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Approximate Controllability of a Functional Differential Equation with Deviated Argument

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Abstract: This paper deals with the approximate controllability of a functional differential equation with deviated argument and finite delay. Sufficient condition for approximate controllability is proved under the assumption that the linear control system is approximately controllable; thereby removing the need to assume the invertibility of a controllability operator which fails to exist in infinite dimensional space if the generated semigroup is compact. Schauder fixed point theorem is used and the C_0 semigroup associated with mild solution has been replaced by the fundamental solution.

Keywords: *deviated argument; approximate controllability; fundamental solution; semilinear control system; delay; reachable set; Schauder fixed point theorem.*

Mathematics Subject Classification (2010): 34K30, 34K35, 93C25.

1 Introduction

In certain real world problems, delay depends not only on time but also on the unknown quantity. The differential equations with deviated arguments are generalization of delay differential equations in which the unknown quantity and its derivative appear in different values of their arguments. Functional differential equations with deviated argument model various control problems arising in the field of engineering, physics and so on. Many partial differential systems can be reduced to functional differential equations with deviated arguments, see for instance [3, 8, 15, 16]. Aftereffect, hereditary systems, equations with deviated arguments, etc. feature in several mathematical models. As a matter of fact delay differential systems are still resistant to many classical controllers.

In recent years, controllability of infinite dimensional systems has been extensively studied for various applications. The papers of Benchohra et al. [10] and Chang [19]

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discuss the exact controllability of functional systems with infinite delay. However, in these papers the invertibility of a controllability operator is assumed. As a consequence their approach fails in infinite dimensional spaces whenever the generated semigroup is compact. Also it is practically difficult to verify their condition directly. This is one of the motivations of our paper.

Controllability results are available in overwhelming majority of investigations for abstract differential delay systems (see [4–6, 9–11, 18–20]); rather than for functional differential equations with deviated arguments. It is interesting to note that approximate controllability problem for nonlinear dynamical systems with deviated argument has not been investigated thoroughly in literature. In an attempt to fill this gap we study the approximate controllability of the following control system using fixed point approach which removes the above restrictions.

However C.G. Gal [1] studied the existence and uniqueness of local and global solutions for initial value problem with deviated argument

$$u'(t) = Au(t) + f(t, u(t), u[\alpha(u(t), t)]), t \in R_+, u(0) = u_0.$$

Muslim and Bahuguna [12] studied a neutral differential equation with the same type of deviated argument as studied by C.G. Gal [1]. Haloi, Pandey and Bahuguna [17] studied a system with the same deviated argument. Fractional operators, analyticity and compactness are mostly used to establish these results which impose more restriction on the semigroup and the nonlinear part of the semilinear system. Thus, in this paper the C_0 semigroup associated with mild solution has been replaced by the fundamental solution.

Several papers studied the approximate controllability of semilinear control systems, see for instance [2, 7, 14] and references therein. Generally these papers proposed conditions on the systems operators by assuming the corresponding linear system is approximately controllable. For instance, Naito [7] proved that a semilinear system is approximately controllable under range condition on the control operator and uniform boundedness of the nonlinear operator. Sukavanam [14] proved sufficient conditions for approximate controllability where the nonlinear function satisfies growth conditions.

Motivated by results in [7] and [14] the purpose of this paper is to study the existence and uniqueness of mild solution and approximate controllability of a functional differential equation with deviated argument and finite delay using Schuader fixed point theorem. However we proceed by establishing a relation between the reachable set of linear control problem and that of the semilinear delay control problem.

In this work we study the approximate controllability of the functional differential equation with finite delay and deviated argument, which is illustrated as follows.

$$\begin{aligned} &\frac{dx(t)}{dt} = Ax(t) + A_1x_t + Bu(t) + f(t, x_t, x(a(x(t), t))), t \in J = [0, \tau], \\ &x(t) = \phi(t), -h \le t \le 0, \end{aligned} \tag{1}$$

where $x(t) \in X$ and $u(t) \in U$, X and U being Hilbert spaces. Let $Z = L_2([0, \tau]; X)$, $Z_h = L_2([-h, \tau]; X)$, $0 < h < \tau$ and $Y = L_2([0, \tau]; U)$ be the corresponding function spaces. $A : D(A) \subset X \to X$ is a closed linear operator which generates a strongly continuous semigroup $T(t)$. A_1 is a bounded linear operator from $C([-h, \tau]; X)$ to $L_2([0, \tau], X)$. $B : Y \to Z$ is a bounded linear operator. When $x : [-h, \tau] \to X$ is a continuous function then $x_t(.)$ is denoted by $x_t(\theta) = x(t + \theta)$, $\theta \in [-h, 0]$ and $\phi \in C([-h, 0]; X)$. $x_t \in$

