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The Duffing–Van der Pol Equation: Metamorphoses of Resonance Curves

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Abstract: We study dynamics of the Duffing–Van der Pol driven oscillator. Periodic steady-state solutions of the corresponding equation are determined within the Krylov-Bogoliubov-Mitropolsky approach to yield dependence of amplitude on forcing frequency as an implicit function, referred to as resonance curve or amplitude profile. Equations for singular points of resonance curves are solved exactly. We investigate metamorphoses of the computed amplitude profiles induced by changes of control parameters near singular points of these curves since qualitative changes of dynamics occur in neighbourhoods of singular points. More exactly, conditions for birth of resonances as well as for attractor crises are found. Bifurcation diagrams are computed to show good agreement with theoretical analysis.

Keywords: *oscillators; resonance curves; singular points.*

Mathematics Subject Classification (2010): 34C15, 70K30, 37G10.

1 Introduction

Nonlinear oscillators have many important applications in various areas of science and engineering [1, 2]. In this paper we investigate Duffing–Van der Pol oscillator which has been extensively studied due to potential applications in physics, chemistry, biology, engineering, electronics, and many other fields, see [3, 4] and references therein.

The periodically forced Duffing – Van der Pol oscillator (DvdP) is written as:

$$\frac{d^2x}{dt^2} - \left(b - cx^2\right)\frac{dx}{dt} + ax + dx^3 = f\cos\omega t. \tag{1}$$

There are three main cases of the Duffing potential $V(x) = \frac{1}{2}ax^2 + \frac{1}{4}dx^4$: (i) single well ($a > 0$, $d > 0$), (ii) double well ($a < 0$, $d > 0$), and (iii) double hump ($a > 0$, $d < 0$).

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In the present paper we consider cases (i), (iii) and it is thus assumed that $a, b, c, f, \omega > 0$ while d is arbitrary.

There are many numerical and analytical methods to solve nonlinear oscillator equations. Analytical methods lead to complicated (approximate) formulae and analysis of such solutions is difficult. In the present work we analyse such approximate analytic solutions using the theory of algebraic curves. More exactly, we determine periodic steady-state solutions of the DvdP equation within the Krylov-Bogoliubov-Mitropolsky approach to get dependence of amplitude on forcing frequency as an implicit function, referred to as resonance curve or amplitude profile. We investigate metamorphoses of the computed amplitude profiles induced by changes of control parameters near singular points of these curves since qualitative changes of dynamics occur in neighbourhoods of singular points, see [5] and references therein. We have learned recently that idea to use Implicit Function Theorem in this context was put forward in [6].

The paper is organized as follows. In Section 2 the Van der Pol – Duffing equation is written in nondimensional form and implicit equation for resonance curves $L(\Omega,\ A) = 0$ is derived via the Krylov-Bogoliubov-Mitropolski (KBM) approach. In Section 3 theory of algebraic curves is applied to compute singular points on the amplitude profiles. Equations for singular points are solved and in Section 4 the solutions are used to study birth of resonances as well as attractor crises. We summarize our results in the last Section 5.

2 Nonlinear Resonances via KBM Method

We apply the Krylov-Bogoliubov-Mitropolsky (KBM) perturbation approach [7, 8] to equation (1). Substituting into (1):

$$x = \sqrt{\tfrac{b}{c}}z,\ t = \tfrac{1}{\sqrt{a}}\tau,\ b = \sqrt{a}\mu,\ \omega = \sqrt{a}\Omega, \tag{2}$$

we get the DvdP in nondimensional form:

$$\begin{gathered} \frac{d^2 z}{d\tau^2} - \mu\left(1 - z^2\right)\frac{dz}{d\tau} + z + \lambda z^3 = G\cos\left(\Omega\tau\right), \\ \mu,\ G,\ \Omega > 0,\ \lambda - \text{arbitrary}, \end{gathered} \tag{3}$$

where $\lambda = \frac{bd}{ac}$, $G = \frac{f}{a}\sqrt{\frac{c}{b}}$.

