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# Act-and-Wait Control Theory for Continuous-Time Systems with Random Feedback Delays

Bo Li[1], Xiaona Song[2*] and Junjie Zhao[1]

[1] *School of Automation, Nanjing University of Science and Technology, Nanjing 210094, China*
[2] *Electronic and Information Engineering College, Henan University of Science and Technology, Luoyang 471003, China*

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**Abstract:** Continuous-time systems with random state feedback delay are difficult to control in general because of its infinite poles. In this paper, the act-and-wait controller is well developed to solve this problem. If the infinite dimensional pole placement problem can be reduced to a finite dimensional one, it would be facility to make the system stable by the aid of pole placement method. The mechanism of the act-and-wait concept is that the state feedback is periodically switched on (act) and off (wait) during the control procedure. By using the act and wait controller, the stability of system can be represented by a finial dimensional monodromy matrix when the interval between two successive act moments is larger than the maximum state feedback delay. The aim of this paper is to design the periodic controller so that a finite number of eigenvalues can describe stability of the delay system, so the stability of the system can be achieved by use of pole placement method. The efficiency of the method is shown by a simulation.

**Keywords:** *act-and-wait controller; random delay; pole placement; stability.*

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* Corresponding author: mailto:xiaona_97@163.com

## 1 Introduction

Pole placement method is a very important tool in control theory [3], which is used in stabilizing plants and improving the performance of the controlled systems [16]. In the real situations, instability and poor performance of system are often led by time delay in the feedback loop of control systems. Many researchers have studied various kinds of delay systems [9, 19, 20, 22, 23]. Classic pole placement technique is well developed and common when it is applied to the systems without delay, while it is complex and crucial in delayed systems [5, 14]. In delayed systems the number of poles to be controlled is much larger than the degrees of freedom in the controller [13], so classical pole placement techniques of ordinary -differential equations can not be applied for delayed systems.

Periodic control method has shown advantages in stabilizing linear time-invariant (LTI) systems [18]. Several papers have been published which have used periodic feedback controller to control systems. Recently much attention [2, 10] has been attracted to the stabilization of continuous LTI systems with feedback delays by applying a periodic controller. In [15] it has been shown that the output feedback controller was used to make the system stable, which contains a periodic gain related to a cosine function.

On the other hand, lots of literature in which the stabilization problems of systems were studied by using act-and-wait concept [4, 6–8, 11, 21] focuses on this problem. The scholar in [4] made a comparison between the act-and-wait control and Intermittent control. Other researchers in [6–8,11,21] used the act-and-control mechanism to deal with the stabilization problems in different system, such as LTI systems, robotics systems, chaotic oscillator systems, and so on. In this approach the controller is periodically switched on (act) and off (wait). If the duration of waiting (switched off) time is longer than time delays in a system, then the problem about stabilization of the system is simplified to pole placement.

This paper discussed the act-and-wait controller applied to linear n-dimensional order system with random feedback delay. In general case, the n-dimensional system with feedback delay has infinite number of poles, which is hard to handle by the finite control parameters. As introduced in this paper, we can make the infinite poles of the n-dimensional system with random feedback delay reduce to n-dimensional one. The act-and-wait control mechanism means that the controller can periodically switch off and switch on, so the stability properties of the system can be described by n eigenvalues decided by an $N \times N$ monodromy matrix. It's assumed that the duration of waiting is larger than the maximum feedback delay.

## 2 Problem Statement

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input, and $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are given constant matrices. Firstly, we consider the autonomous delayed state feedback controller

$$u(t) = Dx(t - \tau_r), \tag{2}$$

where $D \in \mathbb{R}^{m\times n}$ is a constant matrix and $\tau_r$ is the random delay of state feedback. Because of the noise, information transmission, online data processing, computation, the problem of application of actuator and so on, delays always occur in feedback control, which are hardly eliminated or tuned during the control design. In general case, the delay is not a fixed parameter of the system with the changing of the environment. The range of random delay can often be estimated before designing the controller. In this paper, we assumed that $\tau_r$ is random non-negative integer, i.e. $0 \leq \tau_r \leq \tau_{max}$, where $\tau_{max}$ is the maximum delay.