$C([-h,0],X)$ a Banach space of all continuous functions from $[-h,0]$ to X with norm

$$\|x_t\|_C := sup_{\theta\in[-h,0]}\|x_t(\theta)\|_X \quad for\ t\in(0,\tau].$$

$C_L(J,X)=\{u\in C(J,X) : \exists l>0\ such\ that\ \|u(t)-u(s)\|\le l|t-s|, \forall t,s\in J\}$.

Simple Lipschitz conditions are required to study the differential equation with deviated argument in Section 3.

2 Preliminaries and Assumptions

Some basic definitions and lemmas are stated which are used in proving the existence and uniqueness of the mild solution and approximate controllability of (1). In equation (1) if we put $f\equiv 0$ the resulting equation without the delay term is called the corresponding linear system (2)

$$\begin{aligned}\frac{dx(t)}{dt} &= Ax(t)+Bu(t),\ t\in[0,\tau],\\ x(0) &= \phi(0)\ \in[-h,0].\end{aligned}\tag{2}$$

Let us consider the linear delayed system

$$\begin{gathered}\frac{dx(t)}{dt}=Ax(t)+A_1x_t,\ t\in[0,\tau],\\ x_0=\phi\ \in[-h,0].\end{gathered}\tag{3}$$

Let $x^\phi(t)$ be the unique solution of system (3). Define a map $S: J\to\mathcal{L}(X)$ by

$$S(t)\phi(0)=\begin{cases} x^\phi(t), & t\ge 0,\\ 0, & t<0.\end{cases}\tag{4}$$

Then $S(t)$ is called the fundamental solution of (3) satisfying

$$\begin{aligned}&S(t)=T(t)\phi(0)+\int_0^t T(t-s)A_1S(s+\theta)ds,\ t>0,\\ &S(0)=I, S(t)=0,\ -h\le t<0.\end{aligned}\tag{5}$$

It follows from [9] that $S(t)$ is the unique solution of (3). It can be easily shown that

$$S(t)=K_0\exp(K_0||A_1||\tau):=M,$$

where $||T(t)||=K_0$. Therefore the mild solution of semilinear control system (1) is defined as

Definition 2.1 The function $x:(-h,\tau]\to X$ is said to be a mild solution of (1) if $x(.)\in C_L(J,X)$, $x(t)=\phi(t)$ for $t\in[-h,0]$ and it satisfies the integral equation.

$$x(t)=S(t)\phi(0)+\int_0^t S(t-s)Bu(s)ds+\int_0^t S(t-s)f(s,x_s,x(a(x(s),s)))ds,\ t\in J,\tag{6}$$

and the mild solution of the corresponding linear system with delay and control term (7)

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + A_1 x_t + Bu(t),\ t \in [0,\tau], \\ x_0 &= \phi\ \in [-h, 0], \end{aligned} \tag{7}$$

is defined as

$$\begin{aligned} x(t) =& S(t)\phi(0) + \int_0^t S(t-s)Bu(s)ds, t \in [0,\tau], \\ x(t) =& \phi(t), -h \le t < 0. \end{aligned} \tag{8}$$

Definition 2.2 The set given by $K_\tau(f) = \{x(T) \in X : x \in Z_h\}$ is called reachable set of the system (1). $K_\tau(0)$ is the reachable set of the corresponding linear control system (7).

Definition 2.3 The system (1) is said to be approximately controllable if $K_\tau(f)$ is dense in X. The corresponding linear system is approximately controllable if $K_\tau(0)$ is dense in X.

Let us assume that:

(H1) The nonlinear function $f : J \times X \times X \to X$ satisfies Lipschitz condition,

$$\|f(t, x_1, z_1) - f(t, x_2, z_2)\| \le P(\|x_1 - x_2\| + \|z_1 - z_2\|)$$

for all $x_1, x_2, z_1, z_2 \in X,\ t \in (0, \tau]$ and $\exists$ a constant $g > 0$, such that $\|f(s, 0, x(a(x(0), 0)))\| \le g,\ \forall\ s \in J$.

