Nonlinear Dynamics and Systems Theory, **13** (1) (2013) 25–46

Asymptotic Robot Manipulator Generalized Inverse Dynamics

A.H. Bajodah *

Aeronautical Engineering Department, P.O. Box 80204
King Abdulaziz University
Jeddah 21589, Saudi Arabia

Received: January 16, 2012; Revised: January 19, 2013

Abstract: The generalized inverse dynamics methodology is improved to yield asymptotic tracking control of robot manipulator's generalized coordinate trajectories. A scalar kinematic norm measure function of generalized coordinates deviations from their desired trajectories is defined, and a servo-constraint on robot kinematics is prescribed by zeroing the deviation function. A stable linear second-order differential equation in the deviation function is evaluated along trajectory solutions of manipulator's state space model equations, resulting in an algebraic relation that is linear in the control vector. The control law is designed by generalized inversion of the controls coefficient in the algebraic relation using a modified version of the Greville formula. The generalized inverse in the particular part of the modified formula is scaled by a dynamic factor that uniformly decays as steady state response approaches. This yields a uniform convergence of the particular part to its projection on the range space of controls coefficient's generalized inverse, and in asymptotically stable generalized coordinates trajectory tracking. Null-control vector in the auxiliary part of the formula is taken to be linear in manipulator's generalized velocities, and is constructed by means of a positive semidefinite control Lyapunov function that involves controls coefficient's nullprojector, providing asymptotic internal manipulator stability over a predetermined domain of attraction.

Keywords: *generalized inversion control; generalized inverse dynamics; asymptotic tracking; semidefinite Lyapunov function; null-control vector.*

Mathematics Subject Classification (2010): 93D15, 93D20, 93D30.

* Corresponding author: mailto:abajodah@kau.edu.sa

1 Introduction

Generalized inverse dynamics (GID) [1, 2] is an evolving control design paradigm that aims to benefit from non-square inversion in solving the inverse dynamics control problem. In fulfilling that purpose, GID utilizes the nullspace parametrization feature of the generalized inversion-based Greville formula [3, 4]. Nullspace parametrization is the means by which the Greville formula captures solution nonuniqueness of linear algebraic system equations, and it forms the backbone of GID control.

The GID control design methodologies are based on the fact that a prescribed dynamics on a controllable dynamical system can be enforced by infinite number of strategies. Accordingly, a unique robot manipulator's inner loop design in a conventional inverse dynamics solution is quite restrictive. By removing that restriction from the inverse dynamics philosophy, the GID control design reveals the inherent redundancy in the control process [5].

A GID robot control design procedure begins by a coordinate transformation that reduces the size of joint space generalized coordinates error vector to a single dimension. The scalar variable in the transformed coordinate system is named the kinematic deviation function, and it is taken to be the squared Euclidean norm of the joint space error vector. Nullifying the kinematic deviation function is equivalent to bringing manipulator's generalized coordinates to their desired values.

The methodology proceeds by forming a stable second-order time-invariant linear differential equation in the kinematic deviation function. The differential equation is a servo-constraint dynamics that is to be realized by manipulator's control system. Convergence of the differential equation's solution to its steady state zero value implies satisfying the control objective. The differential equation is transformed to an algebraic equation by evaluating the first two time derivatives of the kinematic deviation function along trajectory solutions of the manipulator's state space mathematical model.

The resulting algebraic equation is linear in the control vector. The Greville formula can therefore be utilized to solve the equation for the control variables required to realize the desired servo-constraint dynamics. The solution involves the Moore-Penrose generalized inverse (MPGI) [6, 7] of the row vector formed by the coefficients of control variables in the linear algebraic equation, abbreviated as the controls coefficient [8]. The Greville formula solution is composed of particular and auxiliary parts. The particular part maps to the range space of the controls coefficient's transpose, and it works to realize desired servo-constraint dynamics. The auxiliary part maps to the orthogonal complement nullspace of the controls coefficient, and it works to provide internal manipulator stability [5].

Nevertheless, the Greville formula suffers from the undesirable characteristic of MPGI singularity [9] that hinders the particular part of the formula. The MPGI singularity occurs whenever the generalized-inverted matrix changes rank, causing divergence of the MPGI elements to infinite values. In the present application, MPGI singularity takes place when steady state response approaches as the controls coefficient converges to the zero vector. A technique of MPGI singularity avoidance is presented in Ref. [5], made by replacing the MPGI in the Greville formula by a damped generalized inverse, resulting in a globally uniformly ultimately bounded robot manipulator's generalized coordinate trajectory tracking.

This paper introduces a modified version of the Greville formula. The MPGI in the particular part of the formula is scaled by a dynamic factor that vanishes as closed loop response approaches steady state. The scaling factor is capable of overcoming the MPGI singularity. Simultaneously, the dynamically scaled generalized inverse (DSGI) uniformly converges to the standard MPGI as the dynamic scaling factor decays, resulting in asymptotic realization of desired servo-constraint dynamics, and in a uniformly asymptotically stable tracking of desired robot generalized coordinates.

The auxiliary part of the Greville formula is affine in a free null-vector that is projected onto nullspace of the controls coefficient. Therefore, the null-vector design does not affect realization of the linear algebraic relation, but it affects the manner in which the relation is realized, i.e., it affects how individual state variables evolve in time. In particular, the null-vector affects closed loop internal manipulator's stability. The null-vector in the present context is named the null-control vector, and its design is a crucial step of the GID methodology

Moreover, the design freedom of the null-control vector can be utilized to achieve further requirements, e.g., perturbed feedback linearization of internal closed loop dynamics [5]. The null-control vector is constructed in this work to be linear in manipulator's joint velocity variables. The state dependent proportionality gain matrix is designed via novel positive semidefinite control Lyapunov function and controls coefficient nullprojected Lyapunov equations, resulting in locally asymptotically stable generalized coordinate trajectory tracking. The analysis provides an explicit estimate of the corresponding domain of attraction.

The GID methodology unifies the treatments of inverse kinematics [10] and inverse dynamics by transforming the inverse dynamics problem to an underdetermined problem and utilizing generalized inversion to solve it, overcoming the restrictions of dimensionality and rank that limit the applications of regular inversion.

The contribution of this article is twofold. First, a new GID design element is introduced to robot control applications, namely the dynamically scaled generalized inverse, improving the recently developed GID methodology to yield asymptotic tracking control. Second, a new GID control design methodology is presented. The null-control vector is designed by means of a novel type of control Lyapunov functions and Lyapunov equations.

2 Mathematical Model for Robot Manipulator

The mathematical model of an n degrees of freedom robot manipulator is given by the following system of differential equations

$$M(q,t)\ddot{q} + C(q,\dot{q},t) + G(q,t) = \mathcal{F}, \quad q(t_0) = q_0,\ \dot{q}(t_0) = \dot{q}_0, \tag{1}$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are vectors of manipulator generalized coordinates, velocities, and accelerations, respectively. The vector valued function $C(q,\dot{q},t) : \mathbb{R}^n \times \mathbb{R}^n \times [t_0,\infty) \to \mathbb{R}^n$ contains centrifugal and coriolis forces, the vector valued function $G(q,t) : \mathbb{R}^n \times [t_0,\infty) \to \mathbb{R}^n$ contains gravitational forces, and $\mathcal{F} \in \mathbb{R}^n$ is the vector of control forces acting on the manipulator. The inertia matrix valued function $M(q,t) : \mathbb{R}^n \times [t_0,\infty) \to \mathbb{R}^{n\times n}$ is assumed to be symmetric positive definite for all $q \in \mathbb{R}^n$. Equation (1) can be put in the following state space system of $2n$ kinematical and dynamical differential equations

$$\dot{q} = u, \quad q(t_0) = q_0, \tag{2}$$

$$\dot{u} = -M^{-1}(q,t)[C(q,u,t) + G(q,t)] + \tau, \quad u(t_0) = u_0, \tag{3}$$

where $\tau \in \mathbb{R}^n$ is given by

$$\tau = M^{-1}(q,t)\mathcal{F}. \tag{4}$$

3 Generalized Coordinate Deviation Norm Measure Dynamics

Let $q_r(t) \in \mathbb{R}^n$ be a prescribed desired robot manipulator generalized coordinates vector such that $q_r(t)$ is twice continuously differentiable in t. The manipulator generalized coordinates error vector from $q_r(t)$ is defined as

$$e_q(q,t) := q - q_r(t). \tag{5}$$

Consequently, a scalar positive definite configuration deviation norm measure function $\phi : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}$ is defined to be the squared Euclidean norm of $e_q(q,t)$

$$\phi = \| e_q(q,t) \|^2 = \| q - q_r(t) \|^2 . \tag{6}$$

Therefore, the manipulator is at its desired configuration if and only if the servo-constraint

$$\phi \equiv 0 \tag{7}$$

is realized. The first two time derivatives of ϕ along the manipulator trajectories given by the solution of (2) and (3) are

$$\dot{\phi} = 2e_q^T(q,t)\,[u - \dot{q}_r(t)] \tag{8}$$

and

$$\ddot{\phi} = 2[u - \dot{q}_r(t)]^T [u - \dot{q}_r(t)] + 2e^T(q,t)\Big[\tau - M^{-1}(q,t)[C(q,u,t) + G(q,t)] - \ddot{q}_r(t)\Big]. \tag{9}$$

