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A New Synchronization Scheme for General $3D$ Quadratic Chaotic Systems in Discrete-Time

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Abstract: In this paper, a new general chaos synchronization scheme is proposed for coupled arbitrary 3-D quadratic chaotic dynamical systems in discrete-time. The proposed synchronization method, based on nonlinear controllers and Lyapunov stability theory, is theoretically rigorous. The derived synchronization criterion can be also applicable to a large class of discrete-time chaotic systems. Our control scheme is used to illustrate complete synchronization between the three-dimensional hyperchaotic discrete-time Rössler and Wang systems. Moreover numerical simulations are used to show the effectiveness and the feasibility of the proposed synchronization scheme.

Keywords: *quadratic systems; chaos synchronization; control scheme; discrete-time; Lyapunov stability.*

Mathematics Subject Classification (2010): 93C10, 93C55, 93D05.

1 Introduction

Over the last two decade, many scholars have proposed various control schemes in chaos synchronization [1–6], but the most of works have concentrated on continuous-time rather than discrete-time chaotic systems. In practice, discrete-time chaotic systems play a more important role than their continuous counterparts [7]. In fact, many mathematical models of physical processes [8], biological phenomena [10], chemical reactions [9] and economic systems [11] were defined using discrete-time chaotic systems. Many 3D chaotic and hyperchaotic dynamical systems in discrete-time are founded such as Baier-Klain map [12], Hitzl-Zele map [13], Stefanski map [14], Wang system [15], discrete-time Rössler system [16] and Grassi-Miller map [18], etc.

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Recently, synchronization in discrete-time chaotic systems attracts more and more attention in many areas of science and technology, and has been extensively studied, due to its potential applications in secure communication [19,20,22,23]. Until now, a variety of approaches have been proposed for the synchronization of chaotic systems in discrete-time [24–27] and different types of chaos synchronization have been presented [28–32].

In this paper, using new controller law and Lyapunov stability theory, a general method is proposed to guarantee global synchronization for a special class of chaotic maps. The aim of this paper is to develop a simple criterion for the synchronization between two arbitrary 3D quadratic chaotic systems in discrete-time. In order to verify the effectiveness of the new approach, the proposed scheme is applied between two 3D hyperchaotic maps: the discrete-time Rössler system and the 3D Wang system.

The rest of this paper is organized as follows. In Section 2, a description of the chaotic systems addressed in this paper is provided. In Section 3, a new chaos synchronization approach in discrete-time is introduced and new synchronization criterion is derived. In Section 4, the proposed synchronization scheme is applied to some typical 3D discrete-time hyperchaotic systems and numerical simulations are used to verify the effectiveness of the new approach. In Section 5, conclusion follows.

2 Description of Drive-response Systems

Consider the drive chaotic system in the form of

$$x_i(k+1) = \sum_{j=1}^{3} a_{ij} x_j(k) + \sum_{q=1}^{3}\sum_{p=1}^{3} \alpha_{pq}^{(i)} x_p(k)\, x_q(k) + c_i, \quad 1 \le i \le 3, \tag{1}$$

where $X(k) = (x_i(k))_{1 \le i \le 3} \in \mathbb{R}^3$ is the state vector of the drive system, $(a_{ij}) \in \mathbb{R}^{3\times 3}$, $\left(\alpha_{pq}^{(i)}\right) \in \mathbb{R}^{3\times 3}$ $(i = 1, 2, 3)$, and $(c_i)_{1 \le i \le 3}$ are real numbers.

As the response chaotic system, we consider the following system

$$y_i(k+1) = \sum_{j=1}^{3} b_{ij} y_j(k) + \sum_{q=1}^{3}\sum_{p=1}^{3} \beta_{pq}^{(i)} y_p(k)\, y_q(k) + d_i + u_i, \quad 1 \le i \le 3, \tag{2}$$

where $Y(k) = (y_i(k))_{1 \le i \le 3} \in \mathbb{R}^3$ is the state vector of the response system, $(b_{ij}) \in \mathbb{R}^{3\times 3}$, $\left(\beta_{pq}^{(i)}\right) \in \mathbb{R}^{3\times 3}$ $(i = 1, 2, 3)$, $(d_i)_{1 \le i \le 3}$ are real numbers and $U = (u_i)_{1 \le i \le 3} \in \mathbb{R}^3$ is a vector controller to be determined.

Remark 2.1 3D Quadratic chaotic maps can be written under the form of (1) such as 3D Hénon-like map, Baier-Klein map, 3D generalized Hénon map, Stefanski map, discrete-time Rössler system and Wang system, etc.

