Nonlinear Dynamics and Systems Theory, **13** (3) (2013) 299–315

Numerical Research of Periodic Solutions for a Class of Noncoercive Hamiltonian Systems

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Received: July 30, 2012; Revised: July 11, 2013

Abstract: In this paper, we are interested in the existence of periodic solutions and approximative solutions to the Hamiltonian system $\dot{x} = JH'(t,x)$ when H is non-coercive of the type $H(t,r,p) = G(p - Ar) + h(t) \cdot (r,p)$. For the proof we use the Dual Action Principle and Critical Point Theory.

Keywords: *Hamiltonian systems; periodic solutions; non-coercive; dual action principle; discrete dual action principle; critical point theory; numerical research.*

Mathematics Subject Classification (2010): 34K28, 34K07, 34C25, 35A15.

1 Introduction

Let $G : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function such that $G' : \mathbb{R}^n \longrightarrow G'(\mathbb{R}^n)$ be an homeomorphism. Let A be a matrix of order n and $h : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a continuous $T-$ periodic $(T > 0)$ function with zero mean value. Consider the non-coercive Hamiltonian

$$H(t,r,p) = G(p - Ar) + h(t) \cdot (r,p).$$

Here $x.y$ is the usual inner product of $x, y \in \mathbb{R}^{2n}$. We are interested in the boundary value problem

$$\dot{x} = JH'(t,x) \tag{$\mathcal{H}$}$$

with

$$x(0) = x(T). \tag{$\mathcal{C}$}$$

The goal of this work is to prove the existence of solutions to the problem $(\mathcal{H})(\mathcal{C})$ and to approximate these solutions.

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For T and h given, we define the dual action integral $\varphi : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ as

$$\varphi(v) = \frac{1}{2}\int_0^T Jv \cdot \pi v dt + \int_0^T H_0^*(v-h)dt,$$

where $H_0(r,p) = G(p - Ar)$, H_0^* is the Fenchel's transformation of H_0 and E is the closed subspace of $L^2(0,T;\mathbb{R}^{2n})$ defined by:

$$E = \left\{ v \in L^2(0,T;\mathbb{R}^{2n}) / \int_0^T v(t)dt = 0 \right\}.$$

Under some suitable assumptions on G, we will prove, in Section 2, that the problem $(\mathcal{H})(\mathcal{C})$ has at least one solution and is equivalently to the following problem:

$$find\ v \in E\ such\ that\ 0 \in \bar{\partial}\varphi(v), \tag{$\mathcal{R}$}$$

where we introduce the notation $\bar{\partial}$ to distinguish the sub differential in E and in $L^2(0,T;\mathbb{R}^{2n})$. In Section 3, we will introduce some problems $(\mathcal{H}_N)(\mathcal{C}_N)$, $(\mathcal{R}_N)$, $(\mathcal{P}_N)$ defined in a finite dimensional space and related together by a discret dual action principle. In Section 4, we will study the existence of solutions to problem (P_N), which give solutions to problem $(\mathcal{R}_N)$. In Section 5, we will study some convergence problems related to this discretisation. We want to know if the differences system $(\mathcal{H}_N)$ is near to system $(\mathcal{H})$ for example for a very large integer N. In Section 6, we will give an example of application and in Section 7, we will conclude this work.

2 Existence of Periodic Solutions

Let $G : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable convex function, A be a symmetric matrix of order n and $h : \mathbb{R} \longrightarrow \mathbb{R}^{2n}$ be a continuous T-periodic function with zero mean value on $[0,T]$. Consider the assumptions:

Assumption 2.1

$$\lim_{|x| \longrightarrow \infty} G(x) = +\infty. \tag{G_1}$$

Assumption 2.2 There exist $\alpha \in]0, \frac{\pi}{T(1+|A|^2)}[$ and $\beta \geq 0$ such that

$$\forall x \in \mathbb{R}^n,\ G(x) \leq \frac{\alpha}{2}|x|^2 + \beta, \tag{G_2}$$

where $|A|$ is the usual norm of A. Consider the non-coercive sub-quadratic Hamiltonian:

$$H(t,r,p) = G(p - Ar) + h \cdot (r,p).$$

We are interested in the existence of solutions for the boundary value problem

$$\dot{x} = JH'(t,x) \tag{$\mathcal{H}$}$$

with

$$x(0) = x(T), \tag{$\mathcal{C}$}$$

where H' is the derivative of H with respect to the second variable x and J is the standard $(2n \times 2n)$ symplectic matrix:

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the identity matrix of order n.

Example 2.1 Consider a relativistic particle with a very small charge e and rest mass m_0, subject to a uniform constant magnetic field B and a uniform electric field $E(t)$. The energy expressed as a function of (t, r, p), i.e. a Hamiltonian, is given by

$$H(t,r,p) = c\Big[m_0^2c^2 + \Big|p - \frac{e}{2}B(t) \wedge r\Big|^2\Big]^{\frac{1}{2}} - E(t).r,$$

where c is the velocity of light, p the usual mechanical momentum of particle and r is its position. The particle motion is described by the associated Hamiltonian system $(\mathcal{H})$.

The function

$$H_0(r,p) = G(p - Ar)$$

is convex and its Fenchel's transformation H_0^* is given for all $(s,q) \in \mathbb{R}^n \times \mathbb{R}^n$ by (see [7])

$$H_0^*(s,q) = \begin{cases} G^*(q), \ if \ s + A^*q = 0, \\ +\infty, \ if \ s + A^*q \neq 0. \end{cases}$$

Consider the functional

$$\psi(y) = \int_0^T [\frac{1}{2}J\dot{y}.y + H_0^*(\dot{y} - h)]dt \tag{2.1}$$

defined over the space

$$\{y \in H^1(0,T;\mathbb{R}^{2n})/y(0) = y(T)\}.$$

Note that, from the periodicity condition:

$$\forall \xi \in \mathbb{R}^{2n}, \psi(y + \xi) = \psi(y),$$

the true variable in (2.1) is $\dot{y}$ and we can choose for y any primitive we like. The only condition on $\dot{y}$ is:

$$\dot{y} \in L^2(0,T;\mathbb{R}^{2n}) \ and \ \int_0^T \dot{y}dt = 0.$$

In other terms, we have

$$\psi(y) = \varphi(\dot{y}),$$

where φ is the functional

$$\varphi(v) = \int_0^T [\frac{1}{2}Jv \cdot \pi v + H_0^*(v - h)]dt$$

defined on the space

$$E = \{v \in L^2(0,T;\mathbb{R}^{2n})/\int_0^T v(t)dt = 0\},$$

where πv is the primitive of v with zero mean value:

$$\frac{d}{dt}(\pi v) = v \ and \ \int_0^T (\pi v)(t)dt = 0$$

or also

$$(\pi v)(t) = \int_0^t v(s)ds - \frac{1}{T}\int_0^T \int_0^r v(s)dsdr.$$

This allows to introduce the following problem:

$$find\ v \in E\ such\ that\ 0 \in \bar{\partial}\varphi(v). \tag{$\mathcal{R}$}$$

The problems $(\mathcal{R})$ and $(\mathcal{H})(\mathcal{C})$ are related by a dual action principle.

