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Existence of the Solution for Discontinuous Fuzzy Integro-differential Equations and Strong Fuzzy Henstock Integrals

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Abstract: In this paper, we use convergence theorem and the properties of strong fuzzy Henstock integrals to establish some existence theorems of solution for a kind of the discontinuous fuzzy integro-differential equations. The results are generalizations of earlier investigation for continuous fuzzy systems.

Keywords: *fuzzy number; existence of solution; discontinuous fuzzy integro-differential equations; strong fuzzy Henstock integrals.*

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1 Introduction

Differential equations are used for modeling of various physical phenomena. Unfortunately, many problems are dynamical and too complicated and accurate differential equation model for such problems requires complex and time consuming algorithms hardly implementable in practice. Thus, a usage of fuzzy mathematics seems to be appropriate. In recent years, the fuzzy set theory introduced by Zadeh [5] has emerged as an interesting and fascinating branch of pure and applied sciences. The applications of fuzzy set theory can be found in many branches of science such as physical, mathematical, differential equations and information science.

The Cauchy problems for fuzzy differential equations have been studied by several authors [16,21–23,25,26] on the metric space (E^n, D) of normal fuzzy convex set with the distance D given by the maximum of the Hausdorff distance between the corresponding

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level sets. In [23], the author has proved the Cauchy problem has a uniqueness result if f was continuous and bounded. In [16, 22], the authors presented a result for uniqueness of solution when f satisfies a Lipschitz condition. For a general reference to fuzzy differential equations, see a recent book by Lakshmikantham and Mohapatra [27] and references therein. In 2002, Xue and Fu [28] established solutions to fuzzy differential equations with right-hand side functions satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets. However, there are discontinuous systems in which the right-hand side functions $f : [a, b] \times E^n \to E^n$ are not integrable in the sense of Kaleva [16] on certain intervals and their solutions are not absolute continuous functions. So, in this paper, we will use the strong fuzzy Henstock integral, which is nonabsolute integrable.

It is well known that the Henstock integral is designed to integrate highly oscillatory functions which the Lebesgue integral fails to do. It is known as nonabsolute integration and is a powerful tool. It is well-known that the Henstock integral includes the Riemann, improper Riemann, Lebesgue and Newton integrals [2, 3]. Though such an integral was defined by Denjoy in 1912 and also by Perron in 1914, it was difficult to handle using their definitions. But with the Riemann-type definition introduced more recently by Henstock [2] in 1963 and also independently by Kurzweil [3], the definition is now simple and furthermore the proof involving the integral also turns out to be easy. For more detailed results about the Henstock integral, we refer to [4]. Recently, Wu and Gong [14, 15] have combined the fuzzy set theory and nonabsolute integration theory, and discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva [16] integration. In order to complete the theory of fuzzy calculus and to meet the solving need of transferring a fuzzy differential equation into a fuzzy integral equation, we [17, 18] has defined the strong fuzzy Henstock integrals and discussed some of their properties and the controlled convergence theorem.

In this paper, according to the idea of [1, 29] and using the concept of generalized differentiability [19], the operator j which is the isometric embedding from (E^n, D) onto its range in the Banach space X and the controlled convergence theorems for the fuzzy Henstock integrals, we will deal with the Cauchy problem of discontinuous fuzzy integro-differential equations as following:

$$\begin{cases} x'(t) = \tilde{f}(t, x(t), \int_0^t \tilde{k}(t, s, x(s))\mathrm{d}s), \\ x(0) = x_0, \quad t \in I_a = [0, a], a > 0, x_0 \in E^n, \end{cases} \tag{1}$$

where the integral is taken in the sense of strong fuzzy Henstock integral.

To make our analysis possible, we will first recall some basic results of fuzzy numbers and give some definitions of absolutely continuous fuzzy-number-valued function. In addition, we present the concept of generalized differentiability and we present the concept of fuzzy Henstock integrals and the controlled convergence theorem for the fuzzy Henstock integrals. In Section 3, we deal with the Cauchy problem of discontinuous fuzzy integro-differential equations. And in Section 4, we present some concluding remarks.