(H2) Let $a : X \times R^+ \to R^+$ satisfy the Lipschitz condition $|a(x_1, s) - a(x_2, s)| \le L_a \|x_1 - x_2\|$ and $a(., 0) = 0$.

Lemma 2.1 *The fundamental solution $S(t)$ is bounded.*

Proof. Since

$$\begin{aligned} ||S(t)|| \le& K_0 + K_0||A_1|| \int_0^t ||S(s+\theta)||ds \\ \le& K_0 + k_0||A_1|| \int_0^{t+\theta} ||S(\sigma)||d\sigma \\ \le& K_0 + ||A_1||K_0 \int_{-h}^t ||S(\sigma)||d\sigma \\ \le& K_0 + K_0||A_1|| \int_0^{t+h} ||S(\sigma)||d\sigma \\ ||S(t)|| \le& K_0 \exp K_0||A_1||(t+h) \le K_0(1+d) \exp K(\tau+h) = M \\ & max\{||S(t)|| : t \in [0,\tau]\} = M, \end{aligned} \tag{9}$$

the fundamental solution is bounded.

Lemma 2.2 *If the C_0-semigroup $T(t)$ is compact then the fundamental solution $S(t)$ is compact.*

Proof. Let us define the sequence of operators $S_n(t)$ on $[-h, \tau]$. From the compactness of $T(t)$ and boundedness of $||A_1||$ we conclude that S_n is compact. Let $||A_1|| = K_1$. To prove $S_n(t) \to S(t)$ in $\mathcal{L}(X)$ we first show that $\{S_n(t)\}$ is a Cauchy sequence in $\mathcal{L}(X)$. Let us define

$$\begin{aligned} S_1(t) &= T(t), t \in [0, \tau], \\ &= 0, t \in [-h, 0], \\ S_{n+1}(t) &= T(t) + \int_0^t T(t-s)S_n(s+\theta)ds, t \in (0, \tau], \theta \in [-h, 0], \\ &= 0, t \in [-h, 0], \end{aligned} \tag{10}$$

for $n = 1, 2, ...$

Therefore,

$$\begin{aligned} ||S_2(t) - S_1(t)|| &\leq \int_0^t ||T(t-s)||||A_1||||S(s+\theta)||ds \leq K_0 K_1 M t, \\ ||S_{n+1}(t) - S_n(t)|| &\leq \frac{1}{n!} K_0^n K_1^n M_1 \tau^n \to 0 \ \ as \ \ n \to 0. \end{aligned} \tag{11}$$

Thus $\{S_n(t)\}$ is a Cauchy sequence. As $\mathcal{L}(X)$ is the Banach space of all bounded linear operators on X, $\exists$ an operator $S(t) \in \mathcal{L}(X)$ such that $S_n(t) \to S(t)$ uniformly on $[0, \tau]$ and hence $S(t)$ is compact $\forall t \in [0, \tau]$. It is easy to check that $S(t)$ is unique.

2.1 Existence and uniqueness of mild solution

The equation (6) is verified to be the unique mild solution of the semilinear delay control system (1).

Theorem 2.1 *The system (1) has a unique mild solution in $C_L(J, X)$ for each control $u \in L_2([0, T]; U)$ if assumptions $(H1)$ and $(H2)$ are satisfied.*

Proof. Define the space $C_{L_0}([-h, \tau], X) = \{x \in C([-h, \tau], X) : x \in C_L([0, \tau], X)\}$. Fix $0 < t_1 < T$ such that

$$PMt_1(l + 2lL_a)R < M\|\phi\| + MM_BT\|u\| + MTg + 1.$$

Define the mapping $\Phi : C_{L_0}([-h, t_1], X) \to C_{L_0}([-h, t_1], X)$ as

$$\begin{aligned} (\Phi x)(t) &= S(t)\phi(0) + \int_0^t S(t-s)[Bu(s) + f(s, x_s, x(a(x(s), s)))]ds, \ t \in (0, t_1], \\ &= \phi(\theta), \ \ \theta \in [-h, 0]. \end{aligned} \tag{12}$$

