (6) under a control $u(\cdot)$, denoted by $x(\cdot; u)$, is called the trajectory (state) function of (1) under $u(\cdot)$. The set of all possible trajectories, denoted by

$$K_\alpha(f) := \{x(\cdot; u) \in C([\alpha, T]; X) : \, u \in L^2([0, T]; U), \, 0 < \alpha \leq T\} \tag{7}$$

is called the trajectory reachable set of system (1). In particular, the set of all possible terminal states, denoted by

$$K_T(f) := \{x(T; u) \in X : \, u \in L^2([0, T]; U)\} \tag{8}$$

is called the reachable set of system (1) at terminal time T.

Definition 2.2 System (1) is said to be approximate controllable on $[0, T]$ if $\overline{K_T(f)} = X$, where $\overline{K_T(f)}$ stands for the closure of $K_T(f)$ in X.

Now, we define two functions $F : \mathcal{C}_T \to L^2([0,T]; X)$ and $B_1 : Y \to L^2([0,T]; X]$ as $(Fx)(t) = f(t,\, x(t),\, x_{b(t)})$, $\quad (B_1u)(t) = Bu(t)$.

Theorem 2.2 *Under assumptions (H1)-(H5) and* $R(F) \subseteq \overline{R(B_1)}$*, we have* $\overline{K_\alpha(f)} \supseteq K_\alpha(0)$*.*

Proof. Let $x(\cdot) \in K_\alpha(0)$, there exists a control $u \in Y$ such that

$$x(t) = \begin{cases} \chi(t), & t \in [-\tau, 0], \\ S(t)\chi(0) + \int_0^t S(t-s)Bu(s)\mathrm{d}s, & t \in [0, T]. \end{cases} \tag{9}$$

Due to range condition, for a given $\epsilon > 0$ $\exists$ $w \in Y$ such that

$$\|Fx - B_1 w\|_{L^2([0,T];X)} \leq \epsilon. \tag{10}$$

Now, let $y(\cdot)$ be mild solution of (1) corresponding to control $u - w$. Then

$$\begin{aligned} x(t) - y(t) &= \int_0^t S(t-s)Bw(s)\mathrm{d}s - \int_0^t S(t-s)f(s, y(s), y_{b(t)})\mathrm{d}s \\ &= \int_0^t S(t-s)(B_1 w - Fx)(s)\mathrm{d}s + \int_0^t S(t-s)(Fx - Fy)(s)\mathrm{d}s. \end{aligned} \tag{11}$$

Using (H4) and (10) we have

$$\begin{aligned} \|x(t) - y(t)\| &\leq Me^{\omega T} \int_0^t \|(B_1 w - Fx)(s)\|\mathrm{d}s + Me^{\omega T} \int_0^t \|(Fx - Fy)(s)\|\mathrm{d}s \\ &\leq Me^{\omega T}\sqrt{T}\epsilon + 2Mle^{\omega T} \int_0^t ||x - y||_{\mathcal{C}_T}\mathrm{d}s. \end{aligned} \tag{12}$$

This implies

$$||x - y||_{\mathcal{C}_T} \leq Me^{\omega T}\sqrt{T}\epsilon + 2Mle^{\omega T} \int_0^t ||x - y||_{\mathcal{C}_T}\mathrm{d}s. \tag{13}$$

Now, using Gronwall's inequality it can be shown that

$$||x - y||_{\mathcal{C}_T} \leq Me^{\omega T}\sqrt{T}\epsilon \exp(2lTMe^{\omega T}). \tag{14}$$

From the above inequality it is clear that $||x - y||_{\mathcal{C}_T}$ can be made arbitrary small by choosing suitable w. Hence the theorem is proved. $\square$

Corollary 2.1 *Under assumptions of the above theorem, system (1) is approximate controllable if its corresponding linear system is approximate or exact controllable.*

Proof. The proof is a particular case of Theorem 2.2 at $\alpha = T$. $\square$

Remark 2.1 Fixed-point theory arguments make it necessary to assume uniform boundedness of nonlinear term f with certain inequality condition involving various system parameters, and/or compactness of semigroup $T(t)$. But, these conditions (specially inequality conditions) are not easy to verify in many situations. In this paper these conditions are replaced with a range condition $R(F) \subset \overline{R(B_1)}$. Note that this range condition is satisfied trivially for the system (1) if B is the identity operator. Obviously, Theorem 2.2 gives the controllability of system (1) when $b(t) = t$ in the case of constant delay, and this case is explained in Example 3.1.

3 Application

Example 3.1 Consider the following mathematical model

$$\begin{aligned} \frac{\partial}{\partial t}y(t,x) &= \frac{\partial^2}{\partial x^2}y(t,x) + u(t,x) + (\int_0^1 y(t,x)dx)y(t,x) \\ &+ (\int_0^1 y(t-\tau,x)dx)y(t-\tau,x), 0 \le x \le 1, t \in [0,T], \quad (15) \\ y(t,0) &= y(t,1) = 0, t \in [0,T], \\ \frac{1}{\tau}\int_{-\tau}^0 e^{2s}y(s,x)ds &= y_0(x), 0 \le x \le 1, \end{aligned}$$

where $y(t,.)$, $u(t,.)$, $y_0 \in L^2(0,1)$. If we take

(1) $X = L^2(0,1)$ as the state space and $y(t,\cdot) = \{y(t,x) : 0 \le x \le 1\}$ as the state.