2 Preliminaries

2.1 Fuzzy number theory

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric [30]:

$$d_H(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} \parallel a-b \parallel, \sup_{b\in B}\inf_{a\in A} \parallel b-a \parallel\}.$$

Denote $E^n = \{u : R^n \to [0,1] | u$ satisfies (1)–(4) below$\}$ is a fuzzy number space. where

(1) u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2) u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

(3) u is upper semi-continuous,

(4) $[u]^0 = cl\{x \in R^n | u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n | u(x) \geq \alpha\}$. Then from above (1)-(4), it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha < 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows [30]:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [ku]^\alpha = k[u]^\alpha,$$

where $u, v \in E^n$ and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0, \infty)$

$$D(u,v) = \sup\{d_H([u]^\alpha, [v]^\alpha) : \alpha \in [0,1]\},$$

where d is the Hausdorff metric defined in $P_k(R^n)$. Then it is easy to see that D is a metric in E^n. Using the results [31], we know that:

(1) (E^n, D) is a complete metric space,

(2) $D(u+w, v+w) = D(u,v)$ for all $u, v, w \in E^n$,

(3) $D(\lambda u, \lambda v) = |\lambda| D(u,v)$ for all $u, v, w \in E^n$ and $\lambda \in R$.

The metric space (E^n, D) has a linear structure, it can be imbedded isomorphically as a cone in a Banach space of function $u^* : I \times S^{n-1} \longrightarrow R$, where S^{n-1} is the unit sphere in R^n, with an imbedding function $u^* = j(u)$ defined by

$$u^*(r,x) = \sup_{\alpha\in[u]^\alpha} < \alpha, x >$$

for all $< r, x > \in I \times S^{n-1}$ (see [31]).

Theorem 2.1 [31] *There exists a real Banach space X such that E^n can be imbedding as a convex cone C with vertex 0 into X. Furthermore the following conclusions hold:*

(1) the imbedding j is isometric,

(2) addition in X induces addition in E^n,

(3) multiplication by nonnegative real number in X induces the corresponding operation in E^n,

(4) $C - C$ is dense in X,

(5) C is closed.

It is well-known that the H-derivative for fuzzy-number-functions was initially introduced by Puri and Ralescu [25] and it is based on the condition (H) of sets. We note that this definition is fairly strong, because the family of fuzzy-number-valued functions H-differentiable is very restrictive. For example, the fuzzy-number-valued function $\tilde{f} : [a,b] \to \mathbb{R}_{\mathcal{F}}$ defined by $\tilde{f}(x) = C \cdot g(x)$, where C is a fuzzy number, $\cdot$ is the scalar multiplication (in the fuzzy context) and $g : [a, b] \to \mathbb{R}^+$, with $g'(t_0) < 0$, is not H-differentiable in t_0 (see [19, 20]). To avoid the above difficulty, in this paper we consider a more general definition of a derivative for fuzzy-number-valued functions enlarging the class of differentiable fuzzy-number-valued functions, which has been introduced in [19].

Definition 2.1 [19] Let $\tilde{f} : (a, b) \to E^n$ and $x_0 \in (a, b)$. We say that $\tilde{f}$ is differentiable at x_0, if there exists an element $\tilde{f}'(t_0) \in E^n$, such that:

(1) for all $h > 0$ sufficiently small, there exist $\tilde{f}(x_0+h) -_H \tilde{f}(x_0)$, $\tilde{f}(x_0) -_H \tilde{f}(x_0-h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{\tilde{f}(x_0+h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0-h)}{h} = \tilde{f}'(x_0)$$

or

(2) for all $h > 0$ sufficiently small, there exist $\tilde{f}(x_0) -_H \tilde{f}(x_0+h)$, $\tilde{f}(x_0-h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0+h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0-h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

or

(3) for all $h > 0$ sufficiently small, there exist $\tilde{f}(x_0+h) -_H \tilde{f}(x_0)$, $\tilde{f}(x_0-h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0+h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0-h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0)$$

or

(4) for all $h > 0$ sufficiently small, there exist $\tilde{f}(x_0) -_H \tilde{f}(x_0+h)$, $\tilde{f}(x_0) -_H \tilde{f}(x_0-h)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0+h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0-h)}{h} = \tilde{f}'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}\cdot$ and $-\frac{1}{h}\cdot$, respectively).

2.2 The strong Henstock integrals of fuzzy-number-valued functions in E^n

In this section we define the strong Henstock integrals of fuzzy-number-valued functions in fuzzy number space E^n and we give some properties of this integral.

