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Nonlinear Plane Waves in Rotating Stratified Boussinesq Equations

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Abstract: In this paper we have built special exact solutions to rotating stratified Boussinesq equations in the form of nonlinear plane waves. We also conclude that these solutions grow exponentially in unstable stratifications. Whereas, in the special case of stable stratification these waves are oscillatory in nature. Consequently, we determined internal gravity waves and some sinusoidal wave forms.

Keywords: *plane waves; rotating stratified Boussinesq equation; sinusoidal waves.*

Mathematics Subject Classification (2010): 34A05, 35J35.

1 Introduction

The stratified Boussinesq equations form a system of PDEs modelling the movements of planetary atmospheres. It may be noted that the Boussinesq approximation in the literature is also referred to as the Oberbeck-Boussinesq approximation for which, the reader is referred to an interesting paper of Rajagopal et al [1] providing a rigorous mathematical justification of use of these equations as perturbations of the Navier-Stokes equations. Majda & Shefter [2] have chosen certain special solutions of this system of PDEs to demonstrate onset of instability when the Richardson number is less than 1/4. In their study of instability in stratified fluids at large Richardson number, Majda & Shefter [2] have obtained the exact solutions to stratified Boussinesq equations neglecting the effects of rotations and viscosity. Further, in the absence of strain field Srinivasan et al [3] have shown that the reduced system of ODEs is completely integrable. Desale and Dasre [4] have obtained the numerical solutions of this reduced system of ODEs. For the similar kind of work the reader may refer to Maas [5,6]. In his monograph Majda [7] has

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obtained the special solution of stratified Boussinesq equations excluding the effects of viscosity and finite rotation. Whereas, Desale & Sharma [8] included the effect of rotation. In his earlier study Desale [9] has proved the complete integrability of the system of six coupled ODEs, which arises in the reduction of rotating stratified Boussinesq equations in context to the theory of basin scale dynamics. Since the rotating stratified Boussinesq equations admit the periodic solution near the critical point. On the other hand Desale & Patil [10] deployed the Painlevé test to determine the complete integrability of the same system. Further, the stability criteria can be resolved via Floquet theory. In their paper Slane & Tragesser [11] explained the use of Floquet theory to discuss the stability of homogeneous parametrically excited system.

In this paper we deploy the procedure of Majda & Shefter [2] to build the exact solutions of rotating stratified Boussinesq equations in the form of nonlinear plane waves. In the steady state these solutions increase exponentially. We conclude that the steady state is unstable. Whereas, in the special case of stable stratification these waves are oscillatory in nature. In this case, we also find internal gravity waves as some sinusoidal wave forms.

2 Nondimensional Rotating Stratified Boussinesq Equations

The motion of an incompressible flow of fluid in the atmosphere and in the ocean is considered where, the flow velocities are too slow to account for compressible effects. The flow of fluid is governed by the following rotating stratified Boussinesq equations (we ignore the effects of viscosity and heat dissipation) that involve the interaction of gravity with density stratification about the reference state.

$$\begin{aligned} \frac{D\vec{\mathbf{v}}}{Dt} + f(\hat{\mathbf{e_3}} \times \vec{\mathbf{v}}) &= -\nabla \frac{\tilde{p}}{\rho_b} - \frac{g\rho}{\rho_b}\hat{\mathbf{e_3}}, \\ \operatorname{div} \vec{\mathbf{v}} &= 0, \\ \frac{D\tilde{\rho}}{Dt} &= 0, \end{aligned} \tag{1}$$

where $D/Dt = \partial/\partial t + \vec{\mathbf{v}}\cdot\nabla$, the unit vector in vertical direction is $\hat{\mathbf{e_3}} = (0, 0, 1)$, the space variable $\vec{\mathbf{x}} = (x_1, x_2, x_3)$ and fluid velocity is given by $\vec{\mathbf{v}} = (v_1, v_2, v_3)$. The full density $\tilde{\rho}$ consists of perturbations ρ about the density $\overline{\rho}$ in hydrostatic balance, which itself creates only small deviations from the baseline constant ρ_b, $\tilde{\rho}(\vec{\mathbf{x}}, t) = \rho_b + \overline{\rho}(x_3) + \rho(\vec{\mathbf{x}}, t)$. We make the usual assumption valid for local consideration that $d\overline{\rho}/dx_3$ is constant.