Let us consider the space $B_R = \{x(.) \in C_{L_0}([-h, t_1], X) : \|x\|_{C([-h,t_1],X)} \leq R, \ \ x(0) = \phi(0)\}$ endowed with the norm of uniform convergence. For any $x \in B_R$ and $0 \leq t \leq t_1$,

$$\|x_t\|_C = sup_{-h \leq \theta \leq 0} \|x_t(\theta)\|_X \leq sup_{-h \leq \zeta \leq t_1} \|x(\zeta)\|_X \leq R.$$

Then

$$\begin{aligned}
\|(\Phi x)(t)\| &\leq M\|\phi(0)\| + MM_B T\|u\| \\
&+ \int_0^t M[\|f(s, x_s, x(a(x(s), s))) - f(s, 0, x(a(x(0), 0)))\| \\
&+ \|f(s, 0, x(a(x(0), 0)))\|]ds \\
&\leq M\|\phi\| + MM_B T\|u\| \\
&+ \int_0^t M[P(\|x(s+\theta) - 0\| + lL_a\|x(s) - x(0)\|) + g]ds \\
&\leq M\|\phi(0)\| + MM_B t_1\|u\| \\
&+ \int_{-h}^{t_1} MP(\|x(\sigma)\|d(\sigma) + \int_0^{t_1} [MlL_a\|x(s) - x(0)\| + g]ds \\
&\leq M\|\phi(0)\| + MM_B t_1\|u\| + M(t_1 + h)P\|x\| + 2Mt_1 PlL_a\|x\| + gt_1 \\
&\leq M\|\phi(0)\| + MM_B t_1\|u\| + M(t_1 + h)PR + 2Mt_1 PlL_a R + gt_1.
\end{aligned}$$

Let

$$M\|\phi\| + MM_B t_1\|u\| + M(t_1 + h)PR + 2Mt_1 PlL_a R + gt_1 < R.$$

Then

$$M\|\phi\| + MM_B t_1\|u\| + gt_1 < R(1 - M(t_1 + h)P - 2Mt_1 PlL_a).$$

RHS is positive if

$$\begin{aligned}
t_1(PM + 2MPlL_a) &< M(t_1 + h)P + 2Mt_1 PlL_a < 1, \\
t_1 &< \frac{1}{(PM + 2MPlL_a)}.
\end{aligned} \tag{13}$$

Hence Φ maps B_R into itself when t_1 satisfies (13). Next it is shown that Φ is a contraction. Let $x_1,\ x_2 \in B_R$

$$\begin{aligned}
\|(\Phi x_1)(t) - (\Phi x_2)(t)\| &\leq \int_0^t M\|f(s, (x_1)_s, x_1(a(x_1(s), s))) \\
&- f(s, (x_1)_s, x_1(a(x_2(s), s))) - f(s, (x_2)_s, x_2(a(x_2(s), s))) \\
&+ f(s, (x_1)_s, x_1(a(x_2(s), s)))\|ds \\
&\leq tMP[\|x_1(a(x_1(s), s)) - x_1(a(x_2(s), s))\| \\
&+ (\|(x_2)_s - (x_1)_s\| \\
&+ \|x_2(a(x_2(s), s) - x_1(a(x_2(s), s)))\|)] \\
&\leq tMP[l|a(x_1(s), s) - a(x_2(s), s)| \\
&+ \|x_2(s+\theta) - x_1(s+\theta)\| + (\|x_2 - x_1\|_{C([-h,t_1];X)})] \\
&\leq tM(lPL_a\|x_1(s) - x_2(s)\|_{C([-h,t_1],X)} \\
&+ P\|x_2(t_1) - x_1(t_1)\| + P\|x_2 - x_1\|_{C([-h,t_1],X)}) \\
&\leq Mt(lPL_a + 2P)\|x_2 - x_1\|_{C([-h,t_1],X)}.
\end{aligned} \tag{14}$$

So, $\|\Phi x_1 - \Phi x_2\|_{C([-h,t_1],X)} \leq Mt(lPL_a + 2P)\|x_1 - x_2\|_{C([-h,t_1],X)}$. Thus Φ is a contraction mapping. Therefore, Φ has a fixed point in B_R. Hence (6) is the mild solution on $[-h, t_1]$.

