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On Solutions to a Nonautonomous Neutral Differential Equation with Deviating Arguments

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Abstract: The main objective of this paper is to study solutions of a non-autonomous neutral differential equation of parabolic type with a deviating argument in an arbitrary Banach space. The main results are obtained by the Sobolevskiĭ-Tanabe theory of parabolic equations and the Banach fixed point theorem.

Keywords: *analytic semigroup; neutral differential equation; Banach fixed point theorem.*

Mathematics Subject Classification (2010): 34G20, 34K30, 34K40, 47N20.

1 Introduction

A natural way of generalizing differential equations is allowing the unknown function to appear with different values of the argument. Thus, differential equations with a deviating argument are differential equations in which the unknown function and its derivative appear in different places of the argument. This type of equations arise in many fields such as the theory of automatic control, the theory of self-oscillating systems, the problems of long-term planning in economics, the study of problems related with combustion in rocket motion, a series of biological problems, and many other areas [2]. One of the important examples is the process in fuel injection system for high-speed diesel engines which can be modeled as differential equations with a deviating argument of neutral type (see [2]).

The purpose of this work is to study solutions of the following type of neutral equation in a Banach space $(X, \|\cdot\|)$:

$$\left.\begin{aligned} \frac{d}{dt}[u(t)+g(t,u(a(t)))]+A(t)u(t) &= f(t,u(t),u(h(u(t),t))),\ t>0; \\ u(0) &= u_0, \end{aligned}\right\} \tag{1}$$

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where $u : \mathbb{R}_+ \to X$. Here, we assume that $-A(t)$, for each $t \geq 0$, generates an analytic semigroup of bounded linear operators on X. The continuous functions f, g and h satisfy suitable conditions in their arguments and the function $a : [0,T] \to [0,T]$ satisfies the delay properties.

The plentiful applications motivate the development of the theory of differential equation with deviating arguments (see e.g. [1,6–9,13,14,17–19] and references cited therein).

Fu and Liu [6] have considered the following abstract neutral functional equation with infinite delay:

$$\begin{aligned} \frac{d}{dt}[u(t) + f(t,u_t)] + A(t)u(t) &= g(t,u_t), \quad t \in (0,T], \\ u_0 &= \phi \in \mathcal{C}_0. \end{aligned}$$

Here $u(t)$ takes values in a Banach space X, the family $\{A(t) : t \in [0,T], T \in [0,\infty)\}$ of unbounded linear operators generates a bounded linear evolution operators on X, the function $f : [0,T] \times \mathcal{C}_0 \to X$ is uniformly Lipschitz continuous in both variables, the function $g : [0,T] \times \mathcal{C}_0 \to X$ satisfies suitable conditions (here $\mathcal{C}_0$ is a phase space defined appropriately). The existence of a solution has been obtained by the Sadovskii fixed point principle.

In [8], Haloi *et. al.* have studied the existence of solutions to the following differential equation

$$\begin{aligned} \frac{d}{dt}[u(t) + g(t,u(a(t)))] + A(t)[u(t) + g(t,u(a(t)))] &= f(t,u(t),u(h(u(t),t))), \ t > 0; \\ u(0) &= u_0. \end{aligned}$$

The main results are obtained by the Banach fixed point theorem without any regularity assumption on the function g.

Using the Banach fixed point theorem and the Sobolevskiĭ-Tanabe theory of parabolic equations, we prove the existence, uniqueness and asymptotic stability of a solution to Problem (1). The main results generalize some results of [7], [9], [14] and [19].The work is organized as follows. In Section 2, we provide preliminaries, assumptions and lemmas that will be needed for proving the main results. In Section 3, we prove the main results. Finally, we discuss an example as an application of the abstract results.

2 Preliminaries and Assumptions

This section deals with basic assumptions, preliminaries and lemmas necessary for proving the main results. For more details, we refer to [4,12,15,16].

Let $(X, \|\cdot\|)$ be a complex Banach space. Let $\{A(t) : 0 \leq t \leq T, 0 \leq T < \infty\}$ be a family of linear operators on the Banach space X. We use the following assumptions.

(H_1) For each $t \in [0,T]$, $A(t)$ is closed linear operator with domain $D(A)$ of $A(t)$ independent of t and dense in X.

(H_2) For each $t \in [0,T]$, the resolvent $R(\lambda; A(t))$ exists for all Re $\lambda \leq 0$ and there is a constant $C > 0$ (independent of t and λ) such that

$$\|R(\lambda; A(t))\| \leq \frac{C}{|\lambda| + 1}, \ \text{Re}\ \lambda \leq 0, \ t \in [0,T].$$

(H_3) For each fixed $s \in [0,T]$, there are constants $C > 0$ and $\rho \in (0,1]$ such that

$$\|[A(t) - A(\tau)]A(s)^{-1}\| \leq C|t-\tau|^{\rho}$$

for any $t, \tau \in [0,T]$. Here C and ρ are independent of t, τ and s.

It is well known that the assumption (H_2) implies that for each $s \in [0,T]$, $-A(s)$ generates a strongly continuous analytic semigroup $\{e^{-tA(s)} : t \geq 0\}$ in $\mathfrak{L}(X)$, where $\mathfrak{L}(X)$ denotes the Banach algebra of all bounded linear operators on X. Then there exist positive constants C and δ such that

$$\|e^{-tA(s)}\| \leq Ce^{-\delta t}, \quad t \geq 0; \tag{2}$$

$$\|A(s)e^{-tA(s)}\| \leq \frac{Ce^{-\delta t}}{t}, \quad t > 0, \tag{3}$$

for all $s \in [0,T]$ [4]. In the remainder of this work, C will denote a constant independent of $s,\ t$.

Theorem 2.1 *[4, 15] If the assumptions (H_1)–(H_3) hold, then there exists a unique fundamental solution $\{U(t,s) : 0 \leq s \leq t \leq T\}$ to homogeneous Cauchy problem.*

Now consider the following inhomogeneous Cauchy problem

$$\frac{d}{dt}u(t) + A(t)u(t) = h(t),\ t > t_0 \geq 0, \qquad u(t_0) = u_0. \tag{4}$$

Let $C^{\beta}([t_0,T];X)$ denote the space of all X-valued functions $h(t)$, that are uniformly Hölder continuous on $[t_0,T]$ with exponent β, where $0 < \beta \leq 1$. Then $C^{\beta}([t_0,T];X)$ is a Banach space endowed with the norm

$$\|h\|_{C^{\beta}([t_0,T];X)} = \sup_{t_0 \leq t \leq T} \|h(t)\| + \sup_{t,s \in [t_0,T], t \neq s} \frac{\|h(t)-h(s)\|}{|t-s|^{\beta}}.$$

Then we have the following theorem.