Definition 2.2 [18] A fuzzy-number-valued function $\tilde{f}$ will be termed piecewise additive on $[a, b]$ if there exists a division $T : a = a_0 < a_1 < \cdots < a_n = b$, such that $\tilde{f}(x)$ is additive on each $[a_i, a_{i+1}]$ $(i = 0, 1, \cdots, n-1)$.

Definition 2.3 [17, 18] A fuzzy-number-valued function $\tilde{f}$ is said to be strong Henstock integrable on $[a, b]$ if there exists a piecewise additive fuzzy-number-valued function $\tilde{F}$ on $[a, b]$ such that for every $\varepsilon > 0$ there is a function $\delta(\xi) > 0$ and for any δ-fine division $P = \{[x_{i-1}, x_i], \xi_i\}_{i=1}^n$ of $[a, b]$, we have

$$\sum_{i\in K_n} D(\tilde{f}(\xi_i)(x_i - x_{i-1}), \tilde{F}([x_{i-1}, x_i])) + \sum_{j\in I_n} D(\tilde{f}(\xi_j)(x_j - x_{j-1}), (-1)\cdot\tilde{F}([x_j, x_{j-1}])) < \varepsilon,$$

where $K_n = \{i \in \{1, 2, \cdot\cdot, n\}$ such that $\tilde{F}([x_{i-1}, x_i])$ is a fuzzy number and $I_n = \{j \in \{1, 2, \cdot\cdot, n\}$ such that $\tilde{F}([x_j, x_{j-1}])$ is a fuzzy number. We write $\tilde{f} \in SFH[a, b]$.

Definition 2.4 [18]A fuzzy-number-valued function $\tilde{F}$ defined on $X \subset [a,b]$ is said to be $AC^*(X)$ if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every finite sequence of non-overlapping intervals $\{[a_i, b_i]\}$, satisfying $\Sigma_{i=1}^n |b_i - a_i| < \eta$ where $a_i, b_i \in X$ for all i we have

$$\sum \omega(\tilde{F}, [a_i, b_i]) < \varepsilon,$$

where ω denotes the oscillation of $\tilde{F}$ over $[a_i, b_i]$, i.e.,

$$\omega(\tilde{F}, [a_i, b_i]) = \sup\{D(\tilde{F}(y), \tilde{F}(x)); x, y \in [a_i, b_i]\}.$$

Definition 2.5 [18] A fuzzy-number-valued function $\tilde{F}$ is said to be ACG^* on X if X is the union of a sequence of closed sets $\{X_i\}$ such that on each X_i, $\tilde{F}$ is $AC^*(X_i)$.

For the strong fuzzy Henstock integrable we have the following theorems.

Theorem 2.2 *Let $\tilde{f} : [a,b] \to E^n$. If $\tilde{f} = 0$ a.e. on $[a,b]$, then $\tilde{f}$ is SFH integrable on $[a,b]$ and $\int_a^b \tilde{f}(t)\mathrm{d}t = 0$.*

Theorem 2.3 *Let $\tilde{f} : [a,b] \to E^n$ be SFH integrable on $[a,b]$ and let $\tilde{F}(x) = \int_a^x \tilde{f}(t)\mathrm{d}t$ for each $x \in [a,b]$. Then*
(a) the function $\tilde{F}$ is continuous on $[a,b]$;
(b) the function $\tilde{F}$ is differentiable a.e. on $[a,b]$ and $\tilde{F}' = f$;
(c) $\tilde{f}$ is measurable.

Theorem 2.4 *(Controlled Convergence Theorem)* [18] *Suppose $\{\tilde{f}_n\}$ is a sequence of SFH integrable functions on $[a,b]$ satisfying the following conditions:*
(1) $\tilde{f}_n(x) \to \tilde{f}(x)$ almost everywhere (a.e.) in $[a,b]$ as $n \to \infty$;
(2) the primitives $\tilde{F}_n$ of $\tilde{f}_n$ are ACG^* uniformly in n;
(3) the primitives $\tilde{F}_n$ converge uniformly on $[a,b]$;
then $\tilde{f}$ is also SFH integrable on $[a,b]$ and

$$\lim_{n\to\infty} \int_a^b \tilde{f}_n(x)\mathrm{d}x = \int_a^b \tilde{f}(x)\mathrm{d}x.$$

3 The Existence of Solutions for Discontinuous Fuzzy Integro-differential Equations

In this section we prove the existence theorem for the problem (1).