A desired stable second-order dynamics of ϕ is specified to be of the form

$$\ddot{\phi} + a_1\dot{\phi} + a_2\phi = 0, \quad a_1, a_2 > 0. \tag{10}$$

With ϕ, $\dot{\phi}$, and $\ddot{\phi}$ given by (6), (8), and (9), it is possible to write (10) in the pointwise-linear form

$$\mathcal{A}(q,t)\tau = \mathcal{B}(q,u,t), \tag{11}$$

where the row vector-valued controls coefficient function $\mathcal{A}(q,t) : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^{1\times n}$ is given by

$$\mathcal{A}(q,t) = 2e_q^T(q,t) \tag{12}$$

and the scalar-valued controls load function $\mathcal{B}(q,u,t) : \mathbb{R}^n \times \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}$ is given by

$$\begin{aligned}\mathcal{B}(q,u,t) = -2[u - \dot{q}_r(t)]^T [u - \dot{q}_r(t)] \\ + 2e_q^T(q,t)\left[M^{-1}(q,t)[C(q,u,t) + G(q,t)] + \ddot{q}_r(t)\right] \\ - 2a_1 e_q^T(q,t)[u - \dot{q}_r(t)] - a_2 \| e_q(q,t) \|^2 .\end{aligned} \tag{13}$$

4 Generalized Inverse Dynamics

The infinite set of all manipulator control laws τ realizing the servo-constraint given by (7) via the linear dynamics given by (10) is parameterizable by the Greville formula [3] as [1]

$$\tau(q,u,y,t) = \mathcal{A}^{+}(q,t)\mathcal{B}(q,u,t) + \mathcal{P}(q,t)y, \tag{14}$$

where $\mathcal{A}^{+}(q,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ is the controls coefficient Moore-Penrose generalized inverse (CCGI) given by

$$\mathcal{A}^{+}(q,t) = \begin{cases} \frac{\mathcal{A}^T(q,t)}{\|\mathcal{A}(q,t)\|^2}, & \mathcal{A}(q,t) \neq \mathbf{0}_{1\times n} \\ \\ \mathbf{0}_{n\times 1}, & \mathcal{A}(q,t) = \mathbf{0}_{1\times n} \end{cases} \tag{15}$$

and $\mathcal{P}(q,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^{n\times n}$ is the corresponding controls coefficient nullprojector (CCNP) given by

$$\mathcal{P}(q,t) = I_{n\times n} - \mathcal{A}^{+}(q,t)\mathcal{A}(q,t) \tag{16}$$

and $y \in \mathbb{R}^n$ is an arbitrary null-control vector.

Substituting the control laws expressions given by (14) in manipulator's equations of motion (3) yields the following parametrization of the infinite set of manipulator closed loop system equations that realize the servo-constraint dynamics given by (10)

$$\dot{q} = u \tag{17}$$

$$\begin{aligned} \dot{u} = & -M^{-1}(q,t)[C(q,u,t) + G(q,t)] \\ & + \mathcal{A}^{+}(q,t)\mathcal{B}(q,u,t) + \mathcal{P}(q,t)y. \end{aligned} \tag{18}$$

Different choices of the null-control vector y in the control laws expression given by (14) yield different solutions to (11), and every solution results in closed loop system trajectory solutions for (17) and (18) that satisfy the linear servo-constraint dynamics given by (10). Nevertheless, designing y is a critical issue, because y substantially affects manipulator's internal dynamics given by (3), and an inadequate design of y can destabilize that dynamics [1].

5 Generalized Inversion Singularity

Satisfying the servo-constraint dynamics given by (10) implies from the definition of ϕ given by (6) and the expression of $\mathcal{A}(q,t)$ given by (12) that

$$\lim_{\phi\to 0} \mathcal{A}(q,t) = \mathbf{0}_{1\times 3}. \tag{19}$$

Since the expression of $\mathcal{A}(q,t)$ is continuous in q and t, the definition of $\mathcal{A}^{+}(q,t)$ given by (15) implies that if the initial manipulator configuration condition is such that $\mathcal{A}(q_0,t_0) \neq \mathbf{0}_{1\times n}$, then [5]

$$\lim_{\mathcal{A}(q,t)\to\mathbf{0}_{1\times n}} \|\mathcal{A}^{+}(q,t)\| = \lim_{\mathcal{A}(q,t)\to\mathbf{0}_{1\times n}} \frac{1}{\|\mathcal{A}^T(q,t)\|} = \infty_{n\times 1} \tag{20}$$

causing the particular part in the expression of the control law $\tau(q,u,y,t)$ given by (14) to go unbounded, and driving the closed loop dynamical subsystem given by (18) unstable.

Instability due to generalized inversion singularity is well-known in MPGI applications. A remedy of the problem in the context of generalized inverse control is made by deactivating the particular part of the Greville formula in the vicinity of singularity, resulting in discontinuous control laws [11]. Another remedy is made by modifying the definition of the MPGI by means of a damping factor, resulting in uniformly ultimately bounded control and a tradeoff between generalized inversion stability and closed loop system performance [5]. The concept of dynamically scaled generalized inversion [12] is used in this paper for the purpose of avoiding instability due to CCGI singularity and to guarantee asymptotic generalized coordinate trajectory tracking.

6 Dynamically Scaled Generalized Inverse

A reference (desired) internal dynamics is defined based on the system equations given by (17) and (18) as

$$\dot{u}_r = -M^{-1}(q,t)[C(q,u_r,t)+G(q,t)]+\mathcal{A}^+(q,t)\mathcal{B}(q,u_r,t)+\mathcal{P}(q,t)y_r,\ q(t_0)=q_0, \quad (21)$$

where $u_r, \dot{u}_r \in \mathbb{R}^n$ are reference (desired) velocity and acceleration vectors, and $y_r \in \mathbb{R}^n$ is a reference null-control vector. The reference internal dynamics is obtained by replacing u and y by u_r and y_r in the dynamical subsystem given by (18), and the reference acceleration vector $\dot{u}_r$ is equal to the acceleration vector $\dot{u}$ for all $t \geq t_0$ if $y = y_r$ for all $t \geq t_0$ and $u_r(t_0) = u(t_0)$.

The dynamically scaled generalized inverse (DSGI) of the controls coefficient $\mathcal{A}(q,t)$ is introduced next.

Definition 6.1 [Dynamically scaled controls coefficient generalized inverse] The DSGI $\mathcal{A}_s^+(q,u,t) : \mathbb{R}^n \times \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^{n\times 1}$ is given by

$$\mathcal{A}_s^+(q,u,t) = \frac{\mathcal{A}^T(q,t)}{\mathcal{A}(q,t)\mathcal{A}^T(q,t) + \|e_u(u,u_r)\|_p^p} \quad (22)$$

for some vector p norm, where

$$e_u(u,u_r) = u - u_r. \quad (23)$$

Properties of the DSGI

The following properties can be verified by direct evaluation of the CCGI $\mathcal{A}^+(q,t)$ given by (15) and its dynamic scaling $\mathcal{A}_s^+(q,u,t)$ given by (22):

1. $\mathcal{A}_s^+(q,u,t)\mathcal{A}(q,t)\mathcal{A}^+(q,t) = \mathcal{A}_s^+(q,u,t)$;
2. $\mathcal{A}^+(q,t)\mathcal{A}(q,t)\mathcal{A}_s^+(q,u,t) = \mathcal{A}_s^+(q,u,t)$;
3. $(\mathcal{A}_s^+(q,u,t)\mathcal{A}(q,t))^T = \mathcal{A}_s^+(q,u,t)\mathcal{A}(q,t)$;
4. $\lim_{\|u-u_r\|_p\to 0} \mathcal{A}_s^+(q,u,t) = \mathcal{A}^+(q,t)$.

7 Perturbed Controls Coefficient Nullprojector

Similar to other projective operators, a fundamental property of the CCNP $\mathcal{P}(q,t)$ is that it is rank deficient. A singular perturbation that disencumber rank deficiency of $\mathcal{P}(q,t)$ is provided by the perturbed controls coefficient nullprojector (PCCN) $\widetilde{\mathcal{P}}(q,\delta,t)$ [8].

