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Observer Design for Descriptor Systems with Lipschitz Nonlinearities: an LMI Approach

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Abstract: In this paper, a method is proposed to design asymptotic observers for a class of semilinear descriptor systems satisfying the complete detectability condition on the corresponding linear part. The method is based on the properties of restricted system equivalent, derived here from a given descriptor system by means of simple matrix theory. Using restricted system equivalent form, coefficient matrices of the proposed observer have been synthesized by linear matrix inequality (LMI) approach based on the Lyapunov stability theory.

Keywords: *descriptor system; Lipschitz continuity; observer design; detecability.*

Mathematics Subject Classification (2010): 93C10, 93C35, 93D20.

1 Introduction

In the last three decades, considerable amount of research was focused on the analysis, design, and numerical simulation techniques for descriptor systems, which arise in modeling of many real and practical systems, e.g. electrical network analysis, power systems, constrained mechanics, economic systems, chemical process control, see, [1–7] and the references therein. Depending on the area, descriptor systems are termed by a variety of names, *viz.* differential algebraic equations (DAEs), singular, implicit, generalized state space, noncanonic, degenerate, semi-state and nonstandard systems. In this paper, we consider the following semilinear system:

$$E^*\dot{x} = A^*x + B^*u + D^*f(Hx, u, t), \tag{1a}$$

$$y = Cx, \tag{1b}$$

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where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, are the state vector, the input vector and the output vector, respectively, $E^* \in \mathbb{R}^{n\times n}$ $A^* \in \mathbb{R}^{n\times n}$, $B^* \in \mathbb{R}^{n\times m}$, $D^* \in \mathbb{R}^{n\times n_d}$, $H \in \mathbb{R}^{n_h\times n}$, and $C \in \mathbb{R}^{p\times n}$ are known constant matrices, $\text{rank}(E^*) = r < n$. Without loss of generality, we assume that $\text{rank}(B^*) = m$, $\text{rank}(D^*) = n_d$, $\text{rank}(C) = p$, and $\text{rank}(H) = n_h$. If E is nonsingular or $E \equiv I$, then the system is called normal system.

To design a controller, the knowledge about the states of the system is important. But it is not always possible or necessary to measure all the state variables. In such cases, the states can be estimated from the output of another dynamical system, which is called an observer for the given system. An observer is a mathematical realization which uses the input and output information of a given system and its output asymptotically approaches to the true state values of the given system.

Observer design problem for normal linear systems has received a great attention in the literature [8–11] and the techniques used for them have been extended successfully to descriptor linear systems, see [6, 7, 12, 13] and references therein. For normal nonlinear systems, in general, literature concerned with the design of observers could broadly be classified into two categories based on the solution approach. In the first approach, the states are transformed in such a way that the given nonlinear system converts into a system, where linear theory is applicable [14–17]. In another approach, the observers are designed for nonlinear systems without any state transformation [18–21]. For a comparison of these approaches, we refer to [22]. On the other hand literature on observers for descriptor nonlinear systems is not so rich. However some researchers have extended the approaches mentioned above to descriptor nonlinear systems [23–32].

In [23], authors extended linearization technique to design state observers for descriptor nonlinear systems and illustrated its application to AC/DC converter model. Boutayeb *et al.* [24] extended the results of [23] to the rectangular descriptor systems. In [25], a method for observing the states of continuous quasilinear descriptor systems is developed by casting the given system as an equivalent system of explicit differential equations on a restricted manifold. In [26], authors considered a nonlinear observer for a class of continuous nonlinear descriptor systems with unknown inputs and faults. In last few years, due to availability of computationally fast and reliable algorithms for solving convex optimization problems subjected to LMI constraints (like MATLAB LMI tool box [33]), researchers developed LMI based approaches to design controllers and observers for normal [34, 35] and descriptor systems [27–32]. Semilinear descriptor systems with the Lipschitz nonlinearities and arbitrary unknown inputs with or without disturbances were considered in [26–32] and existence conditions were derived for full-order, reduced-order, minimal-order, or H_∞ observers in the form of LMIs.