Now we consider the following nondimensional form of (1). For more details one may refer to Desale & Sharma [8].

$$\begin{aligned} \frac{D\vec{\mathbf{v}}}{Dt} + \frac{1}{R_0}\vec{\mathbf{u}} &= -\overline{P}\nabla p - \Gamma\rho\hat{\mathbf{e_3}}, \\ \operatorname{div} \vec{\mathbf{v}} &= 0, \\ \frac{D\tilde{\rho}}{Dt} &= \frac{D\rho}{Dt} + \left(\frac{d\overline{\rho}}{dx_3}\right) v_3 = 0. \end{aligned} \tag{2}$$

Here, we have $\vec{\mathbf{u}} = (u_1, u_2, u_3) = \hat{\mathbf{e_3}} \times \vec{\mathbf{v}}$, Γ is the nondimensional number, R_0 is the Rossby number and $\overline{P}$ is the Euler number. Nondimensional density function is

$$\tilde{\rho}(\vec{\mathbf{x}}, t) = \rho_b + \overline{\rho}(x_3) + \rho(\vec{\mathbf{x}}, t). \tag{3}$$

The more elaborative discussion about the nondimensional analysis of rotating stratified Boussinesq equations is also given by Majda in his monograph [7]. In the following section we have obtained exact solutions of (2) in the form of nonlinear plane waves.

3 Nonlinear Plane Waves

In this section we have determined the exact solutions of rotating stratified Boussinesq equations (2) in the form of nonlinear plane waves. These solutions are suggested by the following Theorem 3.1. The following trivial lemma is useful step towards the proof of Theorem 3.1.

Lemma 3.1 *For* $\vec{\mathbf{v}}$ *of the form* $\vec{\mathbf{v}} = \vec{\mathbf{A}}(t)F(\vec{\alpha}(t)\cdot\vec{\mathbf{x}})$, div $\vec{\mathbf{v}} = 0$ *implies*

(i) $\vec{\mathbf{A}}(t)\cdot\vec{\alpha}(t) = 0$ *and*
(ii) $\vec{\mathbf{v}}\cdot\nabla W(\vec{\alpha}(t)\cdot\vec{\mathbf{x}}) = 0,$

for arbitrary W*, where* $\vec{\mathbf{A}}(t) = (A_1(t), A_2(t), A_3(t))$ *and* $\vec{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$.

For the proof of this lemma one may refer to Majda [7], pp. 20.

Theorem 3.1 *The rotating stratified Boussinesq equations (2) have exact solutions in the form of nonlinear plane waves*

$$\vec{\mathbf{v}} = \vec{\mathbf{A}}(t)F(\vec{\alpha}(t)\cdot\vec{\mathbf{x}}), \quad \rho = B(t)F(\vec{\alpha}(t)\cdot\vec{\mathbf{x}}), \quad p = P(t)G(\vec{\alpha}(t)\cdot\vec{\mathbf{x}}), \tag{4}$$

where F *and* G *are arbitrary functions of* $\vec{\alpha}(t)\cdot\vec{\mathbf{v}}$ *with the condition* $G'(s) = F(s)$ *provided that* $\vec{\alpha}(t)$, $\vec{\mathbf{A}}(t)$, $B(t)$ *and* $P(t)$ *satisfy the following ODEs:*