Definition 7.1 [Perturbed controls coefficient nullprojector] The perturbed CCNP $\widetilde{\mathcal{P}}(q,\delta,t) : \mathbb{R}^n \times \mathbb{R}^{1\times 1} \times [0,\infty) \to \mathbb{R}^{n\times n}$ is defined as

$$\widetilde{\mathcal{P}}(q,\delta,t) := I_{3\times 3} - h(\delta)\mathcal{A}^+(q,t)\mathcal{A}(q,t), \tag{24}$$

where $h(\delta) : \mathbb{R}^{1\times 1} \to \mathbb{R}^{1\times 1}$ is any continuous function such that

$$h(\delta) = 1, \quad \text{if and only if} \quad \delta = 0.$$

The perturbed CCNP $\widetilde{\mathcal{P}}(q,\delta,t)$ is of full rank for all $\delta \neq 0$. Additionally, the CCNP $\mathcal{P}(q,t)$ commutes with its perturbation $\widetilde{\mathcal{P}}(q,\delta,t)$ and inverted perturbation $\widetilde{\mathcal{P}}^{-1}(q,\delta,t)$ for all $\delta \neq 0$. Furthermore, their matrix multiplication yields the CCNP itself [8], i.e.,

$$\widetilde{\mathcal{P}}(q,\delta,t)\mathcal{P}(q,t) = \mathcal{P}(q,t)\widetilde{\mathcal{P}}(q,\delta,t) = \mathcal{P}(q,t) \tag{25}$$

and

$$\widetilde{\mathcal{P}}^{-1}(q,\delta,t)\mathcal{P}(q,t) = \mathcal{P}(q,t)\widetilde{\mathcal{P}}^{-1}(q,\delta,t) = \mathcal{P}(q,t). \tag{26}$$

8 Asymptotic Generalized Inverse Dynamics

The dynamically scaled generalized inverse control law is constructed by replacing the CCGI $\mathcal{A}^+(q,t)$ in (14) by the DSGI $\mathcal{A}_s^+(q,t)$ given by (22), resulting in

$$\tau_s(q,u,y,t) = \mathcal{A}_s^+(q,u,t)\mathcal{B}(q,u,t) + \mathcal{P}(q,t)y. \tag{27}$$

The corresponding closed loop system equations of (2) and (3) become

$$\dot{q} = u \tag{28}$$

$$\dot{u} = -M^{-1}(q,t)[C(q,u,t) + G(q,t)] + \mathcal{A}_s^+(q,u,t)\mathcal{B}(q,u,t) + \mathcal{P}(q,t)y. \tag{29}$$

Null-Control Vector Design

This section presents a design of the null-control vector to guarantee asymptotic tracking of desired robot manipulator generalized coordinate trajectories while maintaining asymptotic stability of the closed loop system over a prescribed domain of the joint space.

Proposition 8.1 *If the null-control vector y in the control law expression given by (27) is chosen such that the angular velocity vector u of the closed loop system given by (28) and (29) satisfies*

$$\|e_u(u,u_r)\| < \infty \ \ \forall\, t \geq t_0, \tag{30}$$

then the resulting closed loop attitude trajectory error vector $e_q(q,t)$ remains bounded

$$\|e_q(q,t)\| < \infty \ \ \forall\, t \geq t_0, \tag{31}$$

and the controls coefficient $\mathcal{A}(q,t)$ also remains bounded

$$\|\mathcal{A}(q,t)\| < \infty \;\; \forall\, t \geq t_0. \tag{32}$$

Proof. It is evident from the expression of the controls coefficient $\mathcal{A}(q,t)$ given by (12) that $\mathcal{A}(q,t)$ is bounded if and only if $e_q(q,t)$ is bounded. Therefore, assuming on the contrary that there exists a matrix gain K that causes the closed loop angular velocity vector u to satisfy (30) such that

$$\lim_{t\to\infty} \|e_q(q,t)\| = \infty, \tag{33}$$

then it follows that

$$\lim_{t\to\infty} \|\mathcal{A}(q,t)\| = \infty \tag{34}$$

which implies from (22) and (30) that

$$\lim_{t\to\infty} \mathcal{A}_s^+(q,u,t) = \mathcal{A}^+(q,t). \tag{35}$$

It accordingly follows from the expression of $\tau_s(q,u,y,t)$ given by (27) that

$$\lim_{t\to\infty} \tau_s(q,u,y,t) = \tau(q,u,y,t), \tag{36}$$

where $\tau(q,u,y,t)$ is given by (14), causing the closed loop system trajectories to asymptotically satisfy the stable servo-constraint dynamics given by (10), and resulting in

$$\lim_{t\to\infty} \phi = 0 \tag{37}$$

which contradicts (33). Therefore, the control law $\tau_s(q,u,y,t)$ given by (27) must yield bounded elements of $e_q(q,t)$ and bounded elements of $\mathcal{A}(q,t)$. Let the null-control vector y be chosen as

$$y = Ku, \tag{38}$$

where $K \in \mathbb{R}^{n\times n}$ is a matrix gain that is to be determined. Hence, a control law that realizes the servo-constraint given by (7) via the dynamics given by (10) is obtained by substituting this choice of y in (29), resulting in the closed loop dynamical subsystem

$$\dot{u} = -M^{-1}(q,t)[C(q,u,t) + G(q,t)] + \mathcal{A}_s^+(q,u,t)\mathcal{B}(q,u,t) + \mathcal{P}(q,t)Ku. \tag{39}$$

Also, let the reference null-control vector be defined as

$$y_r = Ku_r. \tag{40}$$

Then the reference internal dynamics given by (21) becomes

$$\dot{u}_r = -M^{-1}(q,t)[C(q,u_r,t) + G(q,t)] + \mathcal{A}^+(q,t)\mathcal{B}(q,u_r,t) + \mathcal{P}(q,t)Ku_r, \;\; q(t_0) = q_0. \tag{41}$$

The derivative of the generalized velocity error vector e_u is

$$\dot{e}_u = \dot{u} - \dot{u}_r \tag{42}$$

and therefore a generalized velocity error dynamics is obtained by subtracting (41) from (39), resulting in

$$\dot{e}_u = -M^{-1}(q,t)C(q,u,t) - \left[-M^{-1}(q,t)C(q,u_r,t)\right] + \mathcal{A}_s^+(q,u,t)\mathcal{B}(q,u,t) - \mathcal{A}^+(q,t)\mathcal{B}(q,u_r,t) + \mathcal{P}(q,t)Ke_u. \quad (43)$$

Asymptotic stability of the above written error dynamics is analyzed by considering the following positive-semidefinite Lyapunov function candidate

$$V(q,e_u,t) = e_u^T\mathcal{P}(q,t)e_u. \quad (44)$$

A gain matrix K that renders $\dot{V}(q,u,e_u,t)$ negative-semidefinite over a domain $\mathcal{D} \subseteq \mathbb{R}^n \times \mathbb{R}^n \times [t_0,\infty)$ guarantees Lyapunov stability of $e_u = \mathbf{0}_{n\times 1}$ over $\mathcal{D}$ if it asymptotically stabilizes $e_u = \mathbf{0}_{n\times 1}$ over the invariant set $\mathcal{D}_{V=0} \subset \mathcal{D}$ on which $V(q,e_u,t) = 0$. Moreover, the same gain matrix asymptotically stabilizes $e_u = \mathbf{0}_{n\times 1}$ over $\mathcal{D}$ if and only if it asymptotically stabilizes $e_u = \mathbf{0}_{n\times 1}$ over the largest invariant set $\mathcal{D}_{\dot{V}=0} \subset \mathcal{D}$ on which $\dot{V}(q,u,e_u,t) = 0$ [13].

Proposition 8.2 *Let $K = K(q,u,t)$ be a full-rank normal matrix gain, i.e., $KK^T = K^TK$ for all $t \geq 0$. Then the equilibrium point $e_u = \mathbf{0}_{n\times 1}$ of the closed loop error dynamics given by (43) is asymptotically stable over the invariant set $\mathcal{D}_{V=0}$.*

Proof. Since the matrix $\mathcal{P}(q,t)$ is idempotent, the function $V(q,e_u,t)$ can be rewritten as

$$V(q,e_u,t) = e_u^T\mathcal{P}(q,t)e_u = e_u^T\mathcal{P}(q,t)\mathcal{P}(q,t)e_u \quad (45)$$

which implies that

$$V(q,e_u,t) = 0 \Leftrightarrow \mathcal{P}(q,t)e_u = \mathbf{0}_{n\times 1}. \quad (46)$$

Therefore,

$$V(q,e_u,t) = 0 \Leftrightarrow e_u \in \mathcal{N}(\mathcal{P}(q,t)), \quad (47)$$

where $\mathcal{N}(\cdot)$ refers to matrix nullspace. Since the matrix $K(q,u,t)$ is normal and of full-rank, it preserves matrix range space and nullspace under multiplication. Accordingly,

$$\mathcal{N}(\mathcal{P}(q,t)) = \mathcal{N}(\mathcal{P}(q,t)K(q,u,t)) \quad (48)$$

which implies from (46) that

$$V(q,e_u,t) = 0 \Leftrightarrow \mathcal{P}(q,t)K(q,u,t)e_u = \mathbf{0}_{n\times 1}. \quad (49)$$