(2) input trajectory $u(t,.)$ as the control and $U = L^2(0,1)$ as the control space. Note that, here $X = U$.

(3) $A : D(A) \subset X \to X$ defined by $A(z) = \frac{d^2z}{dx^2}$ with domain $D(A) = H^2(0,1) \bigcap H_0^1(0,1)$. Then A is an infinitesimal generator of a C_0-semigroup of bounded linear operators; see [8].

(4) $B = I$.

(5) $b(t) = t$, and $y_{b(t)}(\theta) \equiv y(t-\tau,\cdot)$ (so this is a constant time-delay case).

(6) $f : [0,T] \times X \times \mathcal{C}_0 \to X$, $T > 0$ defined by

$$f(t, y(t,\cdot), y_{b(t)}) = (\int_0^1 y(t,x)dx)y(t,\cdot) + (\int_0^1 y(t-\tau,x)dx)y(t-\tau,\cdot),$$

where $0 \le x \le 1, t \in [0,T]$. It is not hard to see that f satisfies (H4).

(7) $h(z)(\theta) = g(z)$ for $z \in \mathcal{C}_0$, $\theta \in [-\tau,0]$; $\phi(\theta) = y_0$. Here, $g : \mathcal{C}_0 \to X$ is such that $g(z)(x) = \frac{1}{\tau}\int_{-\tau}^0 e^{2s}z(s,x)ds$. For this definition of h, we can find a function $\chi \in \mathcal{C}_0$, given by $\chi(\theta) = \frac{1}{k}y_0$ on $[-\tau, 0]$ with $k = \int_0^\tau \frac{1}{\tau}e^{-2s}ds \ne 0$, such that

$$h(\chi)(\theta) = \frac{1}{\tau}\int_{-\tau}^0 e^{2s}(\frac{1}{k}y_0)ds = y_0 = \phi(\theta), \text{ that is, } h(\chi) = \phi.$$

Then (15) resembles control system (1) and has a mild solution (6) on $[-\tau,T]$. Now take $Y := L^2([0, T]; L^2(0,1))$, $B_1 = I : Y \to Y$, $F : \mathcal{C}_T \to Y$ as $(Fz)(t) = f(t, z(t), z_{b(t)})$. Then it is clear that $R(F) \subset \overline{R(B_1)}$. Since the corresponding linear system is approximate controllable; [8], system (15) is approximate controllable due to Theorem 2.2. Mathematical model (15) may be seen as the population dynamics, see [12], where $y(t,.)$ represents the population density at time t and the term $\frac{\partial^2}{\partial x^2}y(t,x)$ describes the internal migration. Moreover, the continuous functions $B, D : [0,T] \to \mathbb{R}_+$ given by $B(t) = \int_0^1 y(t-\tau,x)dx$ and $D(t) = \int_0^1 y(t,x)dx$, represent average birth and death rates, respectively, τ is the delay due to pregnancy, and source term $u(t,x)$ represents a control.

Example 3.2 Consider the control system governed by the following semilinear heat equation

$$\begin{aligned}\frac{\partial y(t,x)}{\partial t} &= \frac{\partial^2 y(t,x)}{\partial x^2} + Bu(t,x) \\ &+ f(t,y,y_{b(t)});\ 0<t<T, 0<x<\pi,\ -\tau \le \theta \le 0, \\ y(t,0) &= y(t,\pi)=0, t\in[0,T], \end{aligned} \tag{16}$$

with the same initial condition as in the above example, where $y(t,.)$, $y_0 \in L^2(0,\pi)$. Then (16) can be converted into (1), if we take:

(1) $X = L^2(0,\pi)$ as the state space and $y(t,.) = \{y(t,x) : 0 \le x \le \pi\}$ as the state.

(2) input trajectory $u(t,.)$ as the control.

(3) $A : D(A) \subset X \to X$ defined by $A(z) = \frac{d^2z}{dx^2}$ with domain $D(A) = H^2(0,\pi)\bigcap H_0^1(0,\pi)$. Then, $\overline{D(A)} = X$ and A is an infinitesimal generator of a C_0-semigroup of bounded linear operators; see [8]. Further, if we take $\{\phi_n(x) = (2/\pi)^{1/2}\sin(nx); 0 \leqslant x \leqslant \pi; n\in\mathbb{N}\}$, then $\{\phi_n\}$ is an orthonormal basis of X and ϕ_n is an eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of operator A. Then the C_0-semigroup generated by A has $e^{\lambda_n t}$ as the eigenvalues and ϕ_n as their corresponding eigenfunctions.

