Nonlinear Dynamics and Systems Theory, **13** (1) (2013) 47–53

A Poiseuille Flow of an Incompressible Fluid with Nonconstant Viscosity

Y. Hataya [1*], M. Ito [2] and M. Shiba [2]

[1] *Department of Mathematics, Yamaguchi University, Yamaguchi, 753-8512, Japan*
[2] *Department of Applied Mathematics, Hiroshima University, Hiroshima 739-8527, Japan*

Received: March 19, 2012; Revised: January 23, 2013

Abstract: The viscosity coefficient in steady Navier-Stokes equations is determined for a particular velocity vector which arises from the study of conformal embedding of a Riemann surface.

Keywords: *Poiseuille flow; conformal embedding; nonconstant viscosity.*

Mathematics Subject Classification (2010): 76D06, 76M40, 35F10.

1 Introduction

Consider the steady Navier-Stokes equations

$$\begin{aligned} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}\nabla p &= \nu\Delta\boldsymbol{u} && \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega, \end{aligned}$$

in a long uniform tube $\Omega = \Omega_0 \times \mathbb{R}$ with the circular section

$$\Omega_0 := \{(x_1, x_2) \in \mathbb{R}^2;\ (x_1 - a)^2 + (x_2 - b)^2 < R^2\}.$$

Here, $\boldsymbol{u} = (u_1, u_2, u_3), p, \nu$ and ρ stand for the velocity field, the pressure, the viscosity and the density, respectively. We assume that ν and ρ are constant.

The solution of this problem with the additional assumption $u_1 = u_2 = 0$ is known as the Poiseuille flow. If this is the case, the pressure has a constant gradient $(0, 0, dp/dx_3)$ and u_3 is given by

$$u_3 = \frac{R^2 - (x_1 - a)^2 - (x_2 - b)^2}{4\nu\rho}\frac{dp}{dx_3}$$

* Corresponding author: mailto:hataya@yamaguchi-u.ac.jp

(see, e.g. [1]).

In the present paper we shall consider another kind of Poiseuille flow; the viscosity ν is not a priori supposed to be constant. The corresponding Navier-Stokes equations are then

$$(\boldsymbol{U}\cdot\nabla)\boldsymbol{U}+\frac{1}{\rho}\nabla p=\nabla\cdot(\nu\mathbb{T}(\boldsymbol{U})) \qquad \text{in } \Omega, \tag{1}$$

$$\nabla\cdot\boldsymbol{U}=0 \qquad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{U}=\boldsymbol{0} \qquad \text{on } \partial\Omega, \tag{3}$$

where $\mathbb{T}(\boldsymbol{U})=(U_{i,x_j}+U_{j,x_i})_{ij}$ stands for the deformation tensor. We assume furthermore that $b>R$, i.e., the section Ω_0 lies entirely in the upper half plane $\{(x_1,x_2)\in\mathbb{R}\,;\,x_2>0\}$ and that the velocity field $\boldsymbol{U}=(U_1,U_2,U_3)$ in (2) satisfy

$$U_1=U_2=0,\ U_3=\frac{R^2-(x_1-a)^2-(x_2-b)^2}{2Rx_2}. \tag{4}$$

This assumption means that the section is a non-euclidean disk and the velocity component U_3 describes a paraboloid in the non-euclidean sense.

We now explain shortly the reason why we are interested in U_3. For this purpose we first note that the function u_3 is closely connected with the theory of conformal mapping of a multiply connected plane domain. To be more precise, let D be an arbitrary but fixed domain in the (finite) complex z plane and $\zeta\in D$ be a fixed point. We consider all the (one-to-one) conformal mapping f of D into the Riemann sphere $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ such that

$$f(z)=\frac{1}{z-\zeta}+\kappa_f(z-\zeta)+\lambda_f(z-\zeta)^2+\cdots, \qquad \kappa_f,\lambda_f,\cdots\in\mathbb{C},$$

about ζ. It is a classical result that κ_f describes a (euclidean) closed disk in the complex plane. If we realize the disk as Ω_0, then $u_3(x_1,x_2)$ represents the maximum area of $\hat{\mathbb{C}}\setminus f(D)$ for the function f with $\kappa_f=x_1+ix_2$. We thus see that the velocity of the classical Poiseuille flow coincides with the (maximum) area function in the theory of conformal mapping of a planar Riemann surface.

