Nonlinear Dynamics and Systems Theory, **13** (3) (2013) 258–269

Global Stability and Synchronization Criteria of Linearly Coupled Gyroscope

O.I. Olusola [1,2*], U.E. Vincent [3,4], A.N. Njah [2] and B.A. Idowu [5]

[1] *CSIR - Indian Institute of Chemical Biology, 4, Raja S.C. Mullick Road, Kolkata 700032, India.*
[2] *Nonlinear Dynamics Research Group, Department of Physics, Federal University of Agriculture,P.M.B. 2240, Abeokuta, Nigeria.*
[3] *Department of Physical Sciences, Redeemer's University, Km 46, Lagos - Ibadan Expressway P.M.B 3005, Redemption City, Nigeria.*
[4] *Department of Physics, Lancaster University, Lancaster LA1 4YB, United Kingdom.*
[5] *Department of Physics, Lagos state University, Ojo, Lagos State, Nigeria.*

Received: August 15, 2012; Revised: July 10, 2013

Abstract: We examine the synchronization transition of a pair of unidirectionally coupled gyroscope. Based on Lyapunov stability theory and linear matrix inequalities (LMI), some necessary and sufficient criteria for stable synchronous behaviour are obtained and an exact analytic estimate of the threshold for complete chaos synchronization is derived. Finally, numerical simulation results are presented to validate the feasibility of the theoretical analysis.

Keywords: *chaos synchronization; nonlinear gyroscope; linear matrix inequality; Lyapunov stability theory.*

Mathematics Subject Classification (2010): 70H14.

1 Introduction

In the last two decades, an intensive research activity has been devoted to the study of dynamics of coupled and driven chaotic systems. Despite the considerable body of knowledge that has already been gained and established, research on coupled nonlinear systems still remains an active field. In view of the importance of the classical results from the dynamics of driven or coupled harmonic oscillators in science and technology, the question of which phenomena emerge when chaotic oscillators are coupled or somehow

* Corresponding author: mailto:olasunkanmi2000@gmail.com

driven or perturbed has been and is still of great interest. The most relevant and widely studied phenomena until now are the synchronization [1–5] and the suppression of chaos [2, 4–8]. Due to the potential applications in various areas of science and technology, synchronization between two dynamical system has stimulated a wide range of research activity and many effective methods have been presented [1, 9–11].

In the past, research on chaos synchronization and its applications has intensively focused on the autonomous chaotic systems such as Lorenz, Chen, Rössler etc, but recently, the dynamics and synchronization of non-autonomous chaotic systems such as Duffing oscillator, gyroscopes, etc have witnessed tremendous research interest due to their potential applications in engineering and life sciences [12–18]. In particular, the gyroscopes, from a purely scientific viewpoint show strange and interesting properties, and from engineering viewpoint, they have great utility in the navigation of rockets, aircrafts, spacecrafts and in the control of complex mechanical system. In the past years, the gyroscope has been found with rich phenomena [12, 19, 20], for example, when subjected to harmonic vertical base excitations, it exhibits a variety of interesting dynamical behaviours that span the range from regular to chaotic motions [11, 12, 20–22].

The synchronization of the symmetric gyroscope model presented in Ref. [12] has been achieved using different methods, for example, four different kinds of one way coupling [12], active control [23], backstepping design [13, 24], fuzzy logic controller [25], sliding mode control [26, 27], sliding based fuzzy control [28] and so on. Very recently, synchronization of uncertain gyros was considered in [29]. Among the above methods, it is well known that linear feedback method provides simple control inputs for synchronization and has lately been employed to achieve stable synchronization in various unidirectionally coupled systems including, double well Duffing oscillators (DDOs) [30], parametrically excited Duffing oscillators [31] and the gyroscope. However, a crucial issue is the assessment of stability analysis for feedback controlled system and the determination of appropriate feedback gains that would guarantee stable synchronization. Since the beginning of the studies on synchronization of chaotic systems, the stability of synchronous motion was considered the most crucial question needed to be addressed, in order to furnish the proper conditions for a laboratory verification of theoretical findings. The problem of stability can be tackled in different ways and different criteria could be established, depending on specific conditions of interest.

One of the most popular and widely used criterion is the conditional Lyapunov exponents, which constitute average measurements of expansion or shrinkage of small displacements along the synchronized trajectory. However, it has been shown that negativity of the conditional Lyapunov exponents is not a sufficient condition for a stable synchronized state due to some unstable invariant sets in the stable synchronization manifold [32]. Whether this condition is necessary or not has remained an open issue (see [33] and references therein), and needs to be studied further. In [30], we proposed a linear state error feedback approach based on Lyapunov stability theory and Linear Matrix Inequality (LMI) [34], to analyze the stability of the synchronized state and also determine sufficient criteria for stable synchronous behaviour. This method is used because, it is known that many engineering optimization problem can be easily translated into linear matrix inequality (LMI) problems and a wide variety of problems arising in system and control theory can be reduced to a few standard convex or quasi-convex optimization problems involving LMI. The resulting optimization problem can be solved numerically with very high efficiency [35]. Moreover, the Lyapunov methods which are traditionally applied to the analysis of system stability, can just as well be used to determine threshold coupling, k_{th}, at which complete synchronization could be reached in master-slave or mutually coupled oscillators. Critical coupling for the on-set of stable synchronization in coupled or driven oscillators is relevant for various scientific and technological applications [36].