$$\begin{aligned}
&\frac{d\vec{\alpha}}{dt} = 0,\\
&\vec{\mathbf{A}}(t)\cdot\vec{\alpha}(t) = 0,\\
&P(t) = -\frac{1}{R_0\overline{P}}\left(\frac{\vec{\alpha}(t)\cdot(\hat{\mathbf{e_3}}\times\vec{\mathbf{A}}(t))}{|\vec{\alpha}(t)|^2}\right) - \frac{\Gamma\alpha_3(t)}{\overline{P}|\vec{\alpha}(t)|^2}B(t),\\
&\frac{d\vec{\mathbf{A}}(t)}{dt} = \frac{-1}{R_0}(\hat{\mathbf{e_3}}\times\vec{\mathbf{A}}(t)) + \left[\frac{\vec{\alpha}(t)\cdot(\hat{\mathbf{e_3}}\times\vec{\mathbf{A}}(t))}{R_0|\vec{\alpha}(t)|^2} + \frac{\Gamma\alpha_3(t)}{|\vec{\alpha}(t)|^2}B(t)\right]\vec{\alpha}(t) - \Gamma B(t)\hat{\mathbf{e_3}},\\
&\frac{dB(t)}{dt} + \frac{d\overline{\rho}}{dx_3}A_3(t) = 0.
\end{aligned}\tag{5}$$

Proof. Now we begin with the first equation of (2)

$$\frac{\partial\vec{\mathbf{v}}}{\partial t} + \vec{\mathbf{v}}\cdot\nabla\vec{\mathbf{v}} = -\frac{1}{R_0}(\hat{\mathbf{e_3}}\times\vec{\mathbf{v}}) - \overline{P}\nabla p - \Gamma\rho\hat{\mathbf{e_3}}.$$

We have $\vec{\mathbf{v}} = \vec{\mathbf{A}}(t)F(\vec{\alpha}(t)\cdot\vec{\mathbf{x}})$, $\rho = B(t)F(\vec{\alpha}(t)\cdot\vec{\mathbf{x}})$, $p = P(t)G(\vec{\alpha}(t)\cdot\vec{\mathbf{x}})$ and div $\vec{\mathbf{v}} = 0$. Hence by substituting $\vec{\mathbf{v}}$, ρ and p in above equation with $G'(s) = F(s)$ and using Lemma 3.1 we get

$$\left(\frac{d\vec{\mathbf{A}}}{dt} + \frac{1}{R_0}(\hat{\mathbf{e_3}}\times\vec{\mathbf{A}}) + \overline{P}P(t)\vec{\alpha}(t) + \Gamma B(t)\hat{\mathbf{e_3}}\right)F(\vec{\alpha}\cdot\vec{\mathbf{x}}) = -\vec{\mathbf{A}}\left(\frac{d\vec{\alpha}}{dt}\cdot\vec{\mathbf{x}}\right)F'(\vec{\alpha}\cdot\vec{\mathbf{x}}). \tag{6}$$

Since F and F' are arbitrary functions, they must be treated as independent terms implying $\frac{d\vec{\alpha}(t)}{dt} = 0$, which is the first equation of (5). Consequently, (6) gives us

$$\frac{d\vec{\mathbf{A}}(t)}{dt} = -\frac{1}{R_0}(\hat{\mathbf{e_3}} \times \vec{\mathbf{A}}(t)) - \overline{P}P(t)\vec{\alpha}(t) - \Gamma B(t)\hat{\mathbf{e_3}}. \tag{7}$$

Lemma 3.1 proves the second equation of (5). Taking time derivative of $\vec{\mathbf{A}}(t) \cdot \vec{\alpha}(t) = 0$ with the validity of the first equation of (5), we have $\frac{d\vec{\mathbf{A}}(t)}{dt} \cdot \vec{\alpha}(t) = 0$. Then we take the dot product of the equation above for $\frac{d\vec{\mathbf{A}}(t)}{dt}$ with $\vec{\alpha}(t)$ and we determine equation for $P(t)$ as in the required form in (5). Plugging back $P(t)$ into (7) and recasting it we get the fourth equation of (5). Finally plugging plane waves into the third equation of (2) we get the differential equation for $B(t)$ as in the form of the last equation of (5). Hence we complete the proof of the theorem. $\square$