We have shown in [3] that an analogous theorem holds for the conformal embeddings of a noncompact Riemann surface S of genus one into (marked compact) tori T. The moduli of T accept the rôle which the coefficients κ_f played in the planar case, and the maximum area $|f(S)|$ of $f(S)$ for various conformal embeddings f of S into a fixed torus T is described by a constant multiple of the function u_3. That is, the function u_3 works for the Riemann surface R of genus one as well as for the plane domain D. In [3] we have proved more: the function U_3 describes the maximum ratio $|f(S)|/|T|$ for the fixed torus T.

Note that the unknown function in (2) is not the velocity $\boldsymbol{U}$ but the viscosity coefficient ν. We shall find a smooth function ν so that the vector $\boldsymbol{U}=(0,0,U_3)$ is the velocity of a steady flow in the tube of an incompressible fluid with the viscosity ν.

Since the viscosity ν is affected by, say, the temperature, it may change point to point in the tube, when the ambient space of the tube is of nonconstant temperature. Hence, the nonconstant character of ν would be expected to be realistically important.

2 Main Theorem

In the following, we assume the density $\rho > 0$ to be constant. The problem with which we are concerned in the present paper is:

Problem. *Find the pressure $p = p(x_3)$ and the smooth viscosity $\nu = \nu(x_1, x_2)$, for which $(\boldsymbol{U}, p)$ satisfies (1)–(3).*

Our goal is the following:

Theorem 2.1 *The system (1)–(3) has a unique smooth solution $(\nu(x_1, x_2), p(x_3))$. The pressure is given by $p = \gamma\rho x_3 + \gamma'$, where γ, γ' are constants with $\gamma < 0$, and ν is given by*

$$\nu(x_1,x_2) = \begin{cases} -\dfrac{\gamma R x_2^2}{(x_1-a)^2}\left[-x_2 + c + \dfrac{(x_1-a)^2 + x_2^2 - c^2}{2(x_1-a)} \right. \\ \qquad\qquad \left. \times \mathrm{Sin}^{-1}\dfrac{2(x_1-a)(x_2+c)}{(x_1-a)^2+(x_2+c)^2}\right], & \text{if } x_1 \neq a, \\ -\dfrac{2}{3}\gamma R \dfrac{x_2^2(x_2+2c)}{(x_2+c)^2}, & \text{if } x_1 = a, \end{cases} \tag{5}$$

where $c = \sqrt{b^2 - R^2}$.

3 Proof of Theorem

In this proof, we shall denote U_3 by U for simplicity. The deformation tensor of the velocity (4) is then written as

$$\mathbb{T}(\boldsymbol{U}) = \begin{pmatrix} 0 & 0 & U_{x_1} \\ 0 & 0 & U_{x_2} \\ U_{x_1} & U_{x_2} & 0 \end{pmatrix}.$$

We thus rewrite equation (1) as

$$\frac{1}{\rho}\nabla p = (0, 0, (\nu U_{x_1})_{x_1} + (\nu U_{x_2})_{x_2}).$$

From this equation we see first of all that $dp/dx_3 = \gamma\rho$ holds with a constant γ. We have then a PDE for ν of the first order:

$$\nu_{x_1} U_{x_1} + \nu_{x_2} U_{x_2} + \nu \Delta U = \gamma. \tag{6}$$

For later use we first note the following basic expressions.

$$U_{x_1}(x_1, x_2) = -\frac{x_1 - a}{R x_2}, \tag{7}$$

$$U_{x_2}(x_1, x_2) = \frac{(x_1-a)^2 - x_2^2 + c^2}{2R x_2^2}, \tag{8}$$

$$\Delta U(x_1, x_2) = -\frac{(x_1-a)^2 + x_2^2 + c^2}{R x_2^3}. \tag{9}$$

Associated with (6) we now consider another equation

$$dx_1/U_{x_1} = dx_2/U_{x_2}, \tag{10}$$

or

$$\left\{(x_1-a)^2 - x_2^2 + c^2\right\} dx_1 + 2(x_1-a)x_2\, dx_2 = 0. \tag{11}$$

A solution of (10) (or (11)) is called a characteristic curve of (6). For general discussion of characteristic curves, see e.g. [2].