In this paper, we consider the synchronization of unidirectionally coupled gyroscopes. We propose a novel stability criterion using Lyapunov stability theory and linear matrix inequality (LMI) to determine the threshold coupling, k_{th}, at which full and stable synchronous behaviour could be reached in the master-slave coupled gyroscope. The advantage of our method is that the coupling parameters of the system can be obtained at the same time by solving the LMI without predetermining them to check the criterion. Furthermore, the LMI can be easily solved by various optimization algorithms. The sufficient criteria can be applied to directly design the coupling strength resulting in the synchronization. The rest of the paper is structured as follows: in the next section, we present the synchronization scheme, while Section 3 is devoted to synchronization threshold and stability criteria, Section 4 is devoted to numerical results and discussions and the paper is concluded in Section 5.

2 Model and Synchronization Preliminaries

Here, we consider the motion of the symmetric gyro with linear-plus-cubic damping given as [12]

$$\ddot{\theta} + \alpha^2 \frac{(1-\cos\theta)^2}{\sin^3\theta} - \beta\sin\theta + c_1\dot{\theta} + c_2\dot{\theta}^3 = (f\sin\omega t)\sin\theta,$$

where $f\sin\omega t$ is a parametric excitation, $c_1\dot{\theta}$ and $c_2\dot{\theta}^3$ are linear and nonlinear damping, respectively and $\alpha^2\frac{(1-\cos\theta)^2}{\sin^3\theta} - \beta\sin\theta$ is a nonlinear resilence force. After necessary transformation, the gyroscope equation in non-dimensional form can be written as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\alpha^2\frac{(1-\cos x_1)^2}{\sin^3 x_1} - c_1x_2 - c_2x_2^3 + (\beta + f\sin\omega t)\sin x_1, \end{aligned} \tag{1}$$

where

$$\alpha = \frac{\beta_\phi}{I_1} = \frac{I_3\omega_z}{I_1}, \quad c_1 = \frac{D_1}{I_1}, \quad c_2 = \frac{D_1}{I_1}, \quad \beta = \frac{M_g l}{I_1}, \quad f = \frac{M_g\bar{l}}{I_1}. \tag{2}$$

The nonlinear gyro given by Eq. (1) exhibits varieties of dynamical behaviour including chaotic motion displayed in Figure 1 for the following parameters $\alpha^2 = 100, \beta = 1, c_1 = 0.5, c_2 = 0.05, \omega = 2$, and $f = 35.5$ as given in [12].

By letting $\eta(t) = \beta + f\sin\omega t$ and using the first two terms of the Taylor series expansion of $\frac{(1-\cos x_1)^2}{\sin^3 x_1}$, system (1) can be written as:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{\alpha^2 x_1}{4} - \frac{\alpha^2 x_1^3}{12} - c_1x_2 - c_2x_2^3 + \eta(t)\sin x_1. \end{aligned} \tag{3}$$

To facilitate the present analysis, we express system (3) in the following vector form:

$$\dot{\mathbf{x}} = A\mathbf{x} - \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x}), \tag{4}$$

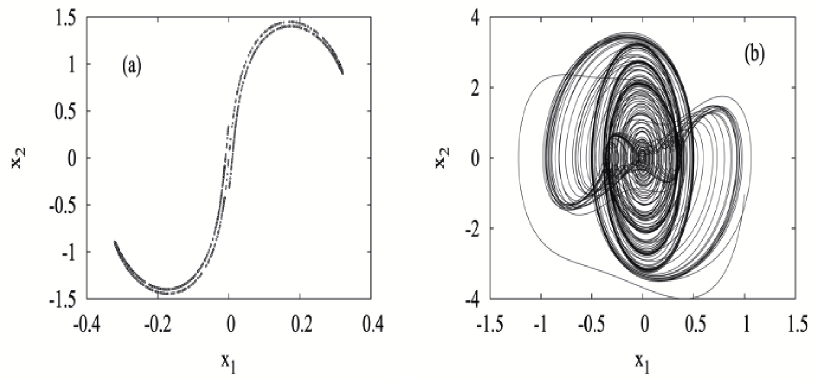

Figure 1: (a) The Poincaré map and (b) phase portrait showing a chaotic attractor of nonlinear gyroscope with the following parameters $\alpha^2 = 100, \beta = 1, c_1 = 0.5, c_2 = 0.05, \omega = 2$, and $f = 35.5$.

where $\mathbf{x} = (x_1, x_2)^T \in \mathbf{R}^2$ are state space variables and

$$A = \begin{pmatrix} 0 & 1 \\ \frac{-\alpha^2}{4} & -c_1 \end{pmatrix}, \quad \mathbf{f}(\mathbf{x}) = \alpha \begin{pmatrix} 0 & 0 \\ -\frac{\alpha^2 x_1^3}{12} & -c_2 x_2^3 \end{pmatrix}, \quad \mathbf{G}(\mathbf{x}) = \eta \begin{pmatrix} 0 \\ \sin x_1 \end{pmatrix}.$$