The first equation in (5) shows that vector $\vec{\alpha}(t)$ is a constant vector and we have $\frac{d\overline{\rho}}{dx_3}$ is constant. We can write $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and the last two equations of (5) can be written in component form as:

$$\begin{aligned}
\frac{dA_1(t)}{dt} &= \frac{1}{R_0}A_2(t) + \left[\frac{A_1(t)\alpha_2 - A_2(t)\alpha_1}{R_0|\vec{\alpha}|^2} + \frac{\Gamma\alpha_3 B(t)}{|\vec{\alpha}|^2}\right]\alpha_1,\\
\frac{dA_2(t)}{dt} &= -\frac{1}{R_0}A_1(t) + \left[\frac{A_1(t)\alpha_2 - A_2(t)\alpha_2}{R_0|\vec{\alpha}|^2} + \frac{\Gamma\alpha_3 B(t)}{|\vec{\alpha}|^2}\right]\alpha_2,\\
\frac{dA_3(t)}{dt} &= \left[\frac{A_1(t)\alpha_2 - A_2(t)\alpha_2}{R_0|\vec{\alpha}|^2} + \frac{\Gamma\alpha_3 B(t)}{|\vec{\alpha}|^2}\right]\alpha_3 - \Gamma B(t),\\
\frac{dB(t)}{dt} &+ \left(\frac{d\overline{\rho}}{dx_3}\right)A_3(t) = 0.
\end{aligned} \tag{8}$$

We see that above system (8) is a linear system with constant coefficients, hence there exists a unique solution passing through the given initial values that satisfy the condition $\vec{\mathbf{A}}(t) \cdot \vec{\alpha} = 0$. Plugging these solutions into plane waves given by (4), we determine the physical terms velocity, density and pressure.

In the following section we classified the fluids in the special case of plane waves in which the vectors $\hat{\mathbf{e_3}}, \vec{\mathbf{A}}(t)$ and $\vec{\alpha}$ are coplanar. Consequently we determined the internal gravity waves and sinusoidal waves.

4 Classification in the Special Case of Plane Waves

In this section we consider the special case of plane waves in which $\hat{\mathbf{e_3}}, \vec{\mathbf{A}}(t)$ and $\vec{\alpha}$ are coplanar. It means we consider $\hat{\mathbf{e_3}} \cdot (\vec{\mathbf{A}}(t) \times \vec{\alpha}) = 0$. So that equations (8) reduce to

$$\begin{aligned}
\frac{dA_1(t)}{dt} &= \frac{1}{R_0}A_2(t) + \frac{\Gamma\alpha_3\alpha_1}{|\vec{\alpha}|^2}B(t),\\
\frac{dA_2(t)}{dt} &= -\frac{1}{R_0}A_1(t) + \frac{\Gamma\alpha_3\alpha_2}{|\vec{\alpha}|^2}B(t),\\
\frac{dA_3(t)}{dt} &= \left(\frac{\alpha_3^2}{|\vec{\alpha}|^2} - 1\right)\Gamma B(t),\\
\frac{dB(t)}{dt} &= \left(-\frac{d\overline{\rho}}{dx_3}\right)A_3(t).
\end{aligned} \tag{9}$$

The scalar function $P(t)$ in pressure term becomes

$$P(t) = -\frac{\Gamma \alpha_3}{\overline{P}|\vec{\alpha}|^2} B(t). \tag{10}$$

Differentiating the last equation of (9) with respective time variable t we get

$$\frac{d^2 B(t)}{dt^2} = \left(-\frac{d\overline{\rho}}{dx_3}\right) \frac{dA_3(t)}{dt}. \tag{11}$$

We recast the above equation by plugging back the equation for $\frac{dA_3(t)}{dt}$ from the third equation of (9) as follows

$$\frac{d^2 B(t)}{dt^2} = \frac{d\overline{\rho}}{dx_3}\left(1 - \frac{\alpha_3^2}{|\vec{\alpha}|^2}\right) \Gamma B(t) = -\omega^2(\vec{\alpha}) B(t). \tag{12}$$

Thus we observe that the behavior of solutions depends on the sign of ω^2. Because the Γ is nondimensional positive number and angular term in parentheses is always positive, the overall sign depends on the sign of the density gradient $\frac{d\overline{\rho}}{dx_3}$.

- **Case(i):** $\frac{d\overline{\rho}}{dx_3} > 0$ **(Heavier fluids on top).** This case will have exponentially growing solutions of the form $e^{|\omega|t}$. We conclude that steady state is unstable.
- **Case(ii):** $\frac{d\overline{\rho}}{dx_3} < 0$ **(Heavier fluids at bottom).** In this case equation (12) suggests that solutions will be oscillatory in nature. Hence we refer to it as stable stratification.