We can solve (11) and obtain the family of curves

$$C_k : \begin{cases} x_2^2 = c^2 - (x_1-a)(x_1-k), & \text{if } k \neq a, \\ x_1 = a,\ x_2 > 0, & \text{if } k = a. \end{cases} \tag{12}$$

It is easy to see that C_a is a characteristic curve. On the other hand for $k \neq a$, the function

$$\Phi(x_1,x_2) := \frac{x_1(x_1-a) + x_2^2 - c^2}{x_1 - a} \quad \left[= x_1 + \frac{x_2^2 - c^2}{x_1 - a}\right] \tag{13}$$

satisfies

$$\frac{\partial \Phi}{\partial x_1} = \frac{(x_1-a)^2 - x_2^2 + c^2}{(x_1-a)^2}, \tag{14}$$

$$\frac{\partial \Phi}{\partial x_2} = \frac{2x_2}{x_1 - a}. \tag{15}$$

Then, along the curve

$$\Phi(x_1,x_2) = k \tag{16}$$

for a constant k, the identity

$$0 = d\Phi = \frac{\partial \Phi}{\partial x_1}dx_1 + \frac{\partial \Phi}{\partial x_2}dx_2 = \frac{(x_1-a)^2 - x_2^2 + c^2}{(x_1-a)^2}dx_1 + \frac{2x_2}{x_1-a}dx_2$$

holds, which shows that (16) is a characteristic curve of (6) for any constant k. That is, (12) are the characteristic curves of (6). We observe that each characteristic curve C_k $(k \neq a)$ represents a half-circle

$$\left(x_1 - \frac{a+k}{2}\right)^2 + x_2^2 = d_k^2, \qquad x_2 > 0, \tag{17}$$

of the radius d_k:

$$d_k := \sqrt{c^2 + \left(\frac{a-k}{2}\right)^2}. \tag{18}$$

We remark that each curve C_k $(k \in \mathbb{R})$ passes through the point (a,c). Furthermore for each (x_1,x_2) other than (a,c), there exists a unique $k \in \mathbb{R}$ such that C_k passes through (x_1,x_2).

We now fix a $k \in \mathbb{R} \setminus \{a\}$ and consider the characteristic curve C_k. On this curve we can express x_2 as a single-valued function of x_1, since $x_2 > 0$ for the present problem. We next consider the function $\tilde{\nu}(x_1) := \nu(x_1, x_2(x_1))$ on (17). Since

$$\frac{d\tilde{\nu}}{dx_1} = \frac{\partial \nu}{\partial x_1} + \frac{\partial \nu}{\partial x_2}\frac{dx_2}{dx_1} = \left(\nu_{x_1}U_{x_1} + \nu_{x_2}U_{x_2}\right)\frac{1}{U_{x_1}},$$

our equation (6) becomes now of the form

$$\frac{d\tilde{\nu}}{dx_1} + \tilde{\nu}\frac{\Delta U}{U_{x_1}} = \frac{\gamma}{U_{x_1}}, \tag{19}$$

or, equivalently

$$\frac{d}{dx_1}\left(\tilde{\nu}(x_1)\exp\int\frac{\Delta U}{U_{x_1}}\,dx_1\right) = \frac{\gamma}{U_{x_1}}\exp\int\frac{\Delta U}{U_{x_1}}\,dx_1. \tag{20}$$

To solve this equation explicitly we first observe that

$$\frac{\Delta U(x_1,x_2)}{U_{x_1}(x_1,x_2)} = \frac{(x_1-a)^2+x_2^2+c^2}{(x_1-a)x_2^2}, \tag{21}$$

which follows immediately from (7) and (9). This, together with equation (12), yields

$$\frac{\Delta U(x_1,x_2)}{U_{x_1}(x_1,x_2)} = \frac{(x_1-a)(x_1-k)-(x_1-a)^2-2c^2}{(x_1-a)\{(x_1-a)(x_1-k)-c^2\}}. \tag{22}$$