Similarly it can be shown that (6) is the mild solution on the interval $[t_1, t_2],\ t_1 < t_2$ Repeating the above process we get that

$$\|\Phi^n x_1 - \Phi^n x_2\|_{C([-h,t_1],X)} \leq \frac{Mt^n}{n!}(lPL_a + 2P)\|x_1 - x_2\|_{C([-h,t_1];X)}.$$

Thus (6) is the mild solution on the maximal existence interval $[-h, t^*],\ t^* < \tau$.

Now it is shown that x is well defined in $[-h, \tau]$.

$$\begin{aligned}
\|x(t)\| &\leq M\|\phi\| + M\int_0^t [M_B\|u(s)\| + P\|x_s - 0\| \\
&+ P|x(a(x(s), s) - x(a(x(0), 0)\| + g]ds \\
&\leq M\|\phi\| + MM_B\tau\|u(s)\| \\
&+ M\int_0^t P[\|x_s\| + lL_a\|x(s) - x(0)\| + g] \\
&\leq M\|\phi\| + MM_B\tau\|u(s)\| \\
&+ M\tau P(\|x(0)\| + g) + M\int_0^t l\|x(s)\|ds. \qquad (15)
\end{aligned}$$

By Gronwall's inequality $\|x(t)\| \leq \|x_t\|_C \leq [M\|\phi\| + MM_B\tau\|u(s)\| + MTP(\|x(0)\| + g)]\exp(M\tau P)$. So $\|x(t)\|$ is bounded on $[-h, t^*]$. Thus x is well defined on $[-h, T]$. To prove the uniqueness of solution let x_1 and x_2 be any two mild solutions of (6) such that for $t \in [-h, 0],\ x_1(t) = x_2(t) = \phi$. For $t \in [0, t^*)$

$$\begin{aligned}
\|x_1(t) - x_2(t)\| &\leq M\int_0^t \|f(s, (x_1)_s, x_1(a(x_1(s), s))) \\
&- f(s, (x_2)_s, x_2(a(x_1(s), s)))\|ds + f(s, (x_2)_s, x_2(a(x_1(s), s))) \\
&- f(s, (x_2)_s, x_2(a(x_2(s), s)))\| \\
&\leq M\int_0^t P\{\|(x_1)_s - (x_2)_s\| + \|x_1(s) - x_2(s)\| \\
&+ lL_a\|x_1(s) - x_2(s)\|\}ds \\
&\leq M\int_{-h}^t P\|x_1(\eta) - x_2(\eta)\|d\eta + M\int_0^t P\|x_1(s) - x_2(s)\|ds \\
&+ M\int_0^t PlL_a\|x_1(s) - x_2(s)\|ds \\
&\leq M\int_{-h}^0 P\|x_1(\eta) - x_2(\eta)\|d\eta + M\int_0^t P(2 + lL_a)\|x_1(s) - x_2(s)\|ds.
\end{aligned}$$

Since uniqueness of the mild solution is proved on $[-h, 0]$, we get

$$\|x_1(t) - x_2(t)\| \leq MP(2 + lL_a)\int_0^t \|x_1(s) - x_2(s)\|ds.$$

Hence by Gronwall's inequality $x_1(t) = x_2(t)$ for all $t \in [-h, \tau]$.

3 Main Result

Define a linear operator L from Z to $C_L([0, \tau], X)$ by $Lx = \int_0^\tau S(t - s)x(s)ds, t \in [0, \tau]$. Let $Kx(t) = \int_0^t S(t - s)x(s)ds, t \in [0, \tau]$.

Z can be decomposed uniquely as $Z = N_0(L) \oplus N_0^{\perp}(L)$ where $N_0(L)$ is the null space of the operator L and $N_0(L)$ is its orthogonal space.

Let us assume

(H3) $\forall\, p \in Z$, $\exists$ a function $q \in \overline{R(B)}$ such that $Lp = Lq$.