Theorem 2.2 *[4, 15] Let the assumptions (H_1)–(H_3) hold. If $h \in C^{\beta}([t_0,T];X)$, then there exists a unique solution to Problem (4). Furthermore, the solution is given by*

$$u(t) = U(t,t_0)u_0 + \int_{t_0}^{t} U(t,s)h(s)ds, \quad t_0 \leq t \leq T,$$

and $u : [t_0,T] \to X$ is a strongly continuously differentiable on $(t_0,T]$.

It follows from the assumptions (H_2) that the negative fractional powers of the operator $A(t)$ is well defined and defined as

$$A(t)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-\tau A(t)} \tau^{\alpha-1} d\tau$$

for $\alpha > 0$. Then $A(t)^{-\alpha}$ is a one-to-one and bounded linear operator on X [4]. We define the positive fractional powers of $A(t)$ by $A(t)^{\alpha} \equiv [A(t)^{-\alpha}]^{-1}$. It can be seen that $A(t)^{\alpha}$

is closed linear operator with domain $D(A(t)^\alpha)$ dense in X and $D(A(t)^\alpha) \subset D(A(t)^\beta)$ if $\alpha > \beta$. For $0 < \alpha \leq 1$, let $X_\alpha = D(A(0)^\alpha)$ and equip the space X_α with the graph norm

$$\|x\|_\alpha = \|A(0)^\alpha x\|.$$

Then $(X_\alpha, \|\cdot\|_\alpha)$ is a Banach space. If $0 < \alpha \leq 1$, the embeddings $X_1 \hookrightarrow X_\alpha \hookrightarrow X$ are dense and continuous. For each $\alpha > 0$, we define $X_{-\alpha} = (X_\alpha)^*$, the dual space of X_α, and endow the space $X_{-\alpha}$ with the natural norm

$$\|x\|_{-\alpha} = \|A(0)^{-\alpha} x\|.$$

Then $(X_{-\alpha}, \|\cdot\|_{-\alpha})$ is a Banach space. The following assumptions are necessary for proving the main results. For $0 < \alpha \leq 1$, let V_α and $V_{\alpha-1}$ be open sets in X_α and $X_{\alpha-1}$ respectively. For each $u \in V_\alpha$ and $u_1 \in V_{\alpha-1}$, there are closed balls such that $B_\alpha \equiv B_\alpha(u, r) \subset V_\alpha$ and $B_{\alpha-1} \equiv B_{\alpha-1}(u_1, r_1) \subset V_{\alpha-1}$ for $r > 0$ and $r_1 > 0$.

(H_4) There exist constants $L_f \equiv L_f(t, u, u_1, r, r_1) > 0$ and $0 < \theta_1 \leq 1$ such that the nonlinear continuous function $f : [0, T] \times V_\alpha \times V_{\alpha-1} \to X$ satisfies

$$\|f(t, x, x_1) - f(s, y, y_1)\| \leq L_f(|t - s|^{\theta_1} + \|x - y\|_\alpha + \|x_1 - y_1\|_{\alpha-1}) \tag{5}$$

for all $x, y \in B_\alpha$, $x_1, y_1 \in B_{\alpha-1}$ and for all $s, t \in [0, T]$.

(H_5) There exist constants $L_h \equiv L_h(t, u, r) > 0$ and $0 < \theta_2 \leq 1$ such that the continuous function $h : V_\alpha \times [0, T] \to [0, T]$ satisfies

$$|h(x, t) - h(y, s)| \leq L_h(\|x - y\|_\alpha + |t - s|^{\theta_2}), \tag{6}$$

$$h(\cdot, 0) = 0 \tag{7}$$

for all $x, y \in B_\alpha$ and for all $s, t \in [0, T]$.

(H_6) There exists constant $L_g \equiv L_g(t, u_1, r_1) > 0$ such that the continuous function $g : [0, T] \times V_{\alpha-1} \to X_1$ satisfies

$$\|g(t, x_1) - g(s, y_1)\|_1 \leq L_g\{|t - s| + \|x_1 - y_1\|_{\alpha-1}\} \tag{8}$$

for all $x_1, y_1 \in B_{\alpha-1}$ and $t, s \in [0, T]$.

(H_7) The function $a : [0, T] \to [0, T]$ has the following properties:

(i) a satisfies the delay property $a(t) \leq t$ for all $t \in [0, T]$.

(ii) The function a is Lipschitz continuous; that is, there exists a positive constant L_a such that

$$\begin{aligned} |a(t) - a(s)| &\leq L_a|t - s| \quad \text{for all } t, s \in [0, T], \\ L_a\|A(0)^{\alpha-2}\| &< 1. \end{aligned}$$

We will use the following lemmas in the subsequent sections.

Lemma 2.1 *[5, Lemma 1.1] Let $h \in C^\beta([t_0, T]; X)$. Define $\mathcal{F} : C^\beta([t_0, T]; X) \to C([t_0, T]; X_1)$ by*

$$\mathcal{F}h(t) = \int_{t_0}^{t} U(t, s)h(s)ds, \ t_0 \leq t \leq T.$$

Then $\mathcal{F}$ is a bounded mapping and $\|\mathcal{F}h\|_{C([t_0,T];X_1)} \leq C\|h\|_{C^\beta([t_0,T];X)}$ for some constant $C > 0$.

Lemma 2.2 *[10, Lemma 2] Let $0 < \alpha \leq 1$ and $f \in C([t_0, T]; X_\alpha)$. Define*

$$w(t) = \int_{t_0}^{t} U(t,s) f(s) ds, \ t_0 \leq t \leq T.$$

Then $w \in C([t_0, T]; X_1) \cap C^1((t_0, T]; X)$ and $\frac{dw(t)}{dt} + A(t)w(t) = f(t), \ t_0 < t \leq T$.