By using controller (2), system (1) yields the closed-loop equation:

$$\dot{x}(t) = Ax(t) + BDx(t - \tau_r). \tag{3}$$

There is an infinite number of characteristic roots in the transcendental characteristic equation with the random time delay:

$$det(\lambda I - A - BDe^{-\tau\lambda}) = 0. \tag{4}$$

When all the poles of this system are located in the left half of the complex plane, the system will be asymptotically stable. Poles optimization method was used to deal with this type of problem [1, 17].

For the given system matrices $A$, $B$ and random feedback delay $\tau_r$, we want to find an appropriate parameters matrix $D$ in order to get satisfied control effect. The specialty of this feedback delay system is that an infinite poles should be placed by use of finite control parameters from $D$. As introduced in the first section of this paper, a special case of periodic feedback controller called act-and-wait controller will be studied here.

## 3 Act-and-Wait Mechanism

The form of the act-and-wait controller is

$$u(t) = g(t)Dx(t - \tau_r), \tag{5}$$

where g(t) is the T-periodic switching function, which is defined as

$$g(t) = \begin{cases} 0, & [0,\ t_w), \\ 1, & [t_w,\ T]. \end{cases} \tag{6}$$

In the above function, $t_w$ represents the switched off period of the controller, and $t_a$ represents the switched on period of the controller. The whole period is

$$T = t_w + t_a, \tag{7}$$

By using the act-and-wait controller (5), the system (1) can be written as

$$\dot{x}(t) = Ax(t) + BDg(t)x(t - \tau_r). \tag{8}$$

In the classic control stability theory, this system with act-and-wait controller will be stable if all the eigenvalues of the transcendental characteristic equation are located in the left half of the complex plane. Now the stable problem is how to find the appropriate control parameters, such as $t_w$, $t_a$, and control matrix D. In this paper, in order to stabilize the system we focus on the optimization of feedback gain matrix $D$.

When $t_w$ is smaller than $\tau_r$, the characteristic equation will still have infinite poles. If $t_w \geq \tau_r$, the monodromy operator of system equation (8) can be presented as an $N \times N$ matrix.

Assumed that $t_w \geq \tau_r$, there are still two cases here which will be discussed separately below.

(**a**) $0 < t_a \leq \tau_r$; Without loss of generality, the first period of the controller will be studied here. When $t \in [0, t_w)$ and $g(t) = 0$, the system equation with the initial state $x(0)$ can be given as

$$x(t) = e^{At}x_0 \; t \in [0, t_w). \tag{9}$$

When $t \in [t_w, T)$, then the controller is switched on (g(t) = 1). In other words, the delayed term is active in system

$$\dot{x}(t) = Ax(t) + BDe^{A(s-\tau)}x(0). \tag{10}$$

The initial state of ordinary differential equation (10) is $x(t_w) = e^{At_w}x(0)$. Solve the equation (10), it is derived that

$$x(T) = \Big(e^{AT} + \int_{t_w}^{T} e^{A(T-s)BDe^{A(s-\tau)}} ds\Big)x(0). \tag{11}$$

Let

$$\Phi = e^{AT} + \int_{t_w}^{T} e^{A(T-s)BDe^{A(s-\tau)}} ds, \tag{12}$$

where $\Phi$ is the transition matrix of the system with $N$ eigenvalues during the acting time of the controller. This means that all the other eigenvalues of the monodromy matrix of (8) are zeros but n eigenvalues in $\Phi$. Actually, $\Phi$ is the monodromy matrix of the system.