If we denote by α and β the roots of the quadratic equation $(x_1-a)(x_1-k)-c^2=0$, we have

$$\frac{\Delta U(x_1,x_2)}{U_{x_1}(x_1,x_2)} = \frac{2}{x_1-a} - \frac{1}{x_1-\alpha} - \frac{1}{x_1-\beta}. \tag{23}$$

As usual, we can ignore an integration constant and obtain

$$\int\frac{\Delta U}{U_{x_1}}dx_1 = \log\frac{(x_1-a)^2}{c^2-(x_1-a)(x_1-k)}. \tag{24}$$

Hence we have

$$\frac{\gamma}{U_{x_1}}\exp\int\frac{\Delta U}{U_{x_1}}\,dx_1 = -\gamma R\cdot\frac{x_1-a}{x_2} \tag{25}$$

along the characteristic curve C_k $(k\neq a)$.

In order to integrate (20), it is convenient to parametrize the curve (17). Namely, for each k, we consider the parametrization

$$\begin{cases} x_1 = -d_k\sin\theta + \dfrac{a+k}{2}, \\ x_2 = d_k\cos\theta, \end{cases} \quad (-\pi/2<\theta<\pi/2) \tag{26}$$

of the curve (12). Then, according to (17), (18) and (26), the function $\tilde{\nu}(x_1) = \nu(x_1, x_2(x_1))$ can be expressed as $\tilde{\nu}(k,\theta) = \nu(x_1(k,\theta), x_2(k,\theta))$. Let θ_k $(-\pi/2<\theta_k<\pi/2)$ be the value of θ for which

$$\begin{cases} a = -d_k\sin\theta_k + \dfrac{a+k}{2}, \\ c = d_k\cos\theta_k, \end{cases} \tag{27}$$

holds.

Because of the relation $dx_1 = -d_k \cos\theta d\theta = -x_2 d\theta$ on C_k we have

$$\int_a^{x_1} \left(\frac{\gamma}{U_{x_1}} \exp \int \frac{\Delta U}{U_{x_1}} \, dx_1 \right) dx_1 = -\gamma R \int_a^{x_1} \frac{x_1 - a}{x_2} \, dx_1$$
$$= \gamma R \left\{ d_k(\cos\theta - \cos\theta_k) + \frac{k-a}{2}(\theta - \theta_k) \right\}.$$

Noting that $\dfrac{k-a}{2} = d_k \sin\theta_k$, we have

$$\int_a^{x_1} \left(\frac{\gamma}{U_{x_1}} \exp \int \frac{\Delta U}{U_{x_1}} \, dx_1 \right) dx_1 = d_k \gamma R \{ (\cos\theta - \cos\theta_k) + (\theta - \theta_k) \sin\theta_k \}. \tag{28}$$

Now, in virtue of equation (25) we obtain

$$\tilde{\nu}(k,\theta) = d_k \gamma R \cos^2\theta \cdot \frac{(\cos\theta - \cos\theta_k) + (\theta - \theta_k)\sin\theta_k}{(\sin\theta - \sin\theta_k)^2}. \tag{29}$$

This is the solution of (19) on C_k $(k \neq a)$.

We shall next solve (6) on the characteristic curve C_a. On the half line $\{(a, x_2)\,;\, x_2 > 0\}$, equations (1)–(3) reduce to

$$\nu'(a, x_2) \frac{c^2 - x_2^2}{2Rx_2^2} - \nu \frac{x_2^2 + c^2}{Rx_2^3} = k.$$

It has a unique continuous solution

$$\nu(a, x_2) = -\frac{2}{3} \gamma R \, \frac{x_2^2(x_2 + 2c)}{(x_2 + c)^2}. \tag{30}$$