4.1 Sinusoidal Waves

In this subsection we determine sinusoidal plane waves in stable stratifications for $\frac{d\overline{\rho}}{dx_3} < 0$. We write the nondimensional form of buoyancy frequency or Brunt-Väisälä frequency

$$N = \left(-\Gamma \frac{d\overline{\rho}}{dx_3}\right)^{1/2}. \tag{13}$$

We use the notation to the general parameter $\vec{\alpha}$ as wave vector $\vec{\mathbf{k}} = (k_1, k_2, k_3) = (\vec{\mathbf{k}}_H, k_3) = \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, so that ω as defined in (12) is given by

$$\omega(\vec{\mathbf{k}}) = N \frac{|\vec{\mathbf{k}}_H|}{|\vec{\mathbf{k}}|}. \tag{14}$$

The general solution of (12) is

$$B(t) = c_1 \sin(\omega(\vec{\mathbf{k}})t) + c_2 \cos(\omega(\vec{\mathbf{k}})t), \tag{15}$$

where c_1 and c_2 are arbitrary constants. The scalar function in pressure terms is given by

$$P(t) = -\frac{\Gamma k_3}{\overline{P}|\vec{\mathbf{k}}|^2}\Big[c_1 \sin(\omega(\vec{\mathbf{k}})t) + c_2 \cos(\omega(\vec{\mathbf{k}})t)\Big]. \tag{16}$$

Substituting (15) into the last equation of (9) we determine $A_3(t)$ as:

$$A_3(t) = \frac{\Gamma |\vec{\mathbf{k}}_H|}{N|\vec{\mathbf{k}}|}\Big[c_1 \cos(\omega(\vec{\mathbf{k}})t) - c_2 \sin(\omega(\vec{\mathbf{k}})t)\Big]. \tag{17}$$

Now to determine $A_1(t)$ and $A_2(t)$ we have the vector $\hat{\mathbf{e_3}}, \vec{\mathbf{A}}(t)$ and $\vec{\mathbf{k}}$ are coplanar. So that they satisfy the equation $A_2(t)k_1 - A_1(t)k_2 = 0$. But $A(t)$ is a function of time variable t and wave vector $\vec{\mathbf{k}}$ is constant. Hence to meet the requirement $A_2(t)k_1 - A_1(t)k_2 = 0$, we consider the following cases that are related with the magnitude $\vec{\mathbf{k}}_H$.

1. $|\vec{\mathbf{k}}_H| \neq 0$**:** In this case we have the following possibilities.

   (a) Suppose that $k_1 \neq 0$, $k_2 = 0$. But this assumption along with $A_2(t)k_1 - A_1(t)k_2 = 0$ implies that $A_2(t) = 0$. If we plug $A_2(t) = 0$ in the first and second equations of (9) and solve these equations we get $A_1(t) = 0$. This is also true in either case $k_1 = 0$ and $k_2 \neq 0$.

   (b) Suppose $k_1 \neq 0$, $k_2 \neq 0$. But we required that $A_2(t)k_1 - A_1(t)k_2 = 0$, so that $A_1(t)$ must be equal to the constant multiple of $A_2(t)$. But in order to satisfy the first and second equations of (9) we conclude that $A_1(t)$ and $A_2(t)$ must be equal to zero.

   In this case we have

$$\begin{aligned}
&A_1(t) = 0, \quad A_2(t) = 0, \\
&A_3(t) = \frac{\Gamma|\vec{\mathbf{k}}_H|}{N|\vec{\mathbf{k}}_H|}\Big(c_1\cos(\omega(\vec{\mathbf{k}})t) - c_2\sin(\omega(\vec{\mathbf{k}})t)\Big), \\
&B(t) = c_1\sin(\omega(\vec{\mathbf{k}})t) + c_2\cos(\omega(\vec{\mathbf{k}})t), \\
&P(t) = -\frac{\Gamma k_3}{\overline{P}|\vec{\mathbf{k}}|^2}\Big[c_1\sin(\omega(\vec{\mathbf{k}})t) + c_2\cos(\omega(\vec{\mathbf{k}})t)\Big].
\end{aligned} \tag{18}$$