(**b**) $t_a > \tau_r$; Assumed that $t_a$ is between $k\tau_r$ and $(k+1)\tau_r$, so the transition matrix $\Phi$ can be obtained by step-by-step integration over every succeeding small interval. For example, the situation about $k = 1$ will be discussed below. Firstly, the solution over $[0, t_w)$ can be determined similarity to equation (8), then the $N \times N$ monodromy matrix can be obtained by the piecewise integration over the consecutive interval $[0, t_w)$, $[t_w, t_w + \tau)$, $[t_w + \tau_r, T)$:

$$\begin{aligned}\Phi &= e^{AT} + \int_{t_w}^{T} e^{A(T-s)BDe^{A(s-\tau)}} ds \\ &\quad + \int_{t_w+\tau}^{T} e^{A(T-s_1)}BD \int_{t_w}^{s_1-\tau} e^{A(s_1-s_2-\tau)}BDe^{A(s_2-\tau)} ds_1 ds_2. \end{aligned} \tag{13}$$

In this way, n eigenvalues of $\Phi$ can be placed using the control parameters $D$, so that the stability of the system can be achieved. But with the company of increasing of the k, the monodromy matrix becomes more and more complex. Because $\Phi$ depends nonlinearly on the control parameters $D$, it's impossible that arbitrary pole placement of the $\Phi$ can be obtained. Therefore, the simply case($t_a \leq \tau_r$) was studied here, and a simulation of the pole optimization of this case was shown in the next section.

## 4 Simulation

There is a second-order system with delayed feedback described by (1), (5), and (6). In the system:

$$A = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$D = \begin{bmatrix} -d_1 \\ -d_2 \end{bmatrix}, \quad \tau \in \{1, \; 0.9, \; 0.8\} \; .$$

If an autonomous controller is used in the system, then the characteristic equation is

$$\lambda^2 - a + d_1 e^{-\lambda} + d_2 \lambda e^{-\lambda} = 0. \tag{14}$$

The number of poles is infinite, so the poles can't be arbitrarily placed using the only two parameters $d_1$ and $d_2$.

So we discussed this system with act-and-wait controller when $a = 0$ and $a = -4$ are applied in system matrix A. Optimal control parameters will be investigated for the act-and-wait case with $t_w = 1.2$ and $t_a = 0.3$.

1. When $a = 0$. Actually this system is a feedback stabilized double integrator with input delay. When the act-and-wait controller is used with $t_w = 1.2s$, $t_a = 0.3s$. The delay called $\tau_r$ which belongs to $\{1 \; 0.9 \; 0.8\}$is a random variable. In order to study the performance of the system with different control parameters, the monodromy matrices are calculated separately with different delays 1s, 0.9s, 0.8s. Firstly, we assume that the delay $\tau_r$ is 1s, then the system can be presented by the $2 \times 2$ monodromy matrix

$$A = \begin{bmatrix} 1 - 0.045d_1 & 1.5 - 0.0135d_1 - 0.045d_2 \\ -0.3d_1 & 1 - 0.105d_1 - 0.3d_2 \end{bmatrix}$$

given by (10). It can be seen that the pole placement problem is now reduced to the placement of the two eigenvalues of $\Phi$ using the $D$ called feedback matrix. By using the appropriate $D$, the both eigenvalues of the $\Phi$ can be moved to zero. By calculating the $\Phi$, it can be obtained that these optimal parameters are $d_1 = 2.2157$ and $d_2 = 5.5588$. The simulation is shown in Figure 1. It can be seen that the system with act-and-wait controller actually converges to zero within period $2T$.

In the same way, the optimal parameters for $\tau_r = 0.9s$ and $\tau_r = 0.8s$ can be achieved:

$$D_{(\tau_r=0.9)} = \begin{bmatrix} -2.2146 \\ -5.3377 \end{bmatrix}^T,$$
$$D_{(\tau_r=0.8)} = \begin{bmatrix} -2.2157 \\ -5.1157 \end{bmatrix}^T.$$

And the simulations are shown in Figures 2 and 3. In Figures 1–3, it implies that the system can converge to zero with the random delay. In period $2T$ the system stops at zero completing the deadbeat convergence.