The approximate controllability of the corresponding linear system (2) follows from the hypothesis $(H3)$. Then it is to be proved that the linear system (7) with finite delay is approximately controllable. Next by assuming that the linear system with delay (7) is approximately controllable, the system (1) is to be proved to be approximately controllable using Schauder fixed point theorem. Define the operator $F : C_{L_0}([0,\tau], X) \to L_2([0,\tau], X)$ as

$$F(x)(t) = f(t, x_t, x(a(x(t), t))); 0 < t \leq \tau.$$

From hypotheses $(H1), (H2)$ we conclude that F is a continuous map. From hypothesis $(H3)$ it follows that for any $p \in Z$, there exists a $q \in \overline{R(B)}$ such that $L(p - q) = 0$. Therefore $p - q = n \in N_0(L)$ which implies that $Z = N_0(L) \oplus \overline{R(B)}$. Therefore, it implies the existence of a linear and continuous mapping Q from $N_0^{\perp}(L)$ into $\overline{R(B)}$ which is defined as $Qu^* = v$ where v is the unique minimum norm element $v \in (u^* + N_0(L)) \bigcap \overline{R(B)}$, i.e. $\|Qu^*\| = \|v\| = min\{\|v\| : v \in \{(u^* + N_0(L)) \bigcap \overline{R(B)}\}$. By $(H3)$, $\forall v \in \{u^* + N_0^{\perp}\} \cap \overline{R(B)}$ is not empty and $\forall z \in Z$ has a unique decomposition $z = n + q$. Hence the operator Q is well defined. Moreover, $\|Q\| = c$ for some constant c.

Let us consider the subspace M_0 of $C_{L_0}([0,\tau], X)$ which is defined as

$$M_0 = \begin{cases} m \in C_{L_0}([0,\tau], X) : m(t) = Kn(t), & n \in N_0(L); 0 \leq t \leq \tau, \\ m(t) = 0, & -h \leq t \leq 0; \end{cases} \tag{16}$$

Let

$$f_x : \overline{M_0} \to \overline{M_0}$$

defined by

$$f_x = \begin{cases} Kn, & 0 < t \leq \tau; \\ 0, & -h \leq t \leq 0; \end{cases} \tag{17}$$

where n is given by the unique decomposition of $F(x + m)(t) = n(t) + q(t)$, $n \in N_0(L)$ and $q \in \overline{R(B)}$.

The following assumption is made

$(A1)$ $\overline{R(A_1)} \subset \overline{R(B)}$.

Theorem 3.1 *The operator f_x has a fixed point in M_0 if $M(1 + c)P\tau < 1$.*

Proof. Since $S(t)$ is compact, K is compact and f_x is compact. Let $z \in Z$ then $z = q + n$, $n \in N_0(L)$, $q \in \overline{R(B)}$. Also $\|n\|_Z \leq (1 + c)\|z\|_Z$ for some constant c. Let

$$B_r = \{v \in \overline{M_0} : ||v|| \leq r\}.$$

Let $m \in B_r$ and $\|f(0,0,(x+m)(a(m(s),0)\| \leq l_f$. Suppose on the other hand

$$
\begin{aligned}
r < \|f_x(m)\| =& \|Kn\| \leq \int_0^t \|S(t-s)n(s)\|ds \\
& \leq \int_0^t M(1+c)\|F(x+m)\|_Z ds \\
& \leq \int_0^t M(1+c)[\|f(s,(x+m)_s,(x+m)(a((x+m)(s),s)))\| \\
& -\|f(0,0,(x+m)(a(m(s),0))))\| + \|f(0,0,(x+m)(a(m(s),0))))\|] \\
& \leq M(1+c)\int_0^t P[\|(x+m)(s+\theta)-0\| \\
& +\|(x+m)(a((x+m)(s),s)) - (x+m)(a((m)(s),0))\| + l_f]ds \\
& \leq M(1+c)\int_0^t P[\|x\| + \|m\| + l|a((x+m)(s),s) - a(m(s),0)| + l_f]ds \\
& \leq M(1+c)\int_0^t P[\|x\| + r + lL_a\|(x+m)(s) - m(s)\| + l_f]ds \\
& \leq M(1+c)\int_0^t P[\|x\| + r + lL_a\|x\| + l_f]ds \\
& \leq M(1+c)P(\|x\|T + r\tau + lL_a\|x\|T + l_f T).
\end{aligned}
\tag{18}
$$

Dividing by r and taking limit as r tends to ∞ we get a contradiction. So f_x maps B_r into itself. Therefore, by Schauder fixed point theorem it has a fixed point.