In order to examine the synchronization between two unidirectional coupled gyroscopes, we construct a master-slave synchronization scheme for two identical chaotic gyroscopes by linear state error feedback controller in the following form:

$$\begin{aligned} M &: \dot{\mathbf{x}} = A\mathbf{x} - \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x}), \\ S &: \dot{\mathbf{y}} = A\mathbf{y} - \mathbf{f}(\mathbf{y}) + \mathbf{G}(\mathbf{x}) + \mathbf{u}(\mathbf{t}), \\ C &: \mathbf{u}(\mathbf{t}) = K(\mathbf{x} - \mathbf{y}), \end{aligned} \tag{5}$$

where $\mathbf{u} = K(\mathbf{x} - \mathbf{y})$ is the linear state feedback control input and $K \in \mathbf{R}^{2\times 2}$ is a constant control matrix that determines the strength of the feedback into the response system. By defining the synchronization error variable as the difference between the relevant dynamical variables given by

$$\mathbf{e} = \mathbf{x} - \mathbf{y}, \tag{6}$$

we obtain the error dynamics for the master-slave system (5) as:

$$\dot{\mathbf{e}} = (A - K + M(\mathbf{x}, \mathbf{y}) + G(x_1, y_1))\mathbf{e}, \tag{7}$$

where

$$\begin{aligned} M(\mathbf{x}, \mathbf{y}) &= \begin{pmatrix} 0 & 0 \\ -\frac{\alpha^2 m_1(x_1, y_1)}{12} & -c_2 m_2(x_2, y_2) \end{pmatrix}, \\ m_1(x_1, y_1) &= x_1^2 + x_1 y_1 + y_1^2, \quad m_2(x_2, y_2) = x_2^2 + x_2 y_2 + y_2^2, \\ G(x_1, y_1) &= \eta \begin{pmatrix} 0 & 0 \\ g(x_1, y_1) & 0 \end{pmatrix}, \quad g(x_1, y_1) = -\frac{(\sin x_1 - \sin y_1)}{x_1 - y_1}. \end{aligned} \tag{8}$$

In the absence of the control matrix K Eq. (7) would have an equilibrium at $(0,0)$. Our aim is to choose the appropriate coupling matrix K such that the trajectories of the master system $x(t)$ and slave one $y(t)$ satisfy

$$\lim_{t\to\infty} ||e|| = \lim_{t\to\infty} ||x(t) - y(t)|| = 0, \tag{9}$$

where $|| * ||$ represents Euclidean norm of a vector.

3 Threshold and Criteria for Synchronization

Here, we have employed the Lyapunov's direct method and linear matrix inequality (LMI) [37] to establish some criteria for global chaos synchronization in the sense of error system (7). The classical method of Lyapunov stability theory which employs Lyapunov functionals was known for the analysis and synthesis of synchronization dynamics of coupled and driven oscillators (e.g see Refs. [38,39]). In addition to the familiar approach of analyzing and synthesizing the synchronization behaviour of coupled systems; the present paper employed the Lyapunov direct method to obtain the threshold coupling at which the two systems become completely synchronized.

To begin with, we have applied the following assumption and lemma to prove the main theorem of this paper.

Assumption. The chaotic trajectory of the master gyroscope (1) is bounded i.e. for any bounded initial condition $x(0)$ within the defining domain of the drive system, there exists a positive real constant, σ, such that $|(x(t)| \le \sigma \ \forall t \ge 0$.

Remark 1 This assumption is reasonable and valid in the context of bounded feature of chaotic attractors [40].

Lemma 1 *For $g(x_1.y_1)$ defined earlier, the inequality*

$$|g(x_1, y_1)| \le 1 \tag{10}$$

holds.

Proof. By the differential mean-value theorem:

$$\sin x_1 - \sin y_1 \quad = \quad (x_1 - y_1)\cos\phi, \ \phi \in (x_1, y_1) \ or \ \phi \in (y_1, x_1) \tag{11}$$

so that,

$$g(x_1, y_1) = \frac{-(\sin x_1 - \sin y_1)}{x_1 - y_1} = -(\cos\phi). \tag{12}$$

Hence, the inequality (10) holds.

Next, we proceed by utilizing the stability theory on time-varied systems [34] to derive sufficient criteria for global chaos synchronization in the sense of the error system (7). The following theorem is related to the general control matrix

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \in \mathbf{R}^{2\times 2}. \tag{13}$$

Theorem 1 *The master-slave system (4) achieves global chaos synchronization if a symmetric positive matrix*

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \tag{14}$$

and a coupling matrix $K \in \mathbf{R}^{2\times 2}$ *defined in (13) are chosen such that for any* $t > 0$

$$\Omega_1 = -p_{11}k_{11} - p_{12}k_{21} + |p_{12}|\omega^{\beta} < 0, \tag{15}$$

$$\Omega_2 = p_{12}(1-k_{12}) - p_{22}(k_{22} + c_1 + 3c_2\sigma_2^2) < 0, \tag{16}$$

$$4\Omega_1.\Omega_2 > L^2, \tag{17}$$

$L = \left[|p_{11}(1-k_{12}) - p_{12}(k_{11}+k_{22}+c_1+3c_2\sigma_2^2) - p_{22}k_{21}|p_{22}\omega^{\beta}\right]$,
where $\omega^{\beta} = \beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}$.