For any bounded subset A of the Banach space X we denote by $\alpha(A)$ the Kuratowski measure of non-compactness of A, i.e. the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter less than ε. For the properties of α we refer to [24] for example.

Lemma 3.1 [24] *Let $H \subset C(I_\gamma, X)$ be a family of strong equicontinuous functions. Then*

$$\alpha(H) = \sup_{t \in I_\gamma} \alpha(H(t)) = \alpha(H(I_\gamma)),$$

where $\alpha(H)$ denotes the Kuratowski measure of non-compactness in $C(I_\gamma, X)$ and the function $t \to \alpha(H(t))$ is continuous.

Theorem 3.1 [24] *Let D be a closed convex subset of X, and let F be a continuous function from D into itself. If for $x \in D$ the implication*

$$\bar{V} = c\bar{o}n(\{x\} \cup F(V)) \Rightarrow V$$

is relatively compact, then F has a fixed point.

We now give some useful definitions and results, which we will use throughout this paper.

Definition 3.1 A fuzzy-number-valued function $\tilde{f} : I_a \times E^n \longrightarrow E^n$ is L^1-Carathéodory if the following conditions hold:
(1) the fuzzy mapping $(x, y) \in E^n \times E^n$ is measurable for all $t \longrightarrow \tilde{f}(t, x, y)$;
(2) the fuzzy mapping $t \in I_a$ is continuous for all $(x, y) \longrightarrow \tilde{f}(t, x, y)$.

Definition 3.2 A fuzzy-number-valued function $\tilde{k} : I_a \times I_a \times E^n \longrightarrow E^n$ is L^1-Carathéodory if the following conditions hold:
(1) the fuzzy mapping $(t, s) \longrightarrow \tilde{k}(t, s, y)$ is measurable for all $y \in E^n$;
(2) the mapping $y \longrightarrow \tilde{k}(t, s, y)$ is continuous for all $(t, s) \in I_a \times I_a$.

Definition 3.3 A fuzzy-valued function $(t, s, z) \rightarrow h(t, s, z)$ defined on $\tilde{k} : I_a \times I_a \times E^n$ is a fuzzy Kamke function if h satisfied Carathéodory conditions, and for each fixed t, s, the function $z \rightarrow h(t, s, z)$ is nondecreasing and for each $q, 0 < q < a$, the function identically equal to zero is the unique continuous of the integral equation $z(t) = \int_0^t h(t, s, z(s))\mathrm{d}s$ defined on $[0, q)$.

Theorem 3.2 *If the fuzzy-number-valued function $\tilde{f} : I_a \longrightarrow E^n$ is (SFH) integrable, then*

$$\int_I \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv}\tilde{f}(I),$$

where $\overline{conv}\tilde{f}(I)$ is the closure of the convex of $\tilde{f}(I)$, I is an arbitrary subinterval of I_a, and $|I|$ is the length of I.

Proof. Because $j \circ \tilde{f}$ is abstract (H) integrable in a Banach space, by using the mean valued theorem of (H) integrals, we have

$$(H)\int_I j \circ \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv} j \circ \tilde{f}(I) = |I| \cdot j \circ \overline{conv}\tilde{f}(t).$$

In additional, there exists $(H)\int_I j \circ \tilde{f}(t)\mathrm{d}t = j \circ \int_I \tilde{f}(t)\mathrm{d}t$.

So, we have $j \circ \int_I \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv} j \circ \tilde{f}(I)$. And the set $\{|I| \cdot \overline{conv}\tilde{f}(I)\}$ is a closed convex set, we have

$$\int_I \tilde{f}(t)\mathrm{d}t \in |I| \cdot \overline{conv}\tilde{f}(I).$$

We shall consider the problem

$$x(t) = x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z$$

or

$$x(t) = x_0 + (-1) \cdot \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \quad t \in I_a, x_0 \in E^n, \tag{2}$$

where integrals are taken in the sense of (SFH).

To obtain the existence results it is necessary to define a notion of a solution.

Definition 3.4 An ACG^* fuzzy-valued function $x : I_a \to E^n$ is said to be a solution of problem (1) if it satisfies the following conditions: (1) $x(0) = x_0$;
(2) $x'(t) = \tilde{f}(t, x(t), \int_0^t \tilde{k}(t, s, x(s))\mathrm{d}s)$ for a.e. $t \in I_a$.