2. When $a = 4$. In this situation, system matrix $A$ is unstable. Because of the complexity of the system, it can't be stabilized using an autonomous controller since $a > 2$. If the act-and-wait control mechanism is applied with $t_w = 1.2s$ and $t_a = 0.3s$, then we can achieve the monodromy matrix (15). For simplicity, only the situation $\Phi$ with $\tau_r = 1$ is calculated and shown.

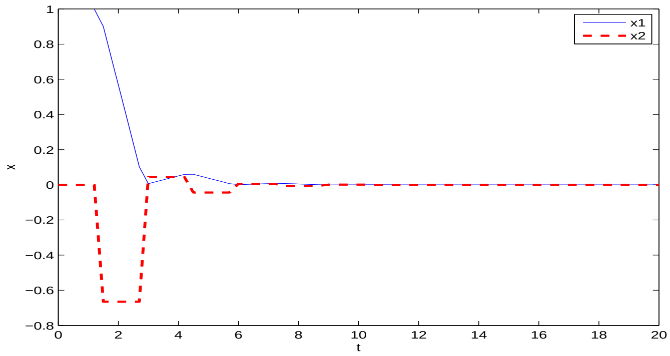


**Figure 1**. $d_1 = 2.2157$; $d_2 = 5.5588$ ($\tau$= $1s$).

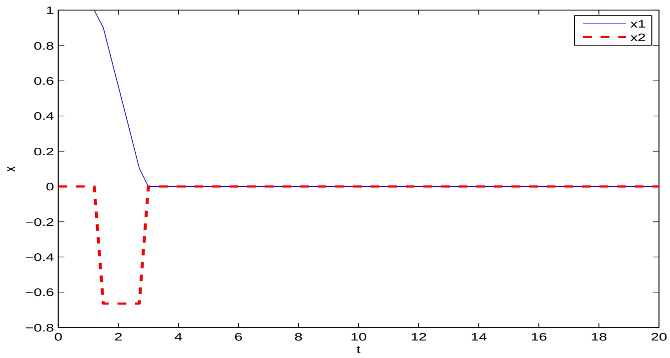


**Figure 2**. $d_1 = 2.2146$; $d_2 = 5.3377$ ($\tau$= $0.9s$).

By applying the act-and-wait controller, the system can be stabilized, and both eigenvalues can be moved to the origin. With this condition $d_1 = 7.4635$ and $d_2 = 9.8639$ can be obtained. In Figure 4 it can be seen that the state $x_1(t)$ and $x_2(t)$ converges to zero at about 10.5s. It's explicit that the state $x_1(t)$ and $x_2(t)$ grows very quickly in wait period in which the controller is switched off. But the growing tendency of $x_1(t)$ and $x_2(t)$ is restrained in act period. The system is stabilized after several periods of the controller.

In order to show the stability region of the system, the decay ratio $\rho = e^{Re(\lambda_1)}$ is introduced, where $\lambda_1$ is rightmost eigenvalue in the pole in the roots figure. In other words, this means $Re(\lambda_1) \geq Re(\lambda_i)$, $i = 2, 3, ..., \infty$. This decay ratio is a measure of the average error decay over a unit period, since $|x(t + 1)| \leq \rho|x(t)|$. Figure 6 shows the stable region of the LTI system with random delay ($a = 0$).

## 5 Conclusions

In this paper, we consider the stability problem in a continuous LTI system with random feedback delays. In the simulation part, a second-order linear time-invariant system with