In this paper, we develop a method for full-order state observer design for a class of Lipschitz nonlinear descriptor systems. Contrary to the results available in [31, 32], the observer presented in this paper has normal system form. The sufficient condition for the stability of error dynamics is given in terms of an LMI. In square system case, the proposed method is simple, easy to understand and implement compared to the methods available in the literature. Numerical examples are provided in the last section to illustrate our results.

2 Problem Description and Design Approach

Let us make the following conditions on the system (1):

(H1) $\text{rank}\begin{bmatrix} E^* \\ C \end{bmatrix} = n$,

(H2) nonlinear function $f(Hx, u, t)$ satisfies the Lipschitz property in its first argument, *i.e.* there exists a $\lambda > 0$ such that

$$\|f(Hx_1, u, t) - f(Hx_2, u, t)\| \leq \lambda \|H(x_1 - x_2)\|, \tag{2}$$

(H3) $\text{rank}\begin{bmatrix} \lambda E^* - A^* \\ C \end{bmatrix} = n \ \forall \ \lambda \ \in \ \bar{\mathbb{C}}^+$, where $\mathbb{C}$ represents the set of complex numbers. $\bar{\mathbb{C}}^+ = \{s | s \in \mathbb{C}, Re(s) \geq 0\}$ is the closed right half complex plane.

The problem is to design the matrices N, L, M, and D of compatible dimensions such that the following normal system becomes a full-order state observer (*i.e.*, $\hat{x} \to x$ as $t \to \infty$) for system (1)

$$\dot{z} = Nz + Bu + Ly + Df(H\hat{x}, u, t), \tag{3a}$$

$$\hat{x} = z + My. \tag{3b}$$

Our approach is as follows.

First, using the algorithm, which we have designed in the Appendix of this paper, a nonsingular matrix $R \in \mathbb{R}^{n \times n}$ is constructed such that the system (1) is restricted system equivalent to the following descriptor system:

$$E\dot{x} = Ax + Bu + Df(Hx, u, t) \tag{4a}$$

$$y = Cx, \tag{4b}$$

where $E = RE^*$, $A = RA^*$, $B = RB^*$, and $D = RD^*$. It is easy to verify that if the system (1) satisfies (H1), then the system (4) satisfies the following property:

$$rank \begin{bmatrix} I - E \\ C \end{bmatrix} = p. \tag{5}$$

It should be noted that this matrix R is not unique. The proof of the existence of such matrix R can be found in [36].

Second, solution of a system does not change by multiplying a nonsingular matrix, observer for the system (4) works for the system (1). From equations (3) and (4) the error

$$\begin{aligned} e &= x - \hat{x} \\ &= x - z - MCx \\ &= (I - MC)x - z \\ &= Ex - z \end{aligned} \tag{6}$$

gives the dynamics

$$\begin{aligned} \dot{e} &= E\dot{x} - \dot{z} \\ &= Ax + Bu + Df(Hx, u, t) \\ &\quad -(Nz + Bu + LCx + Df(H\hat{x}, u, t)) \\ &= (A - LC)x - N(Ex - e) + D\Delta f \\ &= Ne + (A - LC - NE)x + D\Delta f \\ &= Ne + (A - LC - N + NMC)x + D\Delta f, \\ &= Ne + D\Delta f \end{aligned} \tag{7}$$

where $\Delta f = f(Hx, u, t) - f(H\hat{x}, u, t)$. Moreover, in the construction of equations (6) and (7), we have assumed the existence of matrices M, K, N, and L of compatible orders such that

$$
\begin{aligned}
I - MC &= E, && (8)\\
N &= A - KC, && (9)\\
K &= L - NM. && (10)
\end{aligned}
$$

Finally, the problem of designing the state observer (3) boils down to determining the matrices M, K, N, and L such that the equations (8)–(10) are satisfied with the stability of error dynamics (7).