3 Main Results

In this section, we prove the main results on the existence, uniqueness and asymptotic stability of a solution to Problem (1). Let I denote the interval $[0, T_0]$ for some positive number T_0 to be determined later. For $0 \leq \alpha \leq 1$, let $\mathcal{C}_\alpha$ denote the space of all X_α-valued continuous functions on I, endowed with the sup-norm $\| \cdot \|_\infty$, where

$$\|\phi\|_\infty = \sup_{t \in I} \|\phi(t)\|_\alpha, \ \phi \in C(I; X_\alpha).$$

Let

$$Y_\alpha \equiv C_{L_\alpha}(I; X_{\alpha-1}) = \{\psi \in \mathcal{C}_\alpha : \|\psi(t) - \psi(s)\|_{\alpha-1} \leq L_\alpha |t-s| \quad \text{for all } t, s \in I\},$$

where L_α is a positive constant to be specified later. Then Y_α is a Banach space endowed with the sup-norm of $\mathcal{C}_\alpha$.

Definition 3.1 A continuous function $u \colon I \to X_\alpha$ is said to be a mild solution to Problem (1) if

(i) $g(\cdot, \cdot) \in X_1$;

(ii) u satisfies the following integral equation

$$u(t) = U(t,0)[u(0) + g(0, u_0)] - g(t, u(a(t))) + \int_0^t U(t,s) A(s) g(s, u(a(s))) ds + \int_0^t U(t,s) f(s, u(s), u(h(u(s), s))) ds, \ t \in I;$$

(iii) $u(0) = u_0$.

Definition 3.2 A continuous function $u : I \to X$ is said to be a solution to Problem (1) if u satisfies the following:

(i) $u(\cdot) + g(\cdot, u(a(\cdot))) \in C_{L_\alpha}(I; X_{\alpha-1}) \cap C^1((0, T_0); X) \cap C(I; X)$;

(ii) $u(\cdot) \in X_1$ and $g(\cdot, u(a(\cdot))) \in X_1$;

(iii) $\frac{d}{dt}[u(t) + g(t, u(a(t)))] + A(t)u(t) = f(t, u(t), u(h(u(t), t)))$ for all $t \in (0, T_0)$;

(iv) $u(0) = u_0$.

Let $u_0 \in X_\alpha$ and let $r > 0$ be chosen small enough such that the assumptions (H_4)–(H_6) hold for the closed balls $B_\alpha = B_\alpha(u_0, r)$ and $B_{\alpha-1} = B_{\alpha-1}(u_0, r)$. Let $K > 0$ and $0 < \eta < \beta - \alpha$ be fixed constants. Let

$$\begin{aligned}\mathcal{S} = \big\{ & v \in \mathcal{C}_\alpha \cap Y_\alpha : v(0) = u_0, \\ & \sup_{t\in I} \|v(t) - u_0\|_\alpha \le r, \|v(t) - v(s)\|_\alpha \le K|t-s|^\eta \text{ for all } s, t \in I\big\}.\end{aligned}$$

It can be seen that the set $\mathcal{S}$ is a non-empty, closed and bounded subset of $\mathcal{C}_\alpha$. Based on the ideas of Friedman [4], Fu and Liu [6] and Gal [7], we have the following theorem on existence and uniqueness of a local solution to Problem (1).

Theorem 3.1 *For $0 < \alpha < \beta \le 1$, let $u_0 \in X_\beta$. If the assumptions (H_1)–(H_7) hold, then there exist a positive number $T_0 \equiv T_0(\alpha, u_0)$ and a unique solution $u(t)$ to Problem (1) on the interval $[0, T_0]$.*

Proof. For each $v \in \mathcal{S}$ and $t \in I$, we define a map H by

$$\begin{aligned}Hv(t) = {} & U(t,0)[u_0 + g(0,u_0)] - g(t, v(a(t))) + \int_0^t U(t,s)A(s)g(s, v(a(s)))ds \\ & + \int_0^t U(t,s)f_v(s)ds,\end{aligned}$$

where $f_v(t) = f(t, v(t), v(h(v(t), t)))$. If $v \in \mathcal{S}$, then the assumptions (H_4) and (H_5) imply that $f_v(t)$ is Hölder continuous on I of exponent $\gamma = \min\{\theta_1, \theta_2, \eta\}$. Also for $v \in \mathcal{S}$, it is clear from the assumptions (H_6) and (H_7) that $A(t)g(t, v(a(t)))$ is Hölder continuous on I of exponent η. Thus by Lemma 2.1, the map H is well defined and it can be seen that $Hv \in \mathcal{C}_\alpha$. We will claim that H maps from the set $\mathcal{S}$ into $\mathcal{S}$ for sufficiently small $T_0 > 0$. Indeed, if $t_1, t_2 \in I$ with $t_2 > t_1$, then we have

$$\begin{aligned}&\|Hv(t_2) - Hv(t_1)\|_{\alpha-1} \\ &\le \|[U(t_2,0) - U(t_1,0)][u_0 + g(0,u_0)]\|_{\alpha-1} \\ &\quad + \|g(t_2, v(a(t_2))) - g(t_1, v(a(t_1)))\|_{\alpha-1} \\ &\quad + \left\| \int_0^{t_2} U(t_2,s)A(s)g(s, v(a(s)))ds - \int_0^{t_1} U(t_1,s)A(s)g(s, v(a(s)))ds \right\|_{\alpha-1} \\ &\quad + \left\| \int_0^{t_2} U(t_2,s)f_v(s)ds - \int_0^{t_1} U(t_1,s)f_v(s)ds \right\|_{\alpha-1}. \end{aligned} \tag{9}$$

Since the inclusion $X \to X_{\alpha-1}$ is bounded, we get the following estimate for first term on the right hand side of (9) (cf. [4, see Lemma II. 14.1]) as

$$\|[U(t_2,0) - U(t_1,0)][u_0 + g(0,u_0)]\|_{\alpha-1} \le C_1 \|u_0 + g(0,u_0)\|_\alpha (t_2 - t_1), \tag{10}$$

where C_1 is some positive constant.

Similarly, the assumptions (H_6) and (H_7) imply the following estimate

$$\|g(t_2, v(a(t_2))) - g(t_1, v(a(t_1)))\|_{\alpha-1} \le C_2|t_2 - t_1|, \tag{11}$$

where $C_2 = \|A(0)^{\alpha-2}\| L_g(1 + L_a L_\alpha)$.