Definition 3.5 A continuous fuzzy-valued function $x : I_a \to E^n$ is said to be a solution of problem (2) if it satisfies

$$x(t) = x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z$$

or

$$x(t) = x_0 + (-1) \cdot \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z$$

for all $t \in I_a$.

Theorem 3.3 *Each solution $x(t)$ of problem* (1) *is equivalent to the solution of problem* (2).

Proof. Let $x(t)$ be a continuous solution of (1). By the definition, $x(t)$ is ACG^* function and $x(0) = x_0$. Since, for a.e. $t \in I_a$, we have $x'(t) = \tilde{f}((t, x(t), \int_0^t \tilde{k}(t, s, x(s))\mathrm{d}s)$ and the last is (SFH) integrable, it is differentiable *a.e.* Moreover,

$$\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z = \int_0^t x'(s)\mathrm{d}s = x(t) -_H x_0.$$

Thus (2) is satisfied.

In addition, we assume that $y(t)$ is ACG^* function and it is clear that $y(0) = x_0$. By the definition of (SFH) integrals there exists an ACG^* fuzzy-valued function $\tilde{g}$ such that $\tilde{g}(0) = x_0$, and $\tilde{g}'(t) = \tilde{f}((t, y(t), \int_0^t \tilde{k}(t, s, y(s))\mathrm{d}s)$, *a.e.*

Hence

$$\begin{aligned} y(t) &= x_0 + \int_0^t \tilde{f}(z, y(z), \int_0^z \tilde{k}(z, s, y(s))\mathrm{d}s)\mathrm{d}z \\ &= x_0 + \int_0^t \tilde{g}'(s)\mathrm{d}s = x_0 + \tilde{g}(t) -_H \tilde{g}(0) = \tilde{g}(t). \end{aligned}$$

We have $y = \tilde{g}$ and then $y'(t) = \tilde{f}(z, y(z), \int_0^z \tilde{k}(z, s, y(s)\mathrm{d}z)$.

For $x \in C(I_a, E^n)$, we define the metric of x by

$$H(x, \tilde{0}) = \sup_{t \in I_a} D(x, \tilde{0}).$$

Let

$$B = \{x \in C(I_a, E^n) | H(x, \tilde{0}) \leq H(x, \tilde{0}) + p, p > 0\}.$$

Obviously the set B is closed and convex. We define the operator $F : C(I_a, E^n) \to C(I_a, E^n)$ by:

$$F(x)(t) = x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \quad t \in I_a, x_0 \in E^n.$$

Let $\Gamma = \{F(x) \in C(I_a, E^n) | x \in B\}$.

Now we present the existence theorems for the problem (1) in a fuzzy number space E^n.

Theorem 3.4 *Assume that, for each ACG^* fuzzy-valued function $x : I_a \to E^n$, a fuzzy-number-valed function $\tilde{k}(\cdot, s, x(s)), \tilde{f}(\cdot, x(\cdot), \int_0^{(\cdot)} \tilde{k}(\cdot, s, x(s))\mathrm{d}s)$ are (SFH) integrable, $\tilde{f}$ and $\tilde{k}$ is L^1-Carathéodory function. Suppose that there exists a constant d such that*

$$\alpha(j \circ \tilde{f}(t, A, C)) \leq d \cdot \max\{\alpha(j \circ A), \alpha(j \circ C)\} \tag{3}$$

for each bounded subset $A, C \subset E^n$ and $t \in I_a$. Where α denotes the measure of non-compactness. Assume that there exists a continuous $g : I_a \times I_a \to R^+$ such that

$$\alpha(j \circ \tilde{k}(I, I, X)) \leq \sup_{s \in I} g(t, s)\alpha(j \circ X) \tag{4}$$

for each bounded subset $X \subset E^n$, and $t, s \in I, I \subset I_a$, and the zero function is the unique continuous solution of the inequality

$$p(t) \leq d \cdot c \cdot \sup_{z \in I_c} \int_0^c g(z, s)p(s)\mathrm{d}s \quad on \quad I_c. \tag{5}$$

Moreover, let Γ be equicontinuous, equibounded and uniformly ACG^* on I_a. Then there exists a solution of problem (1) on I_c for some $0 < c \leq a, d \cdot c < 1$.