Therefore, the last term in the closed loop error dynamics given by (43) is the zero vector, and the closed loop error dynamics becomes

$$\dot{e}_u = -M^{-1}(q,t)C(q,u,t) - \left[-M^{-1}(q,t)C(q,u_r,t)\right] + \mathcal{A}_s^+(q,u,t)\mathcal{B}(q,u,t) - \mathcal{A}^+(q,t)\mathcal{B}(q,u_r(t),t). \quad (50)$$

On the other hand, since [14]

$$\mathcal{N}(\mathcal{P}(q,t)) = \mathcal{R}(\mathcal{A}^T(q,t)), \quad (51)$$

it follows from (47) that

$$V(q,e_u,t) = 0 \Leftrightarrow e_u \in \mathcal{R}(\mathcal{A}^T(q,t)). \quad (52)$$

Accordingly, $V(q, e_u, t) = 0$ if and only if there exists a continuously bounded scalar function $a(t),\ t \geq 0$ such that

$$e_u = a(t)\mathcal{A}^T(q,t), \;\; a(t) \neq 0. \tag{53}$$

Therefore, assuming that e_u goes unbounded, then $\mathcal{A}^T(q,t)$ must also go unbounded, both expressions of $\mathcal{A}^+(q,t)$ and $\mathcal{A}_s^+(q,u,t)$ given by (15) and (22) must go to zero, and the closed loop error dynamics given by (50) approaches the Lyapunov-stable uncontrolled dynamics

$$\begin{aligned} \dot{e}_u &= -M^{-1}(q,t)C(q,u,t) - \left[-M^{-1}(q,t)C(q,u_r,t)\right] && (54)\\ &= -M^{-1}(q,t)\left[C(q,u,t) - C(q,u_r,t)\right] && (55) \end{aligned}$$

implying boundedness of e_u, in contradiction with the original argument. Therefore, the trajectory of e_u must remain in a finite region. Since a trajectory of the error dynamical system given by (55) does not experience an isolated periodic motion (limit cycle), it follows from the Poincare-Bendixson theorem [15] that the trajectory must go to the equilibrium point $e_u = \mathbf{0}_{n\times 1}$.

Theorem 8.1 *Let the controls coefficient nullprojected gain matrix be given by*

$$\mathcal{P}(q,t)K = -vec^{-1}\Big\{\Big[\widetilde{\mathcal{P}}(q,\delta,t) \oplus \widetilde{\mathcal{P}}(q,\delta,t)\Big]^{-1} vec\Big[\dot{\mathcal{P}}(q,u,t) + \mathcal{P}(q,t)Q - 4\mathcal{P}(q,t)M^{-1}(q,t)C_m(q,u_r,t)\Big]\Big\}, \tag{56}$$

where $\oplus$ denotes the kronecker sum of matrices, "vec" and "vec^{-1}" denote the matrix vectorizing and inverse vectorizing operators, $Q \in \mathbb{R}^{n\times n}$ is an arbitrary positive definite constant matrix, and

$$C_m(q,u,t) = \frac{1}{2}\frac{\partial C(q,u,t)}{\partial u}. \tag{57}$$

Then the equilibrium point $e_u = \mathbf{0}_{n\times 1}$ of the closed loop error dynamics given by (43) is asymptotically stable.

Proof. Since the Coriolis centrifugal forces vector $C(q,u,t)$ is continuously differentiable in the vector u, then expanding the Coriolis centrifugal forces error vector about $e_u = \mathbf{0}_{n\times 1}$ using Taylor series yields

$$C(q,u,t) - C(q,u_r,t) = 2C_m(q,u_r,t)e_u + g(q,u,t), \tag{58}$$

where the following holds true for sufficiently small error vector norms $\|e_u\|$

$$\|g(q,u,t)\| < \|2C_m(q,u_r,t)e_u\| \tag{59}$$

and such that $g(q,u,t)$ satisfies

$$\lim_{\|e_u\|\to \mathbf{0}_{n\times 1}} \frac{\|g(q,u,t)\|}{\|e_u\|} = 0. \tag{60}$$

Therefore, linearizing the first difference in the error dynamics given by (43) about $e_u = \mathbf{0}_{n\times 1}$ yields

$$\dot{e}_{u_l} = -2M^{-1}(q,t)C_m(q,u_r,t)e_u + \mathcal{A}_s^+(q,u,t)\mathcal{B}(q,u,t) - \mathcal{A}^+(q,t)\mathcal{B}(q,u_r,t) + \mathcal{P}(q,t)Ke_u. \tag{61}$$

Evaluating the time derivative of V along solution trajectories of the partially linearized error system given by (61) yields

$$\begin{aligned}\dot{V}_l(q,u,e_u,t) &= 2e_u^T\mathcal{P}(q,t)\left[-2M^{-1}(q,t)C_m(q,u_r,t)e_u\right]\\ &\quad + 2e_u^T\mathcal{P}(q,t)\left[\mathcal{A}_s^+(q,u,t)\mathcal{B}(q,u,t) - \mathcal{A}^+(q,t)\mathcal{B}(q,u_r,t)\right]\\ &\quad + 2e_u^T\mathcal{P}(q,t)\left[\mathcal{P}(q,t)Ke_u\right] + e_u^T\dot{\mathcal{P}}(q,u,t)e_u. \qquad (62)\end{aligned}$$

The second property of $\mathcal{A}_s^+(q,u,t)$ and the nullprojection property of $\mathcal{P}(q,t)$ simplify the above expression to

$$\dot{V}_l(q,u,e_u,t) = e_u^T\left[-4\mathcal{P}(q,t)M^{-1}(q,t)C_m(q,u_r,t) + 2\mathcal{P}(q,t)K + \dot{\mathcal{P}}(q,u,t)\right]e_u. \qquad (63)$$

Since V is only positive-semi definite, it is impossible to design a gain matrix K that renders $\dot{V}_l$ negative definite. Nevertheless, K can be designed to yield $\dot{V}_l$ negative semidefinite by inquiring the existence of a positive semi-definite matrix $\mathcal{Q}(q,u,t)$: $\mathbb{R}^n \times \mathbb{R}^n \times [t_0,\infty) \to \mathbb{R}^{n\times n}$ that satisfies

$$\dot{V}_l(q,u,e_u,t) = -e_u^T\mathcal{Q}(q,u,t)e_u. \qquad (64)$$

Equating (63) and (64) yields the controls coefficient null-projected Lyapunov equation

$$\begin{aligned}&-4\mathcal{P}(q,t)M^{-1}(q,t)C_m(q,u_r,t) + \mathcal{P}(q,t)K\\ &\quad + K^T\mathcal{P}(q,t) + \dot{\mathcal{P}}(q,u,t) + \mathcal{Q}(q,u,t) = \mathbf{0}_{n\times n}. \qquad (65)\end{aligned}$$

The above equation is consistent if every term maps into the range space of $\mathcal{P}(q,t)$. The range space of $\dot{\mathcal{P}}(q,u,t)$ can be shown to be a subset of the range space of $\mathcal{P}(q,t)$ by writing

$$\mathcal{P}(q,t) = \mathcal{P}(q,t)\mathcal{P}(q,t) \Rightarrow \dot{\mathcal{P}}(q,u,t) = 2\mathcal{P}(q,t)\dot{\mathcal{P}}(q,u,t) \qquad (66)$$

so that

$$\mathcal{R}[\dot{\mathcal{P}}(q,u,t)] = \mathcal{R}[\mathcal{P}(q,t)\dot{\mathcal{P}}(q,u,t)] \subseteq \mathcal{R}[\mathcal{P}(q,t)] \qquad (67)$$

where $\mathcal{R}(\cdot)$ refers to matrix range space. Since $\mathcal{Q}(q,u,t)$ is arbitrary positive semi definite, then restricting $\mathcal{Q}(q,u,t)$ to map into the range space of $\mathcal{P}(q,t)$ implies that there is no loss of generality in specifying an arbitrary constant positive definite matrix Q such that a polar decomposition of $\mathcal{Q}(q,u,t)$ is given by

$$\mathcal{Q}(q,u,t) = \mathcal{Q}(q,t) = \mathcal{P}(q,t)Q. \qquad (68)$$

Accordingly, (65) can be written with the aid of the relation given by (25) as

$$\begin{aligned}&-4\mathcal{P}(q,t)M^{-1}(q,t)C_m(q,u_r,t) + \widetilde{\mathcal{P}}(q,\delta,t)\mathcal{P}(q,t)K\\ &\quad + K^T\mathcal{P}(q,t)\widetilde{\mathcal{P}}(q,\delta,t) + \dot{\mathcal{P}}(q,u,t) + \mathcal{P}(q,t)Q = \mathbf{0}_{n\times n}. \qquad (69)\end{aligned}$$