Theorem 2.1 *(Dual action principle). Assume that the function G satisfies (G_1), (G_2) and let $v \in E$. Then the two following assertions are equivalent:*
(i) v is a solution of problem $(\mathcal{R})$,
(ii) there exists a constant ξ in $\mathbb{R}^{2n}$ such that the function $x(t) = J\pi v(t) + \xi$ is a solution of problem $(\mathcal{H})(\mathcal{C})$.

Proof. To prove this theorem, we need the following lemma.
Consider the functional

$$g(v) = \int_T^0 H_0^*(v - h)dt,\ v \in E,$$

we have

Lemma 2.1 *The sub-differential of $g_{|E}$ in a point $v \in E$ where g has finite value, is given by*

$$\bar{\partial}g(v) = \{u \in L^2(0,T;\mathbb{R}^{2n})/\exists\xi \in \mathbb{R}^{2n}, u(t) + \xi \in \partial H_0^*(v(t) - h(t))\ a.e.\}.$$

Proof. If $u \in L^2(0,T;\mathbb{R}^{2n})$, $v \in E$ and $\xi \in \mathbb{R}^{2n}$ are such that $u(t) + \xi \in \partial H_0^*(v(t) - h(t))$ *a.e*, we prove easily that u is in $\bar{\partial}g(v)$. Reversely, it is clear that

$$\bar{\partial}g(v) = \partial(g + \delta_E)(v),$$

where

$$\delta_E(v) = \begin{cases} 0, & if\ v \in E, \\ +\infty, & elsewhere. \end{cases}$$

Since it is clear that $\partial\delta_E(v)$ is the set of constant functions and it is well known that (see [3])

$$\partial g(v) = \{u \in L^2(0,T;\mathbb{R}^{2n})/u(t) \in \partial H_0^*(v(t) - h(t))\ a.e.\},$$

the result will be proved if we have

$$\partial(g + \delta_E)(v) = \partial g(v) + \partial\delta_E(v).$$

Let us establish that $\partial(g + \delta_E)(v) = \partial g(v) + \partial\delta_E(v)$. It is enough to prove that $g^*\nabla\delta_E^*$ is exact (see [1]). By identifying the set of constant functions to $\mathbb{R}^{2n}$, we see that

$$\delta_E^* = \delta_{\mathbb{R}^{2n}}.$$

We deduce that for all u in L^2:

$$(g^*\nabla\delta_E^*)(u) = \inf_{x\in\mathbb{R}^{2n}} \int_0^T H_0(u - h + x)dt$$

and by (G_2), we obtain

$$0 \leq (g^*\nabla\delta_E^*)(u) \leq \int_0^T H_0(u - h)dt \leq \alpha(1 + |A|^2)||u - h||_{L^2}^2 + \beta T. \tag{2.2}$$

By convexity and (2.2), we conclude that $g^*\nabla\delta_E^*$ is continuous (see [3]).
Now, let us write $u = (r, p)$ and $h = (h_1, h_2)$, we have $(g^*\nabla\delta_E^*)(u) = \inf_{\xi\in\mathbb{R}^n} F(\xi)$, where

$$F(\xi) = \int_0^T G(p - h_2 - A(r - h_1) + \xi)dt.$$

By properties of G, it is easy to see that F is continuous and $\lim_{|\xi|\longrightarrow\infty} F(\xi) = +\infty$. Consequently F achieves its minimum on $\mathbb{R}^n$ and then $g^*\nabla\delta_E^*$ is exact. On the other hand, g and δ_E are convex, l.s.c and propers, therefore for all v in E where g is finite, we have

$$\bar{\partial}g(v) = \partial g(v) + \mathbb{R}^{2n}.$$

The proof of Lemma 2.1 is complete.

Let $v \in E$ be such that

$$0 \in \bar{\partial}\varphi(v). \tag{2.3}$$

This is equivalent to

$$0 \in -J\pi v + \bar{\partial}g(v). \tag{2.4}$$

By Lemma 2.1, formula (2.4) is equivalent to the existence of $\xi \in \mathbb{R}^{2n}$ satisfying

$$J(\pi v)(t) + \xi \in \partial H_0^*(v(t) - h(t)) \ a.e. \tag{2.5}$$

Let us put $x(t) = J\pi v(t) + \xi$. By Fenchel's reciprocity, formula (2.5) can be rewritten as

$$v(t) - h(t) = H_0'(x(t))$$

or

$$\dot{x}(t) = JH'(t, x(t))$$

and it is clear that x is $T-$ periodic. Then x is a solution of problem $(\mathcal{H})(\mathcal{C})$ and Theorem 2.1 is proved.

Now, we associate with $(\mathcal{R})$ the problem:

$$find\ \bar{v} \in E\ such\ that\ \inf_{v\in E} \varphi(v) = \varphi(\bar{v}). \tag{$\mathcal{P}$}$$

The problem $(\mathcal{P})$ allows to give a solution of problem $(\mathcal{R})$.

Theorem 2.2 *Assume assumptions (G_1), (G_2) hold, then problem $(\mathcal{H})(\mathcal{C})$ has at least one solution.*

The proof of Theorem 2.2 follows immediately from Lemma 2.1 and the following lemma.