The equation (3) is written in the following form:

$$\frac{d^2 z}{d\tau^2} + \Omega^2 z + \varepsilon\left(\sigma z + g\right) = 0, \tag{4}$$

where

$$\left.\begin{aligned} g =&\ -\Theta_0^2 z - \mu_0\frac{dz}{d\tau} + \mu_0 z^2\frac{dz}{d\tau} + \alpha_0 z + \lambda_0 z^3 - G_0\cos\left(\Omega\tau\right), \\ \Theta_0^2 =&\ \frac{\Theta^2}{\varepsilon},\ \mu_0 = \frac{\mu}{\varepsilon},\ \alpha_0 = \frac{1}{\varepsilon},\ \lambda_0 = \frac{\lambda}{\varepsilon},\ G_0 = \frac{G}{\varepsilon},\ \varepsilon\sigma = \Theta^2 - \Omega^2. \end{aligned}\right\} \tag{5}$$

According to the KBM method we assume for small nonzero ε that the solution for $1:1$ resonance can be written as:

$$z = A\left(\tau\right)\cos\left(\Omega\tau + \varphi\left(\tau\right)\right) + \varepsilon z_1\left(A, \varphi, \tau\right) + \ldots \tag{6}$$

with slowly varying amplitude and phase:

$$\frac{dA}{d\tau} = \varepsilon M_1(A,\varphi) + \dots\ , \tag{7}$$

$$\frac{d\varphi}{d\tau} = \varepsilon N_1(A,\varphi) + \dots\ . \tag{8}$$

Computing now derivatives of z from equations (6), (7), (8) and substituting to equations (4), (5), eliminating secular terms and demanding $M_1 = 0$, $N_1 = 0$ we obtain the following equations for the amplitude and phase of steady states:

$$A^2\left(\mu^2\Omega^2\left(1-\frac{1}{4}A^2\right)^2 + \left(1+\frac{3}{4}\lambda A^2 - \Omega^2\right)^2\right) = G^2. \tag{9}$$

3 General Properties of the Amplitude Profile $A(\Omega)$

After introducing new variables, $\Omega^2 = X$, $A^2 = Y$, the equation (9) defining the amplitude profile reads $L(X,Y) = 0$, where:

$$L(X,Y;\lambda,\mu,G) \stackrel{df}{=} \mu^2 XY\left(1-\tfrac{1}{4}Y\right)^2 + Y\left(1+\tfrac{3}{4}\lambda Y - X\right)^2 - G^2. \tag{10}$$

Singular points of $L(X,Y)$ are computed from equations [9]:

$$L = 0, \tag{11a}$$

$$\tfrac{\partial L}{\partial X} = 0, \tag{11b}$$

$$\tfrac{\partial L}{\partial Y} = 0. \tag{11c}$$

There are several classes of physically acceptable solutions of equations (11), i.e. solutions fulfilling conditions: $X > 0, Y > 0, \mu > 0$.

1. Firstly, if we fix values of X, μ, then the solution reads:

$$\left.\begin{array}{l} 3mY^3 - 36mY^2 + (144m - 64 + 256X)Y - 192m + 256 - 512X = 0, \\ \lambda = \frac{16m-8mY+mY^2-32+32X}{24Y},\ g = -\frac{mY}{48}\left(XY^2 - 16X + 8Y - 16 - Y^2\right), \end{array}\right\} \tag{12}$$

   where $m \equiv \mu^2$, $g \equiv G^2$ and a special solution for the unforced case ($G = 0$) is:

$$Y = 4,\ \lambda = -\tfrac{1}{3} + \tfrac{1}{3}X,\ G = 0 \quad (\mu - \text{arbitrary}). \tag{13}$$

2. In the second case, when λ, μ are fixed, we obtain the special solution (13) again as well as equation for Y:

$$\begin{array}{l} A_3Y^3 + A_2Y^2 + A_1Y + A_0 = 0, \\ A_3 = 5\mu^2, \quad A_2 = -192\lambda - 44\mu^2, \\ A_1 = 112\mu^2 + 384\lambda - 192, \quad A_0 = 256 - 64\mu^2, \end{array} \tag{14}$$

   equation for Z:

