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Stability of Stochastic Interval System with Distributed Delays

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Abstract: In this paper, we study the stability problem of a stochastic interval system with distributed delays. Firstly, we prove that the solution of such system exists and is unique, and then a sufficient criterion of exponential stability is obtained and such result can be generalized to the systems with multiple time delays. Finally, an example is given to illustrate the result.

Keywords: *exponential stability; stochastic interval system.*

Mathematics Subject Classification (2010): 65C30, 60H10.

1 Introduction

Stochastic modelling has come to play an important role in many branches of science and industry, such as neural network and automatic control of stochastic system and so on, see [1–9]. One of the most useful stochastic models which are often used in practice is stochastic differential delay equation [10–15]. However, in many practical models, it is difficult to determine the parameters with a fixed value and instead of obtaining some estimation – the parameters are changed in an interval. Such a system can be described by stochastic interval system.

In the past decades, a lot of work on stochastic differential interval systems could be found in [16–18] and the results are generalized to Markov switched system by [19, 20]. Motivated by these works, in this paper, we study stochastic interval system with distributed delays. Consider the following stochastic system

$$\begin{aligned} dx(t) &= [A_0x(t) + A_1x(t-\tau) + A_2 \textstyle\int_{-\tau}^{0} x(t+\theta)d\mu(\vartheta)]dt \\ &\quad +[B_0x(t) + B_1x(t-\tau) + B_2 \textstyle\int_{-\tau}^{0} x(t+\theta)d\nu(\vartheta)]dB_t, \end{aligned} \tag{1}$$

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where $A_0, A_1, A_2, B_0, B_1, B_2$ are constant matrices and μ, ν denote probability measures, τ is a positive constant. This system is incorporated with time delays, which would be appropriate in circumstances where a process is dependent not only upon the present state but also upon the state at all the times of some interval in the past. In practice, the matrix coefficients must be estimated from empirical data, and chosen as confidence intervals under a statistical method, so we use interval matrices instead of the coefficients of system (1), and this results in an interval system of the form

$$\begin{aligned} dx(t) &= [(A_0+\Delta A_0)x(t) + (A_1+\Delta A_1)x(t-\tau) \\ &+ (A_2+\Delta A_2)\int_{-\tau}^{0} x(t+\theta)d\mu(\vartheta)]dt + [(B_0+\Delta B_0)x(t) \\ &+ (B_1+\Delta B_1)x(t-\tau) + (B_2+\Delta B_2)\int_{-\tau}^{0} x(t+\theta)d\nu(\vartheta)]dB_t, \end{aligned} \quad (2)$$

where $\Delta A_i, \Delta B_i, A_i, B_i$ are constant matrices and $\Delta A_i \in [-A_{im}, A_{im}]$, $\Delta B_i \in [-B_{im}, B_{im}]$, $i = 0,1,2$, where A_{im}, B_{im}, $im = 0,1,2$ are constant matrices and μ, ν denote probability measures, τ is a positive constant.

The study of stochastic interval system (2) becomes more difficult than that of stochastic system (1), since the parameters of (2) belong to intervals. Since the coefficients of such system have the property of uncertainty, we always treat interval systems as uncertain systems. Because the coefficients as interval matrices are not well-performing to preserve the stability properties, so it is useful and helpful to study the stability behaviour of such systems. There are many stability properties to be studied, but this paper will focus on the study of exponential stability of stochastic interval systems (2) with distributed time delay.

In the next section, we will give some notations used throughout this paper. In Section 3, we discuss a particular type of stochastic interval system with distributed delays, a stability criterion is given which will be applied to examine the stability of stochastic interval system with distributed delays, and then we generalize these results to a stochastic interval system with multiple distributed delays. An example is given to illustrate our result.

2 Preliminaries

Let R^n be Euclidean space and $|\bullet|$ be the Euclidean norm in R^n. If A is a matrix, its transpose is denoted by A^T and define a norm of A as $\| A \| = sup\{|Ax| : |x| = 1\} = \sqrt{\lambda_{max}(AA^T)}$. If A is a symmetric matrix, let $\lambda_{max}(A)$ and $\lambda_{min}(A)$ represent its largest and smallest eigenvalue respectively. Obviously, if A is a symmetric matrix, then $\lambda_{max}(A) \le \|A\|$.

If $A^m = [a_{ij}^m]_{n\times n}$ and $A^M = [a_{ij}^M]_{n\times n}$ are matrices and $a_{ij}^m \le a_{ij}^M$, $\forall 1 \le i,j \le n$, the interval matrix $[A^m, A^M]$ is defined by

$$[A^m, A^M] = \{A = [a_{ij}]_{n\times n} : a_{ij}^m \le a_{ij} \le a_{ij}^M,\ \forall 1 \le i,j \le n\}.$$

For $A, A_m \in R^{n\times n}$, where A_m is a nonnegative matrix, we note that any interval matrix $[A^m, A^M]$ has a unique representation of the form $[A - A_m, A + A_m]$, where $A = \frac{1}{2}(A^m + A^M)$, and $A_m = \frac{1}{2}(A^M - A^m)$.

In this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\ge 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}\}_{t\ge 0}$ satisfying the usual conditions. Let B_t denote a Brownian motion defined on the probability space. Let τ be a positive number and $C([-\tau,0]; R^n)$ be the family of all continuous R^n-valued functions on $[-\tau, 0]$ with the values in R^n. We define a norm as $\|y\|_\tau = \sup_{-\tau\le s\le 0}|y(t)|$ for any $y \in C([-\tau,0];R^n)$. Let $L^2(\Omega,\mathcal{F}_{t_0},C([-\tau,0];R^n))$ represent all $\mathcal{F}_{t_0}$-measurable $C([-\tau,0];R^n)$-valued random variables ξ with $E\|\xi\|_\tau^2<\infty$ and we write L^2 for short unless otherwise specified. If $x(t), t\ge t_0-\tau$ is an n-dimensional continuous stochastic process, we denote $x_t = x(t+s) : -\tau\le s\le 0$ as a $C([-\tau,0];R^n)$-valued process on $t\ge 0$. For any initial data $\hat{x}(t_0)=\xi\in L^2(\Omega,\mathcal{F}_{t_0},C([-\tau,0];R^n))$, there exists a unique global solution of (1) which is denoted by $x(t,t_0,\xi)$.