Theorem 3.2 *Suppose the linear control system* (2)

$$
\begin{aligned}
\frac{dx(t)}{dt} &= Ax(t) + Bu(t), \\
x(0) &= \phi(0),
\end{aligned}
\tag{19}
$$

is approximately controllable then the linear delay control system (7)

$$
\begin{aligned}
\frac{dx(t)}{dt} &= Ax(t) + A_1 x_t + Bu(t), \\
x(t) &= \phi(t),\ -h \leq t \leq 0,
\end{aligned}
$$

is controllable if assumptions $(A1)$ *hold.*

Proof. Consider

$$
\begin{aligned}
y'(t) &= Ay(t) + Bu(t), t \in [0,\tau], \\
y(t) &= \phi(t),\ t \in [-h,0].
\end{aligned}
\tag{20}
$$

The mild solution of equation (20) is as follows

$$
\begin{aligned}
y(t) &= T(t)\phi(0) + \int_0^t T(t-s)Bu(s)ds,\ t > 0, \\
y(t) &= \phi(t),\ t \in [-h,0].
\end{aligned}
\tag{21}
$$

Since $\overline{R(A_1)} \subset \overline{R(B)}$, $\forall\ \epsilon > 0$, $\exists\ w \in U$ such that

$$\|A_1 y_s - Bw\|_Z \leq \epsilon.$$

Let $x(t)$ be a solution of linear delay control system corresponding to control $(u - w)$ satisfying

$$\begin{aligned} x(t) &= T(t)\phi(0) + \int_0^t T(t-s)\{B(u-w) + A_1 x_s\}ds, t > 0, \\ x(t) &= \phi(t), t \in [-h, 0]. \end{aligned} \tag{22}$$

If $t \in [-h, 0]$, then

$$x_0(t) - y_0(t) = 0$$

and if $t \in (0, \tau]$ then we get

$$\begin{aligned} x(t) - y(t) &= \int_0^t T(t-s)[-Bw(s) + A_1 x_s] \\ &= \int_0^t T(t-s)[-Bw(s) + A_1 y_s]ds \\ &+ \int_0^t T(t-s)[A_1 x_s - A_1 y_s]ds. \end{aligned} \tag{23}$$

Take the norm on both sides

$$\begin{aligned} \|x(t) - y(t)\| &\leq K_0 \int_0^t \|Bw(s) - A_1 x_s\|ds \\ &+ K_0 \int_0^t \|A_1 x_s - A_1 y_s\|ds \\ &\leq K_0 \tau \|Bw(s) - A_1 x_s\|_Z + K_0 \int_0^t K_1 \|x_s - y_s\|ds \\ &\leq K_0 \epsilon \tau + K_0 \int_0^t K_1 \|x_s - y_s\|ds \\ &\leq K_0 \epsilon \tau + K_0 \int_{-h}^t K_1 \|x(\eta) - y(\eta)\|d\eta, \end{aligned} \tag{24}$$

where $\|A_1\| \leq K_1$, since A_1 is bounded linear operator from $C_{L_0}([-h, \tau], X)$ to $L_2([0, \tau], X)$ and $\widetilde{A} : L_2([0, \tau], X) \to C_0([0, \tau], X)$ defined by $\widetilde{A}(x) = \int_0^t T(t-s)A_1 x_s ds$ This implies

$$||x(t) - y(t)|| \leq K_0 \epsilon \tau + K_0 K_1 \int_{-h}^t \|x(\eta) - y(\eta)\|d\eta. \tag{25}$$

Using Gronwall's inequality

$$||x(t) - y(t)|| \leq K_0 \epsilon \tau \exp(K_0 K_1 \{\tau + h\}).$$

Since RHS depends on ϵ, it can be made as small as possible. This implies that the reachable set of linear delay control system is dense in the reachable set of the linear control system (2) which in turn is dense in X as (7) is apprroximately controllable. Hence the linear delay control system is controllable.

Theorem 3.3 *The semilinear control system (1) is approximately controllable if the linear delay control system (7)*

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + A_1 x_t + Bu(t), \\ x(t) &= \phi(t), \ -h \leq t \leq 0, \end{aligned}$$

is approximately controllable.