Proof. Let us assume a quadratic Lyapunov function of the form:

$$V(e) = \mathbf{e}^T P\mathbf{e}, \tag{18}$$

where P is a positive definite symmetric matrix defined in (14). The derivative of the Lyapunov function with respect to time, t, along the trajectory of the error system (7) is of the form:

$$\dot{V}(e) = \dot{\mathbf{e}}^T P\mathbf{e} + \mathbf{e}^T \mathbf{P}\dot{\mathbf{e}}. \tag{19}$$

Substituting Eq. (7) into the system (19), we have

$$\dot{V}(e) = \mathbf{e}^T\left[(\mathbf{A}-\mathbf{K}+\mathbf{M}+\mathbf{G})^T\mathbf{P} + \mathbf{P}(\mathbf{A}-\mathbf{K}+\mathbf{M}+\mathbf{G})\right]\mathbf{e} \tag{20}$$

$\dot{V}(e) < 0$, if

$$\lambda = (A-K+M+\mathbf{G})^T\mathbf{P} + \mathbf{P}(\mathbf{A}-\mathbf{K}+\mathbf{M}+\mathbf{G}) < 0, \tag{21}$$

that is

$$\lambda = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{pmatrix}, \tag{22}$$

where $\mu_{11} = -2p_{11}k_{11} + 2p_{12}\left(\eta g - \left(\frac{\alpha^2}{4} + \frac{\alpha^2 m_1}{12} + k_{21}\right)\right)$, $\mu_{12} = p_{11}(1-k_{12}) - p_{12}(k_{11} + k_{22} + c_1 + c_2m_2) + p_{22}\left(\eta g - \left(\frac{\alpha^2}{4} + \frac{\alpha^2 m_1}{12} + k_{21}\right)\right)$ and $\mu_{22} = 2p_{12}(1-k_{12}) - 2p_{22}(c_1 + c_2m_2 + k_{22})$ respectively. The symmetric matrix in (22) is negative definite if and only if

$$-2p_{11}k_{11} + 2p_{12}L^{\alpha} < 0, \tag{23}$$

$$2p_{12}(1-k_{12}) - 2p_{22}(c_1+c_2m_2+k_{22}) < 0, \tag{24}$$

$$4L_1L_2 - L_3 > 0, \tag{25}$$

where $L^{\alpha} = \eta g - \left(\frac{\alpha^2}{4} + \frac{\alpha^2 m_1}{12} + k_{21}\right)$,

$$\begin{aligned} L_1 &= \left[p_{12}L^{\alpha} - p_{11}k_{11}\right], \\ L_2 &= \left[p_{12}(1-k_{12}) - p_{22}(c_1+c_2m_2+k_{22})\right], \\ L_3 &= \left[p_{11}(1-k_{12}) - p_{12}(k_{11}+k_{22}+c_1+c_2m_2) + p_{22}L^{\alpha}\right]^2. \end{aligned}$$

It follows from the Assumption that for all $t \geq 0$,

$$\begin{aligned} |m_1(x_1,y_1)| &= |x_1^2 + x_1y_1 + y_1^2| \leq 3\sigma_1^2, \\ |m_2(x_2,y_2)| &= |x_2^2 + x_2y_2 + y_2^2| \leq 3\sigma_2^2 \\ |\eta(t)| &= b|\beta + f\sin\omega t| \leq \beta + |f|. \end{aligned}$$

Since the matrix P is positive definite, we have $p_{11}p_{22} - p_{12}^2 > 0$, so that $-2p_{11}k_{11} + 2p_{12}L^\alpha \leq -2p_{11}k_{11} - 2p_{12}k_{21} + |2p_{12}|L^\alpha \leq 2\Omega_1$,
$|p_{11}(1-k_{12}) - p_{12}(k_{11}+k_{22}+c_1+c_2m_2) + p_{22}\left[\eta g - \left(\frac{\alpha^2}{4} + \frac{\alpha^2 m_1}{12} + k_{21}\right)\right]| \leq |p_{11}(1-k_{12}) - p_{12}(k_{11}+k_{22}+c_1+3c_2\sigma_2^2) - p_{22}k_{21}| + p_{22}\left(\eta - \frac{\alpha^2}{4} + \frac{\alpha^2\sigma_1^2}{4}\right)$.
The inequalities (23)-(25) hold if the inequalities (15)-(17) are satisfied. This completes the proof.

For the purpose of applications, it is necessary that the simplest possible synchronization controllers are employed. Hence, the following corollaries can be obtained from the main theorem of this paper.