By requiring the gain matrix K to be symmetric and of full rank, the unique solution of (69) for $\mathcal{P}(q,t)K$ is given by [14]

$$\begin{aligned}\mathcal{P}(q,t)K = -\mathrm{vec}^{-1}\Big\{&\left[I_{3\times3}\otimes\widetilde{\mathcal{P}}(q,\delta,t) + \widetilde{\mathcal{P}}(q,\delta,t)\otimes I_{3\times3}\right]^{-1}\\ &\mathrm{vec}\left[\dot{\mathcal{P}}(q,u,t) + \mathcal{P}(q,t)Q - 4\mathcal{P}(q,t)M^{-1}(q,t)C_m(q,u_r,t)\right]\Big\}, \qquad (70)\end{aligned}$$

where $\otimes$ denotes the kronecker product of matrices. Equation (70) can be written in the compact form of (56), and $\dot{V}_l(q,u,e_u,t)$ is guaranteed to be globally negative semidefinite. Moreover, since the gain matrix K is symmetric and of full rank, then asymptotic stability of the equilibrium point $e_u = \mathbf{0}_{n\times 1}$ of the error dynamical system given by (43) over the invariant set $\mathcal{D}_{V=0}$ follows from Proposition 8.2. Radial unboundedness of $V(q,e_u,t)$ in e_u together with global negative semidefiniteness of $\dot{V}_l(q,u,e_u,t)$ and asymptotic stability over $\mathcal{D}_{V=0}$ imply that the equilibrium point $e_u = \mathbf{0}_{n\times 1}$ of the partially linearized error dynamics system given by (61) is globally stable in the sense of Lyapunov. Nevertheless, it is evident from the expression of $\dot{V}_l$ given by (63) that $\dot{V}_l = 0$ if and only if $\mathcal{P}(q,t)e_u = \mathbf{0}_{n\times 1}$. Therefore, $\mathcal{D}_{V=0} = \mathcal{D}_{\dot{V}=0}$, and the equilibrium point $e_u = \mathbf{0}_{n\times 1}$ of the system given by (61) is globally asymptotically stable [13]. From Lyapunov's indirect method, asymptotic stability of the system given by (61) implies local stability of the fully nonlinear error system given by (43) [16]. The matrix $\mathcal{Q}(q,t)$ (and the corresponding nullprojected Lyapunov matrix Q) can be designed for guarantee asymptotic stability of $e_u = \mathbf{0}_{n\times 1}$ over a prescribed domain $\mathcal{D}$ of asymptotic stability, as stated by the following theorem.

Theorem 8.2 *Let the controls coefficient nullprojected matrix gain be given by (56), where $Q \in \mathbb{R}^{n\times n}$ is positive definite and satisfying (68). For every prescribed neighborhood $\mathcal{D} \subset \mathbb{R}^n$ of the origin $e_u = \mathbf{0}_{n\times 1}$ there exists a real number $\gamma > 0$ such that if the minimum nonzero eigenvalue of $\mathcal{Q}(q,t)$ denoted by $\bar{\lambda}_{min}(\mathcal{Q}(q,t))$ satisfies*

$$\bar{\lambda}_{min}(\mathcal{Q}(q,t)) > \frac{2\gamma}{\lambda_{min}(M(q,t))} \quad \forall\, t \geq t_0, \tag{71}$$

then the equilibrium point $e_u = \mathbf{0}_{n\times 1}$ of the closed loop error dynamics given by (43) is asymptotically stable over $\mathcal{D}$.

Proof. Evaluating the time derivative of $V(q,u,t)$ along solution trajectories of the fully nonlinear error system given by (43) with the aid of the expansion given by (58) yields

$$\begin{aligned} \dot{V}(q,u,t) &= \dot{V}_l(q,u,t) - 2e_u^T\mathcal{P}(q,t)M^{-1}(q,t)g(q,u,t) && (72)\\ &= -e_u^T\mathcal{Q}(q,t)e_u - 2e_u^T\mathcal{P}(q,t)M^{-1}(q,t)g(q,u,t). && (73)\end{aligned}$$

Additionally, (60) implies that for every real scalar $\gamma > 0$ there exists a vector $e_{u_\gamma} \in \mathbb{R}^n$ such that the following inequality holds [16]

$$\|g(q,u,t)\| < \gamma\|e_u\| \quad \forall\, \|e_u\| < \|e_{u_\gamma}\|. \tag{74}$$

Accordingly, if $\dot{V}_l(q,u,t) \neq 0$ then an upper bound on $\dot{V}$ is obtained from (73) as

$$\begin{aligned} \dot{V}(q,u,t) &\leq -\bar{\lambda}_{min}(\mathcal{Q}(q,t))\|e_u\|^2 \\ &\quad +2\|e_u\|\lambda_{\max}(M^{-1}(q,t))\|g(q,u,t)\| && (75)\\ &\leq -\bar{\lambda}_{min}(\mathcal{Q}(q,t))\|e_u\|^2 + \frac{2\gamma}{\lambda_{min}(M(q,t))}\|e_u\|^2 && (76)\\ &= \left[-\bar{\lambda}_{min}(\mathcal{Q}(q,t)) + \frac{2\gamma}{\lambda_{min}(M(q,t))}\right]\|e_u\|^2 \\ &\qquad \forall\, \|e_u\| < \|e_{u_\gamma}\|. && (77)\end{aligned}$$

Therefore, given a real scalar $\gamma > 0$ and a corresponding vector e_{u_γ}, if $\mathcal{D}$ is defined as the set of all vectors $e_u \in \mathbb{R}^n$ satisfying

$$\|e_u\| < \|e_{u_\gamma}\| \tag{78}$$

and $\mathcal{Q}(q,t)$ is chosen such that $\bar{\lambda}_{min}(\mathcal{Q}(q,t))$ satisfies (71) then $\dot{V}$ is guaranteed to remain negative along solution trajectories of the fully nonlinear error system given by (43) initiated at $e_u(t_0) \in \mathcal{D}_{\dot{V}\neq 0}$ along which

$$\dot{V}_l(q,u,t) \neq 0 \quad \forall\, t \geq t_0. \tag{79}$$

The above mentioned argument together with the arguments of Proposition 8.2 and Theorem 8.1 on global asymptotic stability of $e_u = \mathbf{0}_{n\times 1}$ with respect to trajectories initiated within $\mathcal{D}_{\dot{V}=0}$ prove asymptotic stability of $e_u = \mathbf{0}_{n\times 1}$ over $\mathcal{D}$. A corresponding necessary condition on Q for asymptotic stability of $e_u = \mathbf{0}_{n\times 1}$ can be derived also. Since

$$\bar{\lambda}_{min}(\mathcal{Q}(q,t)) = \lambda_{min}(\mathcal{P}(q,t)Q) \leq \lambda_{\max}(\mathcal{P}(q,t))\lambda_{min}(Q) = \lambda_{min}(Q), \tag{80}$$

the condition given by (71) for asymptotic stability implies that

$$\lambda_{min}(Q) > \frac{2\gamma}{\lambda_{min}(M(q,t))} \quad \forall\, t \geq t_0, \tag{81}$$

provided that

$$e_u(t_0) \in \mathcal{D}. \tag{82}$$

Corollary 8.1 *Let γ be a positive scalar that satisfies*

$$\gamma > 2\sup_{q,t}(\sigma_{\max}(C_m(q,u_r,t))), \tag{83}$$

where $\sup_{q,t}$ denotes the supremum over all admissible values of robot generalized coordinates and over all $t \geq 0$. If $\mathcal{Q}(q,t) : \mathbb{R}^n \times [t_0,\infty) \to \mathbb{R}^{n\times n}$ is positive semidefinite and satisfies the condition given by (71), $Q \in \mathbb{R}^{n\times n}$ is positive definite and satisfies (68), and the controls coefficient nullprojected gain matrix is given by (56), then the equilibrium point $e_u = \mathbf{0}_{n\times 1}$ of the closed loop error dynamics given by (43) is asymptotically stable over a domain of attraction $\mathcal{D} \subset \mathbb{R}^n$ that is given by all vectors $e_u \in \mathbb{R}^n$ satisfying

$$\|e_u\| < \|e_{u_\gamma}\|, \tag{84}$$

where

$$\|e_{u_\gamma}\| = \frac{\|C(q_0,u_0,t_0) - C(q_0,u_r(t_0),t_0) - 2C_m(q_0,u_r(t_0),t_0)e_u(t_0)\|}{2\sup_{q,t}(\sigma_{\max}(C_m(q,u_r,t)))}. \tag{85}$$

Proof. The fact on $g(q,u,t)$ given by (59) implies that for sufficiently small values of $\|e_u\|$, the following inequality holds true

$$\|g(q,u,t)\| < 2\sigma_{\max}(C_m(q,u_r,t))\|e_u\|. \tag{86}$$

Therefore, a particular choice of γ that holds inequality (74) true is found by setting

$$2\sigma_{\max}(C_m(q,u_r,t))\|e_u\| < \gamma\|e_u\| \tag{87}$$

resulting in the expression given by (83) for a lower bound estimate of γ that ensures satisfaction of (74) for sufficiently small values of $\|e_u\|$. To obtain the corresponding estimate of $\|e_{u\gamma}\|$, the two sides of (74) are equated and the above written estimate of γ is substituted in the resulting equation, yielding

$$\|g(q,u,t)\| = 2\sup_{q,t}(\sigma_{\max}(C_m(q,u_r,t)))\|e_u\|. \tag{88}$$

The value of $\|e_u\|$ in the above equation is an estimate of the smallest vector norm $\|e_u\|$ that causes inequality (74) to be violated, i.e., it is an estimate of $\|e_{u_\gamma}\|$. Accordingly, evaluating $g(q,u,t)$ at t_0 and replacing e_u by e_{u_γ} yields

$$\|e_{u_\gamma}\| = \frac{\|g(q_0,u_0,t_0)\|}{2\sup_{q,t}(\sigma_{\max}(C_m(q,u_r,t)))}. \tag{89}$$

Evaluating $g(q_0,u_0,t_0)$ using (58) yields the expression of $\|e_{u_\gamma}\|$ given by (85).