Proof. By equicontinuous and equiboundedness of Γ, there exists a number $c, 0 < c \leq a$ such that

$$\begin{aligned}
&H(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
&= \sup_{t \in I_c} D(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
&\leq p
\end{aligned}$$

for fixed $p > 0, x \in B, t \in I_c$. By the assumption on the operator F, we have

$$\begin{aligned}
H(F(x)(t), \tilde{0}) &= H(x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
&= \sup_{t \in I_c} D(x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
&= \sup_{t \in I_c} D(x_0, \tilde{0}) + \sup D(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0}) \\
&\leq \sup_{t \in I_c} D(x_0, \tilde{0}) + l.
\end{aligned}$$

Using Theorem 2.3 we have F is continuous.

Suppose that $V \subset B$ satisfies the condition $\overline{V} = \overline{conv}(\{x\} \cup F(V))$ for some $x \in B$. Next, we will prove that V is relatively compact.

In fact, let $V(t) = \{v(t) \in E^n | v \in V\}$ for $t \in I_c$. Since V is equicontinuous, by Lemma 3.1, $t \to v(t) = \alpha(j \circ V(t))$ is continuous on I_c. For fixed $t \in I_c$ we divide the interval $[0,t]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = it/m, i = 0,1,2 \cdots, m$. And for fixed $z \in [0,t]$, we divide the interval $[0,z]$ into m parts: $0 = z_0 < z_1 < \cdots < z_m = z$, where $z_j = jz/m, j = 0,1,2,\cdots,m$.

Let $V([z_j, z_{j+1}]) = \{u(s) | u \in V, z_j \le s \le z_{j+1}\}, j = 0,1,2,\cdots,m-1$. By Lemma 3.1 and the continuity of v, there exists $s_j \in I_j = [z_j, z_{j+1}]$ such that

$$\alpha(j \circ V([z_j, z_{j+1}])) = \sup_{t \in I_c} \{\alpha(j \circ V(s)) | z_j \le s \le z_{j+1}\} := v(s_j).$$

By Theorem 3.2 and the properties of the (SFH) integral we have

$$\begin{aligned} F(u)(t) &= x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \tilde{f}(z, u(z), \sum_{j=0}^{m-1} \int_{z_j}^{z_{j+1}} \tilde{k}(z,s,u(s)) \mathrm{d}s) \mathrm{d}z \\ &\le x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i) \overline{conv} \tilde{f}(z, V(I_i), \sum_{j=0}^{m-1} (z_{j+1} - z_j) \overline{conv} \tilde{k}(z, I_j, V([z_j, z_{j+1}]))). \end{aligned}$$

Using (4), (5) and the properties of measure of noncompactness α, we have

$$\begin{aligned} &\alpha(j \circ F(V(t))) \\ &\le \sum_{i=0}^{m-1} (t_{i+1} - t_i) \alpha(j \circ \tilde{f}(z, V(I_i), \sum_{j=0}^{m-1} (z_{j+1} - z_j) \overline{conv} \tilde{k}(z, I_j, V([z_j, z_{j+1}])))) \\ &\le \sum_{i=0}^{m-1} (t_{i+1} - t_i) \cdot d \cdot \max\{\alpha(j \circ V(I_i)), \\ &\alpha(j \circ \sum_{j=0}^{m-1} (z_{j+1} - z_j) \overline{conv} \tilde{k}(z, I_j, V([z_j, z_{j+1}])))\}. \end{aligned}$$

We observe that

(1) if $\alpha(j \circ V(I_i)) > \alpha(j \circ \sum_{j=0}^{m-1} (z_{j+1} - z_j) \overline{conv} \tilde{k}(z, I_j, V([z_j, z_{j+1}])))$, then

$$\alpha(j \circ V) = \alpha(j \circ \overline{conv}(\{x\} \cup g(V))) \le \alpha(j \circ F(V)) < d \cdot c \cdot \alpha(j \circ V),$$

Because $d \cdot c < 1$, so $\alpha(j \circ V) < \alpha(j \circ V)$ is a contradiction;