Corollary 1 *If the coupling matrix is defined by* $\mathbf{K} = diag\{k_1, k_2\}$ *and the symmetric positive definite matrix* $\mathbf{P}$ *is as defined in (24) such that*

$$k_1 > \frac{|p_{12}|\left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}\right)}{p_{11}}, \tag{26}$$

$$k_2 > \frac{p_{12} - (c_1 + 3c_2\sigma_2^2)p_{22}}{p_{22}}, \tag{27}$$

$$\begin{aligned}&4[|p_{12}|\left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}\right) - p_{11}k_1][p_{12} - p_{22}(k_2 + c_1 + 3c_2\sigma_2^2)] > \\ &\left[|(p_{11} - p_{12}(k_1 + k_2 + c_1 + 3c_2\sigma_2^2)| + p_{22}\left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}\right)\right]^2,\end{aligned} \tag{28}$$

then the master-slave system (4) achieves global chaos synchronization.

Proof. The inequalities (26) - (28) can be obtained according to the inequalities (15)-(17) with $k_{11} = k_1$, $k_{22} = k_2$ and $k_{12} = k_{21} = 0$.

Corollary 2 *The master-slave system (4) achieves global chaos synchronization if the coupling matrix* $\mathbf{K} = diag\{k, k\}$ *and the positive symmetric matrix* $\mathbf{P}$ *defined in (14) are chosen such that*

$$k = max\left(\frac{|p_{12}|(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4})}{p_{11}}, \ \frac{p_{12} - (c_1 + 3c_2\sigma_2^2)p_{22}}{p_{22}}\right) \geq 0, \tag{29}$$

$$\begin{aligned}&4(p_{11}p_{22} - p_{12}^2)k^2 - 4k[2p_{22}|p_{12}|\left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}\right) \\ &+p_{11}(p_{12} - (c_1 + 3c_2\sigma_2^2)p_{22})) - |p_{12}(p_{11} - (c_1 + 3c_2\sigma_2^2)p_{12}|] \\ &+4|p_{12}|\left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}\right)(p_{12} - (c_1 + 3c_2\sigma_2^2)p_{22}) \\ &-\left[|p_{11} - (c_1 + 3c_2\sigma_2^2)p_{12}| + p_{22}\left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}\right)\right]^2 > 0.\end{aligned} \tag{30}$$

Proof. Letting $k_1 = k_2 = k$ in the partial synchronization conditions (26) and (27), the inequality (29)) can be obtained.

For $k > 0$ given by (29), we have
$\left[|p_{11} - p_{12}(2k + c_1 + 3c_2\sigma_2^2)| + p_{22}\left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}\right)\right]^2 \leq$
$\left[|p_{11} - (c_1 + 3c_2\sigma_2^2)p_{12}| + 2k|p_{12}| + p_{22}\left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}\right)\right]^2$.

Hence, the inequality (30) can be realised by partial synchronization criterion (28) with $k_1 = k_2 = k$. Since $p_{11}p_{22} - p_{12}^2 > 0$, the solution k to the inequality (30) exists.

Remark 2 We select the elements of the positive symmetric matrix $\mathbf{P}$ as $p_{12} = 0$, $p_{11} = p_{22}\left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}\right)$, and obtain the following algebraic synchronization criterion via the inequalities (29) and (30).

$$\begin{aligned} K &= diag\{k, k\}, \\ k &> \frac{\sqrt{(c_1 + 3c_2\sigma_2^2)^2 + 4\left(\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}\right)} - (c_1 + 3c_2\sigma_2^2)}{2} = k_{th}^1. \end{aligned} \tag{31}$$

Corollary 3 *The synchronization scheme (5) achieves global chaos synchronization if the control matrix $K = diag\{k, 0\}$ and a symmetric positive definite matrix $\mathbf{P}$ given in (14) are selected such that*

$$k > b\frac{|p_{12}|\gamma}{p_{11}}, \tag{32}$$

$$np_{12} - (c_1 + 3c_2\sigma_2^2)p_{22} < 0, \tag{33}$$

$$\begin{gathered} k[|p_{12}(p_{11} - (c_1 + 3c_2\sigma_2^2)p_{12})| + |p_{12}|p_{22}\gamma - 2((c_1 + 3c_2\sigma_2^2)p_{22} - p_{12})p_{11}] \\ p_{12}^2k^22 + 4|p_{12}|((c_1 + 3c_2\sigma_2^2)p_{22} - p_{12})\gamma + [|p_{11} - (c_1 + 3c_2\sigma_2^2)p_{12}| + p_{22}\gamma]^2 < 0, \end{gathered} \tag{34}$$

where $\gamma = \beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}$.

Remark 3 We select the symmetric positive definite matrix

$$\mathbf{P} = p_{22}\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}$$

with $p_{22} > 0$.