3 Main Result

Theorem 3.1 *Suppose the assumptions* $(H1)$ *and* $(H2)$ *hold for the system (1). Then system (3) is observer for the system (1) if the following LMI has a solution for any* $P > 0$

$$
\begin{bmatrix} PA + A^T P - \tilde{K}C - C^T \tilde{K}^T + \lambda^2 H^T H & PD \\ D^T P & -I \end{bmatrix} < 0, \tag{11}
$$

where $\tilde{K} = PK$.

Proof. Equation (5) implies that there exists a matrix M such that (8) is satisfied. Now, we show the existence of matrix K such that matrix N (in equation (9)) and the error dynamics (7) are stable if the LMI (11) has a solution for $P > 0$. Considering a Lyapunov function $V = e^T Pe$, and using (7) and (9) we have

$$
\begin{aligned}
\dot{V} &= \dot{e}^T Pe + e^T P\dot{e} \\
&= (Ne + D\Delta f)^T Pe + e^T P(Ne + D\Delta f) \\
&= e^T (N^T P + PN)e + \Delta f^T D^T Pe + e^T PD\Delta f \\
&\le e^T (N^T P + PN)e + \Delta f^T D^T Pe + e^T PD\Delta f \\
&\quad + \lambda^2 e^T H^T He - \Delta f^T \Delta f \\
&= \begin{bmatrix} e^T & \Delta f^T \end{bmatrix} \begin{bmatrix} N^T P + PN + \lambda^2 H^T H & PD \\ D^T P & -I \end{bmatrix} \begin{bmatrix} e \\ \Delta f \end{bmatrix} \\
&= \begin{bmatrix} e^T & \Delta f^T \end{bmatrix} \begin{bmatrix} N^T P + PN + \lambda^2 H^T H & PD \\ D^T P & -I \end{bmatrix} \begin{bmatrix} e \\ \Delta f \end{bmatrix}.
\end{aligned}
$$

According to the stability theory, the error dynamics (7) is stable if

$$
\begin{aligned}
&\begin{bmatrix} N^T P + PN + \lambda^2 H^T H & PD \\ D^T P & -I \end{bmatrix} < 0 \\
\Rightarrow &\begin{bmatrix} (A - KC)^T P + P(A - KC) + \lambda^2 H^T H & PD \\ D^T P & -I \end{bmatrix} < 0 \\
\Rightarrow &\begin{bmatrix} PA + A^T P - \tilde{K}C - C^T \tilde{K}^T + \lambda^2 H^T H & PD \\ D^T P & -I \end{bmatrix} < 0.
\end{aligned}
$$

Hence by a solution of LMI (11), we can find a matrix K, and hence matrix N, such that the error dynamics (7) is stable. Finally using the equation (10), we can find the matrix L.

Remark 3.1 If the LMI (11) is solvable then it is clear that

$$PA + A^T P - \tilde{K}C - C^T \tilde{K}^T + \lambda^2 H^T H < 0.$$

That implies

$$PA + A^T P - \tilde{K}C - C^T \tilde{K}^T < 0,$$

which is equivalent to the detectability of matrix pair (A, C). It can be proved easily that under assumption $(H1)$, condition $(H3)$ is equivalent to the detectability of matrix pair (A, C). Hence the condition $(H3)$ is a necessary condition for the solvability of LMI (11).