Lemma 2.2 *Problem $(\mathcal{P})$ possesses a solution: there exists a point $\bar{v} \in E$ such that*

$$\min_E \varphi = \varphi(\bar{v}).$$

Proof. By using assumption (G_2) and going through the conjugate, we verify that

$$\forall y \in \mathbb{R}^{2n},\ H_0^*(y) \geq \frac{1}{2\alpha(1 + |A|^2)}|y|^2 - \beta. \tag{2.6}$$

On the other hand, by Wirtinger's inequality and using Fourier expansion, we have

$$\forall v \in E,\ ||\pi v||_{L^2} \leq \frac{T}{2\pi}||v||_{L^2}. \tag{2.7}$$

We deduce from (2.6), (2.7) and Hölder's inequality that

$$\forall v \in E, \varphi(v) \geq \frac{1}{2}[\frac{1}{\alpha(1+|A|^2)} - \frac{T}{2\pi}]||v||_{L^2} - \beta T. \tag{2.8}$$

Now, let (v_k) be a minimising sequence, then by (2.8), (v_k) is bounded. Since the space E is reflexive, then there exists a subsequence (v_{k_p}) weakly convergent to a $\bar{v} \in E$.
It is well known that the functional g introduced above is l.s.c, so we have

$$\liminf_{p\longrightarrow\infty} \int_0^T H_0^*(v_{k_p} - h)dt \geq \int_0^T H_0^*(\bar{v} - h)dt. \tag{2.9}$$

Elsewhere, the operator π is compact, so

$$\pi v_{k_p} \longrightarrow \pi\bar{v},\ in\ \mathrm{L}^2$$

and then

$$\lim_{p\longrightarrow\infty} \int_0^T Jv_{k_p} \cdot \pi v_{k_p} dt = \int_0^T J\bar{v} \cdot \pi\bar{v} dt. \tag{2.10}$$

Consequently, we deduce from (2.9) and (2.10) that

$$\min_E \varphi = \varphi(\bar{v}).$$

3 A Discrete Dual Action Principle

Giving a period $T > 0$ and a forcing h, we have defined in the previous section the space $E = L_0^2(0, T; \mathbb{R}^{2n})$ and the functional $\varphi : E \longrightarrow \bar{\mathbb{R}}$. We will write a problem $(\mathcal{R}_N)$ obtained by writing $(\mathcal{R})$ not in L^2 but in a finite dimensional space. This will allow us, having put a differences system $(\mathcal{H}_N)$ and a constraint $(\mathcal{C}_N)$, to establish a "discrete dual action principle" connecting $(\mathcal{R}_N)$ to $(\mathcal{H}_N)(\mathcal{C}_N)$.

Notations. For $x \in \mathbb{R}^{nN}$, we will adopt the following agreement writing:

$$\begin{cases} x = (x_1, x_2, ..., x_n), where\ x_i \in \mathbb{R}^N, \\ x = (x^1, x^2, ..., x^N), where\ x^j \in \mathbb{R}^n, \\ x_i^j \in \mathbb{R},\ i = 1, 2, ..., n,\ j = 1, 2, ..., N. \end{cases}$$

This allows us to define the space

$$E_N = \{v = (r, p) \in \mathbb{R}^{2nN} / \sum_{j=1}^N r^j = \sum_{j=1}^N p^j = 0\}.$$

Let us define on $\mathbb{R}$ the sequence $(t^j)_{j=1,2,...,N}$ by

$$\begin{cases} t^1 = 0,\ t^{N+1} = T, \\ t^{j+1} - t^j = \delta = \frac{T}{N},\ \forall j = 1, 2, ..., N. \end{cases}$$

With any vector $x \in \mathbb{R}^{nN}$, we can associate a step function $\tilde{x}$ from $\mathbb{R}$ into $\mathbb{R}^n$, which will be, by construction, $T-$ periodic, as follows:

$$\begin{cases} \tilde{x}(t) = x^j, \ \forall t \in [t^j, t^{j+1}[, \ \forall j = 1, 2, ..., N; \\ \tilde{x}(t + kT) = \tilde{x}(t), \ \forall t \in [0, T[, \ \forall k \in \mathbb{Z}. \end{cases}$$

Then we can write φ applied to any element $v = (r, p)$ of $\mathbb{R}^{2nN}$. We will denote by $\varphi_N(v)$ its value (the index N in φ is to recall that we have calculated φ for elements of $\mathbb{R}^{2nN}$). We obtain

$$\varphi_N(v) = \frac{\delta^2}{2} \sum_{j=1}^{N} \sum_{k=1}^{j} Jv^j \cdot v^k + \delta \sum_{j=1}^{N} H_0^*(v^j - h^j).$$

The vector h^j is obtained by discretising h with respect to $(t^j)_{j=1,2,...,N}$, which is possible since h is $T-$ periodic.

Definition 3.1 We recall the problem $(\mathcal{R}_N)$:

$$find \ v \in E_N \ such \ that \ 0 \in \bar{\partial}\varphi_N(v). \qquad (\mathcal{R}_N)$$

Definition 3.2 We will denote by $w_N = (r_N, p_N)$ the continuous piecewise linear functions, defined with respect to $(t^j)_{j=1,2,...,N}$. For these functions, we define the differences system

$$-J\frac{w_N(t^{j+1}) - w_N(t^j)}{t^{j+1} - t^j} = H_0'(\frac{w_N(t^{j+1}) + w_N(t^j)}{2}) + h(t^j), j = 1, ..., N. \qquad (\mathcal{H}_N)$$

Then we look for w_N satisfying $(\mathcal{H}_N)$ and the constraint

$$w_N(0) = w_N(T). \qquad (\mathcal{C}_N)$$

Theorem 3.1 *(Discrete dual action principle). Assume G satisfies (G_1), (G_2). Then for $v \in E_N$ the following two assertions are equivalent:*
(i) v is a solution of $(\mathcal{R}_N)$,
(ii) there exists a constant ξ_N in $\mathbb{R}^{2N}$ such that the function

$$w_N(t) = J\int_0^t \tilde{v}(\tau)d\tau + \xi_N$$

is a solution of $(\mathcal{H}_N)(C_N)$, where $\tilde{v}$ is defined with respect to v as above.

Proof. 1) The function w_N defined in (ii) is a continuous linear piecewise function as in Definition 3.2.
2) Given the definition of E_N, it is clear that w_N satisfies condition $(\mathcal{C}_N)$ if and only if v belongs to this space, since

$$\int_0^T \tilde{v}(\tau)d\tau = \sum_{j=1}^{N} \int_{t^j}^{t^{j+1}} \tilde{v}(\tau)d\tau = \delta \sum_{j=1}^{N} v^j.$$

3) In the following, we will need the next result:

Lemma 3.1 *Let*

$$F(v) = \sum_{j=1}^{N} H_0^*(v^j),$$

then we have

$$\partial F(v) = \{u \in \mathbb{R}^{2nN} / u^j \in \partial H_0^*(v^j),\ \forall j = 1, ..., N\},$$

where $u = (u^1, ..., u^N)$.