Definition 2.1 The system (1) is said to be

(a) exponentially stable in $L^2(\Omega,C([-\tau,0];R^n))$, if there exist positive constants M and γ such that for all $t_0\ge 0$ and $\xi\in L^2(\Omega,\mathcal{F}_{t_0},C([-\tau,0];R^n))$,

$$E\|\hat{x}(t,t_0,\xi)\|_\tau^2\le Me^{-\gamma(t-t_0)}E\|\xi\|_\tau^2.$$

(b) almost surely exponentially stable if

$$\lim_{t\to\infty} sup\frac{1}{t}log|x(t,t_0,\xi)|<0. \qquad a.s.$$

3 Main Results

In this section, we will study the stability properties of system (2). For system (2), we can't make sure that the solution of such system exists and is unique under the condition of linear growth and Lipschitz condition. In this paper, we affirm that the solution of system (2) exists and is unique; here we only need to prove the results for system (1), because the norm of the matrix coefficient of system(2) is bounded as in system (1), which is an essential step in the proof of the following theorems.

Let

$$\Im=\{\phi|\phi(\theta):-\tau\le\theta\le 0 \; is\ random\ variable; \phi\in\mathcal{F}_{t_0}\cap C([-\tau,0];R^n),\ E\|\phi\|^2<\infty\}.$$

Lemma 3.1 *If $x(t)$ is a solution of equation (1), then for any $T>t_0$, $\exists C>0$, such that*

$$E(\sup_{t_0-\tau\le t\le T}|x(t)|^2)<C.$$

In particular, $x(t)$ belongs to $L^2([t_0-\tau,T];R^n))$.

Theorem 3.1 *For any $\xi\in\Im$, there exists a unique solution $x(t)$ of system (1), (2) and $x_{t_0}=\xi$.*

Proof. This can be easily proved by using the method of [21].

In order to study the stability of system (2), firstly, we consider system (1).

Theorem 3.2 *Assume that there exists a symmetric positive-definite matrix Q such that*

$$\begin{aligned}&2\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}A_1^TQA_1Q^{-\frac{1}{2}})}+2\sqrt{\varpi_A(\lambda)\lambda_{max}(Q^{-\frac{1}{2}}A_2^TQA_2Q^{-\frac{1}{2}})}\\&+(\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}B_0^TQB_0Q^{-\frac{1}{2}})}+\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}B_1^TQB_1Q^{-\frac{1}{2}})}\\&+\sqrt{\varpi_B(\lambda)\lambda_{max}(Q^{-\frac{1}{2}}B_2^TQB_2Q^{-\frac{1}{2}})})^2<-\lambda_{max}(Q^{-\frac{1}{2}}(QA_0+A_0^TQ)Q^{-\frac{1}{2}})\triangleq\lambda,\end{aligned}\qquad (3)$$

where $\varpi_A(\lambda) = \int_{-\tau}^{0} e^{-\lambda\theta} d\mu(\theta), \quad \varpi_B(\lambda) = \int_{-\tau}^{0} e^{-\lambda\theta} d\nu(\theta)$.

Then system (1) is exponentially stable in $L^2(\Omega, C([-\tau, 0]; R^n))$ *and moreover, it is almost surely exponentially stable.*

Proof. Firstly, we note that $Q^{-\frac{1}{2}}(QA_0 + A_0^T Q)Q^{-\frac{1}{2}}$ must be negative definite.

Set

$$\lambda = -\lambda_{max}(Q^{-\frac{1}{2}}(QA_0 + A_0^T Q)Q^{-\frac{1}{2}}) > 0. \tag{4}$$

By the condition of Theorem 3.2, we can find a constant $\gamma \in (0, \lambda)$ such that

$$\begin{aligned}
&(1 + e^{\gamma\tau})\sqrt{\lambda_{max}(Q^{-\frac{1}{2}} A_1^T Q A_1 Q^{-\frac{1}{2}})} \\
&+(1 + e^{\gamma\tau})\sqrt{\lambda_{max}(Q^{-\frac{1}{2}} B_0^T Q B_0 Q^{-\frac{1}{2}})\lambda_{max}(Q^{-\frac{1}{2}} B_1^T Q B_1 Q^{-\frac{1}{2}})} \\
&+2\sqrt{\varpi_A(\lambda)\lambda_{max}(Q^{-\frac{1}{2}} A_2^T Q A_2 Q^{-\frac{1}{2}})} \\
&+2\sqrt{\lambda_{max}(Q^{-\frac{1}{2}} B_0^T Q B_0 Q^{-\frac{1}{2}})\varpi_B(\lambda)\lambda_{max}(Q^{-\frac{1}{2}} B_2^T Q B_2 Q^{-\frac{1}{2}})} \\
&+(1 + e^{\gamma\tau})\sqrt{\varpi_B(\lambda)\lambda_{max}(Q^{-\frac{1}{2}} B_2^T Q B_2 Q^{-\frac{1}{2}})\lambda_{max}(Q^{-\frac{1}{2}} B_1^T Q B_1 Q^{-\frac{1}{2}})} \\
&+\lambda_{max}(Q^{-\frac{1}{2}} B_0^T Q B_0 Q^{-\frac{1}{2}}) + \varpi_B(\lambda)\lambda_{max}(Q^{-\frac{1}{2}} B_2^T Q B_2 Q^{-\frac{1}{2}}) \\
&+e^{\gamma\tau}\lambda_{max}(Q^{-\frac{1}{2}} B_1^T Q B_1 Q^{-\frac{1}{2}}) < \lambda - \gamma.
\end{aligned} \tag{5}$$

We claim that there exists a constant $C > 0$ such that

$$\int_{t_0}^{\infty} e^{\gamma t} E(x(t)^T Q x(t)) dt \le C e^{\gamma t_0} E\|\xi^T Q \xi\|, \tag{6}$$

for all $t_0 \ge 0$ and $\xi \in L^2(\Omega, \mathcal{F}_{t_0}, C([-\tau, 0]; R^n)$.

In addition, we also affirm that there exists another constant $C' > 0$ such that

$$E\|x_t^T Q x_t\| \le C' e^{-\gamma(t - t_0)} E\|\xi^T Q \xi\|, \tag{7}$$

which is held in $L^2(\Omega, \mathcal{F}_{t_0}, C([-\tau, 0]; R^n))$. It follows from (6) that (2) is almost surely exponentially stable.

Next, we will give proofs of (6) and (7).