4 Numerical Examples

Example 4.1 Consider the descriptor system (1) described by the following matrices. (This example is taken from [28] with zero disturbance vector.)

$$E^* = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B^* = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T,$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D^* = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T, \quad H = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

$u(t) = sin(2t)$. The nonlinearity function $f(x, u, t) = sin(x_2(t))$. Since rank$\begin{bmatrix} E^* \\ C \end{bmatrix} = 3$ and rank$\begin{bmatrix} I - E^* \\ C \end{bmatrix} \neq 1$, using the algorithm given in the Appendix, we calculate

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T.$$

Now, we can check that rank$\begin{bmatrix} I - E \\ C \end{bmatrix} = 1$ and $M = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

By using MATLAB LMI tool box we solve (11) and find

$$K = \begin{bmatrix} 4.4252 & 4.3573 & 4.5410 \end{bmatrix}^T.$$

Thus

$$N = \begin{bmatrix} -3.4252 & 0 & 1.0000 \\ -3.3573 & 0 & 0 \\ -4.5410 & 1.0000 & 0 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T.$$

If we take

$$x(0) = \begin{bmatrix}-2.8415 & 1 & 2\end{bmatrix}^T,$$

$$z(0) = \begin{bmatrix}2 & 3 & 5\end{bmatrix}^T,$$

then the truth and estimated states are plotted in Figure 1. The graph of the error vector is shown in Figure 2, which clearly shows the efficiency of the proposed observer.

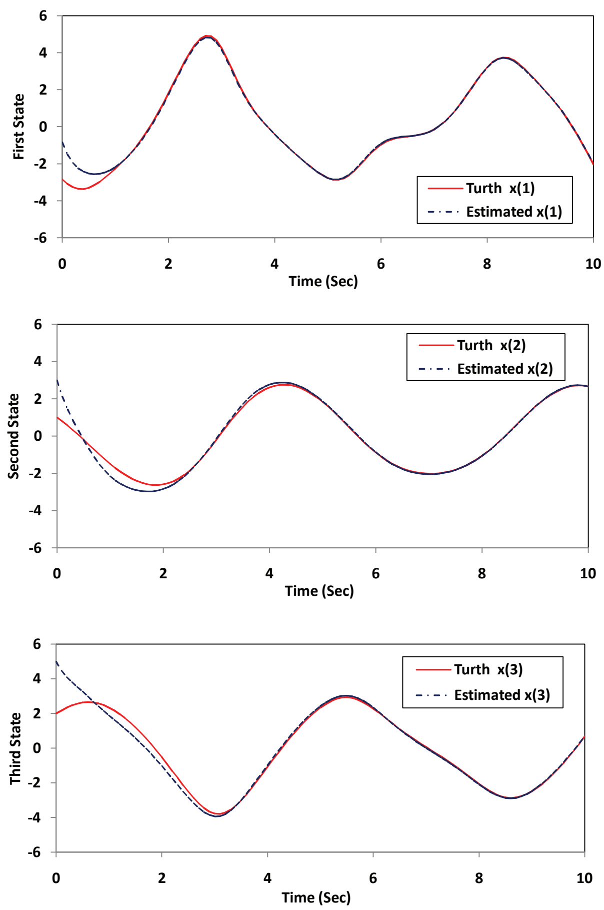

Figure 1: Plot of the true and estimated values of the states in Example 4.1.

Example 4.2 Consider the descriptor system (1) described by the following matrices:

$$E^* = \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \quad A^* = \begin{bmatrix}1 & 2 & 0\\0 & -2 & 0\\1 & 0 & -3\end{bmatrix}, \quad B^* = \begin{bmatrix}0 & 1 & 2\end{bmatrix}^T, \quad C = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix},$$

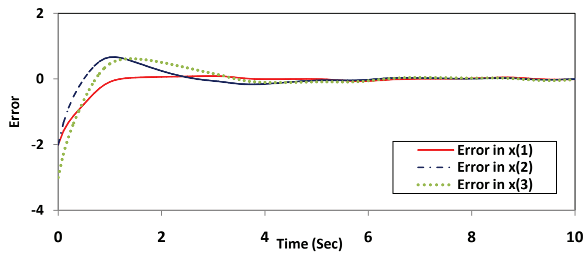

Figure 2: Estimation performance in Example 4.1.

$$D^* = \begin{bmatrix}1 & 2 & 1\end{bmatrix}^T, \quad H = \begin{bmatrix}0 & 0 & 1\end{bmatrix},$$