   In order to write the physical variables, we must merely remember their definitions in Theorem 3.1. Recalling that $G'(s) = F(s)$, we have

$$\begin{aligned}
\vec{\mathbf{v}} &= \frac{\Gamma|\vec{\mathbf{k}}_H|}{N|\vec{\mathbf{k}}|}\Big[c_1\cos(\omega(\vec{\mathbf{k}})t) - c_2\sin(\omega(\vec{\mathbf{k}})t)\Big]F(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}})\hat{\mathbf{e_3}}, \\
\rho &= \Big[c_1\sin(\omega(\vec{\mathbf{k}})t) + c_2\cos(\omega(\vec{\mathbf{k}})t)\Big]F(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}), \\
p &= -\frac{\Gamma k_3}{\overline{P}|\vec{\mathbf{k}}|^2}\Big[c_1\sin(\omega(\vec{\mathbf{k}})t) + c_2\cos(\omega(\vec{\mathbf{k}})t)\Big]G(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}).
\end{aligned} \tag{19}$$

   Equations (19) represent the special case of nonlinear plane waves with $k_3 \neq 0$ and are supported by the stable stratification, so we call them the internal gravity waves.

   In order to find sinusoidal wave forms, we put

$$F(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}) = \sin(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}). \tag{20}$$

   The density function in this case is

$$\begin{aligned}
\rho \;=\; &\frac{c_1}{2}\Big[\cos(\omega(\vec{\mathbf{k}})t - \vec{\mathbf{k}}\cdot\vec{\mathbf{x}}) - \cos(\omega(\vec{\mathbf{k}})t + \vec{\mathbf{k}}\cdot\vec{\mathbf{x}})\Big] \\
+\; &\frac{c_2}{2}\Big[\sin(\omega(\vec{\mathbf{k}})t + \vec{\mathbf{k}}\cdot\vec{\mathbf{x}}) - \sin(\omega(\vec{\mathbf{k}})t - \vec{\mathbf{k}}\cdot\vec{\mathbf{x}})\Big].
\end{aligned} \tag{21}$$

These calculations illustrate that there are waves moving in different directions corresponding to the two branches of dispersion relation. Let us simplify the case $c_2 = 0$ and we write the solutions

$$\begin{aligned}
\rho &= \frac{c_1}{2}\Big[\cos(\omega(\vec{\mathbf{k}})t - \vec{\mathbf{k}}\cdot\vec{\mathbf{x}}) - \cos(\omega(\vec{\mathbf{k}})t + \vec{\mathbf{k}}\cdot\vec{\mathbf{x}})\Big],\\
p &= \frac{c_1}{2}\frac{\Gamma k_3}{\overline{P}|\vec{\mathbf{k}}|^2}\Big[\sin(\omega(\vec{\mathbf{k}})t - \vec{\mathbf{k}}\cdot\vec{\mathbf{x}}) + \sin(\omega(\vec{\mathbf{k}})t + \vec{\mathbf{k}}\cdot\vec{\mathbf{x}})\Big],\\
\vec{\mathbf{v}} &= \frac{c_1}{2}\frac{\Gamma|\vec{\mathbf{k}}_H|}{N|\vec{\mathbf{k}}|}\Big[\sin(\omega(\vec{\mathbf{k}})t + \vec{\mathbf{k}}\cdot\vec{\mathbf{x}}) - \sin(\omega(\vec{\mathbf{k}})t - \vec{\mathbf{k}}\cdot\vec{\mathbf{x}})\Big]\,\hat{\mathbf{e_3}}.
\end{aligned} \tag{22}$$

2. $|\vec{\mathbf{k}}_H| = 0$ **:** In this case the horizontal components of wave vector are $k_1 = k_2 = 0$. The vector $\vec{\mathbf{A}}(t)$ and scalar function $B(t)$ have to satisfy the following differential equations:

$$\begin{aligned}
&\frac{dA_1(t)}{dt} = \frac{1}{R_0}A_2(t), \quad \frac{dA_2(t)}{dt} = -\frac{1}{R_0}A_1(t),\\
&\frac{dA_3(t)}{dt} = 0, \quad \frac{dB(t)}{dt} = \left(-\frac{d\overline{\rho}}{dx_3}\right)A_3(t).
\end{aligned} \tag{23}$$