Proof. We have

$$u \in \partial F(v) \Longleftrightarrow \forall x \in \mathbb{R}^{2nN} / F(x) \leq F(v) + (x - v) \cdot u$$

$$\Longleftrightarrow \forall x \in \mathbb{R}^{2nN},\ \sum_{j=1}^{N} H_0^*(x^j) \leq \sum_{j=1}^{N} H_0^*(v^j) + \sum_{j=1}^{N} (x^j - v^j) \cdot u^j$$

$$\Longrightarrow \forall j = 1, ..., N,\ \forall x^j \in \mathbb{R}^{2n},\ H_0^*(v^1) + ... + H_0^*(x^j) + ... + H_0^*(v^N)$$

$$\leq \sum_{j=1}^{N} H_0^*(v^j) + (x^j - v^j) \cdot u^j$$

$$\Longrightarrow \forall j = 1, ..., N,\ \forall x^j \in \mathbb{R}^{2n},\ H_0^*(x^j) \leq H_0^*(v^j) + (x^j - v^j) \cdot u^j$$

$$\Longrightarrow \forall j = 1, ..., N,\ u^j \in \partial H_0^*(v^j).$$

Reversely, if $\forall j = 1, ..., N, u^j \in \partial H_0^*(v^j)$, then

$$\forall j,\ \forall x^j \in \mathbb{R}^{2n},\ H_0^*(x^j) \leq H_0^*(v^j) + (x^j - v^j) \cdot u^j$$

$$\Longrightarrow \forall x \in \mathbb{R}^{2nN}, \sum_{j=1}^{N} H_0^*(x^j) \leq \sum_{j=1}^{N} H_0^*(v^j) + \sum_{j=1}^{N} (x^j - v^j) \cdot u^j$$

$$\Longrightarrow \forall x \in \mathbb{R}^{2nN},\ F(x) \leq F(v) + (x - v) \cdot u$$

$$\Longrightarrow u \in \partial F(v).$$

Now, consider the functional

$$\varphi_N(v) = Q_N(v) + \delta \sum_{j=1}^{N} H_0^*(v^j - h^j)$$

defined over the space E_N, with

$$Q_N(v) = \frac{\delta^2}{2} \sum_{j=1}^{N} \sum_{k=1}^{j} Jv^j \cdot v^k.$$

We have

$$Q_N(v) = \frac{\delta^2}{2}[Jv^2 \cdot v^1 + ... + Jv^N \cdot v^1] + \ \textit{terms without } v^1,$$

so

$$\frac{\partial Q_N}{\partial v^1} = \frac{\delta^2}{2}[Jv^2 + Jv^3 + ... + Jv^N] = -\frac{\delta^2}{2} Jv^1.$$

Similarly for $2 \leq j \leq N$,

$$Q_N(v) = \frac{\delta^2}{2}[Jv^j.(v^1 + ... + v^{j-1}) + (Jv^{j+1} + ... + Jv^N).v^j] + \ terms\ without\ v^j,$$

so

$$\begin{aligned}\frac{\partial Q_N}{\partial v^j} &= \frac{\delta^2}{2}[-J(v^1 + ... + v^{j-1}) + J(v^{j+1} + ... + v^N)] \\ &= \frac{\delta^2}{2}[-J\sum_{k=1}^{j-1} v^k - J\sum_{k=1}^{j} v^k] = -\frac{\delta^2}{2}J(2\sum_{k=1}^{j-1} v^k + v^j).\end{aligned}$$

Therefore

$$\partial\varphi_N(v) = \{u \in \mathbb{R}^{2nN}/\forall j = 1, ..., N, u^j \in -\frac{\delta^2}{2}(2\sum_{k=1}^{j-1} v^k + v^j) + \delta\partial H_0^*(v^j - h^j)\}.$$

4) By writing

$$\partial\varphi_N(v) = \begin{cases} 0, \ if\ v \in E_N, \\ +\infty, \ elsewhere, \end{cases}$$

we have

$$\bar{\partial}\varphi_N(v) = \partial(\varphi_N + \delta_{E_N})(v),$$

where we introduce the notation $\bar{\partial}$ to distinguish the sub-differentials in E_N and in $\mathbb{R}^{2nN}$.

Lemma 3.2 *We have*

$$\bar{\partial}\varphi_N(v) = \partial\varphi_N(v) + \partial\delta_{E_N}(v).$$

Proof. By writing

$$g_N(v) = \sum_{j=1}^{N} H_0^*(v^j - h^j),$$

it is enough to prove that $\bar{\partial}g_N(v) = \partial g_N(v) + \partial\delta_{E_N}(v)$. It is clear that $\bar{\partial}g_N(v) = \partial(g_N + \delta_{E_N})(v)$. The result will be proved if we have

$$\partial(g_N + \delta_{E_N})(v) = \partial g_N(v) + \partial\delta_{E_N}(v).$$

For this, it is enough to prove that $g_N^*\nabla\delta_{E_N}^*$ is exact. We have $\delta_{E_N}^* = \delta_{E_N^\perp}$. Let us determine $E_N^\perp$. We have

$$u = (r, p) \in E_N^\perp \Longleftrightarrow \forall v \in E_N, \ u \cdot v = 0 \Longleftrightarrow \forall (s, q) \in E_N, \ \sum_{j=1}^{N}(s^j \cdot r^j + q^j \cdot p^j) = 0$$

$$\Longrightarrow [\forall i \neq j = 1, ..., N, \ \forall s^i, s^j \in \mathbb{R}^n, \ s^i + s^j = 0 \Longrightarrow s^i \cdot r^i + s^j \cdot r^j = 0]$$

$$\Longrightarrow [\forall i \neq j = 1, ..., N, \ \forall s^i \in \mathbb{R}^n, \ s^i \cdot (r^i - r^j) = 0]$$

$$\Longrightarrow \forall i, j = 1, ..., N, \ r^i = r^j.$$

Similarly, $\forall i,j=1,...,N,\ p^i=p^j$. Therefore we have

$$(r,p)\in E_N^{\perp}\Longrightarrow r^1=...=r^N,\ p^1=...=p^N.$$

Reversely, if $(r,p)\in\mathbb{R}^{2nN}$ is such that $r^1=...=r^N$ and $p^1=...=p^N$, then

$$\forall(s,q)\in E_N,\ (s,q)\cdot(r,p)=\sum_{j=1}^N s^j\cdot r^j+\sum_{j=1}^N q^j\cdot p^j=(\sum_{j=1}^N s^j)\cdot r^1+(\sum_{j=1}^N q^j)\cdot p^1=0.$$

Therefore, we have

$$E_N^{\perp}=\left\{(r,p)\in\mathbb{R}^{2nN}/r^1=...=r^N,\ p^1=...=p^N\right\}.$$

For u in $\mathbb{R}^{2nN}$, we have

$$(g_N^*\nabla\delta_{E_N}^*)(u)=\inf_{u_1+u_2=u}(g_N^*(u_1)+\delta_{E_N}^*(u_2))=\inf_{\xi\in E_N^{\perp}}g_N^*(u+\xi)=\inf_{\xi\in E_N^{\perp}}\sum_{j=1}^N H_0(u^j+\xi^j)$$

$$=\inf_{(x,y)\in\mathbb{R}^{2n}}\sum_{j=1}^N H_0(u^j+(x,y))=\inf_{(x,y)\in\mathbb{R}^{2n}}\sum_{j=1}^N G(u_2^j-Au_1^j+y-Ax)=\inf_{x\in\mathbb{R}^n}K(x),$$

where $u^j=(u_1^j,u_2^j)$ and

$$K(x)=\sum_{j=1}^N G(u_2^j-Au_1^j+x).$$

Since K is continuous and goes to infinity as $|x|\longrightarrow\infty$, then K achieves its minimum on $\mathbb{R}^n$. The proof of Lemma 3.2 is complete.