Proof. Let $x(.)$ be the mild solution of the linear delay control system (7) given by

$$x(t) = S(t)\phi(0) + KBu(t), t \in (0, \tau],$$

$$x(t) = \phi(t), t \in [-h, 0].$$

We prove

$$y(t) = x(t) + m_0(t)$$

to be mild solution of semilinear problem (1). Since

$$KF_h(x + m_0)(t) = Kn(t) + Kq(t),$$

operating K on both sides at $m = m_0$, fixed point of f_x,

$$\begin{aligned} KF_h(x + m_0)(t) &= Kn(t) + Kq(t) \\ &= m_0(t) + Kq(t). \end{aligned} \tag{26}$$

Add $x(.)$ to both sides and using $y(t) = x(t) + m_0(t)$, we have

$$\begin{aligned} x(t) + KF_h(x + m_0)(t) &= x(t) + m_0(t) + Kq(t), \\ x(t) + KF_h(y)(t) &= y(t) + Kq(t), \\ \Rightarrow y(t) &= x(t) + KF_h(y)(t) - Kq(t), \\ \Rightarrow y(t) &= S(t)\phi(0) + K(Bu - q)(t) + KF_h(y)(t). \end{aligned} \tag{27}$$

This is the mild solution of semilinear problem with control $(Bu - q)$. By following the same proof in [13] we get the following conclusion that since $q \in \overline{R(B)}$, there exists a $v \in U$ such that $\|Bv - q\| < \epsilon$ for any given $\epsilon > 0$. Let x_v be a solution of the given semilinear delay control system (1.1) corresponding to the control v. Then as shown by [7] we have $||y(\tau) - x_v(\tau)|| = ||x(\tau) - x_v(\tau)|| \leq \epsilon$. This implies that $x(\tau) \in \overline{K_\tau(f)}$. Then it follows that $\overline{K_\tau(0)} \subset \overline{K_\tau(f)}$. Thus (1) is approximately controllable, since the corresponding linear system (7) is approximately controllable.

4 Example

Let us consider the heat control system with finite delay

$$\begin{aligned} \frac{\partial y(t,x)}{\partial t} &= \frac{\partial^2 y(t,x)}{\partial x^2} + y(t+\theta, x) + Bu(t,x) + f(t, x(t+\theta), x(a(x(s), s)))ds \\ &\quad 0 < t < T, \ -h < \theta < 0, \ 0 < x < \pi, \\ y(t, 0) &= y(t, \pi) = 0, \ 0 \leq t \leq T, \\ y(t,x) &= \xi(x), \ \ -h \leq t \leq 0, \ \ 0 \leq x \leq \pi. \end{aligned} \tag{28}$$

Let $X = L_2(0,\pi)$ and $A = -\frac{d^2}{dx^2}$. Define

$$D(A) = \{y \in X : y, \frac{dy}{dx} \text{ are absolutely continuous}, \frac{d^2y}{dx^2} \in X \text{ and } y(0) = y(\pi) = 0\}.$$

For $y \in D(A)$, $y = \sum_{n=1}^{\infty} < y, \phi_n > \phi_n$ and $Ay = -\sum_{n=1}^{\infty} n^2 < y, n > \phi_n$. where $\phi_n(x) = \frac{2}{\pi}^{\frac{1}{2}} \sin nx,\ 0 \leq x \leq \pi,\ n = 1,2,3...$ is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. ϕ_n is an orthonormal base. A will generate a compact semigroup $T(t)$ such that $T(t)y = \sum_{n=1}^{\infty} e^{-n^2t} < y, \phi_n > \phi_n,\ n = 1,2,...\ \forall\ y \in X$. Let the infinite dimensional control space be defined as $U = \{u : u = \sum_{n=2}^{\infty} u_n\phi_n,\ \sum_{n=2}^{\infty} u_n^2 < \infty\}$ with norm $\|u\|_U = (\sum_{n=2}^{\infty} u_n^2)^{\frac{1}{2}}$. Thus U is a Hilbert space. Let $\widetilde{B} : U \to X :\ \widetilde{B}u = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n\ \ for\ u = \sum_{n=2}^{\infty} u_n\phi_n \in U$. The bounded linear operator $B : L_2(0,T : U) \to L_2(0,T;X)$ is defined by $(Bu)(t) = \widetilde{B}u(t)$. Then this problem (28) can be reformulated into an abstract semilinear differential equation with deviated argument and finite delay by substituting $I = A_1$. If the hypotheses $(H1) - (H3)$ and assumption $(A1)$ are satisfied then it can be shown that this system (28) is approximately controllable.

5 Conclusion

Thus, we prove the existence and uniqueness and approximate controllability of the functional differential equation (1) with deviated argument and finite delay by using Schuader fixed point theorem and fundamental solution instead of C_0 semigroup.

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