Solving these equations, we get

$$\begin{aligned}
A_1(t) &= c_1\cos(t/R_0) + c_2\sin(t/R_0),\\
A_2(t) &= -c_1\sin(t/R_0) + c_2\cos(t/R_0),\\
A_3(t) &= c_3, \quad B(t) = c_4(-\tfrac{d\overline{\rho}}{dx_3})t + c_5,
\end{aligned} \tag{24}$$

where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants. The scalar function $P(t)$ in pressure term is

$$P(t) = -\frac{\Gamma}{\overline{P}k_3}\left[c_4\left(-\frac{d\overline{\rho}}{dx_3}\right)t + c_5\right]. \tag{25}$$

In this special case of plane waves, the physical terms, namely the velocity, density and pressure involved in (2) are given by the following equations

$$\begin{aligned}
\vec{\mathbf{v}} &= \Big(c_1\cos(\tfrac{t}{R_0}) + c_2\sin(\tfrac{t}{R_0}),\ -c_1\sin(\tfrac{t}{R_0}) + c_2\cos(\tfrac{t}{R_0}),\ c_3\Big)F(k_3x_3),\\
\rho &= \Big(c_4(-\tfrac{d\overline{\rho}}{dx_3})t + c_5\Big)F(k_3x_3),\\
p &= -\tfrac{\Gamma}{\overline{P}k_3}\Big[c_4\left(-\tfrac{d\overline{\rho}}{dx_3}\right)t + c_5\Big]G(k_3x_3).
\end{aligned} \tag{26}$$

Now we put $F(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}) = F(k_3x_3) = \sin(k_3x_3)$ in equations (26) with $G'(s) = F(s)$ to determine the sinusoidal wave forms. These wave forms are:

$$\begin{aligned}
\vec{\mathbf{v}} = \Big(&\tfrac{c_1}{2}\Big[\sin(\tfrac{t}{R_0} + k_3x_3) - \sin(\tfrac{t}{R_0} - k_3x_3)\Big] + \tfrac{c_2}{2}\Big[\cos(\tfrac{t}{R_0} - k_3x_3) - \cos(\tfrac{t}{R_0} + k_3x_3)\Big],\\
&-\tfrac{c_1}{2}\Big[\cos(\tfrac{t}{R_0} - k_3x_3) - \cos(\tfrac{t}{R_0} + k_3x_3)\Big] + \tfrac{c_2}{2}\Big[\sin(\tfrac{t}{R_0} + k_3x_3) - \sin(\tfrac{t}{R_0} - k_3x_3)\Big],\\
&\quad c_3\sin(k_3x_3)\Big),\\
\rho = &\Big[c_3(-\tfrac{d\overline{\rho}}{dx_3})t + c_4\Big]\sin(k_3x_3),\\
p = &\tfrac{\Gamma}{\overline{P}k_3^2}\Big[c_3\left(-\tfrac{d\overline{\rho}}{dx_3}\right)t + c_4\Big]\cos(k_3x_3).
\end{aligned} \tag{27}$$

The sinusoidal waves given by (27) with $k_3 \neq 0$ are supported by stable stratification so termed as internal gravity waves.

5 Conclusion

The special exact solutions of rotating stratified Boussinesq equations (2) in the form of nonlinear plane waves are obtained from the solutions of linear system (8). In the special case of fluids in which $\hat{\mathbf{e}_3}$, $\vec{\mathbf{A}}(t)$, $\vec{\alpha} = \vec{\mathbf{k}}$ are coplanar and $\frac{d\overline{\rho}}{dx_3} > 0$, the nonlinear plane waves given by (9) with $P(t)$ as in (10) grow exponentially. Whereas, if heavier fluids are at the bottom with $k_3 \neq 0$ then the plane waves given by (19) and (26) are oscillatory in nature. These waves are called the internal gravity waves. The exact solutions of (2) in the form of sinusoidal waves are given by (22) and (27).

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