We have $\partial\delta_{E_N}(v)=E_N^{\perp}$ then $\bar{\partial}\varphi_N(v)=\partial\varphi_N(v)+E_N^{\perp}$. Consequently, we have

$$u\in\bar{\partial}\varphi_N(v)\Longleftrightarrow u\in\partial\varphi_N(v)+E_N^{\perp}$$

$$\Longleftrightarrow\exists\xi\in\mathbb{R}^{2n}/\left\{\begin{array}{l}u^1\in\frac{-\delta^2}{2}Jv^1+\xi+\delta\partial H_0^*(v^1-h^1)\\ u^j\in\frac{-\delta^2}{2}J(2\sum_{k=1}^{j-1}v^k+v^j)+\xi+\delta\partial H_0^*(v^j-h^j),\ \forall j=2,...,N.\end{array}\right.$$

5) v is a critical point of φ_N if and only if there exists a constant $\xi_N\in\mathbb{R}^{2n}$ such that

$$\left\{\begin{array}{l}0\in\frac{-\delta^2}{2}J\bar{v}_1-\xi_N+\delta\partial H_0^*(\bar{v}^1-h^1),\\ 0\in\frac{-\delta^2}{2}J(2\sum_{k=1}^{j-1}\bar{v}^k+v^j)-\xi_N+\delta\partial H_0^*(\bar{v}^j-h^j),\ \forall j=2,...,N.\end{array}\right.$$

$\Longleftrightarrow\exists\xi_N\in\mathbb{R}^{2n}$ such that

$$\left\{\begin{array}{l}\xi_N+\frac{\delta}{2}J\bar{v}^1\in\partial H_0^*(\bar{v}^1-h^1),\\ \xi_N+\frac{\delta}{2}J(2\sum_{k=1}^{j-1}\bar{v}^k+v^j)\in\partial H_0^*(\bar{v}^j-h^j),\forall j=2,...,N.\end{array}\right.$$

Let us associate with $v\in\mathbb{R}^{2nN}$, the step function $\tilde{v}$ and the continuous piecewise linear function w_N defined by

$$w_N(t)=J\int_0^t\tilde{v}(\tau)d\tau+\xi_N.$$

In particular, we have

$$w_N(t^{j+1}) = J\int_0^{t^{j+1}} \tilde{v}(\tau)d\tau + \xi_N = J\sum_{k=1}^{j}\int_{T^k}^{t^{k+1}} \tilde{v}(\tau)d\tau + \xi_N = \delta J\sum_{k=1}^{j} v^k + \xi_N,$$

which implies

$$\begin{cases} w_N(t^{j+1}) - w_N(t^j) = \delta J\tilde{v}(t^j), \\ w_N(t^{j+1}) + w_N(t^j) = 2[\frac{\delta}{2}(2J\sum_{k=1}^{j-1}\tilde{v}(t^k) + \tilde{v}(t^j)) + \xi_N]. \end{cases}$$

Therefore we have

$$\begin{cases} w_N(t^{j+1}) - w_N(t^j) = \delta J\tilde{v}(t^j), \\ w_N(t^{j+1}) + w_N(t^j) \in 2\partial H_0^*(v^j - h^j). \end{cases}$$

This yields

$$\frac{w_N(t^{j+1}) + w_N(t^j)}{2} \in \partial H_0^*(-J\frac{w_N(t^{j+1}) - w_N(t^j)}{t^{j+1} - t^j} - h(t^j)).$$

By using Fenchel's reciprocity formula, we obtain

$$-J\frac{w_N(t^{j+1}) - w_N(t^j)}{t^{j+1} - t^j} = H_0^{'}(\frac{w_N(t^{j+1}) + w_N(t^j)}{2}) + h(t^j).$$

4 Existence Results

To resolve the problem $(\mathcal{H}_N)(\mathcal{C}_N)$, it suffices, by using Section 3, to find a point $\bar{v}$ of $\mathbb{R}^{2nN}$ solution of $(\mathcal{R}_N)$, i.e.

$$find\ \bar{v} \in E_N\ such\ that\ 0 \in \bar{\partial}\varphi_N(\bar{v}). \qquad (\mathcal{R}_N)$$

For this, we can study the existence of a minimum to the associate problem

$$find\ \bar{v} \in E_N\ satisfying\ \inf_{v\in E_N} \varphi_N(v) = \varphi_N(\bar{v}). \qquad (\mathcal{P}_N)$$

Assume that G and h satisfy the assumptions of Section 2.

Remark 4.1 In Section 3, we have seen that we can associate with a point v in $\mathbb{R}^{2nN}$ a step function $\tilde{v}$ defined from $\mathbb{R}$ into $\mathbb{R}^{2n}$ by the relations:

$$\begin{cases} (i)\ \tilde{v}(t) = v^j,\ \forall t \in [t^j, t^{j+1}[,\ \forall j = 1, ..., N, \\ (ii)\ \tilde{v}(t + kT) = \tilde{v}(t),\ \forall k \in \mathbb{Z},\ \forall t \in [0, T[. \end{cases} \qquad (4.1)$$

It is easy to see that the restriction $\tilde{v}_{|[0,T]}$ of $\tilde{v}$ to $[0, T]$ is in $L^2(0, T; \mathbb{R}^{2n})$.