The following synchronization criterion is gained based on the criteria (31)-(34).

$$K = diag\{k, 0\}, \quad k > \frac{\beta + |f| + \frac{\alpha^2\sigma_1^2}{4} - \frac{\alpha^2}{4}}{2(c_1 + 3c_2\sigma_2^2)}. \tag{35}$$

4 Results and Discussion

In this section, we present numerical simulation results to confirm the obtained criteria. We utilized the fourth order Runge-Kutta routine with the following initial conditions $(x_1(0), y_1(0)) = (1.0, -1.0)$, $(x_2(0), y_2(0)) = (1.0, -1.2)$, a time-step of 0.001 and fixing the parameter values of $\alpha^2 = 100, \beta = 1, c_1 = 0.5, c_2 = 0.05, \omega = 2$, and $f = 35.5$ as in Figure 1, to ensure chaotic motion, we solved the master-slave system (4) with the control matrices as defined in Eqs. (31) and (35). The simulation results obtained reveal that the trajectory of the master gyroscope depicted in Figure 1 is bounded and the error dynamics shown in Figure 2 oscillate chaotically with time when the two systems are decoupled. The partial variables x_1 and x_2 of the chaotic attractor satisfy $x_1(t) = x_2(t) < 1.25$ for any $t \geq 0$. Thus we find out that the constant $\sigma_1 = \sigma_2 = 1.25$.

The critical coupling at which complete synchronization could be observed is vital for many scientific and technological applications because it provides useful information regarding the operational regime for optimal performance in coupled systems. In Figure 3, we displayed a simulation result of average error, E_{ave}, against coupling, k, and noticed that as k increases and as full synchronization is approached, $E_{ave} \to 0$ asymptotically

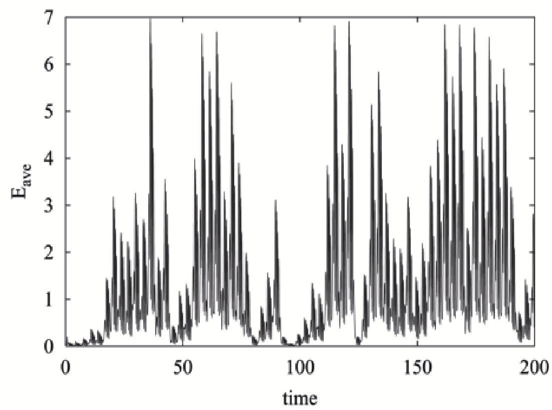

Figure 2: Average error, E_{ave}, as a function of time for the uncoupled systems with the same parameters as in Figure 1.

at the threshold coupling, $k_{th} \approx 5.98$. Then for all $k > k_{th}$, $E_{ave} = 0$ and remains stable as $t \to \infty$ implying that the oscillators are completely synchronized. Interestingly, we noticed that by direct calculations of Eq. (31) for the control matrix $K = diag\{k, k\}$, $k > k_{th} = 6.18$. Thus the obtained criterion is in good agreement with numerical simulation result.

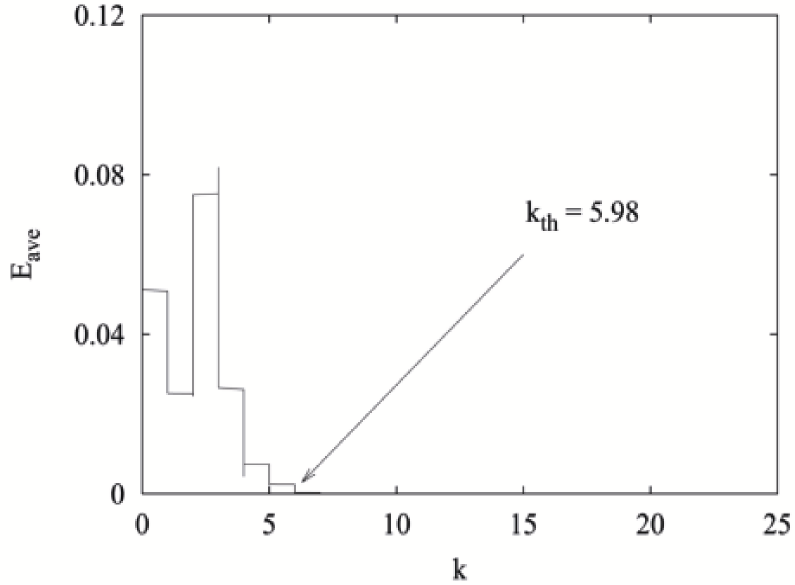

Figure 3: Average Error dynamics, E_{ave}, as a function of the coupling strength, k. Here the parameters of the system are as in Figure 1.

Using the criterion defined by Eq. (31), one readily obtains a coupling matrix $K =$

$diag\{6.18, 6.18\}$ by which the master-slave system (4) achieves chaos synchronization. Figure 4 shows the synchronization for $k = 6.2$. Finally, we depict the simulation results for the second case in which we choose constant control matrix $K = diag\{k, 0\}$, such that $k > 34.43$ which satisfies the condition in Eq. (35). The simulation results displayed in Figure 4 confirmed that complete synchronization is achieved for $k = 35.0 > k_{th}$. Notice that in both cases, the synchronization is already reached at $t = 1.0$, showing an excellent transient performance.