Fix $t_0 \ge 0$ and ξ , write $x(t) = x(t, t_0, \xi)$, then Ito's formula yields that

$$
\begin{aligned}
&e^{\lambda t}E(x(t)^TQx(t)) = e^{\lambda t_0}E(x(t_0)^TQx(t_0)) + \lambda\int_{t_0}^{t} e^{\lambda s}E(x(s)^TQx(s))ds\\
&+2\int_{t_0}^{t} e^{\lambda s}E(x(s)^TQA_0x(s))ds + 2\int_{t_0}^{t} e^{\lambda s}E(x(s)^TQA_1x(s-\tau))ds\\
&+2\int_{t_0}^{t} e^{\lambda s}E(x(s)^TQA_2\int_{-\tau}^{0} x(s+\vartheta)d\mu(\vartheta))ds\\
&+\int_{t_0}^{t} e^{\lambda s}E(x(s)^TB_0^TQB_0x(s))ds + \int_{t_0}^{t} e^{\lambda s}E(x(s)^TB_0^TQB_1x(s-\tau))ds\\
&+\int_{t_0}^{t} e^{\lambda s}E(x(s)^TB_0^TQB_2\int_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds\\
&+\int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^TB_1^TQB_0x(s))ds + \int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^TB_1^TQB_1x(s-\tau))ds\\
&+\int_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^TB_1^TQB_2\int_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds\\
&+\int_{t_0}^{t} e^{\lambda s}E(\int_{-\tau}^{0} x(s+\vartheta)^Td\nu(\vartheta)B_2^TQB_0x(s))ds\\
&+\int_{t_0}^{t} e^{\lambda s}E(\int_{-\tau}^{0} x(s+\vartheta)^Td\nu(\vartheta)B_2^TQB_1x(s-\tau))ds\\
&+\int_{t_0}^{t} e^{\lambda s}E(\int_{-\tau}^{0} x(s+\vartheta)^Td\nu(\vartheta)B_2^TQB_2\int_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds.
\end{aligned}
\tag{8}
$$

We note that by (4)

$$2x(t)^TQA_0x(t) \le -\lambda x(t)^Tx(t).$$

For any $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 > 0$,

$$
\begin{aligned}
& 2\int_{t_0}^{t} e^{\lambda s}E(x(s)^TQA_1x(s-\tau))ds\\
\le\;& \varepsilon_1\int_{t_0}^{t} e^{\lambda s}E(x(s)^TQx(s))ds + \int_{t_0}^{t} \tfrac{e^{\lambda s}}{\varepsilon_1}E(x(s-\tau)^TA_1^TQA_1x(s-\tau))ds,\\
& 2\int_{t_0}^{t} e^{\lambda s}E(x(s)^TQA_2\int_{-\tau}^{0} x(s+\vartheta)d\mu(\vartheta))ds\\
=\;& 2\int_{t_0}^{t} e^{\lambda s}\int_{-\tau}^{0} E(x(s)^TQA_2x(s+\vartheta)d\mu(\vartheta))ds\\
\le\;& \int_{t_0}^{t} e^{\lambda s}\int_{-\tau}^{0} (\varepsilon_3E(x(s)^TQx(s)) + \tfrac{E(x(s+\theta)^TA_2^TQA_2x(s+\theta))}{\varepsilon_3})d\mu(\theta)ds\\
=\;& \varepsilon_3\int_{t_0}^{t} e^{\lambda s}E(x(s)^TQx(s))ds + \int_{t_0}^{t}\int_{-\tau}^{0} \tfrac{e^{\lambda s}}{\varepsilon_3}E(x(s+\theta)^TA_2^TQA_2x(s+\theta))d\mu(\theta)ds\\
\le\;& \varepsilon_3\int_{t_0}^{t} e^{\lambda s}E(x(s)^TQx(s))ds + \int_{t_0-\tau}^{t} \tfrac{e^{\lambda s}}{\varepsilon_3}E(x(s)^TA_2^TQA_2x(s))ds\int_{-\tau}^{0} e^{-\lambda\theta}d\mu(\theta)\\
=\;& \varepsilon_3\int_{t_0}^{t} e^{\lambda s}E(x(s)^TQx(s))ds + \varpi_A(\lambda)\int_{t_0-\tau}^{t_0} \tfrac{e^{\lambda s}}{\varepsilon_3}E(\xi^TA_2^TQA_2\xi)ds\\
+\;& \varpi_A(\lambda)\int_{t_0}^{t} \tfrac{e^{\lambda s}}{\varepsilon_3}E(x(s)^TA_2^TQA_2x(s))ds,
\end{aligned}
\tag{9}
$$

where $\varpi_A(\lambda) = \int\limits_{-\tau}^{0} e^{-\lambda\theta} d\mu(\theta)$.

We can proceed in a similar fashion for other forms in (5).

$$
\begin{aligned}
&e^{\lambda t}E(x(t)^T Qx(t)) \le e^{\lambda t_0}E\left\|\xi^T Q\xi\right\| + \varepsilon_1 \int\limits_{t_0}^{t} e^{\lambda s}E(x(s)^T Qx(s))ds \\
&+\int\limits_{t_0}^{t} \tfrac{e^{\lambda s}}{\varepsilon_1}E(x(s-\tau)^T A_1^T QA_1 x(s-\tau))ds + \varepsilon_3 \int\limits_{t_0}^{t} e^{\lambda s}E(x(s)^T Qx(s))ds \\
&+\varpi_A(\lambda)[\int\limits_{t_0-\tau}^{t_0} \tfrac{e^{\lambda s}}{\varepsilon_3}E(\xi^T A_2^T QA_2\xi)ds + \int\limits_{t_0}^{t} \tfrac{e^{\lambda s}}{\varepsilon_3}E(x(s)^T A_2^T QA_2 x(s))ds] \\
&+(1+\varepsilon_2)\int\limits_{t_0}^{t} e^{\lambda s}E(x(s)^T B_0^T QB_0 x(s))ds \\
&+\int\limits_{t_0}^{t} \tfrac{(1+\varepsilon_2)e^{\lambda s}}{\varepsilon_2}E(x(s-\tau)^T B_1^T QB_1 x(s-\tau))ds \\
&+\varepsilon_4\int\limits_{t_0}^{t} e^{\lambda s}E(x(s)^T B_0^T QB_0 x(s))ds \\
&+\int\limits_{t_0}^{t} \tfrac{e^{\lambda s}}{\varepsilon_4}E(\int\limits_{-\tau}^{0} x(s+\vartheta)^T d\nu(\vartheta)B_2^T QB_2 \int\limits_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds \\
&+\varepsilon_5\int\limits_{t_0}^{t} e^{\lambda s}E(x(s-\tau)^T B_1^T QB_1 x(s-\tau))ds \\
&+\int\limits_{t_0}^{t} \tfrac{e^{\lambda s}}{\varepsilon_5}E(\int\limits_{-\tau}^{0} x(s+\vartheta)^T d\nu(\vartheta)B_2^T QB_2 \int\limits_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds \\
&+\int\limits_{t_0}^{t} e^{\lambda s}E(\int\limits_{-\tau}^{0} x(s+\vartheta)^T d\nu(\vartheta)B_2^T QB_2 \int\limits_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds.
\end{aligned}
\tag{10}
$$