Our aim is to realize synchronization between the drive system (1) and the response system (2) for arbitrary constants a_{ij}, b_{ij}, $\alpha_{pq}^{(i)}$, $\beta_{pq}^{(i)}$, c_i and d_i $(i, p, q = 1, 2, 3)$, and to determine the controllers u_i $(1 \le i \le 3)$, which stabilize the synchronization errors

$$e_i(k) = y_i(k) - x_i(k), \quad 1 \le i \le 3, \tag{3}$$

then the aim of synchronization is to make $\lim_{k\to\infty} e_i(k) = 0$, $(i = 1, 2, 3)$.

3 New Chaos Synchronization Scheme in Discrete-time

The synchronization errors between the drive system (1) and the response system (2), can be derived as follows

$$e_i(k+1) = \sum_{j=1}^{3} b_{ij} e_j(k) + R_i + u_i, \quad 1 \le i \le 3, \tag{4}$$

where

$$\begin{aligned} R_i &= \sum_{j=1}^{3} (b_{ij} - a_{ij}) x_j(k) + \sum_{q=1}^{3}\sum_{p=1}^{3} \beta_{pq}^{(i)} y_p(k) y_q(k) \\ &\quad - \sum_{q=1}^{3}\sum_{p=1}^{3} \alpha_{pq}^{(i)} x_p(k) x_q(k) + d_i - c_i, \quad 1 \le i \le 3. \end{aligned} \tag{5}$$

To achieve synchronization between systems (1) and (2), we choose the vector controller $U = (u_i)_{1\le i\le 3}$ as follows

$$\begin{aligned} u_1 &= l_1 e_1(k) + (b_{22} - b_{12} + l_2) e_2(k) - (b_{13} + b_{33} + l_3) e_3(k) - R_1, \\ u_2 &= -(b_{21} + b_{11} + l_1) e_1(k) + l_2 e_2(k) + (b_{33} - b_{23} + l_3) e_3(k) - R_2, \\ u_3 &= (b_{11} - b_{31} + l_1) e_1(k) - b_{32} e_2(k) + (b_{33} + 2l_3) e_3(k) - R_3, \end{aligned} \tag{6}$$

where $(l_i)_{1\le i\le 3}$ are control constants to be determined later. By substituting Eq. (6) into Eq. (4), the synchronization errors can be written as

$$\begin{aligned} e_1(k+1) &= (b_{11} + l_1) e_1(k) + (b_{22} + l_2) e_2(k) - (b_{33} + l_3) e_3(k), \\ e_2(k+1) &= -(b_{11} + l_1) e_1(k) + (b_{22} + l_2) e_2(k) + (b_{33} + l_3) e_3(k), \\ e_3(k+1) &= (b_{11} + l_1) e_1(k) + 2(b_{33} + l_3) e_3(k). \end{aligned} \tag{7}$$

Now, we have the following result.

Theorem 3.1 *If the control constants $(l_i)_{1\le i\le 3}$ are chosen such that*

$$\begin{cases} -b_{11} - \frac{1}{\sqrt{3}} < l_1 < -b_{11} + \frac{1}{\sqrt{3}}, \\ -b_{22} - \frac{1}{\sqrt{2}} < l_2 < -b_{22} + \frac{1}{\sqrt{2}}, \\ -b_{33} - \frac{1}{\sqrt{6}} < l_3 < \frac{1}{\sqrt{6}}, \end{cases} \tag{8}$$

then the drive system (1) and the response system (2) are globally synchronized under the controller law (6).

Proof. Let us consider the following quadratic Lyapunov function

$$V(e(k)) = \sum_{i=1}^{3} e_i^2(k), \tag{9}$$

then we obtain

$$\begin{aligned}
\Delta V\left(e(k)\right) &= V\left(e(k+1)\right) - V\left(e(k)\right) \\
&= \sum_{i=1}^{3} e_i^2\left(k+1\right) - \sum_{i=1}^{3} e_i^2\left(k\right) \\
&= \left(3\left(b_{11}+l_1\right)^2 - 1\right) e_1^2\left(k\right) + \left(2\left(b_{22}+l_2\right)^2 - 1\right) e_2^2\left(k\right) \\
&\quad + \left(6\left(b_{33}+l_3\right)^2 - 1\right) e_3^2\left(k\right) \\
&\quad + \left[\left(b_{11}+l_1\right)\left(b_{22}+l_2\right) - \left(b_{11}+l_1\right)\left(b_{22}+l_2\right)\right] e_1\left(k\right) e_2\left(k\right) \\
&\quad + \left[-\left(b_{11}+l_1\right)\left(b_{33}+l_3\right) - \left(b_{11}+l_1\right)\left(b_{33}+l_3\right)\right. \\
&\quad \left. +2\left(b_{11}+l_1\right)\left(b_{33}+l_3\right)\right] e_1\left(k\right) e_3\left(k\right) \\
&\quad + \left[-\left(b_{22}+l_2\right)\left(b_{33}+l_3\right) + \left(b_{22}+l_2\right)\left(b_{33}+l_3\right)\right] e_2\left(k\right) e_3\left(k\right) \\
&= \left(3\left(b_{11}+l_1\right)^2 - 1\right) e_1^2\left(k\right) + \left(2\left(b_{22}+l_2\right)^2 - 1\right) e_2^2\left(k\right) \\
&\quad + \left(6\left(b_{33}+l_3\right)^2 - 1\right) e_3^2\left(k\right),
\end{aligned}$$

and by using (8), we get: $\Delta V\left(e(k)\right) < 0$.