$u(t) = t^2$. The nonlinearity function $f(x,u,t) = cos(x_3(t))$. Since rank $\begin{bmatrix} I - E^* \\ C \end{bmatrix} = 2$, $R = I_3$ and $M = \begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$. By using MATLAB LMI tool box we solve (11) and find

$$K = \begin{bmatrix} -172.2387 & 132.1813 \\ -386.6106 & 180.5962 \\ -193.7974 & 103.5193 \end{bmatrix}.$$

Thus,

$$N = \begin{bmatrix} 173.2387 & -130.1813 & 0 \\ 386.6106 & -182.5962 & 0 \\ 194.7974 & -103.5193 & -3.0000 \end{bmatrix}$$

and

$$L = \begin{bmatrix} 1.0000 & 132.1813 \\ 0 & 180.5962 \\ 1.0000 & 103.5193 \end{bmatrix}.$$

If we take

$$x(0) = \begin{bmatrix}-1.5839 & 1 & 2\end{bmatrix}^T,$$
$$z(0) = \begin{bmatrix}10 & 12 & 15\end{bmatrix}^T,$$

then the truth and estimated states are plotted in Figure 3. The graphs of the errors are plotted in Figure 4, which clearly shows that error vector converges to zero.

5 Conclusions

A method has been developed to design the state observers for a class of semilinear descriptor systems. This class is characterized by two properties: (i) the linear part of each member system is completely detectable, and (ii) the nonlinear part satisfies the Lipschitz property. The sufficient condition for the stability of error dynamics is given in terms of an LMI. A new restricted equivalent system which follows the same state representation as the given descriptor system, has been made with the help of an