Using [4, Lemma II. 14.4], we get the following estimates for the third and fourth term on the right hand side of (9) as

$$\begin{aligned}&\left\|\int_0^{t_2} U(t_2,s)A(s)g(s,v(a(s)))ds - \int_0^{t_1} U(t_1,s)A(s)g(s,v(a(s)))ds\right\|_{\alpha-1}\\ &\quad \le C_3 M_g (t_2-t_1)(|\log(t_2-t_1)|+1), \end{aligned}\tag{12}$$

$M_g = \sup_{t\in[0,T]} \|g(t,v(a(t)))\|_1$ and C_3 is some positive constant, and

$$\begin{aligned}&\left\|\int_0^{t_2} U(t_2,s)f_v(s)ds - \int_0^{t_1} U(t_1,s)f_v(s)ds\right\|_{\alpha-1}\\ &\quad \le C_4 N_f (t_2-t_1)(|\log(t_2-t_1)|+1), \end{aligned}\tag{13}$$

where $N_f = \sup_{t\in[0,T]} \|f_v(t)\|$ and C_4 is some positive constant.

Using estimates (10), (11), (12) and (13) in inequality (9), we get

$$\|Hv(t_2) - Hv(t_1)\|_{\alpha-1} \le L_\alpha |t_2 - t_1|, \tag{14}$$

where $L_\alpha = \max\Big\{C_1\|u_0 + g(0,u_0)\|_\alpha, \dfrac{\|A(0)^{\alpha-2}\|L_g}{1-\|A(0)^{\alpha-2}\|L_a}, C_3 M_g(|\log(t_2-t_1)| + 1), C_4 N_f(|\log(t_2-t_1)|+1)\Big\}$.

For sufficiently small $T_0 > 0$, we will show that

$$\sup_{t\in I}\|H(v)(t) - u_0\|_\alpha \le r.$$

Since $u_0+g(0,u_0)\in X_\alpha$, we can choose sufficiently small $T_1>0$ such that (cf. [4, Lemma II.14.1]),

$$\|[U(t,0)-I][u_0+g(0,u_0)]\|_\alpha \le \frac{r}{4} \quad \text{for all } t\in[0,T_1]. \tag{15}$$

Also, it is clear from the assumptions (H_6) and (H_7) that we can choose $T_2 > 0$ small enough such that

$$\|g(t,v(a(t))) - g(0,u_0)\|_\alpha \le \frac{r}{4} \quad \text{for all } t\in[0,T_2]. \tag{16}$$

Let $K_1 := \sup_{0\le t\le T}\|f(t,u_0,u_0)\|$.

We choose $T_3 > 0$ such that

$$\left(\frac{C_5}{1-\alpha}L_f[(1+L_\alpha L_h)r + T_3^{\theta_2}] + \frac{C_5 K_1}{1-\alpha}\right)T_3^{1-\alpha} \le \frac{r}{4}$$

for some positive constant C_5. Now from the assumptions (H_4) and (H_5), we have for $t \in [0, T_3]$

$$
\begin{aligned}
&\left\| \int_0^t U(t,s) f(s, v(s), v(h(v(s), s))) ds \right\|_\alpha \\
&\leq C_5 L_f \int_0^t (t-s)^{-\alpha} \big[\|v(s) - u_0\|_\alpha + \|v([h(v(s), s)]) - u_0\|_{\alpha-1} \big] ds \\
&\quad + C_5 K_1 \int_0^t (t-s)^{-\alpha} ds \\
&\leq C_5 L_f \int_0^t (t-s)^{-\alpha} \big[\|v(s) - u_0\|_\alpha + L_\alpha |h((v(s), s)) - h(u(0), 0)| \big] ds \\
&\quad + C_5 K_1 \int_0^t (t-s)^{-\alpha} ds \\
&\leq C_5 L_f \int_0^t (t-s)^{-\alpha} \big[\|v(s) - u_0\|_\alpha + L_\alpha |h((v(s), s)) - h(u(0), 0)| \big] ds \\
&\quad + \frac{C_5 K_1 \delta^{1-\alpha}}{1-\alpha} \\
&\leq C_5 L_f \int_0^t (t-s)^{-\alpha} [r + L_\alpha L_h (\|v(s) - u_0\|_\alpha + s^{\theta_2})] ds + \frac{C_5 K_1 T_3^{1-\alpha}}{1-\alpha} \\
&\leq C_5 L_f [(1 + L_\alpha L_h) r + T_3^{\theta_2}] \int_0^t (t-s)^{-\alpha} ds + \frac{C_5 K_1 T_3^{1-\alpha}}{1-\alpha} \\
&\leq \left(\frac{C_5}{1-\alpha} L_f [(1 + L_\alpha L_h) r + T_3^{\theta_2}] + \frac{C_5 K_1}{1-\alpha} \right) T_3^{1-\alpha} \qquad (17)
\end{aligned}
$$

for some positive constant C_5. Let $K_2 = \sup_{t \in [0,T]} \|g(t, u_0)\|_1$. We choose $T_4 > 0$ small enough such that

$$
C_6 \big(L_g L_\alpha L_a T_4 + K_2 \big) \frac{T_4^{1-\alpha}}{1-\alpha} \leq \frac{r}{4}
$$

for some positive constant C_6. Using the assumptions (H_6) and (H_7), we get

$$
\begin{aligned}
\left\| \int_0^t U(t,s) A(s) g(s, v(a(s))) ds \right\|_\alpha &\leq C_6 \int_0^t (t-s)^{-\alpha} \big(L_g (1 + L_\alpha L_a) s + \|g(s, u_0)\|_1 \big) ds \\
&\leq C_6 \big(L_g (1 + L_\alpha L_a) T_4 + K_2 \big) \frac{T_4^{1-\alpha}}{1-\alpha}, \qquad (18)
\end{aligned}
$$

where C_6 is a positive constant. Combining estimates (15), (16), (17) and (18), we obtain

$$
\sup_{t \in [0, T_5]} \|Hv(t) - u_0\|_\alpha \leq r,
$$

where $T_5 = \min\{T_1, T_2, T_3, T_4\}$.