The function

$$\nu(x_1, x_2) := \begin{cases} \tilde{\nu}(k(x_1, x_2), \theta(x_1, x_2)), & \text{for } (x_1, x_2) \in C_k, \ k \neq a, \\ \nu(a, x_2), & \text{for } (x_1, x_2) \in C_a \end{cases}$$

is now well-defined on $\Omega_0 \setminus (a, c)$, since for each $(x_1, x_2) \neq (a, c)$ we can find a unique $k \in \mathbb{R}$ with $(x_1, x_2) \in C_k$. If (x_1, x_2) approaches to (a, c) along a characteristic curve C_k, the function $\nu(x_1, x_2)$ has a finite limit which is independent of k. To show this, we first discuss the case $k \neq a$. We can then apply the de l'Hôpital theorem to obtain

$$\lim_{\theta \to \theta_k} \frac{(\cos\theta - \cos\theta_k) + (\theta - \theta_k)\sin\theta_k}{(\sin\theta - \sin\theta_k)^2} = \lim_{\theta \to \theta_k} \frac{-\sin\theta + \sin\theta_k}{2(\sin\theta - \sin\theta_k)\cos\theta} = -\frac{1}{2\cos\theta_k}.$$

Consequently along each C_k, we have

$$\lim_{(x_1, x_2) \to (a, c)} \nu(x_1, x_2) = -\frac{c\gamma R}{2}.$$

If $k = a$, it is easy to see $\nu(a, x_2) \to -c\gamma R/2$ as $x_2 \to c$ along C_a. Hence, $\nu(x_1, x_2)$ is a continuous function on Ω_0.

We next rewrite the function ν explicitly in terms of the euclidean coordinates (x_1, x_2). In virtue of (26), we have

$$\sin(\theta - \theta_k) = -\frac{1}{2d_k^2}\frac{(x_2+c)\{(x_1-a)^2+(x_2-c)^2\}}{x_1-a}.$$

Since

$$d_k^2 = \frac{\{(x_1-a)^2+(x_2+c)^2\}\{(x_1-a)^2+(x_2-c)^2\}}{4(x_1-a)^2},$$

by (16) and (18), we obtain

$$\sin(\theta - \theta_k) = \frac{-2(x_1-a)(x_2+c)}{(x_1-a)^2+(x_2+c)^2}. \tag{31}$$

If we substitute (26), (27) and (31) into (29), we conclude

$$\nu(x_1,x_2) = -\frac{\gamma R x_2^2}{(x_1-a)^2}\left[-x_2+c+\frac{(x_1-a)^2+x_2^2-c^2}{2(x_1-a)} \times \mathrm{Sin}^{-1}\frac{2(x_1-a)(x_2+c)}{(x_1-a)^2+(x_2+c)^2}\right]$$

for $x_1 \neq a$.

Finally we shall show ν is $C^1(\Omega_0)$. In fact, putting $y = \dfrac{2(x_1-a)(x_2+c)}{(x_1-a)^2+(x_2+c)^2}$ in the Maclaurin series

$$\mathrm{Sin}^{-1}y = \sum_{j=0}^{\infty}\frac{(2j)!}{2^{2j}\,(j!)^2}\,\frac{y^{2j+1}}{2j+1},$$

we have the expansion

$$-\frac{\nu(x_1,x_2)}{\gamma R x_2^2} = \frac{2}{3}\frac{x_2+2c}{(x_2+c)^2} + O(|x_1-a|^2).$$

Thus $\dfrac{\partial \nu}{\partial x_1}(a, x_2)$ exists and equals to 0. Since the regularity of $\nu(x_1,x_2)$ is obvious except the half line $\{(a,x_2)\,;\, x_2>0\}$, we conclude that ν is continuously differentiable in Ω_0.

Remark. From (5), we can see $\gamma < 0$ if the viscous constant ν is positive.

References

[1] Batchelor, G. K. An Introduction to Fluid Dynamics. Cambridge University Press, Cambridge, 1970.

[2] Garabedian, P. Partial Differential Equations. Wiley, New York, 1964.

[3] Shiba, M. The euclidean, hyperbolic, and spherical spans of an open Riemann surface of low genus and the related area theorems. *Kodai Math. J.* **16** (1993) 118–137.