9 Outer (Kinematic) Closed Loop Stability

Let ϕ_s be a norm measure function of the attitude deviation obtained by applying the control law $\tau_s(q,u,y,t)$ given by (27) to the manipulator's equations of motion (2) and (3) in place of τ, and let $\dot{\phi}_s$, $\ddot{\phi}_s$ be its first two time derivatives. Therefore,

$$\begin{aligned}
\phi_s &:= \phi_s(q,t) = \phi(q,t) &(90)\\
\dot{\phi}_s &:= \dot{\phi}_s(q,u,t) = \dot{\phi}(q,u,t) &(91)\\
\ddot{\phi}_s &:= \ddot{\phi}_s(q,u,\tau_s,t) = \ddot{\phi}(q,u,\tau,t) + \mathcal{A}(q,t)\tau_s(q,u,y,t)\\
&\quad -\mathcal{A}(q,t)\tau(q,u,y,t), &(92)
\end{aligned}$$

where $\tau(q,u,y,t)$ is given by (14). Adding $c_1\dot{\phi}_s + c_2\phi_s$ to both sides of (92) yields

$$\begin{aligned}
\ddot{\phi}_s + c_1\dot{\phi}_s + c_2\phi_s &= \ddot{\phi} + c_1\dot{\phi} + c_2\phi + \mathcal{A}(q,t)\tau_s(q,u,y,t)\\
&\quad -\mathcal{A}(q,t)\tau(q,u,y,t) &(93)\\
&= \mathcal{A}(q,t)[\tau_s(q,u,y,t) - \tau(q,u,y,t)]. &(94)
\end{aligned}$$

With the controls coefficient nullprojected matrix gain be given by (56), the generalized inversion feedback control law given by (27) yields asymptotically stable generalized coordinate trajectory tracking, as stated by the following theorem.

Theorem 9.1 *Let the controls coefficient nullprojected matrix gain $\mathcal{P}(q,t)K$ be given by (56), and the matrix $\mathcal{Q}(q,t)$ satisfies (71) for some real number $\gamma > 0$ and a corresponding domain of asymptotic stability $\mathcal{D} \subset \mathbb{R}^n$. Then the closed loop generalized coordinate deviation dynamics given by (94) is asymptotically stable.*

Proof. Multiplying both sides of the control law $\tau_s(q,u,y,t)$ given by (27) by $\mathcal{A}(q,t)$ yields

$$\mathcal{A}(q,t)\tau_s(q,u,y,t) = \mathcal{A}(q,t)\mathcal{A}_s^+(q,u,t)\mathcal{B}(q,u,t), \tag{95}$$

where

$$\mathcal{A}(q,t)\mathcal{A}_s^+(q,u,t) = \frac{\mathcal{A}(q,t)\mathcal{A}^T(q,t)}{\mathcal{A}(q,t)\mathcal{A}^T(q,t) + \|e_u(u,u_r)\|_p^p}. \tag{96}$$

Therefore, if $\mathcal{A}(q,t) \neq \mathbf{0}_{1\times n}$ then it follows from (96) that

$$0 < \mathcal{A}(q,t)\mathcal{A}_s^+(q,u,t) \leq 1. \tag{97}$$

Dividing (95) by $\mathcal{A}(q,t)\mathcal{A}_s^+(q,u,t)$ yields

$$\mathcal{A}(q,t)\bar{\tau}(q,u,y,t) = \mathcal{B}(q,u,t), \tag{98}$$

where $\mathcal{A}(q,t)$ and $\mathcal{B}(q,u,t)$ are the same controls coefficient and controls load in (11), and

$$\bar{\tau}(q,u,y,t) = \frac{\tau_s(q,u,y,t)}{\mathcal{A}(q,t)\mathcal{A}_s^+(q,u,t)}. \tag{99}$$

Therefore, the algebraic system given by (98) recovers the algebraic system given by (11) via the control law $\bar{\tau}(q,u,y,t)$ for all $\mathcal{A}(q,t) \neq \mathbf{0}_{1\times n}$. Equivalently, the effective control law $\bar{\tau}(q,u,y,t)$ enforces the asymptotically stable second-order system given by (10) on the robot manipulator system given by (2) and (3) whenever $\mathcal{A}(q,t) \neq \mathbf{0}_{1\times n}$. Nevertheless, it is noticed from (6) and (12) that $\mathcal{A}(q,t) = \mathbf{0}_{1\times n}$ if and only if $\phi = \phi_s = 0$. This in addition to the local asymptotic stability of $e_u = \mathbf{0}_{n\times 1}$ concluded from Theorem (8.1) imply that the second order generalized coordinate deviation dynamics given by (94) is asymptotically stable over the domain $\mathcal{D}$. Theorem 9.1 states that employing the DSGI $\mathcal{A}_s^+(q,u,t)$ in the generalized inversion attitude control law yields the same asymptotic attitude tracking property that is obtained by employing the CCGI $\mathcal{A}^+(q,t)$, provided that manipulator's internal asymptotic stability is achieved by a proper design of the null-control vector y.

Remark 9.1 The second order generalized coordinate deviation dynamics given by (94) can be put in the state space form by defining the state vector $\Phi \in \mathbb{R}^2$ as

$$\Phi = \begin{bmatrix}\Phi_1 & \Phi_2\end{bmatrix}^T = \begin{bmatrix}\phi_s & \dot{\phi}_s\end{bmatrix}^T. \tag{100}$$

The two state equations become

$$\dot{\Phi}_1 = \Phi_2 \tag{101}$$

$$\dot{\Phi}_2 = -c_1\Phi_2 - c_2\Phi_1 + \mathcal{A}(q,t)[\tau_s(q,u,y,t) - \tau(q,u,y,t)]. \tag{102}$$

Asymptotic stability of $e_u = \mathbf{0}_{n\times 1}$ over the domain $\mathcal{D}$ inferred from Theorem 8.1 in addition to boundedness of $\mathcal{A}(q,t)$ over the same domain inferred from Proposition 8.1 imply that the limit of the forcing term in (102) as $t \to \infty$ is

$$\lim_{t\to\infty}\Big[\mathcal{A}(q,t)[\tau_s(q,u,y,t) - \tau(q,u,y,t)]\Big] = 0 \tag{103}$$

so that $\dot{\Phi}$ converges to the asymptotically stable canonical part of the dynamics given by (101) and (102), and results in

$$\lim_{t\to\infty}\phi_s = \lim_{t\to\infty}\dot{\phi}_s = 0 \tag{104}$$

over the domain $\mathcal{D}$, verifying the attraction property of $\Phi = \mathbf{0}_{2\times 1}$, i.e.,

$$\lim_{t\to\infty} q = q_r(t) \tag{105}$$

and

$$\lim_{t\to\infty} \dot{q} = \dot{q}_r(t). \tag{106}$$

Remark 9.2 Inequalities (97) imply that

$$\lim_{e_u(u,u_r)\to \mathbf{0}_{n\times 1}} \mathcal{A}(q,t)\mathcal{A}_s^{+}(q,u,t) = 1. \tag{107}$$

Hence, (99) yields

$$\lim_{e_u(u,u_r)\to \mathbf{0}_{n\times 1}} \bar{\tau}(q,u,y,t) = \lim_{e_u(u,u_r)\to \mathbf{0}_{n\times 1}} \tau_s(q,u,y,t) = \tau(q,u,y,t). \tag{108}$$

10 Damped Controls Coefficient Nullprojector

The damped CCNP is a modified controls coefficient nullprojector with vanishing dependency on steady state kinematic state vector q.