(4) $U = \{u : u = \sum_{n=2}^{\infty} u_n\phi_n : \sum_{n=2}^{\infty} u_n^2 < \infty\}$, with norm $|u|_U = (\sum_{n=2}^{\infty} u_n^2)^{1/2}$ as the control space. B is a continuous linear map from U to X defined as

$$Bu = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n \text{ for } u = \sum_{n=2}^{\infty} u_n\phi_n \in U.$$

(5) $b(t) = k|\sin t|$, $k\in(0,1)$ or $b(t) = \frac{t^2}{1+t^2}$.

(6) h and χ are the same as in Example 3.1.

It shows that (16) has a mild solution (6) on $[-\tau, T]$ provided f is Lipschitz continuous. Although, the same example has been discussed in [9, 18, 27] (with or without delay and under local conditions), but approximate controllability was proved under restrictions such as the uniform boundedness on f or some inequality constraints. This paper shows that the approximate controllability also follows for non-uniform bounded function f without having to satisfy any inequality constraint and without using the compactness of C_0-semigroup. For example, consider the function f given by $f(t,z,z_{b(t)}) = \alpha(\|z\|_{\mathcal{C}_T} + \|z_{b(t)}\|_{\mathcal{C}_0})(\phi_3(x)+\phi_4(x))$, where α is a positive constant. Here f is Lipschitz and $R(F) \subseteq R(B_1)$. Moreover, this example shows that time-varying aftereffect and generalized nonlocal conditions can also be handled by the theorem proved in the previous section. In the above example $b(t) = k|\sin t|$, $k\in(0,1)$ or $b(t) = \frac{t^2}{1+t^2}$ is a theoretical construction but many physical and biological processes include time-varying aftereffect phenomena in their inner dynamics, see [20].

Example 3.3 Consider the system of infinite ordinary differential equations:

$$\frac{dx(t)}{dt} = Ax(t) + u(t) + f(t,x(t),x_{b(t)}), \quad \sum_{i=1}^{l} c_i x(\theta_i) = x_0, \tag{17}$$

where $x(t) = (x_1(t), x_2(t), \dots) \in l^2$. Then (17) resembles control system (1), if we take

(1) $X = l^2$ as the state space and $x(t)$ as the state.

(2) input $u(t) = (u_1(t), u_2(t), \ldots)$ as the control and $U = l^2$ as the control space. Note that, here $X = U$.

(3) A is a self-adjoint operator on X defined by $Ae_i = \lambda_i e_i$ where $\{e_i\}$ is an orthonormal basis of X and $\{\lambda_i\}$is a decreasing sequence of positive numbers such that $\lim_{i\to\infty} \lambda_i = \lambda_0 > 0$. Then A is an infinitesimal generator of a C_0-semigroup of bounded linear operators defined by $T(t)x = \Big(e^{\lambda_1 t}x_1, e^{\lambda_2 t}x_2, \ldots\Big)$.

(4) $B = I$ and b is the same as in Example 3.2.

(5) f is defined by $f(t, x(t), x_{b(t)}) = (f_1(t, x(t), x_{b(t)}), f_2(t, x(t), x_{b(t)}), \ldots), \quad 0 \le t \le T$.

(6) $h(z)(\theta) = g(z)$ for $z \in \mathcal{C}_0$, $\theta \in [-\tau, 0]$; $\phi(\theta) = x_0$. Here, $g : \mathcal{C}_0 \to X$ is such that $g(z) = \sum_{i=1}^{l} c_i z(\theta_i)$; $-\tau \le \theta_1 < \theta_2 < \cdots < \theta_l \le 0$. For this definition of h, we can find a function $\chi \in \mathcal{C}_0$, given by $\chi(\theta) = \frac{1}{k}x_0$ on $[-\tau,\, 0]$ with $k = \sum_{i=1}^{l} c_i$.

The approximate controllability of the linear system corresponding to (17) has been proved by Triggiani [25]. In [22], the approximate controllability of (17) (without delay and with local Cauchy condition) has been shown via the solvability of some operator equations under the following conditions:
(i) The linear system is approximate controllable,
(ii) A generates a compact semigroup $T(t)$,
(iii) The nonlinear operator $f(t, x)$ satisfies the Lipschitz condition,
(iv) The operator f satisfies the growth condition $\|f(x(t))\|_X \le a\|x(t)\|_X + b$,
(v) System constants satisfy the constraint $\frac{e^{\lambda_1 T}\sqrt{T}}{2} \cdot \sqrt{2MbT(e^{2MbT} - 1)} < \frac{e^{2T\lambda_0} - 1}{2e^{\lambda_1 T}\sqrt{T}\lambda_0}$, where $\|T(t)\| \le e^{\lambda_1 \tau} = M$ for $0 \le t \le T$.

But due to Theorem 2.2, it follows that the system (17) is approximate controllable only under the above conditions (i) and (iii) for nonlinear operators those satisfy the range condition, *e.g.* f is defined as $f_1(t, x(t), x_{b(t)}) = a\|x\| + b\|x_{b(t)}\| + c$; a, b, and c are positive constants and $f_i(t, x(t), x_{b(t)}) = 0$ for all $i = 2, 3, \ldots$. This shows that the inequalities such as (v) above, assumed by earlier author are not required to be considered.

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