(2) if $\alpha(j \circ V(I_i)) < \alpha(j \circ \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}\tilde{k}(z, I_j, V([z_j, z_{j+1}])))$, then

$$\begin{aligned}
\alpha(j \circ V) &< \sum_{i=0}^{m-1}(t_{i+1} - t_i) \cdot d \cdot \alpha(j \circ \sum_{j=0}^{m-1}(z_{j+1} - z_j)\overline{conv}\tilde{k}(z, I_j, V([z_j, z_{j+1}]))) \\
&\leq \sum_{i=0}^{m-1}(t_{i+1} - t_i) \cdot d \cdot \sum_{j=0}^{m-1}(z_{j+1} - z_j)\alpha(j \circ \tilde{k}(z, I_j, V([z_j, z_{j+1}]))) \\
&\leq \sum_{i=0}^{m-1}(t_{i+1} - t_i) \cdot d \cdot \sum_{j=0}^{m-1}(z_{j+1} - z_j) \sup_{s \in I_j} g(z, s)\alpha(j \circ V([z_j, z_{j+1}])) \\
&= d \cdot c \cdot \sum_{j=0}^{m-1}(z_{j+1} - z_j) \cdot g(z, p_j) j \circ (v(s_j)) \\
&= d \cdot c \cdot [\sum_{j=0}^{m-1}(z_{j+1} - z_j) \cdot g(z, p_j) j \circ (v(p_j)) \\
&+ \sum_{j=0}^{m-1}(z_{j+1} - z_j)(g(z, p_j)(j \circ v(s_j) - j \circ v(p_j)))].
\end{aligned}$$

By continuity of v we have $j \circ v(s_j) - j \circ v(p_j) < \varepsilon$ and $\varepsilon \to 0$ if $m \to \infty$, so

$$j \circ v(t) = \alpha(j \circ V(t)) \leq d \cdot c \cdot \sup_{z \in I_c} \int_0^c g(z, s) j \circ v(s) \mathrm{d}s.$$

By (6) we have $j \circ v(t) = \alpha(j \circ V(t)) = 0$ for $t \in I_c$.

Using Arzel*á*-Ascoli theorem, we have V is relatively compact. By Theorem 3.1 the operator F has a fixed point. This means that there exists a solution of the problem (1).

Theorem 3.5 *Assume that, for each ACG^* fuzzy-valued function $x : I_a \to E^n$, a fuzzy-number-valed function $\tilde{k}(\cdot, s, x(s)), \tilde{f}(\cdot, x(\cdot), \int_0^{(\cdot)} \tilde{k}(\cdot, s, x(s))\mathrm{d}s)$ are (SFH) integrable, $\tilde{f}$ and $\tilde{k}$ is L^1-Carathéodory function. Suppose that there exists a constant d such that*

$$\alpha(j \circ \tilde{f}(t, A, C)) \leq d \cdot \max\{\alpha(j \circ A), \alpha(j \circ C)\} \tag{6}$$

for each bounded subset $A, C \subset E^n$ and $t \in I_a$. Where α denotes the measure of noncompactness. Assume that

$$\alpha(j \circ \tilde{k}(t, s, j \circ X)) \leq h(t, s, \alpha(j \circ X)) \tag{7}$$

for each bounded subset $X \subset E^n$, and $0 \leq s \leq t \leq a$, where h is a Kamke function.

Moreover, Let Γ be equicontinuous, equibounded and uniformly ACG^* on I_a. Then there exists at least one solution of problem (1) on I_c for some $0 < c \leq a, d \cdot c < 1$.

Proof. By equicontinuous and equiboundedness of Γ, there exists a number $c, 0 <$

$c \leq a$ such that

$$\begin{aligned}&H(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})\\&= \sup_{t \in I_c} D(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})\\&\leq p\end{aligned}$$

for fixed $p > 0, x \in B, t \in I_c$. By the assumption on the operator F, we have

$$\begin{aligned}H(F(x)(t), \tilde{0}) &= H(x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})\\&= \sup_{t \in I_c} D(x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})\\&= \sup_{t \in I_c} D(x_0, \tilde{0}) + \sup D(\int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z, \tilde{0})\\&\leq \sup_{t \in I_c} D(x_0, \tilde{0}) + p.\end{aligned}$$

Using Theorem 2.3 we have F is continuous.

Suppose that $V \subset B$ satisfies the condition $\overline{V} = \overline{conv}(\{x\} \cup F(V))$ for some $x \in B$. Next, we will prove that V is relatively compact.