Thus, from the Lyapunov stability theory, it is immediate that $\lim_{k\to\infty} e_i(k) = 0$, $(i = 1, 2, 3)$. Therefore, the systems (1) and (2) are globally synchronized.

4 Illustrative Example

In this example, we consider the discrete-time Rössler system as the drive system and the controlled Wang system as the response system. The discrete-time Rössler system [16], is described by

$$\begin{aligned}
x_1\left(k+1\right) &= \alpha x_1\left(k\right)\left(1 - x_1\left(k\right)\right) - \beta\left(x_3\left(k\right) + \gamma\right)\left(1 - 2x_2\left(k\right)\right), \\
x_2\left(k+1\right) &= \delta x_2\left(k\right)\left(1 - x_2\left(k\right)\right) + \varsigma x_3\left(k\right), \\
x_3\left(k+1\right) &= \eta\left(\left(x_3\left(k\right) + \gamma\right)\left(1 - 2x_2\left(k\right)\right) - 1\right)\left(1 - \theta x_1\left(k\right)\right),
\end{aligned} \tag{10}$$

where $\alpha = 3.8$, $\beta = 0.05$, $\gamma = 0.35$, $\delta = 3.78$, $\varsigma = 0.2$, $\eta = 0.1$, $\theta = 1.9$. The hyperchaotic attractor of the 3D discrete-time Rössler system is shown in Fig. 1.

The controlled Wang system can be described as

$$\begin{aligned}
y_1\left(k+1\right) &= a_3 y_2\left(k\right) + \left(a_4 + 1\right) y_1\left(k\right) + u_1, \\
y_2\left(k+1\right) &= a_1 y_1\left(k\right) + y_2\left(k\right) + a_2 y_3\left(k\right) + u_2, \\
y_3\left(k+1\right) &= \left(a_7 + 1\right) y_3\left(k\right) + a_6 y_2\left(k\right) y_3\left(k\right) + a_5 + u_3,
\end{aligned} \tag{11}$$

where $U = \left(u_1,\ u_2,\ u_3\right)^T$ is the vector controller. The 3D hyperchaotic Wang system (i.e., the system (11) with $u_1 = 0$, $u_2 = 0$, $u_3 = 0$) is chaotic when the parameter values are taken as $(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (-1.9, 0.2, 0.5, -2.3, 2, -0.6, -1.9)$ [15]. The hyperchaotic attractor of the 3D Wang system is shown in Fig. 2. To achieve global synchronization between the discrete-time Rössler system and the controlled Wang system, according to our approach presented in Section 2, the vector controller can be

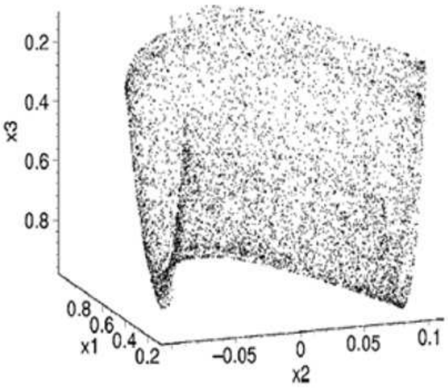

Figure 1: The hyperchaotic attractor of the discrete-time Rossler system.

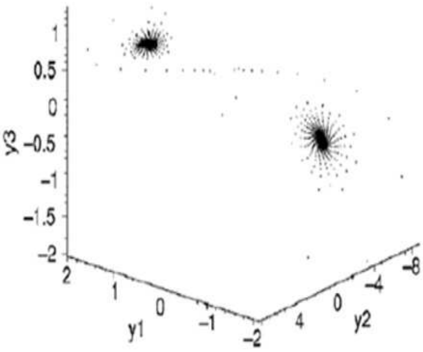

Figure 2: Hyperchaotic attractor of Wang system when $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, \delta) = (-1.9, 0.2, 0.5, -2.3, 2, -0.6, -1.9, 1)$.