$$\begin{array}{l} Z^2 = B_6\mu^6 + B_4\mu^4 + B_2\mu^2 + B_0, \\ B_6 = -4(Y-4)^2, \\ B_4 = (39\lambda + 20)Y^2 - (144\lambda + 152)Y + 336\lambda + 416, \\ B_2 = \left(2232\lambda^2 + 660\lambda\right)Y^2 - \left(6048\lambda^2 + 192\lambda - 480\right)Y \\ +2304\lambda^2 - 1344\lambda - 640, \\ B_0 = 2304\lambda^2\left(3\lambda Y^2 + (-6\lambda + 3)Y - 4\right), \end{array} \tag{15}$$

and, finally, after solving equations (14), (15), we can compute X, G:

$$X = -\tfrac{1}{32}\mu^2 Y^2 + \tfrac{1}{4}\left(\mu^2 + 3\lambda\right) Y + 1 - \tfrac{1}{2}\mu^2, \quad G = \tfrac{1}{10\mu} Z. \tag{16}$$

3. On the other hand, if we fix values of $X,\ Y$, then we get:

$$\begin{aligned} G^2 &= \tfrac{4}{9}\left(YX + 4X - Y + 4\right)\left(4YX - 8X - Y + 4\right)\tfrac{Y}{(Y-4)^2}, \\ \mu^2 &= -\tfrac{64}{3}\tfrac{4YX-8X-Y+4}{(Y-4)^3}, \\ \lambda &= -\tfrac{4}{9}\tfrac{5YX+Y-4X-4}{Y(Y-4)}. \end{aligned} \tag{17}$$

Necessary conditions: $\mu^2 > 0,\ G^2 > 0$ lead to:

Y	X	λ
$0 < Y < 2$	$0 < X < \frac{Y-4}{4(Y-2)}$	$\mathrm{sign}(\lambda) = \mathrm{sign}\,(s)$
$2 < Y < 4$	$0 < X$	$\mathrm{sign}(\lambda) = \mathrm{sign}\,(s)$
$4 < Y$	$0 < X < \frac{Y-4}{4(Y-2)}$	$\lambda < 0$

(18)

where $s = X + \frac{Y-4}{5Y-4}$.

4 Computational Results

In this Section singular points of amplitude profiles – solutions of equations (10), (11) – are studied. In the first Subsection we shall consider the special solution (13), corresponding to the amplitude profile with isolated point. On the other hand, this solution describes birth of a limit cycle in the unforced case ($G = 0$). In Subsection 4.2 we consider the second class of solutions defined by equations (14), (15), (16). In this case metamorphosis of the resonance corresponds to change of attractor size (crisis).

4.1 Birth of resonances from isolated points

The solution (13) yields for $X = \Omega^2 = 4$ the following values: $Y = A^2 = 4$, $\lambda = 1$ and we choose $\mu = 0.5$. Since $G = 0$, this solution corresponds to a resonance in the unforced case living exactly at this critical value $X = \Omega^2 = 4$. On the plot $L\left(\Omega^2, A^2\right) = 0$ this resonance is represented by an isolated point. For increasing values of G this point gives rise to growing ovals and thus the corresponding resonances (limit cycles) exist in a broader and broader range of Ω. In Fig. 1 implicit plots $L\left(\Omega^2, A^2; \lambda, \mu, G\right) = 0$ are shown for $\mu = 0.5$, $\lambda = 1$ and $G = 0.01,\ 0.10,\ 0.20,\ 0.50$.

We have computed bifurcation diagrams for $\mu = 0.5$, $\lambda = 1$ and $G = 0.01,\ 0.05, 0.10, 0.20$ to show birth and growth of the resonance. This scenario is shown in Fig. 2. It can be seen that for decreasing values of G the resonance shrinks around $\Omega \cong \sqrt{4} = 2$ with amplitude $A \cong \sqrt{4} = 2$, in good agreement with (13). More exactly, the resonance appears at $\Omega = 1.92$ rather than $\Omega = 2$ thus providing estimate of the KBM method's error.