It remains to show

$$
\|Hv(t+h) - Hv(t)\|_\alpha \leq K h^\eta
$$

for some $K > 0$ and $0 < \eta < 1$. Let $T_6 > 0$ be a sufficiently small number. If $0 \leq \alpha < \beta \leq 1, 0 \leq t \leq t+h \leq T_6$, then we have for $t \in [0, T_6]$

$$\begin{aligned}
&\|Hv(t+h) - Hv(t)\|_\alpha \\
&\leq \|[U(t+h,0) - U(t,0)][u_0 + g(0,u_0)\|_\alpha \\
&+ \|g(t+h, v(a(t+h))) - g(t, v(a(t)))\|_\alpha \\
&+ \left\| \int_0^{t+h} U(t+h,s)A(s)g(s,v(a(s)))ds - \int_0^t U(t,s)A(s)g(s,v(a(s)))ds \right\|_\alpha \\
&+ \left\| \int_0^{t+h} U(t+h,s)f(s,v(s),v(h(v(s),s)))ds - \int_0^t U(t,s)f(s,v(s),v(h(v(s),s)))ds \right\|_\alpha .
\end{aligned} \tag{19}$$

The bellow estimates follow from [4, Lemma II.14.1 and Lemma II.14.4],

$$\|[U(t+h,0) - U(t,0)][u_0 + g(0,u_0)]\|_\alpha \leq C_7 \|u_0 + g(0,u_0)\|_\beta h^{\beta-\alpha}; \tag{20}$$

$$\begin{aligned}
&\left\| \int_0^{t+h} U(t+h,s)A(s)g(s,v(a(s)))ds - \int_0^t U(t,s)A(s)g(s,v(a(s)))ds \right\|_\alpha \\
&\leq C_8 M_g h^{1-\alpha}(1 + |\log h|);
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\left\| \int_0^{t+h} U(t+h,s)f(s,v(s),v(h(v(s),s)))ds - \int_0^t U(t,s)f(s,v(s),v(h(v(s),s)))ds \right\|_\alpha \\
&\leq C_9 N_f h^{1-\alpha}(1 + |\log h|),
\end{aligned} \tag{22}$$

where $C_7,\ C_8$ and C_9 are some positive constants. Again form the assumption (H_6) and (H_7), it is clear that

$$\|g(t+h, v(a(t+h))) - g(t, v(a(t)))\|_\alpha \leq C_{10} L_g (1 + L_\alpha L_a) h \tag{23}$$

for some constant C_{10}. Combining estimates (20), (21), (22) and (23), we get for $t \in [0, T_6]$,

$$\begin{aligned}
&\|Hv(t+h) - Hv(t)\|_\alpha \\
&\leq h^\eta \big[C_7 \|u_0 + g(0,u_0)\|_\beta T_6^{\beta-\alpha-\eta} + C_{10} L_g (1 + L_\alpha L_a) h^{1-\eta} \\
&\quad + C_8 M_g T_6^{1-\alpha-\eta}(1 + |\log h|) + C_9 N_f T_6^\nu h^{1-\alpha-\eta-\nu}(|\log h| + 1) \big]
\end{aligned}$$

for any $\nu > 0, \nu < 1 - \alpha - \eta$. Hence, for sufficiently small $T_6 > 0$, we have

$$\|Hv(t+h) - Hv(t)\|_\alpha \leq K h^\eta$$

for $t \in [0, T_6]$ and for some $K > 0$. Thus, we have shown that H maps from the set $\mathcal{S}$ into $\mathcal{S}$.

We will now claim that the map H is a strict contraction. We choose $T_7 > 0$ such that

$$L_g \|A(0)^{-1}\| + C L_g \|A(0)^{-1}\| \frac{T_7^{1-\alpha}}{1-\alpha} + C L_f (2 + L_\alpha L_h) \frac{T_7^{1-\alpha}}{1-\alpha} \leq \frac{1}{2}$$

for some positive constant C. Using the assumptions (H_4)–(H_7) and [15, inequality (1.65), page 23], we have for $t \in [0, T_7]$ and $v_1, v_2 \in \mathcal{S}$,

$$\begin{aligned}
&\|Hv_1(t) - Hv_2(t)\|_\alpha \\
&\le L_g\|A(0)^{-1}\|\|v_1 - v_2\|_\infty \\
&+ CL_g\|A(0)^{-1}\| \int_0^t (t-s)^{-\alpha}\|v_1(a(s)) - v_2(a(s))\|_\alpha ds \\
&+ CL_f \int_0^t (t-s)^{-\alpha}(\|v_1(s) - v_2(s)\|_\alpha + \|v_1([h(v_1(s), s)]) - v_2([h(v_2(s), s)])\|_{\alpha-1})ds \\
&\le L_g\|A(0)^{-1}\|\|v_1 - v_2\|_\infty \\
&+ CL_g\|A(0)^{-1}\|\|v_1 - v_2\|_\infty \frac{T_7^{1-\alpha}}{1-\alpha} + CL_f(2 + L_\alpha L_h)\frac{T_7^{1-\alpha}}{1-\alpha}\|v_1 - v_2\|_\infty
\end{aligned} \tag{24}$$

for a positive constant C. Thus, the choice of T_7 implies that the map H is a strict contraction. Since $\mathcal{S}$ is a complete metric space, by the Banach fixed-point theorem, there exists $v \in \mathcal{S}$ such that $Hv = v$. Thus Problem (1) has a unique mild solution on $[0, T_0]$ where $T_0 = \min\{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}$

From Lemma 2.1 and Theorem 2.2, it follows that $v \in C^1((0, T_0); X)$. Thus v is a solution to Problem (1) on $[0, T_0]$. □

Next we will prove the following theorem that gives the existence of a global solution to Problem (1).

Theorem 3.2 *Let the assumptions (H_1)–(H_7) hold. If there are continuous nondecreasing real valued functions $k_1(t)$, $k_2(t)$ and $k_3(t)$ such that*

$$\|f(t, x, y)\| \le k_1(t)(1 + \|x\|_\alpha + \|y\|_{\alpha-1}), \tag{25}$$

$$|h(x, t)| \le k_2(t)(1 + \|x\|_\alpha), \tag{26}$$

$$\|g(t, y)\|_1 \le k_3(t)(1 + \|y\|_{\alpha-1}), \tag{27}$$

for all $t \ge 0$, $x \in X_\alpha$ and $y \in X_{\alpha-1}$, then Problem (1) has a unique solution and the solution exists for all $t \in [0, T]$, $T \in [0, \infty)$ for each $u_0 \in X_\beta$, where $0 < \alpha < \beta \le 1$.

Proof. It follows from Theorem 3.1 that there exists a $T_0 \in (0, T]$ and a unique local solution $u(t)$ on $t \in [0, T_0]$ to Problem (1) is given by

$$\begin{aligned}
u(t) = U(t, 0)[u_0 + g(0, u_0)] - g(t, u(a(t))) + \int_0^t U(t, s)A(s)g(s, u(a(s)))ds \\
+ \int_0^t U(t, s)f(s, u(s), u(h(u(s), s)))ds, \quad t \in [0, T_0].
\end{aligned}$$

If

$$\|u(t)\|_\alpha \le \tilde{C}$$

for all $t \in [0, T_0]$ and for some constant $\tilde{C}$ that is independent of t, then the solution $u(t)$ to Problem (1) may be continued further to the right of T_0. Thus to show global existence of the solution $u(t)$, it is enough to show that $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$.