Here we mention that

$$
\begin{aligned}
&\int\limits_{t_0}^{t} e^{\lambda s}E(\int\limits_{-\tau}^{0} x(s+\vartheta)^T d\nu(\vartheta)B_2^T QB_2 \int\limits_{-\tau}^{0} x(s+\vartheta)d\nu(\vartheta))ds \\
&=\int\limits_{t_0}^{t} e^{\lambda s}\int\limits_{-\tau}^{0}\int\limits_{-\tau}^{0} E(x(s+\phi)^T B_2^T QB_2 x(s+\vartheta))d\nu(\phi)d\nu(\vartheta)ds \\
&\le \int\limits_{t_0}^{t} \tfrac{e^{\lambda s}}{2}\int\limits_{-\tau}^{0}\int\limits_{-\tau}^{0}[E(x(s+\phi)^T B_2^T QB_2 x(s+\phi)) \\
&+E(x(s+\vartheta)^T B_2^T QB_2 x(s+\vartheta))]d\nu(\phi)d\nu(\vartheta)ds \\
&\le \int\limits_{t_0}^{t} e^{\lambda s}\int\limits_{-\tau}^{0} E(x(s+\vartheta)^T B_2^T QB_2 x(s+\vartheta))d\nu(\vartheta)ds \\
&\le \varpi_B(\lambda)[\int\limits_{t_0-\tau}^{t_0} e^{\lambda s}E(\xi^T B_2^T QB_2\xi)ds + \int\limits_{t_0}^{t} e^{\lambda s}E(x(s)^T B_2^T QB_2 x(s))ds],
\end{aligned}
\tag{11}
$$

where $\varpi_B(\lambda) = \int\limits_{-\tau}^{0} e^{-\lambda\theta} d\nu(\theta)$. Hence,

$$
\begin{aligned}
&e^{\lambda t}E(x(t)^T Qx(t)) \\
&\le e^{\lambda t_0}E\left\|\xi^T Q\xi\right\| + \int\limits_{t_0-\tau}^{t_0} \tfrac{\varepsilon_4+\varepsilon_5+\varepsilon_4\varepsilon_5}{\varepsilon_4\varepsilon_5}\varpi_B(\lambda)e^{\lambda s}E(\xi^T B_2^T QB_2\xi)ds \\
&+\int\limits_{t_0-\tau}^{t_0} \tfrac{\varpi_A(\lambda)}{\varepsilon_3}e^{\lambda s}E(\xi^T A_2^T QA_2\xi)ds + (\varepsilon_1+\varepsilon_3)\int\limits_{t_0}^{t} e^{\lambda s}E(x(s)^T Qx(s))ds
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&+\int_{t_0}^{t} \tfrac{e^{\lambda s}}{\varepsilon_1} E(x(s-\tau)^T A_1^T Q A_1 x(s-\tau))ds \\
&+\int_{t_0}^{t} \tfrac{\varpi_A(\lambda)}{\varepsilon_3} e^{\lambda s} E(x(s)^T A_2^T Q A_2 x(s))ds \\
&+(1+\varepsilon_2+\varepsilon_4)\int_{t_0}^{t} e^{\lambda s} E(x(s)^T B_0^T Q B_0 x(s))ds \\
&+\int_{t_0}^{t} \tfrac{1+\varepsilon_2+\varepsilon_2\varepsilon_5}{\varepsilon_2} e^{\lambda s} E(x(s-\tau)^T B_1^T Q B_1 x(s-\tau))ds \\
&+\varpi_B(\lambda)\int_{t_0}^{t} \tfrac{\varepsilon_4+\varepsilon_5+\varepsilon_4\varepsilon_5}{\varepsilon_4\varepsilon_5} e^{\lambda s} E(x(s)^T B_2^T Q B_2 x(s))ds \\
&\le C_1 e^{\lambda t_0} E\left\|\xi^T Q\xi\right\| + C_2\int_{t_0}^{t} e^{\lambda s} E(x(s)^T Q x(s))ds \\
&+C_3\int_{t_0}^{t} e^{\lambda s} E(x(s-\tau)^T Q x(s-\tau))ds,
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= 1+(\tfrac{1}{\varepsilon_4}+\tfrac{1}{\varepsilon_5}+1)\varpi_B(\lambda)\tau\lambda_{\max}(Q^{-\frac12}B_2^T Q B_2 Q^{-\frac12}) \\
&\quad +\tfrac{1}{\varepsilon_3}\varpi_A(\lambda)\tau\lambda_{\max}(Q^{-\frac12}A_2^T Q A_2 Q^{-\frac12}), \\
C_2 &= \varepsilon_1+\varepsilon_3+\tfrac{1}{\varepsilon_3}\varpi_A(\lambda)\lambda_{\max}(Q^{-\frac12}A_2^T Q A_2 Q^{-\frac12}) \\
&\quad +(1+\varepsilon_2+\varepsilon_4)\lambda_{\max}(Q^{-\frac12}B_0^T Q B_0 Q^{-\frac12}) \\
&\quad +(\tfrac{1}{\varepsilon_4}+\tfrac{1}{\varepsilon_5}+1)\varpi_B(\lambda)\lambda_{\max}(Q^{-\frac12}B_2^T Q B_2 Q^{-\frac12}), \\
C_3 &= \tfrac{1}{\varepsilon_1}\lambda_{\max}(Q^{-\frac12}A_1^T Q A_1 Q^{-\frac12})+(1+\tfrac{1}{\varepsilon_2}+\varepsilon_5)\lambda_{\max}(Q^{-\frac12}B_1^T Q B_1 Q^{-\frac12}).
\end{aligned}
$$

Then

$$
\begin{aligned}
E(x(t)^T Q x(t)) &\le C_1 e^{-\lambda(t-t_0)} E\left\|\xi^T Q\xi\right\| + C_2\int_{t_0}^{t} e^{-\lambda(t-s)} E(x(s)^T Q x(s))ds \\
&+ C_3\int_{t_0}^{t} e^{-\lambda(t-s)} E(x(s-\tau)^T Q x(s-\tau))ds.
\end{aligned}
$$

For any $T>t_0$,

$$
\begin{aligned}
&\int_{t_0}^{T} e^{\gamma t} E(x(t)^T Q x(t))dt \\
&\le C_1\int_{t_0}^{T} e^{\gamma t} e^{-\lambda(t-t_0)} E\left\|\xi^T Q\xi\right\| dt + C_2\int_{t_0}^{T} e^{\gamma t}\int_{t_0}^{t} e^{-\lambda(t-s)} E(x(s)^T Q x(s))dsdt \\
&\quad +C_3\int_{t_0}^{T} e^{\gamma t}\int_{t_0}^{t} e^{-\lambda(t-s)} E(x(s-\tau)^T Q x(s-\tau))dsdt \\
&\le \tfrac{C_1+C_3\tau e^{\gamma\tau}}{\lambda-\gamma} e^{\gamma t_0} E\left\|\xi^T Q\xi\right\| + \tfrac{C_2+C_3 e^{\gamma\tau}}{\lambda-\gamma}\int_{t_0}^{T} e^{\gamma t} E(x(t)^T Q x(t))dt.
\end{aligned}
$$

If we let

$$
\frac{1}{\lambda-\gamma}(C_2+C_3 e^{\gamma\tau}) < 1, \tag{13}
$$

which implies that there exists a constant $C > 0$ such that (6) holds.