In fact, let $V(t) = \{v(t) \in E^n | v \in V\}$ for $t \in I_c$. Since V is equicontinuous, by Lemma 3.1, $t \to v(t) = \alpha(j \circ V(t))$ is continuous on I_c. For fixed $t \in I_c$ we divide the interval $[0, t]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = it/m, i = 0, 1, 2 \cdots, m$. We denote $T_i = [t_i, t_{i+1}]$ and fix $z \in I_c$. Let $\int_0^z \tilde{K}(s)\mathrm{d}s = \{\int_0^z x(s) : x \in \tilde{K}\}$ for any $\tilde{K} \subset C(I_c, E^n)$ and let $\tilde{k_z}$ denote the mapping defined by $\tilde{k_z}(x(s)) = \tilde{k}(z, s, x(s))$ for each $x \in B$ and $s \in I_c$. Obviously, $\tilde{k_z}(j \circ V(s)) = \tilde{k}(z, s, j \circ V(s))$.

Let

$$\begin{aligned}\tilde{F}(j \circ V(t)) &= \{\tilde{F}(x)(t) \in C(I_c, E^n) : x \in V, t \in I_c\}\\&= \{x_0 + \int_0^t \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z : x \in V, t \in I_c\}.\end{aligned}$$

By Theorem 3.2 and the properties of the (SFH) integral we have

$$\begin{aligned}F(x)(t) &= x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \tilde{f}(z, x(z), \int_0^z \tilde{k}(z, s, x(s))\mathrm{d}s)\mathrm{d}z\\&\in x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{conv}\tilde{f}(z, V(T_i), \int_0^z \tilde{k_z}(j \circ V(s))\mathrm{d}s).\end{aligned}$$

Therefore, $\tilde{F}(j \circ V(t)) \subset x_0 + \sum_{i=0}^{m-1}(t_{i+1} - t_i)\overline{conv}\tilde{f}(z, V(T_i), \int_0^z \tilde{k_z}(j \circ V(s))\mathrm{d}s)$.

Using (6), (7) and the properties of measure of noncompactness α, we have

$$\begin{aligned}
&\alpha(j \circ F(V(t)))\\
&\le \sum_{i=0}^{m-1}(t_{i+1}-t_i)\alpha j \circ (\tilde{f}(z, V(T_i)), \int_0^z \tilde{k_z}(j \circ V(s))\mathrm{d}s))\\
&\le \sum_{i=0}^{m-1}(t_{i+1}-t_i) \cdot d \cdot \max\{\alpha(j \circ V(T_i)), \alpha(\int_0^z \tilde{k_z}(j \circ V(s))\mathrm{d}s)\}.
\end{aligned}$$

We observe that

(1) if $\alpha(j \circ V(T_i)) > \alpha(j \circ \int_0^z \tilde{k_z}(V(s))\mathrm{d}s)$, then

$$\alpha(j \circ V) = \alpha(j \circ \overline{conv}(\{x\} \cup F(V))) \le \alpha(j \circ F(V)) < d \cdot c \cdot \alpha(j \circ V),$$

Because $d \cdot c < 1$, so $\alpha(j \circ V) < \alpha(j \circ V)$ is a contradiction;

(2)if $\alpha(j \circ V(T_i)) < \alpha(j \circ \int_0^z \tilde{k_z}(V(s))\mathrm{d}s)$, then

$$\begin{aligned}
\alpha(j \circ V) &< \sum_{i=0}^{m-1}(t_{i+1}-t_i) \cdot d \cdot \alpha(j \circ \int_0^z \tilde{k_z}(V(s))\mathrm{d}s) \le 2dc\alpha(j \circ \int_0^z \tilde{k_z}(V(s))\mathrm{d}s)\\
&\le 2dc \int_0^Z \alpha(j \circ \tilde{k}(z, s, V(s))\mathrm{d}s) \le 2dc \int_0^z h(z, s, V(s))\mathrm{d}s,
\end{aligned}$$

since $V = \overline{conv}(\{x\} \cup j \circ F(V))$, we have

$$v(t) = 2dc \int_0^z h(z, s, v(s))\mathrm{d}s.$$

Now, we apply a theorem of differential inequalities. We have $v(t) = \alpha(j \circ V(t)) = 0$. By Arzelá-Ascoli theorem, we have V is relatively compact. By Theorem 3.1 the operator F has a fixed point. This means that there exists a solution of the problem (1).

4 Conclusion

In this paper, we deal with the Cauchy problem of discontinuous fuzzy integro-differential equations involving the strong fuzzy Henstock integral in fuzzy number space. The function governing the equations is supposed to be discontinuous with respect to some variables and satisfy nonabsolute fuzzy integrability. Our result improves the result given in [16, 23, 26] and [28] (where uniform continuity was required), as well as those referred therein.

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