Let $k_1(T) = \sup_{t\in[0,T]} k_1(t)$, $k_2(T) = \sup_{t\in[0,T]} k_2(t)$ and $k_3(T) = \sup_{t\in[0,T]} k_3(t)$. Form the assumptions $(H_4)-(H_7)$, (25), (26) and (27), we get for $t \in [0, T_0]$,

$$\begin{aligned}
\|u(t)\|_\alpha \leq\; & \|U(t,0)[u_0 + g(0,u_0)]\|_\alpha \\
& + \|g(t,u(a(t)))\|_\alpha + \left\| \int_0^t U(t,\tau)A(\tau)g(\tau,u(a(\tau)))d\tau \right\|_\alpha \\
& + \left\| \int_0^t U(t,\tau)f(\tau,u(\tau),u(h(u(\tau),\tau)))d\tau \right\|_\alpha \\
\leq\; & \|A^\alpha(0)A^{-\beta}(t)A^\beta(t)U(t,0)A(0)^{-\beta}A(0)^\beta[u_0 + g(0,u_0)] \\
& + k_3(T)\|A(0)^{\alpha-1}\|\Big(1 + \|A(0)^{-1}\| \sup_{s\in[0,t]} \|u(s)\|_\alpha\Big) \\
& + k_3(T)\int_0^t (t-\tau)^{-\alpha}\Big(1 + \|A(0)^{-1}\| \sup_{\zeta\in[0,\tau]} \|u(\zeta)\|_\alpha\Big)d\tau \\
& + k_1(T)\int_0^t (t-\tau)^{-\alpha}\Big[(1 + \|u(\tau)\|_\alpha + L_\alpha|h(u(\tau),\tau) - h(u_0,0)| + \|u_0\|_{\alpha-1}\Big]d\tau \\
\leq & \Big(C\|u_0 + g(0,u_0)\|_\beta + k_3(T)\|A(0)^{\alpha-1}\|\tilde{L} + k_1(T)\|u_0\|_{\alpha-1}\int_0^t (t-\tau)^{-\alpha}d\tau\Big) \\
+ & \Big(k_3(T)\|A(0)^{\alpha-1}\|\tilde{L} + k_1(T)(1 + L_\alpha k_2(T))\Big)\int_0^t (t-\tau)^{-\alpha}\Big(1 + \sup_{\zeta\in[0,\tau]} \|u(\zeta)\|_\alpha\Big)d\tau,
\end{aligned}$$

where $\tilde{L} = \max\{1, \|A(0)^{-1}\|\}$. Thus we have

$$\sup_{s\in[0,t]} \|u(s)\|_\alpha \leq \tilde{L}_1 + \tilde{M}_1 \int_0^t (t-\tau)^{-\alpha}(1 + \sup_{\zeta\in[0,\tau]} \|u(\zeta)\|_\alpha))d\tau,$$

where

$$\tilde{L}_1 = \frac{\Big(C\|u_0 + g(0,u_0)\|_\beta + k_3(T)\|A(0)^{\alpha-1}\|\tilde{L} + k_1(T)\|u_0\|_{\alpha-1}\int_0^t (t-\tau)^{-\alpha}d\tau\Big)}{(1 - k_3(T)\|A(0)^{\alpha-2}\|)},$$

$$\tilde{M}_1 = \frac{\Big(k_3(T)\|A(0)^{\alpha-1}\|\tilde{L} + k_1(T)(1 + L_\alpha k_2(T))\Big)}{(1 - k_3(T)\|A(0)^{\alpha-2}\|)}.$$

Applying Gronwall's Lemma, we get that $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$. □

Next we give a theorem of existence of solutions to Problem (1) under more smoothness condition on the function f and u_0. Denote $D(A(0))$ by X_1. Equipped this space X_1 with the graph norm

$$\|x\|_1 := (\|x\|^2 + \|A(0)x\|^2)^{\frac{1}{2}}.$$

Then $\|\cdot\|_1$ that is equivalent to the usual norm $\|A(0)\cdot\|$.

Let V_1 and V be open sets in X_1 and X, respectively. For each $u \in V_1$ and $u_1 \in V$, there are closed balls $B_1 \equiv B_1(u,r)$ and $B \equiv B(u_1,r_1)$ such that $B_1 \subset V_1$ and $B \subset V$ for some $r, r_1 > 0$. We make the following stronger assumptions.

$(H_4)'$ There exist constants $L_f \equiv L_f(t,u,u_1,r,r_1) > 0$ and $0 < \theta_1 \leq 1$, such that the nonlinear function $f : [0,T] \times V_1 \times V \to X_\alpha$ satisfies

$$\|f(t,x,x_1) - f(s,y,y_1)\|_\alpha \leq L_f(|t-s|^{\theta_1} + \|x-y\|_1 + \|x_1 - y_1\|), \tag{28}$$

for all $x, y \in B_1$, $x_1, y_1 \in B$, for all $s, t \in [0, T]$ and $\alpha \in (0, 1)$.

$(H_5)'$ There exist constants $L_h \equiv L_h(t, u_1, r_1) > 0$ and $0 < \theta_2 \leq 1$, such that $h : V_1 \times [0, T] \to [0, T]$ satisfies

$$|h(x,t) - h(y,s)| \leq L_h(\|x - y\|_1 + |t - s|^{\theta_2}), \tag{29}$$

$$h(\cdot, 0) = 0, \tag{30}$$

for all $x, y \in B_1$ and for all $s, t \in [0, T]$.

$(H_6)'$ There exists constant $L_g \equiv L_g(t, u_1, r_1) > 0$ such that the continuous function $g : [0, T] \times V \to X_1$ satisfies

$$\|g(t,x) - g(s,y)\|_1 \quad \leq \quad L_g\{|t - s| + \|x - y\|\}, \tag{31}$$

for all $x, y \in B$ and $t, s \in [0, T]$.

Then we have the following theorem on the existence and uniqueness of a solution to Problem (1).