Next, we will give a proof of (7).

Using Ito's formula, we have

$$\begin{aligned}
&e^{\gamma(t-\tau)} E \left\| \hat{x}(t)^T Q \hat{x}(t) \right\| \leq E(\sup_{t-\tau \leq r \leq t} e^{\gamma r} x(r)^T Q x(r)) \\
&\leq C_1 e^{\gamma t_0} E \left\| \xi^T Q \xi \right\| + C_2 \int_{t_0}^{t} e^{\gamma s} E(x(s)^T Q x(s)) ds \\
&\quad + C_3 \int_{t_0}^{t} e^{\gamma s} E(x(s-\tau)^T Q x(s-\tau)) ds \\
&\quad + 2E(\sup_{t-\tau \leq r \leq t} \int_{t_0}^{r} e^{\gamma s} x^T(s) Q B_0 x(s) dB_s) \\
&\quad + 2E(\sup_{t-\tau \leq r \leq t} \int_{t_0}^{r} e^{\gamma s} x^T(s) Q B_1 x(s-\tau) dB_s) \\
&\quad + 2E(\sup_{t-\tau \leq r \leq t} \int_{t_0}^{r} e^{\gamma s} [x^T(s) Q B_2 \int_{-\tau}^{0} x(s+\vartheta) d\mu(\vartheta)] dB_s).
\end{aligned}$$

Here we used B-D-G inequality [21]. For any $\varepsilon_6, \varepsilon_7, \varepsilon_8 > 0$,

$$\begin{aligned}
&2E(\sup_{t-\tau \leq r \leq t} \int_{t_0}^{r} e^{\gamma s} x^T(s) Q B_0 x(s) dB_s) \\
&\leq 2\sqrt{32} E(\int_{t-\tau}^{t} e^{2\gamma s} \left| x^T(s) Q x(s) \right| \left| x^T(s) B_0^T Q B_0 x(s) \right| ds)^{1/2} \\
&\leq 2\sqrt{32} E(\|x_s^T Q x_s\|_\tau \int_{t-\tau}^{t} \lambda_{\max}(Q^{-\frac{1}{2}} B_0^T Q B_0 Q^{-\frac{1}{2}}) e^{2\gamma s} \|x_s^T Q x_s\|_\tau ds)^{\frac{1}{2}} \\
&\leq \varepsilon_6 e^{\gamma(t-\tau)} E \left\| x_s^T Q x_s \right\|_\tau \\
&\quad + \tfrac{32}{\varepsilon_6} e^{-\gamma(t-\tau)} \lambda_{\max}(Q^{-1/2} B_0^T Q B_0 Q^{-1/2}) \int_{t-\tau}^{t} e^{2\gamma s} \|x_s^T Q x_s\|_\tau ds, \\
&2E(\sup_{t-\tau \leq r \leq t} \int_{t_0}^{r} e^{\gamma s} x^T(s) Q B_1 x(s-\tau) dB_s) \\
&\leq 2\sqrt{32} E(\int_{t-\tau}^{t} e^{2\gamma s} |x^T(s) Q x(s)| |x^T(s-\tau) B_1^T Q B_1 x(s-\tau)| ds)^{\frac{1}{2}} \\
&\leq 2\sqrt{32} E(\|x_s^T Q x_s\|_\tau \int_{t-\tau}^{t} \lambda_{\max}(Q^{-\frac{1}{2}} B_1^T Q B_1 Q^{-\frac{1}{2}}) e^{2\gamma s} \|x_s^T Q x_s\|_\tau ds)^{\frac{1}{2}} \\
&\leq \varepsilon_7 e^{\gamma(t-\tau)} E \left\| x^T{}_s Q x_s \right\|_\tau \\
&\quad + \tfrac{32}{\varepsilon_7} e^{-\gamma(t-\tau)} \lambda_{\max}(Q^{-\frac{1}{2}} B_1^T Q B_1 Q^{-\frac{1}{2}}) \int_{t-\tau}^{t} e^{2\gamma s} \left\| x_s{}^T Q x_s \right\|_\tau ds, \\
&2E(\sup_{t-\tau \leq r \leq t} \int_{t_0}^{r} e^{\gamma s} [x^T(s) Q B_2 \int_{-\tau}^{0} x(s+\theta) d\mu(\theta)] dB_s) \\
&\leq 2\sqrt{32} E(\int_{t-\tau}^{t} e^{2\gamma s} \left| x^T(s) Q x(s) \right| \left| \int_{-\tau}^{0} x^T(s+\theta) d\mu(\theta) B_2^T Q B_2 \int_{-\tau}^{0} x(s+\theta) d\mu(\theta) \right| ds)^{\frac{1}{2}} \\
&\leq 2\sqrt{32} E(\left\| x_s{}^T Q x_s \right\|_\tau \int_{t-\tau}^{t} \lambda_{\max}(Q^{-\frac{1}{2}} B_2^T Q B_2 Q^{-\frac{1}{2}}) e^{2\gamma s} \left\| x^T{}_s Q x_s \right\|_\tau ds)^{\frac{1}{2}} \\
&\leq \varepsilon_8 e^{\gamma(t-\tau)} E \left\| x^T{}_s Q x_s \right\|_\tau \\
&\quad + \tfrac{32}{\varepsilon_8} e^{-\gamma(t-\tau)} \lambda_{\max}(Q^{-\frac{1}{2}} B_2^T Q B_2 Q^{-\frac{1}{2}}) \int_{t-\tau}^{t} e^{2\gamma s} \left\| x^T{}_s Q x_s \right\|_\tau ds.
\end{aligned}$$