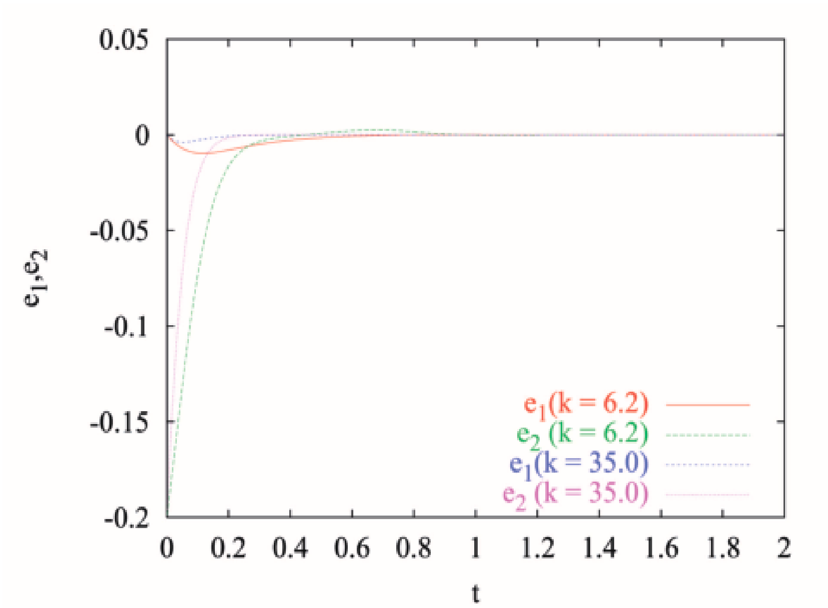

Figure 4: Chaos synchronization of two linearly coupled gyroscopes with the coupling strength $K = diag\{6.20, 6.20\}$ and $K = diag\{35.0, 0\}$.

5 Conclusions

In this paper an analytical method based on Lyapunov stability theory and linear matrix inequality (LMI) have been utilized to examine the stability of synchronized dynamics and thus determine the threshold coupling, k_{th}, at which stable synchronization regime could be observed in master-slave parametrically excited gyroscope. The criteria obtained in this paper are in algebraic form and could be easily employed for designing the feedback control gains that would guarantee complete and stable synchronization. Finally, we have presented numerical simulation results to verify the effectiveness of the obtained criteria.

Acknowledgement

O.I. Olusola gratefully acknowledges the Federation of Indian Chamber of Commerce and Industry and Department of Science and Technology, India for the award of CV Raman International Fellowship for African Researchers. He is also grateful to Federal University of Agriculture, Abeokuta, Nigeria, for granting the study leave. Finally, he acknowledges the hospitality of Dr. S.K. Dana of Indian Institute of Chemical Biology. The work of U.E. Vincent is financially supported by the Royal Society of London, United Kingdom, under the Newton International Fellowship Alumni Scheme.

References

[1] Pecora L. M. and Carroll T. L. Synchronization in chaotic systems. *Phys Rev. Lett.* **84** (1990) 821–824.

[2] Chen, G. and Dong, X. *From Chaos to Order*. World Scientific, Singapore, 1998.

[3] Pikovsky, A. S., Rosenblum, M. G. and Kurths, J. *Synchronization – A Unified Approach to Nonlinear Science*. Cambridge University Press, Cambridge, 2001.

[4] Lakshmanan M. and Murali K. *Chaos in Nonlinear Oscillators: Controlling and Synchronization*. World Scientific, Singapore, 1996.

[5] Fradkov, A. L. and Pogromsky, A. Yu. *Introduction to Control of Oscillations and Chaos*. World Scientific, Singapore, 1996.

[6] Ott, E., Grebogi, C. and Yorke J. A. Controlling chaos. *Phys. Rev. Lett.* **64** (1990) 1196–1199.

[7] Kapitaniak, T. *Controlling Chaos – Theoretical* & *Practical Methods in Non-linear Dynamics*. Academic Press, London, 1996.

[8] Rosas, D., Alvarez, J. and Alvarez, E. Robust synchronization of arrays of uncertain nonlinear second-order dynamical systems. *Nonlinear Dynamics* **67** (2012) 2735–2746.

[9] Ghosh, D. Generalized projetive synchronization in time-delayed systems: Nonlinear observer approach. *Chaos* **19** (2009) 013102(1–9).

[10] Chen G. and Dong X. *From Chaos to Order: Methodologies, Perspectives and Applications*. World Scientific, Singapore, 1998.

[11] Filali, R.L., Hammami, S., Benrejeb, M. and Borne, P. On Synchronization, Anti-synchronization and Hybrid Synchronization of 3D Discrete Generalized Henon Map. *Nonlinear Dynamics and Systems Theory* **12** (2012) 81–95.

[12] Chen, H. K. Chaos and chaos synchronization of a symmetric gyro with linear-plus-cubic damping. *Journal of Sound and Vibration* **255** (2002) 719–740.

[13] Idowu B. A., Vincent U. E. and Njah A. N. Control and Synchronization of chaos in nonlinear gyros via backstepping design. *International Journal of Nonlinear Science* **5** (2008) 11–19.

[14] Lei, Y., Xu, W., Xu, Y. and Fang, T. Chaos control by harmonic excitation with proper random phase. *Chaos Solitons* & *Fractals* **21** (2004) 1175–1181.

[15] Ge ZM and Yu TC. Chaos synchronization of a horizontal platform system.*Journal of Sound and Vibration* **268** (2003) 731–749.