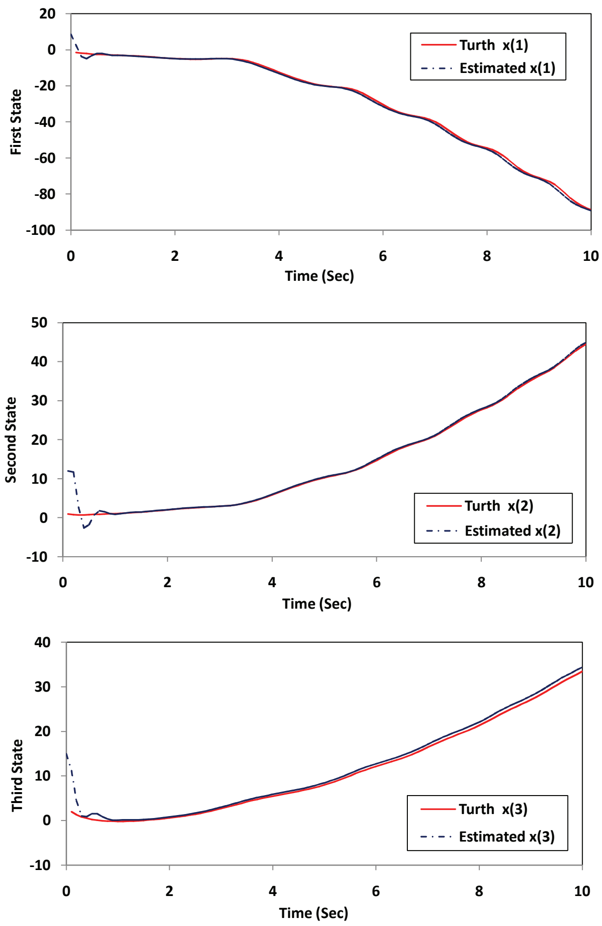

Figure 3: Plot of the true and estimated values of the states in Example 4.2

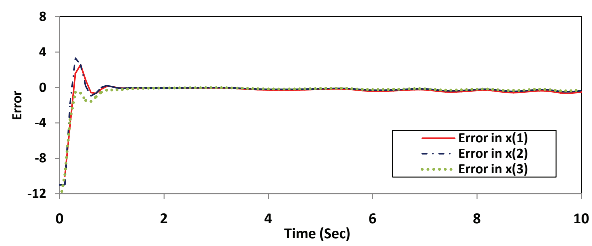

Figure 4: Estimation performance in Example 4.2

invertible matrix R. The advantage of using this equivalent system is the fact that the detectability of its corresponding normal system is equivalent to the detectability of the given descriptor system, and this fact gave necessary condition for the solution of the proposed LMI (see Remark 3.1). The extension of this work to rectangular semilinear and nonlinear descriptor system is under construction.

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Appendix

Algorithm to find the matrix R:

1. Determine
$p :=$ rank of matrix C,
$n :=$order of matrix E^*.

2. Check
(i) If rank $\begin{bmatrix} I - E^* \\ C \end{bmatrix} = p$. Take $R = I_n$ and stop.
(ii) If rank $\begin{bmatrix} E^* \\ C \end{bmatrix} = n$, then go to steps 3-8.

3. Carry out the singular value decomposition (SVD) of matrix $C = U_1 \begin{bmatrix} D_1 & 0 \end{bmatrix} V_1^T$.

4. Calculate $P = V_1 \begin{bmatrix} D_1^{-1} U_1^T & 0 \\ 0 & I_{n-p} \end{bmatrix}$.

5. Calculate $\tilde{E} = E^* P \begin{bmatrix} 0 \\ I_{n-p} \end{bmatrix}$.

6. Carry out the SVD of matrix $\tilde{E} = U_2 \begin{bmatrix} D_2 \\ 0 \end{bmatrix} V_2$.

7. Calculate $R_0 = \begin{bmatrix} 0 & I_p \\ V_2^T D_2^{-1} & 0 \end{bmatrix} U_2^T$.

8. Calculate $R = P R_0$.

References

[1] Campbell, S. L. *Singular Systems of Differential Equations – II*. Pitman, London, 1982.

[2] Luenberger, D. G. Non-linear descriptor systems. *J. Econ. Dynam. Control* **1**(3) (1979) 219–242.

[3] Lewis, F. L. A survey of linear singular systems. *Circuits, Systems and Signal Processing* **5**(1) (1986) 3–36.

[4] Brenan, K. E., Campbell, S. L. and Petzold, L. R. *Numerical Solution of Initial-value Problems in Differential-algebraic Equations*. SIAM, 1996.

[5] Campbell, S. L. *Singular Systems of Differential Equations*. Pitman, 1980.

[6] Dai, L. *Singular Control Systems*. Springer-Verlag, 1989.

[7] Duan, G.-R. *Analysis and Design of Descriptor Linear systems*. Springer, 2010.

[8] Luenberger, D. G. An introduction to observers. *IEEE Trans. Autom. Control* **16**(6) (1971) 596–602.

[9] Carroll, R. L. and Lindorff, D. An adaptive observer for single-input single-output linear systems. *IEEE Trans. Autom. Control* **18**(5) (1973) 428–435.

[10] O'Reilly, J. *Observers for Linear Systems*. Academic Press, 1983.

[11] Spurgeon, S. K. Sliding mode observers: a survey. *Int. J. Systems Science* **39**(8) (2008) 751–764.

[12] Ren, J. and Zhang, Q. PD Observer design for descriptor system: An LMI approach. *Int. J. Control Autom. Syst.* **8**(4) (2010) 735–740.

[13] Chen, X. and Zhai, G. Observation for the descriptor systems with disturbances. *Nonlinear dynam. syst. theory* **7**(2) (2007) 121–139.

[14] Krener, A. J. and Respondek, W. Nonlinear observers with linearizable error dynamics. *SIAM J Control Optim.* **23**(2) (1985) 197–216.

[15] Keller, H. Non-linear observer design by transformation into a generalized observer canonical form. *Int. J. Control* **46**(6) (1987) 1915–1930.