If $t \geq t_0 + \tau$,

$$(1-\varepsilon_6-\varepsilon_7-\varepsilon_8)e^{\gamma(t-\tau)}E\|x_t^TQx_t\|_\tau$$
$$\leq C_1e^{\gamma t_0}E\|\xi^TQ\xi\|_\tau + (C_2+C_3)\int_{t_0}^{t} e^{\gamma s}E\|x_s^TQx_s\|_\tau ds$$
$$+C_4e^{-\gamma(t-\tau)}\int_{t_0}^{t} e^{2\gamma s}E\|x_s^TQx_s\|_\tau ds,$$

where

$$C_4 = 32(\tfrac{1}{\varepsilon_6}\lambda_{\max}(Q^{-1/2}B_0^TQB_0Q^{-\frac{1}{2}}) + \tfrac{1}{\varepsilon_7}\lambda_{\max}(Q^{-\frac{1}{2}}B_1^TQB_1Q^{-\frac{1}{2}})$$
$$+\tfrac{1}{\varepsilon_8}\lambda_{\max}(Q^{-\frac{1}{2}}B_2^TQB_2Q^{-\frac{1}{2}})).$$

If $t_0 \leq t \leq t_0 + \tau$.

$$e^{\gamma(t-\tau)}E\|x_t^TQx_t\|_\tau \leq e^{\gamma t_0}E(\|\xi^TQ\xi\|_\tau + \sup_{t_0\leq r\leq t}\left|x^T(r)Qx(r)\right|)$$
$$\leq e^{\gamma t_0}E\left\|\xi^TQ\xi\right\|_\tau + E\sup_{t_0\leq r\leq t}(e^{\gamma r}\left|x^T(r)Qx(r)\right|)$$
$$\leq (1+C_1)e^{\gamma t_0}E\|\xi^TQ\xi\|_\tau + (C_2+C_3)\int_{t_0}^{t} e^{\gamma s}E\|x_s^TQx_s\|_\tau ds$$
$$+C_4e^{-\gamma(t-\tau)}\int_{t_0}^{t} e^{2\gamma s}E\|x_s^TQx_s\|_\tau ds + C_4e^{-\gamma(t_0-\tau)}\int_{t_0-\tau}^{t_0} e^{2\gamma s}E\|x_s^TQx_s\|_\tau ds$$
$$+(\varepsilon_6+\varepsilon_7+\varepsilon_8)e^{\gamma(t-\tau)}E\|x_t^TQx_t\|_\tau.$$

Set $\varepsilon_6 = \varepsilon_7 = \varepsilon_8 = 1/6$, we obtain

$$e^{\gamma(t-\tau)}E\|x_t^TQx_t\|_\tau \leq M_1 + M_2\int_{t_0}^{t} e^{\gamma s}E\|x_s^TQx_s\|ds \leq M,$$

where

$$M_1 = \begin{cases} 2(1+C_1)e^{\gamma t_0}E\|\xi^TQ\xi\|_\tau, & t_0 \leq t \leq t_0+\tau, \\ 2(1+C_1)e^{\gamma t_0}E\left\|\xi^TQ\xi\right\|_\tau & \\ +2C_4e^{-\gamma(t_0-\tau)}\int_{t_0-\tau}^{t_0} e^{2\gamma s}E\|x_s^TQx_s\|ds, & t \geq t_0+\tau, \end{cases}$$

$$M_2 = 2(C_2+C_3+C_4e^{\gamma\tau}),$$

$$M = M_1 + M_2Ce^{\gamma t_0}E\left\|\xi^TQ\xi\right\|_\tau.$$

And right now we complete the proof of this theorem.

Apply Theorem 3.2. Now we are capable to cope with such system with interval matrix coefficient, just for all matrices belong to the interval the sufficient condition of Theorem (3.2) must be satisfied and here we prove the following result.

Theorem 3.3 *If there exists a symmetric positive-definite matrix Q such that*

$$\begin{aligned}
&2[\lambda_{max}(Q^{-\frac{1}{2}}A_1^TQA_1Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{min}(Q)}(2\|A_1\|\|A_{1m}\|+\|A_{1m}\|^2)]^{\frac{1}{2}}\\
&+2[(\lambda_{max}(Q^{-\frac{1}{2}}A_2^TQA_2Q^{-\frac{1}{2}}))+\tfrac{\|Q\|}{\lambda_{min}(Q)}(2\|A_2\|\|A_{2m}\|+\|A_{2m}\|^2)\varpi_A(\lambda)]^{\frac{1}{2}}\\
&+\{[\lambda_{max}(Q^{-\frac{1}{2}}B_0^TQB_0Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{min}(Q)}(2\|B_0\|\|B_{0m}\|+\|B_{0m}\|^2)]^{\frac{1}{2}}\\
&+[\lambda_{max}(Q^{-\frac{1}{2}}B_1^TQB_1Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{min}(Q)}(2\|B_1\|\|B_{1m}\|+\|B_{1m}\|^2)]^{\frac{1}{2}}\\
&+[(\lambda_{max}(Q^{-\frac{1}{2}}B_2^TQB_2Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{min}(Q)}(2\|B_2\|\|B_{2m}\|+\|B_{2m}\|^2))\varpi_B(\lambda)]^{\frac{1}{2}}\}^2\\
&\le -\lambda_{max}(Q^{-\frac{1}{2}}(QA_0+A_0^TQ)Q^{-\frac{1}{2}})-\tfrac{2\|A_{0m}\|\|Q\|}{\lambda_{min}(Q)}.
\end{aligned} \tag{14}$$

Then equation (2) is exponentially stable in $L^2(\Omega, C([-\tau,0];R^n))$ and moreover, it is almost surely exponentially stable.

Before proving the theorem, we first give some lemmas [18].

Lemma 3.2 *Let Q be a positive-definite symmetric matrix. Then*

$$\|Q^{-\frac{1}{2}}\|\|Q^{\frac{1}{2}}\| \le \frac{\|Q\|}{\lambda_{min}(Q)}.$$

Lemma 3.3 *Let Q be a positive-definitive matrix and A be an $n\times n$ matrix. Then*

$$\lambda_{max}(Q^{-\frac{1}{2}}(QA+A^TQ)Q^{-\frac{1}{2}}) \le \frac{2\|A\|\|Q\|}{\lambda_{min}(Q)}.$$

Lemma 3.4 *If $\Delta A\in[-A_m, A_m]$, then $\|\Delta A\|\le\|A_m\|$.*

Lemma 3.5 *Let Q be a positive-definitive matrix, B be an $n\times n$ matrix and $\Delta B\in[-B_m, B_m]$. Then*

$$\begin{aligned}
&\lambda_{\max}(Q^{-1/2}(B^TQ\Delta B+(\Delta B)^TQB+(\Delta B)^TQ(\Delta B))Q^{-1/2})\\
&\le \tfrac{2\|B\|\|Q\|\|B_m\|}{\lambda_{\min}(Q)}+\tfrac{\|Q\|\|B_m\|^2}{\lambda_{\min}(Q)}.
\end{aligned}$$