Definition 4.1 1) Denote by F_N the subset of $L^2(0, T; \mathbb{R}^{2n})$ defined by

$$F_N = \{\omega \in L^2(0, T; \mathbb{R}^{2n})/\omega\ verifies\ (4.1)\},$$

where

$$\begin{cases} (i)\ \omega\ is\ defined\ for\ all\ t \in [0, T], \\ (ii)\ \omega(t) = \omega^j,\ \forall t \in [t^j, t^{j+1}[,\ \forall j = 1, ..., N, \\ (iii)\ \omega(T) = \omega^1 = \omega(0). \end{cases}$$

Firstly, remark that F_N is a closed subspace of $L^2(0,T;\mathbb{R}^{2n})$.
2) Denote by η_N the function defined from $\mathbb{R}^{2nN}$ into F_N

$$\eta_N(v) = \tilde{v}_{|[0,T]},\ v \in \mathbb{R}^{2nN}.$$

Remark that

$$\varphi_N(v) = \varphi(\eta_N(v)).$$

Lemma 4.1 *The function η_N establishes a diffeomorphism between F_N and $\mathbb{R}^{2nN}$, so we can identify $\mathbb{R}^{2nN}$ with F_N.*

Proof. Since the partition $(t^j)_{j=1,\ldots,N}$ is fixed, then η_N is a differentiable linear map and we can verify easily that it is invertible.

Lemma 4.2 *$\mathbb{R}^{2nN}$ can be provided with the topology obtained by diffeomorphism from the topology induced from $L^2(0,T;\mathbb{R}^{2n})$ on F_N.*

Proof. It is a consequence from the fact that F_N is a closed subspace of $L^2(0,T;\mathbb{R}^{2n})$.

Remark 4.2 By denoting $\|.\|_2$ the norm in $L^2(0,T;\mathbb{R}^{2n})$ and $|.|_{2n}$ the norm in $\mathbb{R}^{2n}$, we have the equality

$$\|\eta_N(v)\|_2 = [\frac{1}{N}\sum_{j=1}^{N}\left|v^j\right|_{2n}^2]^{\frac{1}{2}}.$$

The right quantity defines a norm in $\mathbb{R}^{2nN}$, we will denote it by $|.|_{2,N}$. With these notations, η_N appears as an isometry from $(L^2(0,T;\mathbb{R}^{2n}), \|.\|_2)$ into $(\mathbb{R}^{2nN}, |.|_{2,N})$.

Theorem 4.1 *Under assumptions (G_1), (G_2), the problem $(\mathcal{P}_N)$ has, for all integer N, a solution v_N.*

Proof. By identifying $\mathbb{R}^{2nN}$ to F_N, the proof is the same as that of the general case $(\mathcal{P})$. It is based on the following estimate:

$$\forall v \in \mathbb{R}^{2nN}, \varphi_N(v) \leq \frac{1}{2}[\frac{1}{\alpha(1+|A|^2} - \frac{T}{2\pi}]\,\|\eta_N(v)\|_2^2 - \beta T$$

or also

$$\forall v \in \mathbb{R}^{2nN}, \varphi_N(v) \leq \frac{1}{2}[\frac{1}{\alpha(1+|A|^2)} - \frac{T}{2\pi}]\,|v|_{2,N}^2 - \beta T.$$

The previous theorem permits to assert that if assumptions (G_1), (G_2) are satisfied, then for all integer N, we can find a minimum for φ_N on E_N which is also a solution of $(\mathcal{R}_N)$. Therefore, by the discrete dual action principle introduced in Section 3, the problem $(\mathcal{H}_N)(\mathcal{C}_N)$ has a solution.

Now we define a sequence $(v_1)_{l\in\mathbb{N}^*}$ by setting

$$\begin{cases} (i)\ N = 2^l, \\ (ii)\ v_l \textit{ is a solution of } (\mathcal{P}_N). \end{cases}$$

The estimate of the previous theorem permits to state the following lemma:

Lemma 4.3 *Under assumption* (G_2)*, there exists a constant* $M > 0$ *such that*

$$\forall l \in \mathbb{N}^*, \|\eta_N(v_1)\|_2^2 = |v_1|_{2,N}^2 \leq M.$$

Proof. Note that, from the previous results, we have

$$\forall l \in \mathbb{N}^*, \varphi_N(v_l) = \varphi(\eta_N(v_l)) \leq \frac{k_l}{2} \|\eta_N(v_l)\|_2^2 - k_2$$

where $k_1 = \frac{1}{\alpha(1+|A|^2)} - \frac{T}{2\pi}$ and $k_2 = \beta T$. We have also

$$\forall l' \leq l, \ \varphi_N(v_l) \leq \varphi_N(v_{l'}) \ with \ N = 2^l.$$

Since

$$\varphi_N(v_{l'}) = \varphi_{N'}(v_{l'}) \ with \ N' = 2^{l'},$$

we get

$$\forall l' \leq l, \ \varphi(\eta_N(v_l)) = \varphi_N(v_l) \leq \varphi_{N'}(v_{l'}).$$

Therefore, we have

$$\forall l \in \mathbb{N}^*, \frac{1}{2} k_1 \|\eta_N(v_l)\|_2^2 - k_2 \leq \varphi_N(v_l) \leq \varphi_1(v_1).$$

Since $\varphi_1(v_1)$ is a constant with respect to l, the proof of Lemma 4.3 is complete.

5 Convergence Results

Under assumptions (G_1), (G_2), we have proved in the previous section that there exists a sequence $(v_l)_{l\in\mathbb{N}^*}$ of solutions for the problems $(\mathcal{P}_N)$ with $N = 2^l$. Consider the sequence $(w_l)_{l\in\mathbb{N}^*}$ of piecewise linear functions defined by

$$w_l(t) = \int_0^t \tilde{v}_l(\tau)d\tau + \xi_l$$

with $\xi_l \in \mathbb{R}^{2n}$ such that

$$\xi_l \in -\frac{\delta_l}{2} J\tilde{v}_l(0) + \partial H_0^*(\tilde{v}_l(0) - h_l(0)), \quad \delta_l = \frac{T}{2^l}.$$

Remark 5.1 Giving the definition of H_0, we can assume that ξ_l is of the type $(0, \lambda_l)$ with $\lambda_l \in \mathbb{R}^n$. In fact, we have

$$(r,p) \in (a,b) + \partial H_0^*(s,q) \Longleftrightarrow (s,q) = H_0^{'}((r,p) - (a,b))$$
$$\Longleftrightarrow (s,q) = (-A^*G'(p-b-A(r-a)), G'(p-b-A(r-a)))$$
$$= (-A^*G'(p-Ar-b+Aa), G'(p-Ar-b+Aa))$$
$$\Longleftrightarrow (s,q) = H_0^{'}(-a, p-Ar-b)$$
$$\Longleftrightarrow -(a,b) + (0, p-Ar) \in \partial H_0^*(s,q)$$
$$\Longleftrightarrow (0, p-Ar) \in (a,b) + \partial H_0^*(s,q).$$

In the following, we will take ξ_l of the form $(0, \lambda_l)$, $\lambda_l \in \mathbb{R}^n$, and we will prove that the associated sequence (w_l) has a subsequence strongly convergent in $L^2(0,T;\mathbb{R}^{2n})$ to a solution $\bar{w}$ of $(\mathcal{H})(\mathcal{C})$.