4.2 Metamorphoses of resonances in the neighbourhood of self-intersection

Let us consider the second class of solutions described in Section 3. For example, for $\lambda = 1$, we compute from equation (13) $X = 4$, $Y = 4$, $G_{cr}^{(1)} = 0$ and we choose $\mu = 0.5$. Now for $\lambda = 1$ and $\mu = 0.5$ we get from equations (14), (15), (16) one physical solution

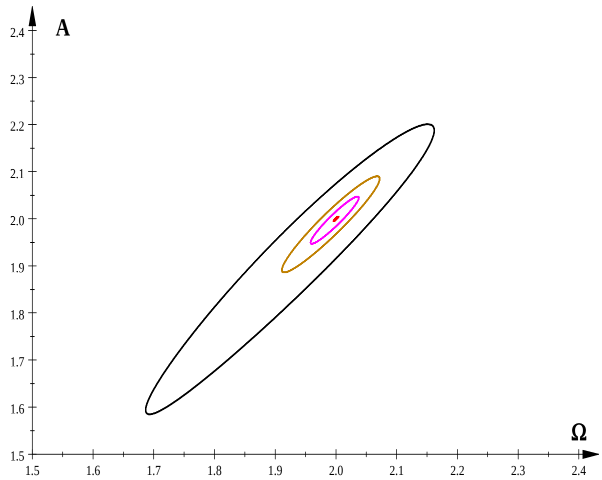

Figure 1: Implicit plots $L\left(\Omega^2, A^2; \lambda, \mu, G\right) = 0$ where $\mu = 0.5$, $\lambda = 1$ and $G = 0.01$ (red line), 0.10 (magenta), 0.20 (sienna), 0.50 (black).

$X = 2.289\,217\,833\,44$, $Y = 1.770\,724\,499\,36$, $G_{cr}^{(2)} = 0.563\,412\,500\,579$. The first solution represents isolated point and describes birth of a resonance in the unforced case ($G = 0$), while the second solution corresponds to a self-intersection of the resonance curve. Both solutions are shown in Fig. 3.

Now, for $G = 0$ and increasing we have to do with scenario described in Subsection 4.1. Black curve in Fig. 3 corresponds to $G = 0.5 < G_{cr}^{(2)}$. The attractor is shown in Fig. 4 (left figure). There is only one stable solution corresponding to black oval in Fig. 3, the lower black branch being unstable. Then for $G = 1.5 > G_{cr}^{(2)}$ (the corresponding amplitude profile is represented by green line in Fig. 3) the attractor increases its size, cf. Fig. 4 where bifurcation diagrams are shown. Let us add here that for smaller values of G but greater than $G_{cr}^{(2)}$, say for $G = 0.75$, the resonance is stable on a smaller interval of Ω.

5 Summary and Discussion

In this work we have studied dynamics of the periodically forced Duffing-van der Pol equation. Steady-state nonlinear resonances have been determined within the Krylov-Bogoliubov-Mitropolsky approach. We have applied theory of algebraic curves [9] to determine singular points of the computed resonance profiles since qualitative changes of dynamics occur in neighbourhood of singular points [5]. Resonance curves (9) have two classes of singular points: isolated points as well as self-intersections. We have found that a family of resonances (limit cycles) of the unforced DvdP equation ($G = 0$) is born when the amplitude profile of the forced equation, computed according to the KBM method has singular point for $G = 0$, see Subsection 4.1. It is possible to control value of Ω at which the resonance appears for $G > 0$, see equation (13). For growing values of forcing amplitude G stability range in the Ω space is growing as well. On the other hand, in the neighbourhood of self-intersections of resonance curves there are crises – the corresponding attractor changes its size. We have computed several bifurcation diagrams