Proof of Theorem 3.3. In order to guarantee the exponential stability of the interval system (2), we should show that the condition (3) of Theorem 3.2 holds for all the matrix coefficients $\Delta A_1\in[-A_{1m}, A_{1m}]$, $\Delta A_2\in[-A_{2m}, A_{2m}]$, $\Delta B_0\in[-B_{0m}, B_{0m}]$, $\Delta B_1\in[-B_{1m}, B_{1m}]$, $\Delta B_2\in[-B_{2m}, B_{2m}]$, i.e.

$$2\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}(A_1+\Delta A_1)^T Q(A_1+\Delta A_1)Q^{-\frac{1}{2}})}$$

$$+2\sqrt{\varpi_A(\lambda)\lambda_{max}(Q^{-\frac{1}{2}}(A_2+\Delta A_2)^T Q(A_2+\Delta A_2 Q^{-\frac{1}{2}})}$$

$$+(\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}(B_0+\Delta B_0)^T Q(B_0+\Delta B_0)Q^{-\frac{1}{2}})}$$

$$+\sqrt{\lambda_{max}(Q^{-\frac{1}{2}}(B_1+\Delta B_1)^T Q(B_1+\Delta B_1)Q^{-\frac{1}{2}})}$$

$$+\sqrt{\varpi_B(\lambda)\lambda_{max}(Q^{-\frac{1}{2}}(B_2+\Delta B_2)^T Q(B_2+\Delta B_2)Q^{-\frac{1}{2}})})^2$$

$$< -\lambda_{max}(Q^{-\frac{1}{2}}(Q(A_0+\Delta A_0)+(A+\Delta A_0)^T Q)Q^{-\frac{1}{2}}),$$

where $\varpi_A(\lambda)=\int_{-\tau}^{0} e^{-\lambda\theta}d\mu(\theta), \quad \varpi_B(\lambda)=\int_{-\tau}^{0} e^{-\lambda\theta}d\nu(\theta)$.

According to Lemma 3.3 and Lemma 3.4, we note that

$$-\lambda_{max}(Q^{-\frac{1}{2}}(Q(A_0+\Delta A_0)+(A+\Delta A_0)^T Q)Q^{-\frac{1}{2}})$$

$$\geq -\lambda_{max}(Q^{-\frac{1}{2}}(QA_0+A_0^T Q)Q^{-\frac{1}{2}})-\tfrac{2\|A_{0m}\|\|Q\|}{\lambda_{min}(Q)} \triangleq I_1.$$

Using Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned}&\lambda_{max}(Q^{-\frac{1}{2}}(A_1+\Delta A_1)^T Q(A_1+\Delta A_1)Q^{-\frac{1}{2}})\\&<\lambda_{max}(Q^{-\frac{1}{2}}A_1^T QA_1Q^{-\frac{1}{2}})+\lambda_{max}(Q^{-\frac{1}{2}}(A_1^T Q\Delta A_1+(\Delta A_1)^T QA_1\\&\quad+(\Delta A_1)^T Q\Delta A_1)Q^{-\frac{1}{2}})\\&\leq \lambda_{max}(Q^{-\frac{1}{2}}A_1^T QA_1Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{min}(Q)}(2\|A_1\|\|A_{1m}\|+\|A_{1m}\|^2).\end{aligned}$$

Then

$$\begin{aligned}&2\sqrt{\lambda_{\max}(Q^{-\frac{1}{2}}(A_1+\Delta A_1)^T Q(A_1+\Delta A_1)Q^{-\frac{1}{2}})}\\&+2\sqrt{\varpi_A(\lambda)\lambda_{\max}(Q^{-\frac{1}{2}}(A_2+\Delta A_2)^T Q(A_2+\Delta A_2)Q^{-\frac{1}{2}})}\\&+(\sqrt{\lambda_{\max}(Q^{-\frac{1}{2}}(B_0+\Delta B_0)^T Q(B_0+\Delta B_0)Q^{-\frac{1}{2}})}\\&+\sqrt{\lambda_{\max}(Q^{-\frac{1}{2}}(B_1+\Delta B_1)^T Q(B_1+\Delta B_1)Q^{-\frac{1}{2}})}\\&+\sqrt{\varpi_B(\lambda)\lambda_{\max}(Q^{-\frac{1}{2}}(B_2+\Delta B_2)^T Q(B_2+\Delta B_2)Q^{-\frac{1}{2}})})^2\end{aligned}$$

$$\leq 2[\lambda_{\max}(Q^{-\frac{1}{2}}A_1^T QA_1Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_1\|\|A_{1m}\|+\|A_{1m}\|^2)]^{\frac{1}{2}}$$

$$+2[(\lambda_{\max}(Q^{-\frac{1}{2}}A_2^T QA_2Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_2\|\|A_{2m}\|+\|A_{2m}\|^2))\varpi_A(\lambda)]^{\frac{1}{2}}$$

$$+\{[\lambda_{\max}(Q^{-\frac{1}{2}}B_0^T QB_0Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_0\|\|B_{0m}\|+\|B_{0m}\|^2)]^{\frac{1}{2}}$$

$$+[\lambda_{\max}(Q^{-\frac{1}{2}}B_1^T QB_1Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_1\|\|B_{1m}\|+\|B_{1m}\|^2)]^{\frac{1}{2}}$$

$$+[(\lambda_{\max}(Q^{-\frac{1}{2}}B_2^T QB_2Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_2\|\|B_{2m}\|+\|B_{2m}\|^2))\varpi_B(\lambda)]^{\frac{1}{2}}\}^2 \triangleq I_2.$$

If $I_1 \geq I_2$, we can conclude that the matrix coefficients of interval type satisfy the condition of Theorem 3.2, which leads to the exponential stability of the stochastic interval system.

It is not hard to generalize the result of the theorems to the multiple time delays case. Without any details of proof, we directly present the conclusion as follows.

Theorem 3.4 *Consider the following system*

$$\begin{aligned} dx(t) &= [(A_0+\Delta A_0)x(t)+\sum_{i=1}^{N}(A_{1i}+\Delta A_{1i})x(t-\tau_i) \\ &\quad +(A_{2i}+\Delta A_{2i})\int_{-\tau_i}^{0} x(t+\theta)d\mu(\theta)]dt+[(B_0+\Delta B_0)x(t) \\ &\quad +\sum_{i=1}^{N}(B_{1i}+\Delta B_{1i})x(t-\tau_i)+(B_{2i}+\Delta B_{2i})\int_{-\tau_i}^{0} x(t+\theta)d\nu(\theta)]dB_t. \end{aligned} \tag{15}$$