Lemma 5.1 *[7] The operator π from $L^2(0,T;\mathbb{R}^{2n})$ into itself, introduced in Section 2, is a Hilbert-Schmidt operator: it transforms quickly convergent sequences to strongly convergent sequences.*

Lemma 5.2 *Under assumptions (G_1), (G_2), there exists a subsequence (w_{l_k}) of (w_l) strongly convergent in $L^2(0,T;\mathbb{R}^{2n})$ to $\bar{w}$. Moreover $\bar{w}$ is defined in* 0 *and* T *and satisfies* $\bar{w}(0)=\bar{w}(T)$.

Proof. It is easy to verify that the sequence (w_l) is included in $L^2(0,T;\mathbb{R}^{2n})$. By Lemma 4.3, the sequence $(\tilde{v}_l)$ is bounded in $L^2(0,T;\mathbb{R}^{2n})$, then it possesses a subsequence $(\tilde{v}_{l_k})$ weakly convergent in $L^2(0,T;\mathbb{R}^{2n})$ to a point $\bar{v}$. In particular $(\tilde{v}_{l_k})$ being defined for all integer k and for all $t\in[0,T]$, the sequence $(\tilde{v}_{l_k}(t))$ is convergent in $\mathbb{R}^{2n}$ to $\bar{v}(t)$ for all $t\in[0,T]$. Recall that we have defined ξ_l by

$$\xi_l \in -\frac{\delta_l}{2}J\tilde{v}_l(0)+\partial H_0^*(\tilde{v}_l(0)-h_l(0)), \delta_l=\frac{T}{2^l}.$$

We have

$$\xi_l+\frac{\delta_l}{2}J\tilde{v}_l(0)\in\partial H_0^*(\tilde{v}_l(0)-h_l(0))$$

$$\Longleftrightarrow \tilde{v}_l(0)-h_l(0)=H_0^{'}(\xi_l+\frac{\delta_l}{2}J\tilde{v}_l(0))$$

$$=H_0^{'}\Big((0,\lambda_l)+\frac{\delta_l}{2}J(\tilde{v}_l^1(0),\tilde{v}_l^2)(0))\Big)=H_0^{'}(\frac{\delta_l}{2}\tilde{v}_l^2(0),\lambda_l-\frac{\delta_l}{2}\tilde{v}_l^1(0))\Longleftrightarrow$$

$$\tilde{v}_l(0)-h_l(0)=\Big(-A^*G'(\lambda_l-\frac{\delta_l}{2}(\tilde{v}_l^1(0)+A\tilde{v}_l^2(0))),G'(\lambda_l-\frac{\delta_l}{2}(\tilde{v}_l^1(0)+A\tilde{v}_l^2(0)))\Big).$$

Since G' is an homeomorphism from $\mathbb{R}^n$ into $G'(\mathbb{R}^n)$ and since (δ_l) goes to zero in $\mathbb{R}$ as l goes to infinity and $(\tilde{v}_{l_k}(0))$ is bounded and converges to $\bar{v}(0)$, the sequence (λ_{l_k}) converges to $\bar{\lambda}$ in $\mathbb{R}^n$ with

$$\bar{\lambda}=(G')^{-1}(\bar{v}^2(0)-h^2(0)).$$

By previous Remarks and Lemma 5.1, we deduce that the sequence (w_{l_k}) converges strongly to $\bar{w}$ in $L^2(0,T;\mathbb{R}^{2n})$. Moreover

$$\bar{w}(t)=J\int_0^t\bar{v}(\tau)d\tau+\bar{\xi}\ \mathit{with}\ \bar{\xi}=(0,\bar{\lambda})$$

and then, in particular, we have $\bar{w}(0)=\bar{w}(T)$.

Lemma 5.3 *The sequence (y_{l_k}) defined by*

$$y_{l_k}=\tilde{v}_{l_k}-Jh_{l_k}\in L^2(0,T;\mathbb{R}^{2n})$$

converges strongly in $L^2(0,T;\mathbb{R}^{2n})$ to $\bar{y}=\bar{v}-Jh$.

Proof. It is an immediately consequence of previous lemma's proof.

Lemma 5.4 *With the point w_l of $L^2(0,T;\mathbb{R}^{2n})$, we associate the element ω_l of the same space defined by*

$$\begin{cases}\omega_l(t^j)=\frac{1}{2}(w_l(t^{j+1})+w_l(t^j)),\forall j=1,...,N,\\ \omega_l(0)=\omega_l(T),\\ \omega_l(t)=\omega_l(t^j),\forall t\in[t^j,t^{j+1}[,\forall j=1,...,N.\end{cases}$$

Under assumptions (G_1), (G_2), the subsequence (ω_{l_k}) of (ω_l) converges strongly in $L^2(0,T;\mathbb{R}^{2n})$ to $\bar{w}$.

Proof. It suffices to prove

$$\lim_{k\longrightarrow\infty} \|\omega_{l_k} - w_{l_k}\|_2 = 0. \tag{5.1}$$

Then we will use the inequality

$$\|\omega_{l_k} - \bar{w}\| \le \|\omega_{l_k} - w_{l_k}\|_2 + \|w_{l_k} - \bar{w}\|_2$$

and we conclude by using Lemma 5.2.
We have

$$\|\omega_{l_k} - w_{l_k}\|_2^2 = \int_0^T |\omega_{l_k}(t) - w_{l_k}(t)|^2 dt,$$

where $|.|$ denotes $|.|_{2n}$. On the other hand, we have

$$\|\omega_{l_k} - w_{l_k}\|_2^2 = \sum_{j=1}^{N_k} \int_{t^{j+1}}^{t^j} |\omega_{l_k} - w_{l_k}|^2 \, dt, \tag{5.2}$$

where $N_k = 2^{l_k}$. In $[t^j, t^{j+1}[$, $w_{l_k}(t)$ can be written

$$\forall t \in [t^j, t^{j+1}[, w_{l_k}(t) = w_{l_k}(t^j) + (t - t^j)\tilde{v}_{l_k}(t^j).$$

Then equality (5.2) becomes

$$\|\omega_{l_k} - w_{l_k}\|_2^2 = \sum_{j=1}^{N_k} \int_{t^{j+1}}^{t^j} \left|\omega_{l_k}(t^j) - w_{l_k}(t^j) - (t - t^j)\tilde{v}_{l_k}(t^j)\right|^2 dt.$$