Definition 10.1 [Damped controls coefficient nullprojector] The damped CCNP $\mathcal{P}_d(q,\beta,t)$ is defined as

$$\mathcal{P}_d(q,\beta,t) = \begin{cases} \mathcal{P}(q,t) & : \ \|\mathcal{A}(q,t)\| \geq \beta, \\ I_{n\times n} - \frac{\mathcal{A}^T(q,t)\mathcal{A}(q,t)}{\beta^2} & : \ \|\mathcal{A}(q,t)\| < \beta. \end{cases} \tag{109}$$

The above definition implies that

$$\lim_{e_q(q,t)\to \mathbf{0}_{n\times 1}} \mathcal{P}_d(q,\beta,t) = I_{n\times n}. \tag{110}$$

11 Control System Design Procedure

Starting from the standard mathematical model given by (1) for a rigid robot manipulator, the procedure of asymptotic generalized inverse dynamics for tracking a twice continuously differentiable reference trajectory vector $q_r(t)$ is summarized in the following steps.

1. The robot manipulator mathematical model given by (1) is written in its equivalent state space model form given by (2) and (3).

2. The coefficients a_1 and a_2 in the servo-constraint dynamics equation (10) are chosen such that the dynamics of ϕ is asymptotically stable. This implies that both a_1 and a_2 are strictly positive. To avoid oscillatory closed loop transient response induced by underdamped servo-constraint dynamics, the coefficient a_1 is chosen sufficiently large compared to a_2 such that the linear second-order servo-constraint dynamics given by Equation (10) is overdamped.

3. The expressions given by (12) and (13) for $\mathcal{A}(q,t)$ and $\mathcal{B}(q,u,t)$ are obtained, where $e_q(q,t)$ is given by (5).

4. The expression given by (83) is solved for the positive scalar γ, where $C_m(q,u,t)$ is given by (57).

5. The positive semidefinite matrix $\mathcal{Q}(q,t)$ is obtained from (71), and is used to solve (68) for a positive definite constant matrix Q, where $\mathcal{P}(q,t)$ is given by (16).

6. The control law τ_s is given by

$$\tau_s(q,u,y,t) = \mathcal{A}_s^+(q,u,t)\mathcal{B}(q,u,t) + \mathcal{P}(q,t)Ku. \tag{111}$$

   The dynamically scaled controls coefficient generalized inverse $\mathcal{A}_s^+(q,u,t)$ in the above written control law is given by (22) for some vector p norm, where $e_u(u,u_r)$ is given by (23). The controls coefficient nullprojected gain matrix $\mathcal{P}(q,t)K$ in the above written control law is given by (56), where the perturbed controls coefficient nullprojection matrix $\widetilde{\mathcal{P}}(q,\delta,t)$ is given by (24), $C_m(q,u,t)$ is given by (57), and $\dot{\mathcal{P}}(q,u,t)$ is obtained by time differentiating $\mathcal{P}(q,t)$ along solution trajectories of the system equations given by (2).

7. The control law τ_s is used in (2) and (3) in place of τ, and the two sets of equations are integrated to obtain the trajectories of $q(t)$ and $u(t)$. If the initial state vector u_0 is such that $\|e_u(t_0)\| < \|e_{u_\gamma}\|$ where $\|e_{u_\gamma}\|$ is given by (85), then the closed loop robot manipulator control system is asymptotically stable, the resulting trajectory tracking error vectors $e_q(q,t)$ and $e_u(q,u,t)$ are asymptotically decaying to the zero vectors, and the generalized coordinates vector q asymptotically tracks $q_r(t)$.

12 Example: RP Robot Manipulator

The RP robot manipulator shown in Fig. 1 consists of two rigid arms A_1 and A_2 having masses m_1 and m_2, respectively. The two arms are constrained to move in the vertical plane, and A_1 is attached to an inertial reference frame at point O. The body moments of inertia of A_1 and A_2 about the axes normal to their plane of rotation and passing through their mass centers c_1 and c_2 are I_{zz1} and I_{zz2}, respectively. The manipulator is equipped with a revolute joint at point O and a prismatic joint along the longitudinal axis L_c. The revolute joint is actuated by a motor that generates a torque $\mathbf{M}$, and the prismatic joint is actuated by a motor that generates a force $\mathbf{F}$. It is required to design $\mathbf{M}$ and $\mathbf{F}$ such that A_1 oscillates about the left part of the horizontal line passing through O at a frequency of $\pi/6$ Hz according to the harmonic relation

$$\theta = \pi \sin\left(\frac{\pi}{6}t\right). \tag{112}$$

Based on the orientation of A_1, A_2 is required to translate simultaneously along L_c according to

$$d = 2l(1 - 0.5\cos\theta). \tag{113}$$

Choosing the generalized coordinates to be $q_1 = \theta$ and $q_2 = d$, the desired generalized coordinates $q_{r_1}(t)$ and $q_{r_1}(t)$ are given by

$$q_{r_1}(t) = \pi \sin\left(\frac{\pi}{6}t\right) \tag{114}$$

and

$$q_{r_2}(t) = 2l(1 - 0.5\cos q_{r_1}(t)) = 2l\left(1 - 0.5\cos\left(\pi \sin\left(\frac{\pi}{6}t\right)\right)\right). \tag{115}$$

The matrices forming the manipulator state space mathematical model given by (2) and (3) are

$$M(q,t) = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 & 0 \\ 0 & m_2 \end{bmatrix}, \tag{116}$$

$$C(q,u,t) = \begin{bmatrix} 2m_2 q_2 u_1 u_2 \\ -m_2 q_2 u_1^2 \end{bmatrix}, \tag{117}$$

$$G(q,t) = \begin{bmatrix} (m_1 l_1 + m_2 q_2) g \cos q_1 \\ m_2 g \sin q_1 \end{bmatrix}, \tag{118}$$

$$\mathcal{F} = \begin{bmatrix} M \\ F \end{bmatrix}, \tag{119}$$

where M and F are the magnitudes of $\mathbf{M}$ and $\mathbf{F}$, positives in the directions indicated by the arrows in Fig. 1. The two components of the generalized coordinates error vector $e_q(q,t)$ given by (5) are

$$e_1(q_1,t) = q_1 - q_{r_1}(t) = q_1 - \pi \sin\left(\frac{\pi}{6}t\right) \tag{120}$$

and

$$e_2(q_2,t) = q_2 - q_{r_2}(t) \tag{121}$$

$$= q_2 - 2l\left(1 - 0.5\cos\left(\pi \sin\left(\frac{\pi}{6}t\right)\right)\right). \tag{122}$$

Hence, the kinematic deviation norm measure function ϕ given by (6) is

$$\phi = \| e_q(q,t) \|^2 = e_1^2(q_1,t) + e_2^2(q_2,t). \tag{123}$$

The matrix $C_m(q,u_r,t)$ is given by

$$C_m(q,u_r,t) = m_2 q_2 \begin{bmatrix} u_{r_2} & u_{r_1} \\ -u_{r_1} & 0 \end{bmatrix}. \tag{124}$$

The maximum singular value of $C_m(q,u_r,t)$ is found to be

$$\sigma_{max}(C_m(q,u_r,t)) = m_2|q_2|\sqrt{2u_{r_1}^2 + u_{r_2}^2 + \sqrt{u_{r_2}^4 + 4u_{r_1}^2 u_{r_2}^2}}. \tag{125}$$

The manipulator geometric and inertia constants are taken to be $l_1 = 1$ m, $m_1 = 10.5$ kg, $m_2 = 7.0$ kg, I_{zz1} =30 kg.m^2 and I_{zz2} =15 kg.m^2. Upper bounds on the variables u_{r_1}, u_{r_2} are obtained by time differentiating the expressions of q_{r_1} and q_{r_2} given by (114) and (115) as $\pi^2/6$ rad/sec and $\pi^2/6$ m/sec, respectively. A sufficiently conservative upper bound on q_2 is obtained from (113) as 3.5 m. Therefore, a value of γ that satisfies the condition given by (83) is taken to be 10^2, and a matrix Q that satisfies (68) and (71) is taken to be

$$Q = 60 I_{2\times 2}. \tag{126}$$

The servo-constraint dynamics constants in (10) are chosen to be $a_1 = 7$, $a_2 = 4$. With initial conditions $q_0 = [-\pi/2 \;\; 2.8]^T$ and $u_0 = [0.4 \;\; -0.2]^T$, the values of $\|e_u(t_0)\|$ and $\|e_{u_\gamma}\|$ are 1.26 and 1.3, respectively. Fig. 2 shows time history of generalized coordinates θ and d, where p, β, and δ are taken 4, 0.6, and 0.1, respectively. Excellent asymptotically stable trajectory tracking performance is noticed. Figs. 3 and 4 show time histories of the corresponding angular velocity $\dot{\theta}$ and linear velocity $\dot{d}$, and the control variables M and F, respectively.