If there exists a symmetric positive-definite matrix Q such that

$$\begin{aligned} &2\sum_{i=1}^{N}\{[\lambda_{\max}(Q^{-\frac{1}{2}}A_{1i}^TQA_{1i}Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_{1i}\|\|A_{1im}\|+\|A_{1im}\|^2)]^{\frac{1}{2}} \\ &+[(\lambda_{\max}(Q^{-\frac{1}{2}}A_{2i}^TQA_{2i}Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|A_{2i}\|\|A_{2im}\|+\|A_{2im}\|^2))\varpi_A(\lambda)]^{\frac{1}{2}}\} \\ &+\{[\lambda_{\max}(Q^{-\frac{1}{2}}B_0^TQB_0Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_0\|\|B_{0m}\|+\|B_{0m}\|^2)]^{\frac{1}{2}} \\ &+\sum_{i=1}^{N}[\lambda_{\max}(Q^{-\frac{1}{2}}B_{1i}^TQB_{1i}Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_{1i}\|\|B_{1im}\|+\|B_{1im}\|^2)]^{\frac{1}{2}} \\ &+[(\lambda_{\max}(Q^{-\frac{1}{2}}B_{2i}^TQB_{2i}Q^{-\frac{1}{2}})+\tfrac{\|Q\|}{\lambda_{\min}(Q)}(2\|B_{2i}\|\|B_{2im}\|+\|B_{2im}\|^2))\varpi_B(\lambda)]^{\frac{1}{2}}\}^2 \\ &\leq -\lambda_{\max}(Q^{-\frac{1}{2}}(QA_0+A_0^TQ)Q^{-\frac{1}{2}})-\tfrac{2\|A_{0m}\|\|Q\|}{\lambda_{\min}(Q)}. \end{aligned}$$

Then the system (15) is exponentially stable in $L^2(\Omega, C([-\tau,0],R^n))$ and moreover, it is almost surely exponentially stable.

4 Example

In this section, we'll give a simple example to illustrate our result of Theorem 3.3.

Consider the following system

$$\begin{aligned} dx(t) &= [(A_0+\Delta A_0)x(t)+(A_1+\Delta A_1)x(t-\tau) \\ &+ (A_2+\Delta A_2)\int_{-\tau}^{0} x(t+\theta)d\mu(\vartheta)]dt+[(B_0+\Delta B_0)x(t) \\ &+ (B_1+\Delta B_1)x(t-\tau)+(B_2+\Delta B_2)\int_{-\tau}^{0} x(t+\theta)d\nu(\vartheta)]dB_t, \end{aligned} \tag{16}$$

where $\Delta A_0 \in [-A_{0m}, A_{0m}]$, $\Delta A_1 \in [-A_{1m}, A_{1m}]$, $\Delta A_2 \in [-A_{2m}, A_{2m}]$, $\Delta B_0 \in [-B_{0m}, B_{0m}]$, $\Delta B_1 \in [-B_{1m}, B_{1m}]$, $\Delta B_2 \in [-B_{2m}, B_{2m}]$. In order to simplify the computation, we set $Q = I$, in consequence, condition (14) can be simplify as

$$\begin{aligned} &2[\lambda_{max}(A_1^T A_1) + (2\|A_1\|\|A_{1m}\| + \|A_{1m}\|^2)]^{1/2} \\ &+2[(\lambda_{max}(A_2^T A_2) + (2\|A_2\|\|A_{2m}\| + \|A_{2m}\|^2)\varpi_A(\lambda)]^{1/2} \\ &+\{[\lambda_{max}(B_0^T B_0) + (2\|B_0\|\|B_{0m}\| + \|B_{0m}\|^2)]^{1/2} \\ &+[\lambda_{max}(B_1^T B_1) + (2\|B_1\|\|B_{1m}\| + \|B_{1m}\|^2)]^{1/2} \\ &+[(\lambda_{max}(B_2^T B_2) + (2\|B_2\|\|B_{2m}\| + \|B_{2m}\|^2))\varpi_B(\lambda)]^{1/2}\}^2 \\ &\leq -\lambda_{max}(A_0 + A_0^T) - 2\|A_{0m}\|, \end{aligned} \tag{17}$$

where

$$A_0 = \begin{pmatrix} -50 & -3 \\ 3 & -22 \end{pmatrix}, A_{0m} = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0.5 \\ 0.25 & 0.75 \end{pmatrix},$$
$$A_{1m} = \begin{pmatrix} 0 & 0 \\ 0.5 & 0.5 \end{pmatrix}, A_2 = \begin{pmatrix} 0.75 & 0.5 \\ 0.25 & 0.25 \end{pmatrix}, A_{2m} = \begin{pmatrix} 0.25 & 0.5 \\ 0.25 & 0.25 \end{pmatrix},$$
$$B_0 = \begin{pmatrix} 0.75 & 0.5 \\ 0.25 & 0.25 \end{pmatrix}, B_{0m} = \begin{pmatrix} 0.25 & 0.5 \\ 0.25 & 0.25 \end{pmatrix}, B_1 = \begin{pmatrix} 0.25 & 0.75 \\ 0.25 & 0.25 \end{pmatrix},$$
$$B_{1m} = \begin{pmatrix} 0.25 & 0.25 \\ 0.5 & 0.25 \end{pmatrix}, B_2 = \begin{pmatrix} 0.75 & 0.5 \\ 0.25 & 0.25 \end{pmatrix}, B_{2m} = \begin{pmatrix} 0.25 & 0.5 \\ 0.25 & 0.25 \end{pmatrix}.$$

It can easily be computed that $\lambda_{\max}(A_0 + A_0^T) = -44, \|A_{0m}\| = 1.281, \|A_1\| = 1.279, \|A_{1m}\| = 0.354, \|A_2\| = 0.966, \|A_{2m}\| = 0.655, \|B_0\| = 0.966, \|B_{0m}\| = 0.655, \|B_1\| = 1.189, \|B_{1m}\| = 0.655, \|B_2\| = 0.966, \|B_{2m}\| = 0.655$, and we set $\mu(\theta) = \nu(\theta) = \frac{\theta}{\tau}$, then $\varpi_A(\lambda) = \varpi_B(\lambda) = \int_{-\tau}^{0} e^{-\lambda\theta} d(\frac{\theta}{\tau}) = \frac{e^{\lambda\tau}-1}{\lambda\tau}$, here we note that $\lambda = -\lambda_{\max}(A_0 + A_0^T) = 44$, so we can choose a sufficient small delay τ such that $\varpi_A(\lambda) = \varpi_B(\lambda) = 1.1$. Hence condition (14) is satisfied. Therefore we can conclude that (16) is exponentially sable in $L^2(\Omega, C([-\tau, 0]; R^n))$ and moreover, it is almost surely exponentially stable.

5 Conclusion

In this paper, exponential stability of stochastic interval systems with time delays is studied. Using Itô formula and inequality techniques, some sufficient conditions are derived and at last, a simple example is given to illustrate our result.

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