Theorem 3.3 *Let $u_0 \in X_1$ and let the assumptions (H_1)–(H_3), $(H_4)'$ –$(H_6)'$ and (H_7) hold. Then there exist a positive number T_0 and a unique solution $u(t)$ to Problem (1) on the interval $I \equiv [0, T_0]$ such that $u \in C_L(I; X) \cap C^1((0, T_0); X) \cap C(I; X)$, where*

$$C_L(I; X) = \{\psi \in C(I; X_1) : \|\psi(t) - \psi(s)\| \leq L|t - s| \text{ for all } t, s \in I\}$$

for some constant $L > 0$. Moreover, we assume that there are positive constants $k_4(t)$, $k_5(t)$ and $k_6(t)$ such that

$$\|f(t,x,y)\|_\alpha \quad \leq \quad k_4(t)(1 + \|x\|_1 + \|y\|) \quad \text{for } \ 0 < \alpha < 1, \tag{32}$$

$$|h(x,t)| \quad \leq \quad k_5(t)(1 + \|x\|_1), \tag{33}$$

$$\|g(t,y)\|_1 \quad \leq \quad k_6(t)(1 + \|y\|) \tag{34}$$

for all $t \geq 0$, $x \in X_1$ and $y \in X$. Then the unique solution of (1) exists for all $t \geq 0$.

Proof. We define a map P by

$$Pv(t) = U(t,0)[u_0 + g(0,u_0)] - g(t, v(a(t))) + \int_0^t U(t,s)g(s, v(a(s)))ds$$
$$+ \int_0^t U(t,s)f(s, v(s), v(h(v(s), s)))ds$$

for each $t \in I = [0, T_0]$ and for each $v \in C(I, B_1)$. By Lemma 2.2, the map P from $C(I, B_1)$ into $C(I; X_1)$ is well defined.

Let

$$\mathcal{S} = \Big\{ y \in C(I; X_1) \cap C_L(I; X) : y(0) = u_0, \quad \sup_{t \in I} \|y(t) - u_0\|_1 \leq r \Big\}.$$

It is clear that $\mathcal{S}$ is nonempty, closed, and bounded subset of $C(I; X_1) \cap C_L(I; X)$. Thus $\mathcal{S}$ is a complete metric space. It can be proved that the map $P : \mathcal{S} \to \mathcal{S}$ is a contraction mapping. The proof can be obtained by the same argument as in the proof of Theorem 3.1 and Theorem 3.2, so we omit the details of the proof. □

We now prove the asymptotic stability of a solution to Problem (1) that is based on ideas of Friedman [3] and Webb [19].

Theorem 3.4 *Let the assumptions (H_1)–(H_7) hold and $u_0 \in X_\beta$, where $0 < \alpha < \beta \le 1$. Then there exists a continuous solution $u(t)$ to Problem (1) on $[0, T_0]$ for some $T_0 > 0$.*

In addition, suppose that there exist continuous functions ϵ_1 and ϵ_2 that map $[0,\infty)$ into $[0,\infty)$, and there exist constants $c_4 > 0$ and $c_5 > 0$ such that

$$\|f(t, u(t), u(h(u(t), t)))\| \le c_4(\epsilon_1(t) + \|u(t)\|_\alpha + \|u(t)\|_{\alpha-1}) \ \ \text{for } \ 0 < \alpha < 1, \tag{35}$$

$$\|g(t, u(a(t)))\|_1 \le c_5(\epsilon_2(t) + \|u(t)\|_{\alpha-1}), \tag{36}$$

for $t \ge 0$. Then

(i) if $\epsilon_1(t)$ and $\epsilon_2(t)$ are bounded on $[0,\infty)$, then $\|u(t)\|_\alpha$ is bounded on $[0,\infty)$;

(ii) if $\epsilon_1(t)$ and $\epsilon_2(t)$ are of $O(e^{\sigma t})$ for some $-1 < \sigma < 0$, then $\|u(t)\|_\alpha = O(e^{\sigma t})$;

(iii) if $\epsilon_1(t)$ and $\epsilon_2(t)$ are of $o(1)$, then $\|u(t)\|_\alpha = o(1)$.

Proof. It can be seen that there exists $0 < \theta < \delta$ (cf. [4, see page 176]) such that

$$\|A(t)^\gamma U(t, 0)\| \le \frac{C}{t^\gamma} e^{-\theta t}, \ \text{if} \ \ t > 0, \tag{37}$$

for any $0 \le \gamma \le 1$ and some constant $C > 0$. The solution to Problem (1) is given by

$$\begin{aligned} u(t) = U(t,0)[u_0 + g(0,u_0)] - g(t, u(a(t))) + \int_0^t U(t,s)A(s)g(s,u(a(s)))ds \\ + \int_0^t U(t,s) f(s, u(s), u(h(u(s), s)))ds, \end{aligned}$$

for $t \in I$. Now, for $t > 0$, put $\varphi(t) = e^{\theta t}\|u(t)\|_\alpha$. Using (37) in the solution of (1), we obtain

$$\begin{aligned} \varphi(t) &\le Ct^{-\alpha}\|u_0 + g(0,u_0)\| + c_5\|A(0)^{\alpha-1}\|\Big(\|A(0)^{-1}\|\varphi(t) + e^{\theta t}\epsilon_2(t)\Big) \\ &\quad + Cc_5\int_0^t e^{\theta s}(t-s)^{-\alpha}\Big(\epsilon_2(s) + \|A(0)^{-1}\|\|u(s)\|_\alpha\Big)ds \\ &\quad + Cc_4\int_0^t e^{\theta s}(t-s)^{-\alpha}\Big[\epsilon_1(s) + \|u(s)\|_\alpha + \|u(s)\|_{\alpha-1}\Big]ds \\ &\le Ct^{-\alpha}\|u_0 + g(0,u_0)\| + c_5\|A(0)^{\alpha-1}\|\Big(\|A(0)^{-1}\|\varphi(t) + e^{\theta t}\epsilon_2(t)\Big) \\ &\quad + C\int_0^t \Big[c_4\epsilon_1(s) + c_5\epsilon_2(s)\Big]e^{\theta s}(t-s)^{-\alpha}ds \\ &\quad + C\Big[c_4(1 + \|A(0)^{-1}\|) + c_5\|A(0)^{-1}\|\Big]\int_0^t (t-s)^{-\alpha}\varphi(s)ds. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \varphi(t) &\le \{C_0t^{-\alpha}\|u_0 + g(0,u_0)\| + C_0e^{\theta t}\epsilon_2(t) + C_0\int_0^t e^{\theta s}(t-s)^{-\alpha}\Big[c_4\epsilon_1(s) + c_5\epsilon_2(s)\Big]ds\} \\ &\quad + C_0\int_0^t (t-s)^{-\alpha}\varphi(s)ds, \end{aligned} \tag{38}$$

where $C_0 = \dfrac{\max\Big\{C, c_5\|A(0)^{\alpha-1}\|, C\big[c_4(1+\|A(0)^{-1}\|)+c_5\|A(0)^{-1}\|\big]\Big\}}{(1-c_5\|A(0)^{\alpha-1}\|\|A(0)^{-1}\|)}$. Denote

$$\chi(t) = C_0 t^{-\alpha}\|u_0 + g(0,u_0)\| + C_0 e^{\theta t}\epsilon_2(t) + C_0\int_0^t e^{\theta s}(t-s)^{-\alpha}\Big[c_4\epsilon_1(s) + c_5\epsilon_2(s)\Big]ds.$$