[16] Zeitz, M. The extended Luenberger observer for nonlinear systems. *Syst. Control Lett.* **9**(2) (1987) 149–156.

[17] Röbenack, K. Observer design for a class of nonlinear systems with non-full relative degree. *Nonlinear Dynamics and Systems Theory* **7**(4) (2007) 399–408.

[18] Thau, F. Observing the state of non-linear dynamic systems. *Int. J. Control* **17**(3) (1973) 471–479.

[19] Kou, S. R., Elliott, D. L. and Tarn, T. J. Exponential observers for nonlinear dynamic systems. *Inf. Control* **29**(3) (1975) 204–216.

[20] Arcak, M. and Kokotović, P. Nonlinear observers: a circle criterion design and robustness analysis. *Automatica* **37**(12) (2001) 1923–1930.

[21] Aloui, R. and Braiek, N. B. Optimal State Observer Design for Nonlinear Dynamical Systems. *Nonlinear Dynamics and Systems Theory* **12**(1) (2012) 37–48.

[22] Walcott, B., Corless, M. and Żak, S. Comparative study of non-linear state-observation techniques. *Int. J. Control* **45**(6) (1987) 2109–2132.

[23] Kaprielian, S. and Turi, J. An observer for a nonlinear descriptor system. In: *Decision and Control*, IEEE, 1992, 975–976.

[24] Boutayeb, M., Darouach, M., Rafaralahy, H. and Krzakala, G. Asymptotic observers for a class of nonlinear singular systems. In: *American Control Conference,* IEEE, 1994, 1440–1441.

[25] Zimmer, G. and Meier, J. On observing nonlinear descriptor systems. *Syst. Control Lett.* **32**(1) (1997) 43–48.

[26] Shields, D. N. Observer design and detection for nonlinear descriptor systems. *Int. J. Control* **67**(2) (1997) 153–168.

[27] Darouach, M. and Boutat-Baddas, L. Observers for a class of nonlinear singular systems. *IEEE Trans. Autom. Control* **53**(11) (2008) 2627–2633.

[28] Zerrougui, M., Boutat-Baddas, L. and Darouach, M. H_∞ observers design for a class of continuous time nonlinear singular systems. *J. Control Sci. Eng.* **2011** (2011) 1–8.

[29] Darouach, M., Boutat-Baddas, L. and Zerrougui, M. H_∞ observers design for a class of nonlinear singular systems. *Automatica* **47**(11) (2011) 2517–2525.

[30] Koenig, D. Observer design for unknown input nonlinear descriptor systems via convex optimization. *IEEE Trans. Autom. Control* **51**(6) (2006) 1047–1052.

[31] Yang, C., Zhang, Q., Chou, J.-H. and Yin, F. H_∞ Observer design for descriptor systems with solpe-restricted nonlinearities. *Asian J. Control* **14** (2012) 1–8.

[32] Lu, G. and Ho, D. W. Full-order and reduced-order observers for Lipschitz descriptor systems: the unified LMI approach. *IEEE Trans. Circuits Syst. Express Briefs* **53**(7) (2006) 563–567.

[33] Gahinet, P., Nemirovskii, A., Laub, A. J. and Chilali, M. The LMI control toolbox. In: *Decision and Control,* IEEE, 1994, 2038–2038.

[34] Kanev, S., Scherer, C., Verhaegen, M. and De Schutter, B. Robust output-feedback controller design via local BMI optimization. *Automatica* **40**(7) (2004) 1115–1127.

[35] Chen, W. and Saif, M. Unknown input observer design for a class of nonlinear systems: An LMI approach. In: *American Control Conference,* IEEE, 2006, 834–838.

[36] Gupta, M. K., Tomar, N. K. and Bhaumik, S. Detectability and observer design for linear descriptor system. In: *22nd Mediterranean Conference on Control and Automation,* IEEE, 2014, 1094–1098.