This yields

$$\|\omega_{l_k} - w_{l_k}\|_2^2 = \sum_{j=1}^{N_k} \int_{t^j}^{t^{j+1}} \left|\omega_{l_k}(t^j) - w_{l_k}(t^j)\right|^2 dt$$

$$+2\sum_{j=1}^{N_k} \int_{t^j}^{t^{j+1}} \left|t - t^j\right|\left|\tilde{v}_{l_k}(t^j)\right| \left|\omega_{l_k}(t^j) - w_{l_k}(t^j)\right| dt + \sum_{j=1}^{N_k} \int_{t^j}^{t^{j+1}} \left|t - t^j\right|^2 \left|\tilde{v}_{l_k}(t^j)\right|^2 dt$$

$$\le \sum_{j=1}^{N_k} \int_{t^j}^{t^{j+1}} \left|\omega_{l_k}(t^j) - w_{l_k}(t^j)\right|^2 + 2\frac{T}{N_k} \sum_{j=1}^{N_k} [\int_{t^j}^{t^{j+1}} \left|\tilde{v}_{l_k}(t^j)\right|^2 dt]^{\frac{1}{2}} [\int_{t^j}^{t^{j+1}} \left|\omega_{l_k}(t^j) - w_{l_k}(t^j)\right|^2 dt]^{\frac{1}{2}}$$

$$+(\frac{T}{N_k})^2 \sum_{j=1}^{N_k} \int_{t^j}^{t^{j+1}} |\tilde{v}_{l_k}(t^j)|^2 dt. \tag{5.3}$$

The expression $\omega_{l_k}(t^j) - w_{l_k}(t^j)$ can be written

$$\omega_{l_k}(t^j) - w_{l_k}(t^j) = \frac{w_{l_k}(t^j) + w_{l_k}(t^{j+1})}{2} - w_{l_k}(t^j) = \frac{w_{l_k}(t^{j+1}) - w_{l_k}(t^j)}{2}.$$

But we know that

$$\frac{w_{l_k}(t^{j+1}) - w_{l_k}(t^j)}{2} = \frac{1}{2}\delta_{l_k}\tilde{v}_{l_k}(t^j).$$

Therefore the inequality (5.3) becomes

$$\|\omega_{l_k} - w_{l_k}\|_2^2 \leq \frac{9}{4}(\delta_{l_k})^2 \int_0^T \left|\tilde{v}_{l_k(t)}\right|^2 dt. \tag{5.4}$$

Since $\delta_{l_k} = \frac{T}{N_k} = T2^{-l_k}$ goes to zero as k goes to infinity and $\tilde{v}_{l_k}$ is bounded in $L^2(0,T;\mathbb{R}^{2n})$, the relation (5.1) is proved.

If assumption (G_2) is satisfied, Lemma 4.3 permits to write

$$\delta_{l_k} \int_0^T |v_{l_k}(t)|^2 \, dt = \frac{T}{2^{l_k}}[2^{l_k} \sum_{j=1}^{N_k} \left|v_{l_k}^j\right|^2] \leq \frac{T}{2^{l_k}} M.$$

Therefore we can state the following convergence result:

Theorem 5.1 *Under assumptions (G_1), (G_2) and Lemma 5.2 notations, the subsequence (ω_{l_k}) converges strongly in $L^2(0,T;\mathbb{R}^{2n})$ to a solution $\bar{w}$ of $(\mathcal{H})(\mathcal{C})$.*

Proof. To prove this theorem, we will need the following theorem:

Theorem 5.2 *[4] Let A be a monotone maximal operator from its domain $D(A) \subset L^2(0,T;\mathbb{R}^{2n})$ into $L^2(0,T;\mathbb{R}^{2n})$. Let (x_l) and (y_l) be two sequences satisfying*

$$(i)\ x_l \in DomA, \forall l \geq l_0,$$
$$(ii)\ y_l = A(x_l), \forall l \geq l_0,$$
$$(iii)\ (x_l)\ \textit{converges weakly to}\ \bar{x}\ \textit{in}\ L^2(0,T;\mathbb{R}^{2n}),$$
$$(iv)\ (y_l)\ \textit{converges weakly to}\ \bar{y}\ \textit{in}\ L^2(0,T;\mathbb{R}^{2n}),$$
$$(v)\ \limsup_{l \longrightarrow \infty}(x_l y_l) \leq \bar{x}\bar{y}.$$

Then

$$(j)\ \bar{x} \in DomA,$$
$$(jj)\ \bar{y} = A(\bar{x}).$$

By Section 3, we know that for all integer l, the following system is verified:

$$\begin{cases} (i)\ \frac{w_N(t^{j+1}) - w_N(t^j)}{t^{j+1} - t^j} = J[H_0^{'}\frac{w_N(t^{j+1}) + w_N(t^j)}{2}) + h^j], \forall j = 1, ..., 2^l \\ \textit{and} \\ (ii)\ \frac{w_N(t^{j+1}) - w_N(t^j)}{t^{j+1} - t^j} = v_l^j, \forall j = 1, ..., 2^l. \end{cases}$$

By using the notations of Lemma 5.3, equation (i) can be rewritten

$$\forall t \in [0,T], -Jy_l(t) = H_0^{'}(\omega(t)).$$

Since the operator "$-J$" from $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$ is an isometry, we deduce from the previous Lemmas that the sequences $(-Jy_{l_k})$ and (ω_{l_k}) as the operator $H_0^{'}$ verify assumptions of the previous Theorem, therefore we can assert that

$$\forall t \in [0,T], -J\bar{y}(t) = H_0^{'}(\bar{w}(t))$$

or also

$$\forall t \in [0,T],\ \bar{v}(t) = J(H_0^{'}(\bar{w}(t)) + h(t)),$$

where

$$\bar{w}(t) = \int_0^t \bar{v}(\tau)d\tau + (0, \bar{\lambda}).$$

Therefore $\bar{w}$ is a solution of $(\mathcal{H})(\mathcal{C})$.

6 Conclusion

In this paper, we first prove the existence of solutions of a problem of non-coercive convex Hamiltonian systems $(\mathcal{H})(\mathcal{C})$ through the theory of critical point theory and the dual action principle. Then we associate with $(\mathcal{H})(\mathcal{C})$ a sequence of problems $(\mathcal{H}_N)(\mathcal{C}_N)$, (R_N), (P_N) defined in a finite dimensional space and related together by a discrete dual action principle. We prove that problems $(\mathcal{H}_N)(\mathcal{C}_N)$ possess a sequence of solutions which converges to a solution of problem $(\mathcal{H})(\mathcal{C})$.

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