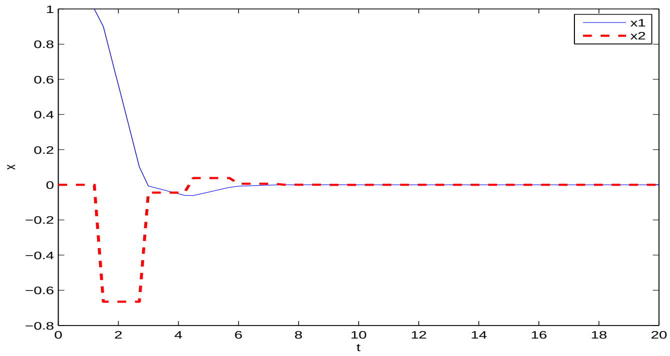


**Figure 3**. $d_1 = 2.2157$; $d_2 = 5.1157$. ($\tau$= 0.8$s$).

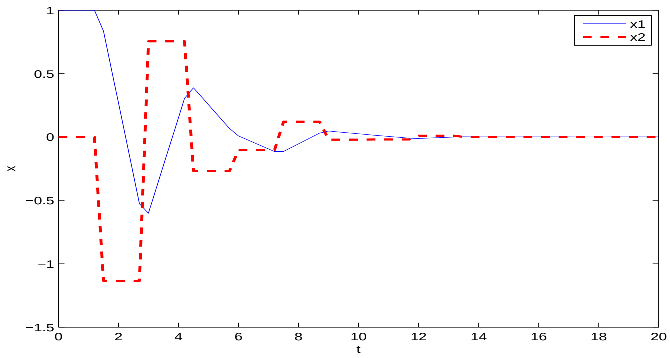


**Figure 4**. State of the system ($a = 4$).

random feedback delay is introduced to verify availability of the act-and-wait controller. By applying the periodic controller the monodromy matrix only have 2 eigenvalues which are easily placed to original by the control parameters. And the periodic controller can still stabilize the system in some cases while the autonomous one can't work.

Generally speaking, large gain in the controller can result in quick convergence, but continued large gain input may make the system become unstable. So in a control period, large gain can only be used in the acting period. In this way, the controller can keep higher performance and the stability. In the future research work, the algorithm about how to design control parameters and how to select the period of the controller should be investigated in order to obtain optimal performance of the system.

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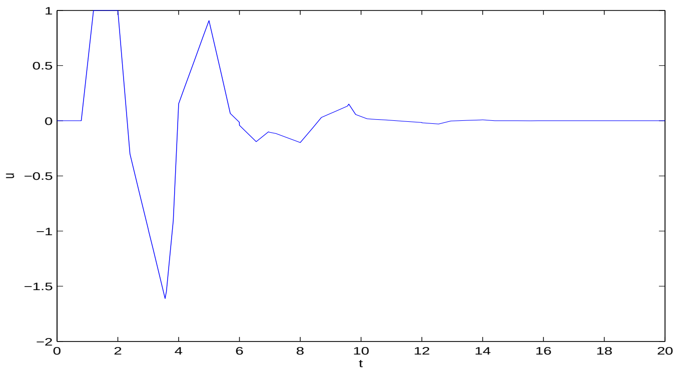


**Figure 5**. Output of the system ($a = 4$).

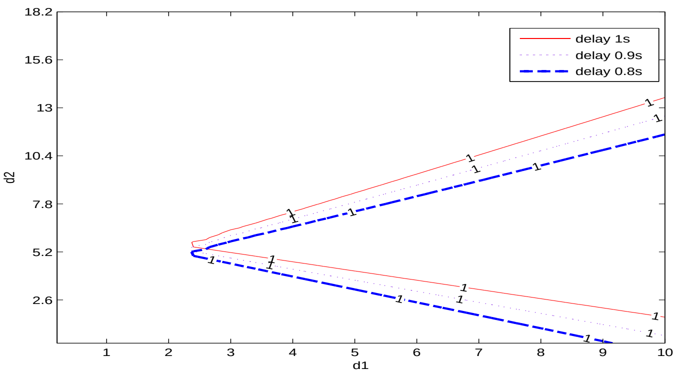


**Figure 6**. Deterministic stability for three different delays $\tau = 1$, $\tau = 0.9$, $\tau = 0.8$. The intersecting region of these triangular boundaries is stability region in deterministic sense.

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