Then it is clear that

$$\chi(t) \le C_0 t^{-\alpha}\|u_0 + g(0,u_0)\| + C_0 e^{\theta t}\epsilon_2(t) + \tilde{C}e^{\theta t}\sup_{0\le s<\infty}\{c_4\epsilon_1(s) + c_5\epsilon_2(s)\},$$

for some constant $\tilde{C} > 0$. By the method of iteration, we get from (38) that

$$\varphi(t) \le \chi(t) + \int_0^t \left[\sum_0^\infty \frac{(t-s)^{j-1-j\alpha}[\Gamma(1-\alpha)]^j}{\Gamma(j-j\alpha)}\right]\chi(s)ds.$$

Since the series in the bracket is bounded by $D_1(t-s)^{-\alpha}\exp[D_2(t-s)^{1-\alpha}]$ for some constants $D_1, D_2 > 0$, it follows that, for $t \ge 1$ and for any $\lambda > 0$,

$$\varphi(t) \le D_3 e^{\lambda t}\|u_0 + g(0,u_0)\| + D_4 e^{\theta t}\epsilon_2(t) + D_5 e^{\theta t}\sup_{0\le s<\infty}\{c_4\epsilon_1(s) + c_5\epsilon_2(s)\},$$

where D_3, D_4 and D_5 are some positive constants. Thus, for any $0 < \theta_0 < \theta$, we get

$$\|u(t)\|_\alpha \le D_3 e^{-\theta_0 t}\|u_0 + g(0,u_0)\| + D_4\epsilon_2(t) + D_5\sup_{0\le s<\infty}\{c_4\epsilon_1(s) + c_5\epsilon_2(s)\}. \tag{39}$$

Thus the proof follows from the inequality (39). □

Remark 3.1 If $A(t)$ is a self adjoint positive definite operator in a Hilbert space X, then Theorem 3.1 and Theorem 3.2 can be strengthened. The assumptions (H_1), (H_2) and (H_3) imply that for $0 \le \gamma \le 1$ and for all $s, t \in [0, T]$ [11, page 185],

$$\|A(t)^\gamma A(s)^{-\gamma}\| \le C\|A(t)A(s)^{-1}\|^\gamma \le \widetilde{C_1}, \tag{40}$$

where $C, \widetilde{C_1} > 0$ are constants. Then Theorem 3.1 and Theorem 3.2 can be proved with less regularity assumption on u_0.

4 Example

Consider the following problem with a deviating argument

$$\left.\begin{array}{rcl} \dfrac{\partial}{\partial t}[w(t,x) & + & g(t, w(a(t),x))] + \dfrac{\partial^2}{\partial x^2}w(t,x) + b(t,x)w(t,x) \\ & & = H(x, w(t,x)) + G(t,x,w(t,x)); \\ w(t,0) & = & w(t,1), \quad t > 0; \\ w(0,x) & = & w_0(x), \quad x \in (0,1), \end{array}\right\} \tag{41}$$

where $b(t,x)$ is a continuous function in x and uniformly Hölder continuous function in t. Here $H(x, w(t,x)) = \displaystyle\int_0^x K(x,y)w(\widetilde{g}(t)|w(t,y)|, y)dy$ for all $(t,x) \in (0,\infty)\times(0,1)$. Assume that $\widetilde{g} : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in t with $\widetilde{g}(0) = 0$ and $K \in$

$C^1([0,1]\times[0,1];\mathbb{R})$. The function $G:\mathbb{R}_+\times[0,1]\times\mathbb{R}\to\mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u, uniformly in x.

Let $X = L^2((0,1);\mathbb{R})$, $A(t)u(t)(x) = -\frac{\partial^2}{\partial x^2}u(t,x) - b(t,x)u(t,x)$. Then $X_1 = D(A(0)) = H^2(0,1)\cap H_0^1(0,1)$ and $X_{1/2} = D((A(0))^{1/2}) = H_0^1(0,1)$. Then the family $\{A(t): t>0\}$ satisfies the assumptions (H_1)–(H_3) on each bounded interval $[0,T]$ (see [4,6]).

Put $w(t,\cdot)\equiv u(t)$, then Problem (41) can be written as

$$\left.\begin{aligned} \frac{d}{dt}[u(t)+g(t,u(a(t)))] + A(t)u(t) &= f(t,u(t),u(h(u(t),t))),\ t>0;\\ u(0) &= u_0. \end{aligned}\right\} \tag{42}$$

We define $f\colon \mathbb{R}_+\times H_0^1(0,1)\times H^{-1}(0,1)\to L^2(0,1)$ by

$$f(t,\phi,\psi) = H(x,\psi) + G(t,\phi)$$

for $\phi\in H^{-1}(0,1)\equiv H_0^1(0,1)$ and $\psi\in H_0^1(0,1)$ Here $H\colon H_0^1(0,1)\to L^2(0,1)$ is defined as $H(x,\psi(x,t)) = \int_0^x K(x,y)\psi(y,t)dy$ for $x\in(0,1)$ and $\psi\in H_0^1(0,1)$. Then it can be proved that f satisfies the assumption (H_4) for $\alpha=\frac{1}{2}$. We assume $h: H_0^1(0,1)\times\mathbb{R}_+\to\mathbb{R}_+$ defined by $h(\phi(x,t),t) = \widetilde{g}(t)|\phi(x,t)|$ satisfies the assumption (H_5) for $\alpha=\frac{1}{2}$(see Gal [7]). We also assume that the function $g:\mathbb{R}_+\times L^2(0,1)\to H_0^1(0,1)$ satisfies the assumption (H_6) for $\alpha=\frac{1}{2}$. We can take the function $a(t)$ where $a(t)=kt$ for $t\in[0,T]$ and $0<k\le 1$. Thus, we can apply our the results to study the existence, uniqueness and asymptotic stability of a solution to Problem (41).

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