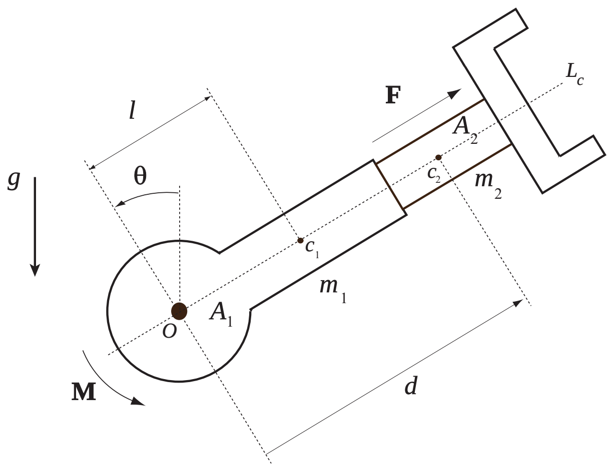

Figure 1: Schematic for RP robot manipulator.

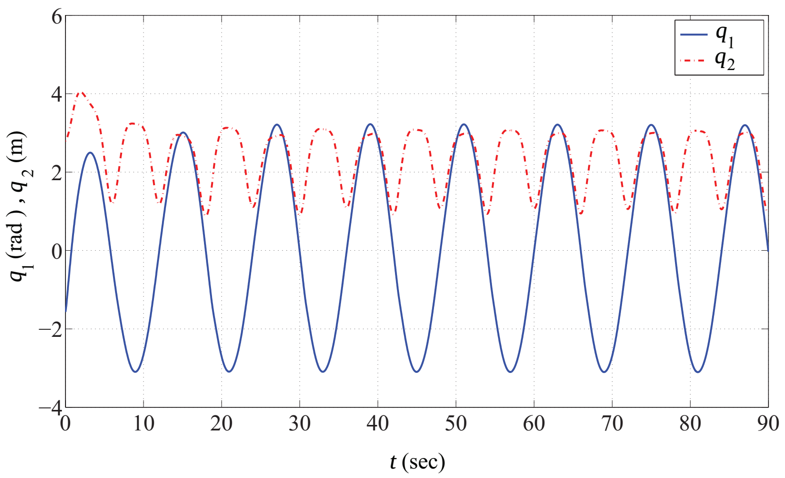

Figure 2: Generalized coordinates

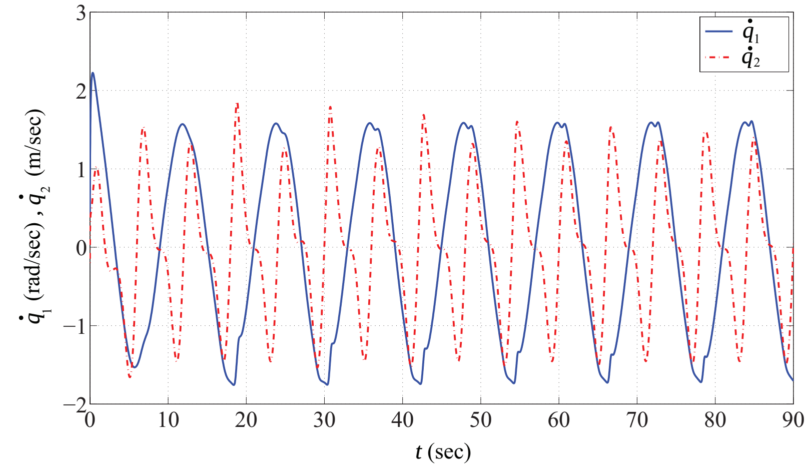

Figure 3: Generalized velocities.

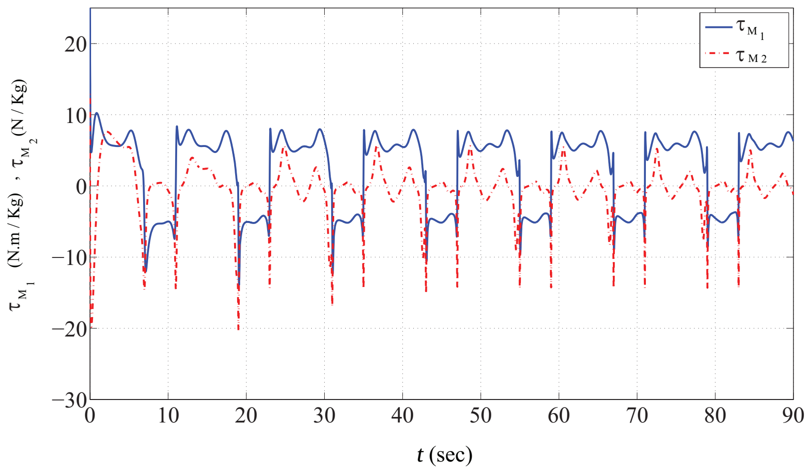

Figure 4: Scaled control forces.

13 Conclusions

The paper presents an approach that unifies the treatments of inverse kinematics and inverse dynamics, which have ever been made distinctive in the robot control literature. The vectorial representation of kinematical errors and their time derivatives in classical inverse dynamics is unfavorable, because the kinematical error is a scalar variable. More importantly, modeling the kinematical error as a vector that has the same number of elements as the number of manipulator's degrees of freedom restricts the inner loop design problem to have a unique solution, and hence it causes the methodology to lose a useful design freedom and makes it susceptible to dynamic inversion singularity. By observing that a control law that realizes any dynamic process on a controllable dynamical system is not unique, this paper removes the restriction on inverse dynamics by redefining the kinematical error as a deviation norm measure scalar. The paper applies the GID control paradigm to robot arm tracking of desired smooth trajectories. The outer loop design is made by generalized inversion of a stable servo-constraint dynamics differential equation in the kinematic deviation norm. The dynamically scaled generalized inverse in the particular part of the control law is capable of overcoming controls coefficient generalized inversion singularity, and it converges to the standard Moore-Penrose generalized inverse as closed loop steady state response approaches. The inner loop design is made by constructing the null-control vector in the auxiliary part of the control law. The null-control vector is designed to be linear in the internal states by means of a quadratic positive semidefinite control Lyapunov function and a controls coefficient nullprojected Lyapunov equation. Future works include utilizing the nullspace parametrization feature associated with generalized inversion and provided by the null-control vector in performing secondary objectives on top of generalized inverse dynamics.

References

[1] Bajodah, A.H., Hodges, D.H., and Chen, Ye-Hwa Inverse dynamics of servo-constraints based on the generalized inverse. *Nonlinear Dynamics* **39** (1-2) (2005) 179–196.

[2] Bajodah, A.H. Generalized Dynamic Inversion Spacecraft Control Design Methodologies. *IET Control Theory & Applications* **3** (2) (2009) 239–251.

[3] Greville, T.N.E. The pseudoinverse of a rectangular or singular matrix and its applications to the solutions of systems of linear equations. *SIAM Review* **1** (1) (1959) 38–43.

[4] Ben-Israel, A. and Greville, T.N.E. *Generalized Inverses, Theory and Applications.* Springer-Verlag New York, Inc., 2003, Second Edition.

[5] Bajodah, A.H. Servo-constraint Generalized Inverse Dynamics for Robot Manipulator Control Design. *International Journal of Robotics and Automation* **25** (1) (2010) 38–47.

[6] Moore, E.H. On the reciprocal of the general algebraic matrix. *Bulletin of the American Mathematical Society* **26** (1920) 394–395.

[7] Penrose, R. A generalized inverse for matrices. *Proceedings of the Cambridge Philosophical Society* **51** (1955) 406–413.

[8] Bajodah, A.H. Singularly Perturbed Feedback Linearization with Linear Attitude Deviation Dynamics Realization. *Nonlinear Dynamics* **53** (4) (2008) 321–343.

[9] Siciliano, B. and Khatib, O. *Springer Handbook of Robotics.* Springer, 2008, 245–268.

[10] Siciliano, B., Sciavicco, L., Villani, L., and Oriolo, G. *Robotics: Modeling, Planning, and Control.* Springer-Verlag London Limited, 2009.

[11] Bajodah, A.H. Stabilization of underactuated spacecraft dynamics via singularly perturbed feedback linearization. *Journal of King Abdulaziz University: Engineering Sciences* **16** (2) (2006) 35–53.

[12] Bajodah, A.H. Asymptotic Perturbed Feedback Linearization of Underactuated Euler's Dynamics. *International Journal of Control* **82** (10) (2009) 1856–1869.

[13] Iqqidr, A., Kalitine, B., and Outbib, R. Semidefinite Lyapunov functions stability and stabilization. *Mathematics of Control, Signals and Systems* **9** (2) (1996) 95–106.

[14] Bernstein, D.S. *Matrix mathematics: theory, facts, and formulas with application to linear system theory*. Princeton university press, 2009, Second Edition.

[15] Slotine, J.E. and Li, W. *Applied Nonlinear Control*. Prentice-Hall, Englewood Cliffs, NJ, 1991.

[16] Khalil, H.K. *Nonlinear Systems*. Prentice-Hall, Inc., 2002, Third Edition.