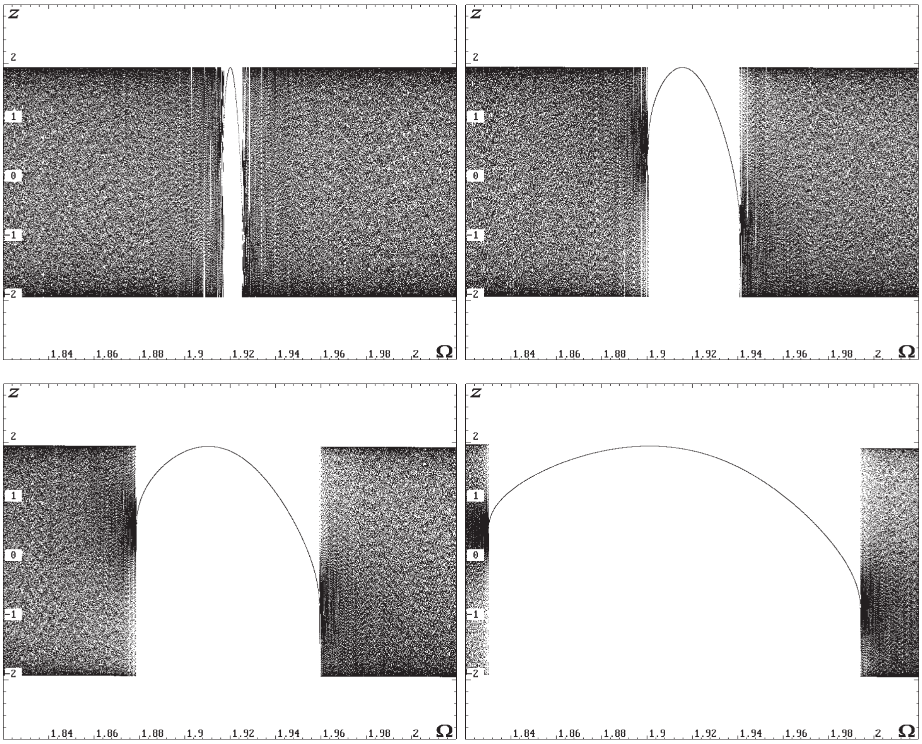

Figure 2: Bifurcation diagrams, $\mu = 0.5$, $\lambda = 1$ and $G = 0.01$, $G = 0.05$, $G = 0.10$, $G = 0.20$ (left to right, top to bottom).

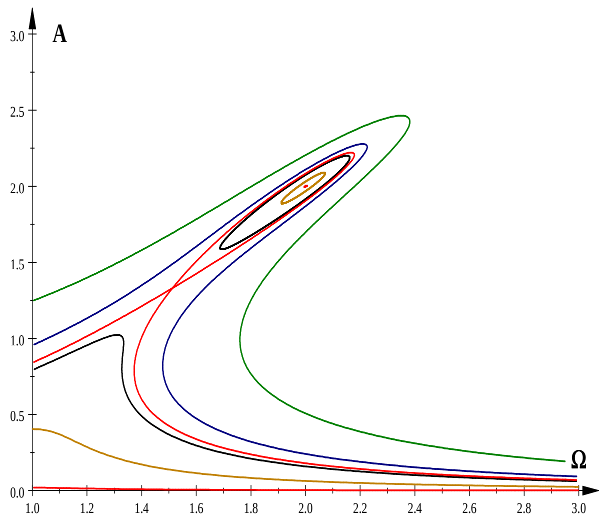

Figure 3: Implicit plots $L\left(\Omega^2, A^2; \lambda, \mu, G\right) = 0$ where $\mu = 0.5$, $\lambda = 1$ and $G = 0.01$ (red), $G = 0.20$ (sienna), $G = 0.50$ (black), $G = G_{cr}^{(2)}$ (red), $G = 0.75$ (blue), $G = 1.50$ (green).

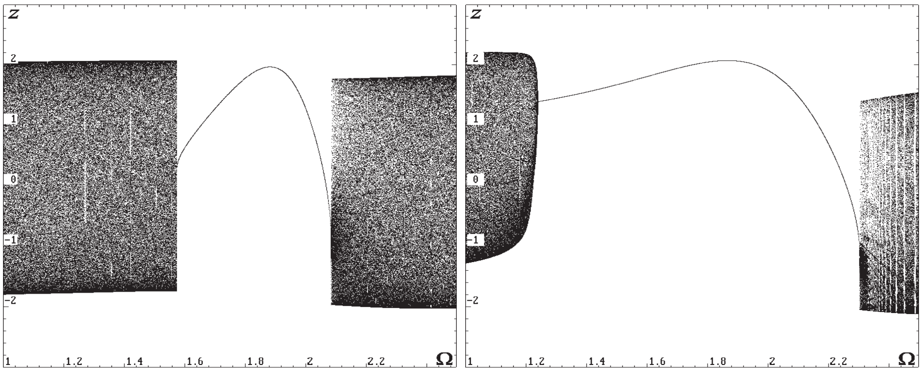

Figure 4: Bifurcation diagrams, $\mu = 0.5$, $\lambda = 1$, and $G = 0.5$ (left figure), $G = 1.5$ (right figure).

documenting qualitative changes of dynamics in the neighbourhood of metamorphoses of resonance curves.

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