constructed as follows

$$\begin{aligned} u_1 &= l_1 e_1(k) + (1 - a_3 + l_2) e_2(k) - (a_7 + 1 + l_3) e_3(k) - R_1, \\ u_2 &= -(a_1 + a_4 + 1 + l_1) e_1(k) + l_2 e_2(k) + (a_7 + 1 - a_2 + l_3) e_3(k) - R_2, \\ u_3 &= (a_4 + 1 + l_1) e_1(k) + (a_7 + 1 + 2l_3) e_3(k) - R_3, \end{aligned} \tag{12}$$

where the control constants $(l_i)_{1 \le i \le 3}$ are chosen as follows

$$\begin{cases} -a_4 - 1 - \frac{1}{\sqrt{3}} < l_1 < -a_4 - 1 + \frac{1}{\sqrt{3}}, \\ -1 - \frac{1}{\sqrt{2}} < l_2 < -1 + \frac{1}{\sqrt{2}}, \\ -a_7 - 1 - \frac{1}{\sqrt{6}} < l_3 < -a_7 - 1 - \frac{1}{\sqrt{6}} \end{cases} \tag{13}$$

and

$$R_i = L_i + N_i, \quad i = 1, 2, 3, \tag{14}$$

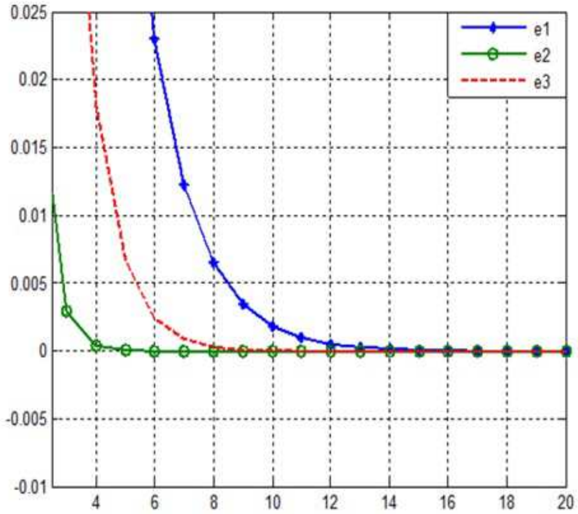

Figure 3: Time evolution of synchronization errors between the drive system (10) and the response system (11).

where

$$\begin{aligned} L_1 &= (a_4+1-\alpha)\,x_1(k)+(a_3-\beta\gamma 2)\,x_2(k)+\beta x_3(k)+\beta\gamma, \\ L_2 &= a_1x_1(k)+(1-\beta\gamma 2)\,x_2(k)+(a_2-\varsigma)\,x_3(k), \\ L_3 &= -\theta(1-\eta\gamma)\,x_1(k)+2\eta\gamma x_2(k)+(a_7+1-\eta)\,x_3(k)+a_5-\eta\gamma+1 \end{aligned} \tag{15}$$

and

$$\begin{aligned} N_1 &= \alpha x_1^2(k)-2\beta x_3(k)\,x_2(k), \\ N_2 &= \delta x_2^2(k), \\ N_3 &= a_6y_2(k)\,y_3(k)-2\eta\gamma\theta x_1(k)\,x_2(k)+\eta\theta x_1(k)\,x_3(k) \\ &\quad +2\eta x_2(k)\,x_3(k)-2\eta\theta x_1(k)\,x_2(k)\,x_3(k). \end{aligned} \tag{16}$$

It is easy to show that all conditions of Theorem 3.1 are satisfied. Therefore, the drive system (10) and the response system (11) are globally synchronized.

Using controllers (12), the error functions can be described as:

$$\begin{aligned} e_1(k+1) &= (a_4+1+l_1)\,e_1(k)+(1+l_2)\,e_2(k)-(a_7+1+l_3)\,e_3(k), \\ e_2(k+1) &= -(a_4+1+l_1)\,e_1(k)+(1+l_2)\,e_2(k)+(a_7+1+l_3)\,e_3(k), \\ e_3(k+1) &= (a_4+1+l_1)\,e_1(k)+2\,(a_7+1+l_3)\,e_3(k). \end{aligned} \tag{17}$$

Corollary 4.1 *For two coupled systems: the hyperchaotic discrete-time Rössler system and the hyperchaotic Wang system, if we choose the control constants $(l_i)_{1\leq i\leq 3}$ such that: $l_1=1$, $l_2=-\frac{1}{2}$ and $l_3=0.8$. Then, they are globally synchronized, see Fig. 3.*

5 Conclusion

In this paper, a new control scheme has been designed to achieve synchronization between 3-D quadratic drive-response chaotic systems in discrete-time. Based on nonlinear controllers and Lyapunov stability theory, a synchronization criterion has been obtained and new conditions have been derived. It was shown that the proposed controllers guarantee the asymptotic convergence to zero of the errors between the drive and the response systems. Finally, numerical example and computer simulations were used to verify the effectiveness of the proposed approach.

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