[16] Chen LJ and Li JB. Chaotic behaviour and subharmonic bifurcations for a rotating pendulum equation. *Int. J. Bifur. Chaos* **14** (2004) 3477–3488.

[17] Yamapi, R. and Woafo, P. Dynamics and synchronization of coupled self-sustained electromechanical devices. *Journal of Sound and Vibration* **285** (2005) 1151–1170.

[18] Vincent, U. E. and Guo, R. Adaptive Synchronization for Oscillators in ϕ^6 Potentials. *Nonlinear Dynamics and Systems Theory* **13**(1) (2012) 93–106.

[19] Dooren, R. V. Comments on Chaos and chaos synchronization of a symmetric gyro with linear-plus-cubic damping. *Journal of Sound and Vibration* **268** (2003) 632–634.

[20] Ge, Z. M. and Chen, H. K. Bifurcations and chaos in a rate gyro with harmonic excitation. *Journal of Sound and Vibration* **194** (1996) 107–117.

[21] López-Mancilla, D. and Cruz-Hernandez, C. Output synchronization of chaotic systems: Model-matching approach with application to secure communication. *Nonlinear Dynamics and Systems Theory* **5** (2005) 141–156.

[22] Boukabou, A. On Nonlinear Control and Synchronization Design for Autonomous Chaotic Systems. *Nonlinear Dynamics and Systems Theory* **8** (2008) 151–167.

[23] Lei, Y., Xu, W. and Zheng, H. Synchronization of two chaotic nonlinear gyros using active control. *Physics Letter A* **343** (2005) 153–158.

[24] Farivar, F., Shoorehdeli, M. A., Nekoui, M. A. and Teshnehlab M. Chaos control and modified projective synchronization of unknown heavy symmetric chaotic gyroscope systems via Gaussian radial basis adaptive backstepping control. *Nonlinear Dynamics* **67** (2012) 1913–1941.

[25] Yau, H.-T. Nonlinear rule-based controller for chaos synchronization of two gyros with linear-plus-cubic damping. *Chaos Solitons & Fractals* **34** (2007) 1357–1365.

[26] Salarieh, H. and Alasty, A. Chaos synchronization of nonlinear gyros in presence of stochastic excitation via sliding mode control. *Journal of Sound and Vibration* **313** (2008) 760–771.

[27] Yau, H.-T., Kuo, C.-L., and Yan, J.-J. Fuzzy Sliding Mode Control for a Class of Chaos Synchronization with Uncertainties. *International Journal of Nonlinear Sciences and Numerical Simulation* **7** (2006) 333–338.

[28] Farivar, F., Shoorehdeli, M.A., Nekoui, M.A., and Teshnehlab, M. Generalized Projective Synchronization of Chaotic Heavy Gyroscope Systems via Sliding Rule-Based Fuzzy Control. *ISRN Artificial Intelligence* **2012** (2012) 576873. doi:10.5402/2012/576873.

[29] Aghababa, M. P. and Aghababa H. P. Synchronization of nonlinear chaotic electromechanical gyrostat systems with uncertainties. *Nonlinear Dynamics* **67** (2012) 2689-2701.

[30] Olusola, O. I., Vincent, U. E. and Njah, A. N. Bistability in coupled oscillators exhibiting synchronized dynamics. *Comm. in Theor. Phus.* **53** (2010) 815–824.

[31] Wu, X. Cai, J. and Wang, M. Global chaos synchronization of the parametrically excited Duffing oscillators by linear state error feedback control. *Chaos, Solitons* & *Fractals* **36** (2008) 121–128.

[32] Heagy, J. F., Carroll, T. L. and Pecora, L. M. Desynchronization by periodic orbits. *Phys. Rev. E* **52** (1995) 1253-1256.

[33] Shuai, J. W., Wong, K. W. and Cheng, L. M. Synchronization of spatiotemporal chaos with positive conditional Lyapunov exponents. *Phys. Rev. E* **56** (1997) 2272.

[34] Liao, X., Wang, L. and P. Yu. Stability of Dynamical Systems: In: *Monograph series on nonlinear science and complexity* (Eds. A. C. J. Luo and G. Zaslavsky). Elsevier B.V., Netherlands, 2007.

[35] Boyd, S., Ghaoui, L.EI., Feron, E. and Balakrishnan, V. *Linear Matrix Inequalities in System and Control Theory.* Studies in Applied Mathematics, SIAM, Philadelphia, PA, 1994.

[36] Stefański, A. *Determination of Thresholds for Complete Synchronization and Application.* World scientific, Singapore, 2009.

[37] Horn, R. A. and Johnson, C. R. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1991.

[38] Parekh, N., Kumar, V. R., and Kulkarni, B. D. Control of Spatiotemporal Chaos: A case with an autocatalytic reaction diffusion system. *Pramana J. Phys.* **48** (1997) 303–323.

[39] He, R. and Vaidya, P. G. Analysis and Synthesis of Synchronous Periodic and Chaotic Systems. *Phys. Rev. A* **46** (1994) 7387–7392.

[40] Curran, P.F. and Chua, I.O. Absolute stability theory and synchronization problem. *Int. J. of bifurcation and chaos* **7** (1997) 1357–1382.